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Estimates of the Convergence Rate in the Generalized Rényi Theorem with a Structural Digamma Distribution Using Zeta Metrics

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Abstract: This paper considers a generalization of the Rényi theorem to the case of a structural distribution with a scale parameter. In terms of the zeta metric, some estimates of the convergence rate in the generalized Rényi theorem are obtained when the structural mixed Poisson distribution of the summation index is a scale mixture of the generalized gamma distribution. Estimates of the convergence rate for the structural digamma distribution are given as a special case. The paper extends the results previously obtained for the generalized gamma distribution.

Keywords: generalized Rényi theorem; estimates of the convergence rate; generalized gamma distribution; digamma distribution; zeta metrics

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1. Introduction

Beta and gamma classes of distributions traditionally play an important role in modeling real processes with the use of probability theory techniques. The properties of infinite divisibility and stability inherent to some types of generalized gamma and beta distributions make it possible to use them as adequate asymptotic approximations in various popular limit theorems.

In 1925, the Italian researcher L. Amoroso, who studied the problems of dynamic equilibrium, considered the distribution [1], a special form of which is usually called the generalized gamma distribution $GG(\nu, p, \delta)$ with the density

$$g_{\nu,p,\delta}(x) = \frac{|\nu| x^{\nu p-1} e^{-(x/\delta)^{\nu}}}{\delta^{\nu p}\Gamma(p)}, \quad \nu \neq 0, \quad p > 0, \quad \delta > 0, \quad x > 0. \tag{1}$$

This distribution, along with its special cases, has found wide application in the study of many applied problems.

A natural generalization of distributions from the gamma class is the digamma distribution proposed in [2,3].

Definition 1. *A random variable $\rho_\delta \equiv \rho_{r,\nu,p,q,\delta}$ has a digamma distribution $DiG(r, \nu, p, q, \delta)$ with the characteristic index $r \in \mathbb{R}$ and the parameters of shape $\nu \neq 0$, concentration $p, q > 0$ and scale $\delta > 0$, if its Mellin transform is*

$$\mathcal{M}_{\rho_\delta}(z) = \frac{\delta^z \Gamma(p + z/\nu)\Gamma(q - rz/\nu)}{\Gamma(p)\Gamma(q)}, \quad p + \frac{\mathrm{Re}(z)}{\nu} > 0, \quad q - \frac{r\mathrm{Re}(z)}{\nu} > 0. \tag{2}$$

In addition to the generalized gamma distribution (1), the special cases of digamma distribution (2) also include [2,3] the generalized beta distribution of the second kind (McDonald distribution) [4] used primarily in econometrics and regression analysis, as well as the gamma-exponential distribution [5,6], proposed as a link between the gamma and beta classes.

The possibility of representing the digamma distribution as a scale mixture of generalized gamma laws

$$\rho_\delta \stackrel{d}{=} \delta \left(\frac{\lambda}{\mu^r} \right)^{1/\nu}, \tag{3}$$

where independent random variables λ and μ have gamma distributions $GG(1, p, 1)$ and $GG(1, q, 1)$, respectively, makes it possible to use the digamma distribution in the analysis of balance models [7,8], particulary for studying the asymptotic properties of the integral balance index [9].

One of the first and most important limit theorems related to the gamma family is the Rényi theorem [10] about the convergence of random sums with a geometric summation index to the standard exponential distribution. The classical Rényi theorem has a number of generalizations. In particular, it can be shown [9] that (2) can arise as a limiting distribution in the case when a mixed Poisson index is used instead of a geometric one.

The main accompanying task in the study of the asymptotic behavior of random sums is to estimate the rate of convergence to the limit law [11–15]. In particular, in Refs. [16–18], the estimation of the convergence rate in the Rényi theorem and some of its generalizations was carried out using the zeta metric, which was proposed in 1976 by V.M. Zolotarev [19]. The introduction of the zeta metric was motivated by the following considerations. Since in the presence of convergence there is always a question about its rate, it is necessary to have some metric that can be used to evaluate the accuracy of the approximation. When considering a weak convergence, it would also be desirable to have some "natural" metric. However, the class of continuous bounded functions present in the definition of weak convergence is too wide for presenting some convenient metric. For this reason, Zolotarev proposed to narrow the consideration to the class of Lipschitz differentiable bounded functions.

The paper proves a generalization of the Rényi theorem to the case of structural distributions that have a scale parameter. The results extend the approaches of [16–18], proposed for generalized gamma distributions, and are devoted to estimating the convergence rate in the generalized Rényi theorem with structural mixed generalized gamma distributions. In particular, some results for the structural digamma distribution are given.

2. Representations for Generalized Gamma and Negative Binomial Distributions

By $N_{p,0}$, we denote a random variable with the geometric distribution supported on non-negative integers:

$$\mathsf{P}(N_{p,0} = n) = p(1-p)^n, \ \ n = 0, 1, \ldots, \ \ p \in (0, 1).$$

Let $S_{p,0}$ be the corresponding geometric random sum

$$S_{p,0} = \sum_{j=1}^{N_{p,0}} X_j.$$

Denote

$$S_n = \sum_{i=1}^{n} X_i, \ \ S_0 = 0.$$

By $G_{\nu,p,\delta}$, we denote a random variable having a generalized gamma distribution $GG(\nu, p, \delta)$ with the density (1). In what follows, a special form of (1) will be of particular interest, namely, the exponential distribution $GG(1, 1, \delta)$.

Let $N_1(t)$ be the standard Poisson process. Let $N_1(t)$ and $G_{\nu,p,\delta}$ be independent for all t. We say that the random variable $N_{\nu,p,\delta} \equiv N_1(G_{\nu,p,\delta})$ has a generalized negative binomial distribution $GNB(\nu,p,\delta)$ [16]. Note that $N_{\nu,p,\delta t} \stackrel{d}{=} N_1(G_{\nu,p,\delta}t)$.

In Ref. [16], the following statement was proved.

Lemma 1. *If $\nu \in (0,1)$ and $p \in (0,1]$, then the generalized negative binomial distribution is the $Y_{\nu,p,\delta}$-mixed geometric distribution:*

$$\mathsf{P}(N_{\nu,p,\delta} = k) = \int_0^1 y(1-y)^k \, d\mathsf{P}(Y_{\nu,p,\delta} < y), \ \ k = 0,1,\ldots,$$

where the random variable $Y_{\nu,p,\delta}$ has the density

$$h_{\nu,p,\delta}(y) = \frac{\delta^2}{\Gamma(1-p)\Gamma(p)} \cdot \frac{1}{(1-y)^2} \int_1^\infty \frac{f_{\nu,1}(\delta^\nu y(1-y)^{-1} x^{-1/\nu}) \, dx}{(x-1)^p x^{1+2/\nu}}, \ \ 0 < y < 1, \tag{4}$$

where $f_{\nu,1}(x)$, $0 < \nu < 1$, is the density of a one-sided strictly stable law supported on the positive half-line, with a characteristic function

$$\phi_{\nu,1}(t) = \exp\left\{-|t|^\nu \exp\left\{-\frac{1}{2} i\pi\nu \mathrm{sgn} t\right\}\right\}.$$

In Ref. [17], the following statement for continuous analogues of generalized negative binomial and geometric distributions is proved.

Lemma 2. *If $\nu \in (0,1)$ and $p \in (0,1]$, then the generalized gamma distribution is a mixed exponential distribution:*

$$g_{\nu,p,\delta}(z) = \int_0^1 \frac{y}{1-y} e^{-\frac{y}{1-y}z} \cdot h_{\nu,p,\delta}(y) \, dy, \ \ z > 0,$$

where the density $h_{\nu,p,\delta}(y)$ is defined in (4).

3. Generalization of the Rényi Theorem for Distributions with a Scale Parameter

This section presents a generalization of the classical Rényi theorem [10] for structural distributions with a scale parameter [20]. The following theorem weakens the convergence requirements compared to the generalized Rényi theorem proved in Ref. [9].

Definition 2. *A random variable Λ has a distribution $D(\ldots,\delta)$ with the scale parameter $\delta > 0$, if $\hat{\Lambda} \stackrel{d}{=} \Lambda/\delta$ has a distribution $D(\ldots,1)$, independent of δ.*

Note that all continuous distributions listed above have a scale parameter.

Let $\Lambda \sim D(\ldots,\delta)$ be a non-negative random variable with the scale parameter δ. Consider the standard Poisson process $N_1(t)$ and a sequence of identically distributed random variables $X_1, X_2, \ldots$ with a finite mathematical expectation $\mathsf{E}X_1 = a \neq 0$. Assume that $N_1(t), \Lambda, X_1, X_2, \ldots$ are independent for any $t \geq 0$.

Theorem 1. *Let $\hat{\Lambda} \stackrel{d}{=} \Lambda/\delta$. Then,*

$$\frac{S_{N_1(\Lambda t)}}{a\delta t} \Longrightarrow \hat{\Lambda}, \ \ \delta t \to \infty. \tag{5}$$

Proof of Theorem 1. Note that

$$\lim_{\delta t \to \infty} \mathsf{E} \exp\left\{ is \left(\frac{N_1(\Lambda t)}{\delta t} - \frac{\Lambda}{\delta} \right) \right\} = 1.$$

Hence,

$$\frac{N_1(\Lambda t)}{\delta t} - \frac{\Lambda}{\delta} \xrightarrow{\mathsf{P}} 0, \quad \delta t \to \infty.$$

Since the distribution of $\hat{\Lambda}$ does not depend on δ, according to the Slutsky theorem [21], we conclude that

$$\frac{N_1(\Lambda t)}{\delta t} \Longrightarrow \hat{\Lambda}, \quad \delta t \to \infty.$$

The relation (5) follows from the transfer theorem for random sums, e.g., Theorem 2.2.1 from Ref. [22]. The theorem is proved. □

Remark 1. *Theorem 1 does not require that the parameters δ and t simultaneously tend to infinity.*

4. Estimation of the Rate of Convergence in the Classical Rényi Theorem Using the Zeta Metric

The Rényi theorem is a classical limit theorem. In the study of asymptotic approximations, the rate of convergence to the limit law is of particular interest. One of the approaches to analyzing the convergence rate is based on the use of an ideal metric.

Consider the ζ-metric proposed by V.M. Zolotarev. To demonstrate the importance of this metric, recall that the sequence of random variables Y_n weakly converges to the random variable Y if

$$\Delta_n = \mathsf{E}(f(Y_n) - f(Y)) \longrightarrow 0$$

as $n \to \infty$ for all $f \in \mathcal{F}$, where $\mathcal{F}$ is the set of all bounded and continuous functions. However, it is inconvenient to use the values of Δ_n to construct the convergence rate boundaries, since $\mathcal{F}$ is too large. V.M. Zolotarev proposed a definition of the so-called ideal ζ-metric, which narrows the class $\mathcal{F}$ to a subclass of Lipschitz differentiable bounded functions.

Let us introduce a formal definition of this metric. Let s be a positive number. Then, $s = m + \varepsilon$, where m is a non-negative integer and $\varepsilon \in (0, 1]$. Let $\mathcal{F}_s$ be the set of all m times differentiable real-valued bounded functions f for which

$$\left| f^{(m)}(x) - f^{(m)}(y) \right| \le |x - y|^{\varepsilon}.$$

The ζ-metric $\zeta_s(X, Y) \equiv \zeta_s(F_X, F_Y)$ [19] is defined as

$$\zeta_s(X, Y) \equiv \zeta_{\mathcal{F}_s}(X, Y) = \sup_{f \in \mathcal{F}_s} |\mathsf{E}(f(X) - f(Y))|;$$

see also Refs. [23,24].

Note that the ζ-metric has the following property [24]:

$$\zeta_s(cX, cY) = c^s \zeta_s(X, Y), \quad c > 0. \tag{6}$$

According to the classical Rényi theorem, a geometric random sum, normalized by its mathematical expectation, weakly converges to the standard exponential distribution:

$$\frac{pS_{p,0}}{a(1-p)} \Longrightarrow G_{1,1,1}. \tag{7}$$

In the Refs. [18,25], the following estimates of the convergence rate in (7) were obtained in terms of ζ-metrics.

Lemma 3. *Suppose that $p \in (0, 1)$. Let $X_1, X_2, \ldots$ be a sequence of identically distributed random variables independent of $N_{p,0}$ with $\mathsf{E}X_1 = a \neq 0$ and $\mathsf{E}X_1^2 < \infty$. Then,*

$$\zeta_1\left(\frac{p\sum_{i=1}^{N_{p,0}}X_i}{a(1-p)}, G_{1,1,1}\right) \le \frac{p}{1-p}\cdot\frac{\mathsf{E}X_1^2}{a^2}.$$

Lemma 4. *Suppose that* $p\in(0,1)$. *Let* $X_1, X_2, \dots$ *be a sequence of identically distributed random variables independent of* $N_{p,0}$ *with* $\mathsf{E}X_1 = a \neq 0$ *and* $\mathsf{E}X_1^2 < \infty$. *Then, for* $1 \le s \le 2$

$$\zeta_s\left(\frac{p\sum_{i=1}^{N_{p,0}}X_i}{a(1-p)}, G_{1,1,1}\right) \le \frac{1}{s}\left[\frac{p}{1-p}\cdot\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2},$$

in particular,

$$\zeta_2\left(\frac{p\sum_{i=1}^{N_{p,0}}X_i}{a(1-p)}, G_{1,1,1}\right) \le \frac{p}{2(1-p)}\cdot\frac{\mathsf{E}X_1^2}{a^2}.$$

5. Convergence Rate Estimates in the Generalized Rényi Theorem with a Structural Generalized Gamma Distribution

The statements of this section are the generalization of the results proved in Refs. [16,18] for the structural gamma distribution.

Lemma 5. *Suppose that* $\nu\in(0,1]$ *and* $p\in(0,1)$. *Let* $X_1, X_2, \dots$ *be a sequence of identically distributed random variables with* $\mathsf{E}X_1 = a \neq 0$ *and* $N_1(t), G_{\nu,p,\delta}, X_1, X_2, \dots$ *be independent for all* $t \ge 0$. *Then,*

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})}X_i}{a\delta t}, G_{\nu,p,1}\right) \le \frac{1}{(\delta t)^s}\int_0^1 \frac{(1-y)^s}{y^s}\zeta_s\left(\frac{yS_{y,0}}{(1-y)a}, G_{1,1,1}\right)\cdot h_{\nu,p,\delta t}(y)\,dy$$

where the density $h_{\nu,p,\delta t}(y)$ *is defined in (4).*

Proof of Lemma 5. According to the property (6) of ζ-metrics

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})}X_i}{a\delta t}, G_{\nu,p,1}\right) = \frac{1}{(a\delta t)^s}\zeta_s\left(\sum_{i=1}^{N_1(G_{\nu,p,\delta t})}X_i, aG_{\nu,p,\delta t}\right).$$

Using Lemma 1, we get

$$\mathsf{P}\left(\sum_{i=1}^{N_1(G_{\nu,p,\delta t})}X_i < x\right) = \sum_{n=0}^{\infty}\int_0^1 y(1-y)^n h(y;\nu,p,\delta t)\,dy\,\mathsf{P}(S_n < x)$$

$$= \int_0^1\sum_{n=0}^{\infty} y(1-y)^n\mathsf{P}(S_n < x)h(y;\nu,p,\delta t)\,dy = \int_0^1 \mathsf{P}\big(S_{y,0} < x\big)h(y;\nu,p,\delta t)\,dy.$$

Therefore, for any continuous bounded function f

$$\mathsf{E}f\left(\sum_{i=1}^{N_1(G_{\nu,p,\delta t})}X_i\right) = \int_0^1 \mathsf{E}f\big(S_{y,0}\big)h(y;\nu,p,\delta t)\,dy.$$

Similarly, using Lemma 2,

$$\mathsf{E}f(aG_{\nu,p,\delta t}) = \int_0^\infty f(az)g_{\nu,p,\delta t}(z)\,dz = \int_0^1 \mathsf{E}f\left(aG_{1,1,\frac{1-y}{y}}\right)\cdot h_{\nu,p,\delta t}(y)\,dy.$$

Hence,

$$\zeta_s\left(\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i, aG_{\nu,p,\delta t}\right) = \sup_{f\in\mathcal{F}_s}\left|\mathsf{E}f\left(\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i\right) - \mathsf{E}f(aG_{\nu,p,\delta t})\right|$$

$$= \sup_{f\in\mathcal{F}_s}\left|\int_0^1 \mathsf{E}f(S_{y,0})h_{\nu,p,\delta t}(y)\,dy - \int_0^1 \mathsf{E}f\left(aG_{1,1,\frac{1-y}{y}}\right)\cdot h_{\nu,p,\delta t}(y)\,dy\right|$$

$$\leq \int_0^1 \zeta_s\left(S_{y,0}, aG_{1,1,\frac{1-y}{y}}\right)\cdot h_{\nu,p,\delta t}(y)\,dy$$

Thus,

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i}{a\delta t}, G_{\nu,p,1}\right) \leq \frac{1}{(a\delta t)^s}\int_0^1 \zeta_s\left(S_{y,0}, aG_{1,1,\frac{1-y}{y}}\right)\cdot h_{\nu,p,\delta t}(y)\,dy$$

$$= \frac{1}{(\delta t)^s}\int_0^1 \frac{(1-y)^s}{y^s}\zeta_s\left(\frac{yS_{y,0}}{(1-y)a}, G_{1,1,1}\right)\cdot h_{\nu,p,\delta t}(y)\,dy.$$

The lemma is proved. □

The following statement is a generalization of Lemmas 3 and 4 to the case of a structural generalized gamma distribution.

Lemma 6. *Suppose that $\nu \in (0,1]$ and $p \in (0,1)$. Let $X_1, X_2, \ldots$ be a sequence of identically distributed random variables with $\mathsf{E}X_1 = a \neq 0$, $\mathsf{E}X_1^2 < \infty$ and $N_1(t), G_{\nu,p,\delta}, X_1, X_2, \ldots$ be independent for all $t \geq 0$. Then,*

$$\zeta_1\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i}{a\delta t}, G_{\nu,p,1}\right) \leq \frac{\mathsf{E}X_1^2}{a^2\delta t};$$

for $1 \leq s \leq 2$

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i}{a\delta t}, G_{\nu,p,1}\right) \leq \frac{1}{(\delta t)^{s/2}}\cdot\frac{\Gamma(p+s/(2\nu))}{s\Gamma(p)\Gamma(1+s/2)}\cdot\left[\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2},$$

in particular,

$$\zeta_2\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i}{a\delta t}, G_{\nu,p,1}\right) \leq \frac{1}{\delta t}\cdot\frac{\Gamma(p+1/\nu)}{\Gamma(p)}\cdot\frac{\mathsf{E}X_1^2}{2a^2}.$$

Proof of Lemma 6. From Lemma 3 we get

$$\zeta_1\left(\frac{yS_{y,0}}{(1-y)a}, G_{1,1,1}\right) \leq \frac{y}{1-y}\cdot\frac{\mathsf{E}X_1^2}{a^2}.$$

Hence,

$$\zeta_1\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i}{a\delta t}, G_{\nu,p,1}\right) \leq \frac{1}{\delta t}\int_0^1 \frac{1-y}{y}\zeta_1\left(\frac{yS_{y,0}}{(1-y)a}, G_{1,1,1}\right)\cdot h_{\nu,p,\delta t}(y)\,dy \leq \frac{\mathsf{E}X_1^2}{a^2\delta t}.$$

Let $1 \leq s \leq 2$. Then, by Lemma 4

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i}{a\delta t}, G_{\nu,p,1}\right) \le \frac{1}{(\delta t)^s}\int_0^1 \frac{(1-y)^s}{y^s}\zeta_s\left(\frac{yS_{y,0}}{(1-y)a}, G_{1,1,1}\right)\cdot h_{\nu,p,\delta t}(y)\,dy$$

$$\le \frac{1}{(\delta t)^s}\int_0^1 \frac{(1-y)^s}{y^s}\frac{1}{s}\left[\frac{y}{1-y}\cdot\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2}\cdot h_{\nu,p,\delta t}(y)\,dy$$

$$= \frac{1}{(\delta t)^s}\cdot\frac{1}{s}\cdot\left[\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2}\mathsf{E}\frac{(1-Y_{\nu,p,\delta t})^{s/2}}{Y_{\nu,p,\delta t}^{s/2}}.$$

Since by Lemma 2

$$G_{\nu,p,\delta}\stackrel{d}{=}\frac{1-Y_{\nu,p,\delta}}{Y_{\nu,p,\delta}}\cdot G_{1,1,1},$$

where $Y_{\nu,p,\delta}$ and $G_{1,1,1}$ can be considered independent, we obtain

$$\mathsf{E}\big(G_{\nu,p,\delta}\big)^{s/2} = \mathsf{E}G_{1,1,1}^{s/2}\mathsf{E}\left(\frac{1-Y_{\nu,p,\delta t}}{Y_{\nu,p,\delta t}}\right)^{s/2}.$$

Thus,

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i}{a\delta t}, G_{\nu,p,1}\right) \le \frac{1}{(\delta t)^s}\cdot\frac{1}{s}\cdot\left[\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2}\frac{\mathsf{E}\big(G_{\nu,p,\delta}\big)^{s/2}}{\mathsf{E}G_{1,1,1}^{s/2}}$$

$$= \frac{1}{(\delta t)^{s/2}}\cdot\frac{\Gamma(p+s/(2\nu))}{s\Gamma(p)\Gamma(1+s/2)}\cdot\left[\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2}.$$

The lemma is proved. □

6. Convergence Rate Estimates in a Generalized Rényi Theorem with a Structural Mixed Generalized Gamma Distribution

This section provides an estimate of the convergence rate in the generalized Rényi theorem with a structural distribution that is a scale mixture of the generalized gamma distribution.

Let Q be a non-negative random variable.

Theorem 2. *Suppose that $\nu \in (0,1]$ and $p \in (0,1)$. Let $X_1, X_2, \ldots$ be a sequence of identically distributed random variables with $\mathsf{E}X_1 = a \neq 0$ and $Q, N_1(t), G_{\nu,p,\delta}, X_1, X_2, \ldots$ be independent for all $t \ge 0$. Assume that there is an estimate*

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta t})} X_i}{a\delta t}, G_{\nu,p,1}\right) \le \Delta_s(\delta t).$$

Then,

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(Q\cdot G_{\nu,p,\delta t})} X_i}{a\delta t}, Q\cdot G_{\nu,p,1}\right) \le \int_0^\infty y^s\Delta_s(y\delta t)\,dF_Q(y).$$

Proof of Theorem 2. Averaging over the distribution of Q, we obtain

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(Q\cdot G_{\nu,p,\delta}t)} X_i}{a\delta t}, Q\cdot G_{\nu,p,1}\right) = \int_0^\infty \zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta}yt)} X_i}{a\delta t}, yG_{\nu,p,1}\right) dF_Q(y)$$

$$\leq \int_0^\infty y^s \Delta_s(y\delta t)\, dF_Q(y).$$

The theorem is proved. □

Lemma 6 and Theorem 2 imply the validity of the following statements.

Corollary 1. *Suppose that $\nu \in (0,1]$ and $p \in (0,1)$. Let $X_1, X_2, \ldots$ be a sequence of identically distributed random variables with $\mathsf{E}X_1 = a \neq 0$, $\mathsf{E}X_1^2 < \infty$, and $Q, N_1(t), G_{\nu,p,\delta}, X_1, X_2, \ldots$ be independent for all $t \geq 0$. Then,*

$$\zeta_1\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta}t)} X_i}{a\delta t}, G_{\nu,p,1}\right) \leq \frac{\mathsf{E}X_1^2}{a^2\delta t},$$

and hence,

$$\zeta_1\left(\frac{\sum_{i=1}^{N_1(Q\cdot G_{\nu,p,\delta}t)} X_i}{a\delta t}, Q\cdot G_{\nu,p,1}\right) \leq \frac{\mathsf{E}X_1^2}{a^2\delta t}.$$

Corollary 2. *Suppose that $\nu \in (0,1]$ and $p \in (0,1)$. Let $X_1, X_2, \ldots$ be a sequence of identically distributed random variables with $\mathsf{E}X_1 = a \neq 0$, $\mathsf{E}X_1^2 < \infty$, and $Q, N_1(t), G_{\nu,p,\delta}, X_1, X_2, \ldots$ be independent for all $t \geq 0$. Then by Lemma 4 and Theorem 3 for $1 \leq s \leq 2$*

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(G_{\nu,p,\delta}t)} X_i}{a\delta t}, G_{\nu,p,1}\right) \leq \frac{1}{(\delta t)^{s/2}} \cdot \frac{\Gamma(p+s/(2\nu))}{s\Gamma(p)\Gamma(1+s/2)} \cdot \left[\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2},$$

and hence,

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(Q\cdot G_{\nu,p,\delta}t)} X_i}{a\delta t}, Q\cdot G_{\nu,p,1}\right) \leq \frac{\mathsf{E}Q^{s/2}}{(\delta t)^{s/2}} \cdot \frac{\Gamma(p+s/(2\nu))}{s\Gamma(p)\Gamma(1+s/2)} \cdot \left[\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2}.$$

In particular,

$$\zeta_2\left(\frac{\sum_{i=1}^{N_1(Q\cdot G_{\nu,p,\delta}t)} X_i}{a\delta t}, Q\cdot G_{\nu,p,1}\right) \leq \frac{\mathsf{E}Q}{\delta t} \cdot \frac{\Gamma(p+1/\nu)}{\Gamma(p)} \cdot \frac{\mathsf{E}X_1^2}{2a^2}.$$

As a special case of a scale mixture of the generalized gamma distribution, consider the digamma distribution: $\rho_\delta \sim DiG(r,\nu,p,q,\delta)$.

Since the representation (3) in the form of a scale mixture of generalized gamma distributions is valid for the digamma distribution, the following statements hold.

Corollary 3. *Suppose that $\nu \in (0,1]$, $p \in (0,1)$ or $-\nu/r \in (0,1]$, $q \in (0,1)$. Let $X_1, X_2, \ldots$ be a sequence of identically distributed random variables with $\mathsf{E}X_1 = a \neq 0$, $\mathsf{E}X_1^2 < \infty$, and $N_1(t), \rho_\delta, X_1, X_2, \ldots$ be independent for all $t \geq 0$. Then,*

$$\zeta_1\left(\frac{\sum_{i=1}^{N_1(\rho_\delta t)} X_i}{a\delta t}, \rho_1\right) \leq \frac{\mathsf{E}X_1^2}{a^2\delta t}.$$

Corollary 4. *Suppose that* $\nu \in (0,1]$, $p \in (0,1)$ *or* $-\nu/r \in (0,1]$, $q \in (0,1)$. *Let* $X_1, X_2, \ldots$ *be a sequence of identically distributed random variables with* $\mathsf{E}X_1 = a \neq 0$, $\mathsf{E}X_1^2 < \infty$, *and* $N_1(t), \rho_\delta, X_1, X_2, \ldots$ *be independent for all* $t \geq 0$. *Then, for* $1 \leq s \leq 2$

$$\zeta_s\left(\frac{\sum_{i=1}^{N_1(\rho_\delta t)} X_i}{a\delta t}, \rho_1\right) \leq \frac{1}{(\delta t)^{s/2}} \cdot \frac{\Gamma(q - rs/(2\nu))}{\Gamma(q)} \cdot \frac{\Gamma(p + s/(2\nu))}{s\Gamma(p)\Gamma(1+s/2)} \cdot \left[\frac{\mathsf{E}X_1^2}{a^2}\right]^{s/2}.$$

In particular,

$$\zeta_2\left(\frac{\sum_{i=1}^{N_1(\rho_\delta t)} X_i}{a\delta t}, \rho_1\right) \leq \frac{1}{\delta t} \cdot \frac{\Gamma(q - r/\nu)}{\Gamma(q)} \cdot \frac{\Gamma(p + 1/\nu)}{\Gamma(p)} \cdot \frac{\mathsf{E}X_1^2}{2a^2} = \frac{\mathsf{E}\rho_1}{\delta t} \cdot \frac{\mathsf{E}X_1^2}{2a^2}.$$

7. Conclusions

In this paper, a generalization of the Rényi theorem is obtained for a class of structural distributions with a scale parameter, which includes a generalized gamma distribution and a generalized beta distribution of the second kind. For the case when the distribution of the summation index is a scale mixture of the generalized gamma distribution, the estimates of the convergence rate in the generalized Rényi theorem are obtained, expressed in terms of zeta metrics. In particular, such estimates are obtained for the structural digamma distribution that arises in the study of Bayesian balance models. The paper extends the results previously obtained only for the generalized gamma distribution.

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