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An Investigation of a Nonlinear Delay Functional Equation with a Quadratic Functional Integral Constraint

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Abstract: This research paper focuses on investigating the solvability of a constrained problem involving a nonlinear delay functional equation subject to a quadratic functional integral constraint, in two significant cases: firstly, the existence of nondecreasing solutions in a bounded interval $L_1[0,T]$ and, secondly, the existence of nonincreasing solutions in unbounded interval $L_1(R_+)$. Moreover, the paper explores various qualitative properties associated with these solutions for the given problem. To establish the validity of our claims, we employ the De Blasi measure of noncompactness (MNC) technique as a basic tool for our proofs. In the first case, we provide sufficient conditions for the uniqueness of the solution $\psi \in L_1[0,T]$ and rigorously demonstrate its continuous dependence on some parameters. Additionally, we establish the equivalence between the constrained problem and an implicit hybrid functional integral equation (IHFIE). Furthermore, we delve into the study of Hyers–Ulam stability. In the second case, we examine both the asymptotic stability and continuous dependence of the solution $\psi \in L_1(R_+)$ on some parameters. Finally, some examples are provided to verify our investigation.

Keywords: constrained problem; Hyers–Ulam stability; measure of noncompactness; implicit hybrid functional integral equation; asymptotic stability and dependency

MSC: 93B05; 47H10; 26A33; 93C27; 47H08; 34K401

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1. Introduction

The study of integral equations serves as a significant mathematical tool in both pure and applied analysis. This is particularly notable in fields such as mechanical vibrations, engineering, and mathematical physics. Previous research efforts dedicated to addressing these types of problems are available in the literature (see [1–4]), especially Volterra equations with linear functionals and a small parameter, which are considered in [5].

The principle tools applied in our study are Darbo's fixed-point theorem [6] and the strategy of MNC.

The MNC and Darbo's fixed-point theorem are useful techniques to discuss the nonlinear functional integral equations that appear in many real-world problems [1–3,7,8]. The application of the MNC technique within the Banach space L_1 has proven to be highly effective in establishing the existence and stability of solutions for a wide class of functional equations. Prominently, J. Banaś has successfully utilized this technique in his research (e.g., [9,10]). Moreover, some literature explores the implementation of this approach in studying various functional equations (see [11,12]).

Constrained problems play a crucial role in the mathematical representation of life problems. By converting these problems into mathematical models [13,14]. The importance of dealing with problems involving constraints or control variables is due to the unexpected factors ceaselessly disturbing biological systems within the genuine world; this may result

in alterations to biological characteristics, such as rates of survival. Ecology has a practical interest in the question of whether an ecosystem can withstand those unpredictable, disruptive events that proceed for a brief period of time. In the context of control variables, the disturbance functions are what we refer to as control variables.

In [15], Chen built up a few averaging conditions for a nonautonomous Lotka–Volterra system that is controlled through criticism by creating an appropriate Lyapunov function (Lyapunov functional).

A family of feedback-controlled nonlinear functional integral equations exist, is asymptotically stable, and is globally attractive as demonstrated by Nasertayoob, utilizing the MNC in conjunction with Darbo's fixed-point theorem [16]. Additionally, under suitable circumstances, it was investigated in [17] whether a nonlinear neutral delay population system with a feedback control has a positive periodic solution. The existence of a positive periodic solution for a nonlinear neutral delay population system with a feedback control is considered in [17].

El-Sayed et al. [18] conducted a research study on a constraint functional equation. Further investigations on the existence of solutions can be found in [19], where researchers examined a nonlinear functional integral equation under the constraint of a parameter functional equation.

The authors in [20] extensively investigated its solvability, asymptotic stability, and continuous dependence of the solution on some parameters. They utilized the technique of (MNC) within the space $BC(R_+)$.

In this study, our focus is on examining the constrained problem of the delay functional equation,

$$\psi(\mathfrak{r}) = \hbar_1\ (\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \ \ \Upsilon\ (\mathfrak{r})\ \hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))), \tag{1}$$

subject to the quadratic integral constraint

$$\Upsilon(\mathfrak{r}) = f\left(\mathfrak{r},\ \ g_1(\mathfrak{r}, \Upsilon(\mathfrak{r})),\ \ \succ \int_0^{\mathfrak{r}} f_1(s, \Upsilon(s))ds,\ \ g_2(\mathfrak{r}, \Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s, \Upsilon(s))ds\right). \tag{2}$$

Our aim here is to investigate the existence of nondecreasing solutions, $\psi \in L_1[0, T]$, and nonincreasing solutions, $\psi \in L_1(R_+)$, by the De Blasi (MNC) and Darbo's fixed-point theorem [6]. Sufficient conditions for the uniqueness of the solution and the continuous dependence of the unique solution on the parameter $\succ \geq 0$ and functions g_1 and g_2 of problems (1) and (2) will be studied. Next, we explore the equivalence between problem (1) and (2) and the implicit hybrid functional integral equation (IHFIE).

$$\begin{aligned}
&\frac{\psi(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))} = \\
&f\left(\mathfrak{r},\ g_1\left(\mathfrak{r}, \frac{\psi(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))}\right), \succ \int_0^{\mathfrak{r}} f_1\left(s, \frac{\psi(s) - \hbar_1(s, \psi(\phi_1(s)))}{\hbar_2(s, \psi(\phi_2(s)))}\right)ds,\right. \\
&\left. g_2\left(\mathfrak{r}, \frac{\psi(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))}\right).\int_0^{\mathfrak{r}} f_2\left(s, \frac{\psi(s) - \hbar_1(s, \psi(\phi_1(s)))}{\hbar_2(s, \psi(\phi_2(s)))}\right)ds\right)
\end{aligned} \tag{3}$$

is established. The Hyers–Ulam stability of problem (1) and (2) and of the IHFIE (3) will be studied. The asymptotic stability and the continuous dependence of the solution $\psi \in L_1(R_+)$ on the parameter $\succ \geq 0$ and functions g_1 and g_2 will be proved. Finally, some examples are given to illustrate our results.

The importance of examining hybrid functional integral and quadratic functional integral problems is found in the reality that this category includes distinctive energetic frameworks in specific cases. This class of hybrid differential equations includes the perturbations of original differential equations in several ways.

Theorem 1 ([6]). *Let U be a nonempty, bounded, closed, and convex subset of a Banach space $\mathbb{C}$ and let $\aleph : U \longrightarrow U$ be a continuous mapping. Assume that there exists a constant $\Bbbk \in [0,1)$ such that $\eta(\aleph\wedge) \leq \Bbbk\eta(\wedge)$ for any nonempty subset $\wedge$ of U, where η is an MNC defined in $\mathbb{C}$. Then, $\aleph$ has at least one fixed point in U.*

2. Solvability in Bounded Interval $L_1[0,T]$

Let $L_1 = L_1(I)$ be the class of Lebesgue integrable functions on $I = [0,T]$, $T < \infty$, with the standard norm

$$\|\psi\| = \int_0^T |\psi(t)|dt.$$

2.1. Quadratic Functional Integral Constraint

Let us examine Equation (2) within the framework of the following assumptions:

(i) $f : I \times R^3 \to R$, $g_i : I \times R \to R$ and $f_i : I \times R \to R^+$, $i = 1,2$ are Carathéodory functions [21], and there exist the integrable functions m, m_i, $k_i : I \to R$ and nonnegative constants b, b_i, and c_i such that

$$|f(t,\xi,\vartheta,\omega)| \leq |m(t)| + b(|\xi| + |\vartheta| + |\omega|),$$
$$|g_i(t,\vartheta)| \leq |m_i(t)| + b_i|\vartheta| \text{ and } |f_i(t,\vartheta)| \leq |k_i(t)| + c_i|\vartheta|.$$

(ii) f and g_i, $i = 1,2$ are nondecreasing for every nondecreasing argument such that $t_1 < t_2$ and for all $\xi_1 < \xi_2$, $\vartheta_1 < \vartheta_2$, $\omega_1 < \omega_2$ implies

$$f(t_1,\xi_1,\vartheta_1,\omega_1) \leq f(t_2,\xi_2,\vartheta_2,\omega_2) \text{ and } g_i(t_1,\vartheta_1) \leq g_i(t_2,\vartheta_2).$$

(iii) There exists a positive root r_1 of the algebraic equation

$$bb_2c_2r_1^2 + (bb_1 + b\nwarrow c_1T + bb_2\|k_2\| + bc_2\|m_2\| - 1)r_1 + \|m\| + b\|m_1\| + b\nwarrow\|k_1\|T + b\|k_2\|\|m_2\| = 0.$$

Theorem 2. *Assume that (i)–(iii) are met. If $(bb_1 + bb_2(\|k_2\| + c_2r_1)) < 1$, then (2) has at least one nondecreasing solution $\Upsilon \in L_1(I)$.*

Proof. Let Q_{r_1} be a closed ball of all the nondecreasing functions

$$Q_{r_1} = \{\Upsilon \in L_1(I) : \|\Upsilon\| \leq r_1\}$$

and the operator

$$F\Upsilon(\mathfrak{r}) = f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \nwarrow\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds,\ g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\right),$$

Then, we deduce that F transforms the nondecreasing functions into functions of the same type.

Let $\Upsilon \in Q_{r_1}$, then

$$\begin{aligned}
|F\Upsilon(\mathfrak{r})| &= \left|f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \nwarrow\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds,\ g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\right)\right| \\
&\leq |m(\mathfrak{r})| + b\left(|m_1(\mathfrak{r})| + b_1|\Upsilon(\mathfrak{r})|\right) + b\nwarrow\left(\int_0^{\mathfrak{r}}(k_1(s) + c_1|\Upsilon(s)|)ds\right) \\
&+ b\left(|m_2(\mathfrak{r})| + b_2|\Upsilon(\mathfrak{r})|\right).\int_0^{\mathfrak{r}}(k_2(s) + c_2|\Upsilon(s)|)ds \\
&\leq |m(\mathfrak{r})| + b(|m_1(\mathfrak{r})| + b_1|\Upsilon(\mathfrak{r})|) + b\nwarrow(\|k_1\| + c_1\|\Upsilon\|) \\
&+ b(|m_2(\mathfrak{r})| + b_2|\Upsilon(\mathfrak{r})|).(\|k_2\| + c_2\|\Upsilon\|)
\end{aligned}$$

and

$$\begin{aligned} & \int_0^T |F\,\Upsilon\,(\mathfrak{r})|d\mathfrak{r} \\ \leq\ & \int_0^T |m(\mathfrak{r})|d\mathfrak{r} + b(\int_0^T |m_1(\mathfrak{r})|d\mathfrak{r} + b_1\int_0^T |\,\Upsilon\,(\mathfrak{r})|d\mathfrak{r}) + b\succ\,(\|k_1\| + c_1r_1)\int_0^T d\mathfrak{r} \\ +\ & b(\|k_2\| + c_2r_1).(\int_0^T |m_2(\mathfrak{r})|d\mathfrak{r} + b_2\int_0^T |\,\Upsilon\,(\mathfrak{r})|d\mathfrak{r}). \end{aligned}$$

Hence,

$$\|F\,\Upsilon\,\| \leq \|m\| + b\|m_1\| + bb_1r_1 + b\succ\,(\|k_1\| + c_1r_1)T + b(\|k_2\| + c_2r_1).(\|m_2\| + b_2r_1) = r_1$$

and $F : Q_{r_1} \to Q_{r_1}$.

Now, let $\{\Upsilon_n\} \subset Q_{r_1}$ and $\Upsilon_n \to \Upsilon$, then

$$\begin{aligned} F\,\Upsilon_n\,(\mathfrak{r}) &= f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_n(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon_n(s))ds,\ g_2(\mathfrak{r},\Upsilon_n(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_n(s))ds\right), \\ \lim_{n\to\infty} F\,\Upsilon_n\,(\mathfrak{r}) &= \lim_{n\to\infty} f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_n(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon_n(s))ds,\ g_2(\mathfrak{r},\Upsilon_n(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_n(s))ds\right). \end{aligned}$$

Applying the Lebesgue Theorem [12], we obtain

$$\begin{aligned} & \lim_{n\to\infty} F\,\Upsilon_n\,(\mathfrak{r}) \\ =\ & f\left(\mathfrak{r},\ g_1(\mathfrak{r},\lim_{n\to\infty}\Upsilon_n(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\lim_{n\to\infty}\Upsilon_n(s))ds,\ g_2(\mathfrak{r},\lim_{n\to\infty}\Upsilon_n(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\lim_{n\to\infty}\Upsilon_n(s))ds\right) \\ =\ & f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds,\ g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\right) = F\,\Upsilon\,(\mathfrak{r}). \end{aligned}$$

This means that $F\,\Upsilon_n\,(\mathfrak{r}) \to F\,\Upsilon\,(\mathfrak{r})$. Hence, the operator F is continuous.

Now, let Λ be a nonempty subset of Q_{r_1}. Fix $\epsilon > 0$ and take a measurable set $D \subset I$ such that $D \leq \epsilon$. Then, for any $\Upsilon \in \Lambda$, we have

$$\begin{aligned} \|F\,\Upsilon\,\|_{L_1(D)} &\leq \int_D |F\,\Upsilon\,(\mathfrak{r})|d\mathfrak{r} \\ &\leq \int_D |m(\mathfrak{r})|d\mathfrak{r} + b(\int_D |m_1(\mathfrak{r})|d\mathfrak{r} + b_1\int_D |\,\Upsilon\,(\mathfrak{r})|d\mathfrak{r}) + b\succ\,(\|k_1\| + c_1r_1)\int_D d\mathfrak{r} \\ &+ b(\|k_2\| + c_2r_1).(\int_D |m_2(\mathfrak{r})|d\mathfrak{r} + b_2\int_D |\,\Upsilon\,(\mathfrak{r})|d\mathfrak{r}), \end{aligned}$$

then

$$\begin{aligned} \beta(\Upsilon(\mathfrak{r})) &= \lim_{\epsilon\to 0}\|F\,\Upsilon\,\|_{L_1(D)} \\ &\leq bb_1\beta(\Upsilon(\mathfrak{r})) + bb_2(\|k_2\| + c_2r_1)\beta(\Upsilon(\mathfrak{r})) \end{aligned}$$

and

$$\beta(F\Lambda) \leq (bb_1 + bb_2(\|k_2\| + c_2r_1)).\beta(\Lambda).$$

This implies

$$\chi(F\Lambda) \leq (bb_1 + bb_2(\|k_2\| + c_2r_1))\chi(\Lambda).$$

where χ is the Hausdorff MNC [6,9,22,23].

Since $(bb_1 + bb_2(\|k_2\| + c_2r_1)) < 1$, it follows from Darbo's theorem [6] that F is a contraction and that it has at least one fixed point in Q_{r_1}. Then, there exists at least one nondecreasing solution $\Upsilon \in L_1(I)$ to (2). □

Corollary 1. *Let the assumptions of Theorem 2 hold; then, the solution* $\Upsilon \in L_1(I)$ *to (2) satisfies* $|\Upsilon(\mathfrak{r})| \leq M$.

Proof. The results follow from the monotonicity of the solution Υ on I. □

2.2. The Delay Functional Equation

Now, consider (1) under these assumptions:

(iv) $\phi_i : I \to I, \phi_i(t) \leq t, i = 1, 2$ are increasing and absolutely continuous, and there exist two constants $N_i > 0$ and $i = 1,\ 2$ such that $\phi_i^{`}(t) \geq N_i$ a.e on I.

(v) $\hbar_1 : I \times R \to R$ and $\hbar_2 : I \times R \to R \setminus \{0\}$ are Carathéodory functions [21], and there exist two integrable functions $u_i : I \to R$ and two constants $a_i \in R^+$ $i = 1, 2$ such that

$$|\hbar_i(t, \xi)| \leq |u_i(t)| + a_i|\xi|.$$

Moreover, $\hbar_i$ is nondecreasing for every nondecreasing argument.

(vi) $(\frac{a_1}{N_1} + \frac{Ma_2}{N_2}) < 1$.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied. Assume assumptions (iv)–(vi) are met; then, there exists at least one nondecreasing solution* $\psi \in L_1(I)$ *to (1).*

Proof. Let Q_{r_2} be a closed ball of all nondecreasing functions

$$Q_{r_2} = \{\psi \in L_1(I) : \|\psi\| \leq r_2\}$$

and

$$A\psi(\mathfrak{r}) = \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \ \Upsilon(\mathfrak{r})\ \hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))),$$

Then, we deduce that A transforms the nondecreasing functions into functions of the same type.

Let $\psi \in Q_{r_2}$. Then,

$$\begin{aligned} |A\psi(\mathfrak{r})| &= \left|\hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \ \Upsilon(\mathfrak{r})\ \hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))\right| \\ &\leq \left(|u_1(\mathfrak{r})| + a_1|\psi(\phi_1(\mathfrak{r}))|\right) + |\Upsilon(\mathfrak{r})|\left(|u_2(\mathfrak{r})| + a_2|\psi(\phi_2(\mathfrak{r}))|\right) \end{aligned}$$

and

$$\int_0^T |A\psi(\mathfrak{r})|d\mathfrak{r} \leq \int_0^T |u_1(\mathfrak{r})|d\mathfrak{r} + \frac{a_1}{N_1}\int_0^T |\psi(\theta_1)|d\theta_1 + M\left(\int_0^T |u_2(\mathfrak{r})|d\mathfrak{r} + \frac{a_2}{N_2}\int_0^T |\psi(\theta_2)|d\theta_2\right),$$

Then,

$$\|A\psi\| \leq \|u_1\| + \frac{a_1}{N_1}r_2 + M(\|u_2\| + \frac{a_2}{N_2}r_2) = r_2,\ \ r_2 = \frac{\|u_1\| + M\|u_2\|}{1 - (\frac{a_1}{N_1} + M\frac{a_2}{N_2})}.$$

Hence, $A : Q_{r_2} \longrightarrow Q_{r_2}$.

Now, let $\{\psi_n\} \subset Q_{r_2}$ and $\psi_n \to \psi$. Then,

$$A\psi_n(\mathfrak{r}) = \hbar_1(\mathfrak{r}, \psi_n(\phi_1(\mathfrak{r}))) + \ \Upsilon(\mathfrak{r})\ \hbar_2(\mathfrak{r}, \psi_n(\phi_2(\mathfrak{r})))$$

and

$$\lim_{n\to\infty} A\psi_n(\mathfrak{r}) = \lim_{n\to\infty}\left(\hbar_1(\mathfrak{r}, \psi_n(\phi_1(\mathfrak{r}))) + \Upsilon(\mathfrak{r})\ \hbar_2(\mathfrak{r}, \psi_n(\phi_2(\mathfrak{r})))\right).$$

Apply the Lebesgue Theorem [12]. Then,

$$\begin{aligned}\lim_{n\to\infty} A\psi_n(\mathfrak{r}) &= \hbar_1(\mathfrak{r}, \lim_{n\to\infty} \psi_n(\phi_1(\mathfrak{r}))) + \Upsilon(\mathfrak{r})\ \hbar_2(\mathfrak{r}, \lim_{n\to\infty} \psi_n(\phi_2(\mathfrak{r})))\\ &= \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \Upsilon(\mathfrak{r})\ \hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) = A\psi(\mathfrak{r}).\end{aligned}$$

This means that $A\psi_n(\mathfrak{r}) \to A\psi(\mathfrak{r})$. Hence, the operator A is continuous.

Now, let $\mho$ be nonempty subset of Q_{r_2}. Fix $\epsilon > 0$ and take a measurable set $D \subset I$ such that $D \leq \epsilon$. Then, for any $\psi \in \mho$, we obtain

$$\begin{aligned}\|A\psi\|_{L_1(D)} &\leq \int_D |A\psi(\mathfrak{r})|d\mathfrak{r}\\ &\leq \int_D |u_1(\mathfrak{r})|d\mathfrak{r} + \frac{a_1}{N_1}\int_D |\psi(\phi_1(\mathfrak{r}))|\phi_1\grave{}(\mathfrak{r})d\mathfrak{r} + M\Big(\int_D |u_2(\mathfrak{r})|d\mathfrak{r}\\ &+ \frac{a_2}{N_2}\int_D |\psi(\phi_2(\mathfrak{r}))|\phi_1\grave{}(\mathfrak{r})d\mathfrak{r}\Big)\\ &\leq \int_D |u_1(\mathfrak{r})|d\mathfrak{r} + \frac{a_1}{N_1}\int_{\phi_1(D)} |\psi(\mathfrak{r})|d\mathfrak{r} + M\int_D |u_2(\mathfrak{r})|d\mathfrak{r}\\ &+ \frac{Ma_2}{N_2}\int_{\phi_2(D)} |\psi(\mathfrak{r})|d\mathfrak{r}.\end{aligned}$$

Then,

$$\begin{aligned}\beta(A\psi(\mathfrak{r})) &= \lim_{\epsilon\to 0}\|A\psi\|_{L_1(D)}\\ &\leq \frac{a_1}{N_1}\beta(\psi(\mathfrak{r})) + \frac{Ma_2}{N_2}\beta(\psi(\mathfrak{r}))\end{aligned}$$

and

$$\beta(A\mho) \leq (\frac{a_1}{N_1} + \frac{Ma_2}{N_2})\beta(\mho).$$

This implies

$$\chi(A\mho) \leq (\frac{a_1}{N_1} + \frac{Ma_2}{N_2})\chi(\mho).$$

where χ is the Hausdorff MNC [6,9,22,23].

Since $(\frac{a_1}{N_1} + \frac{Ma_2}{N_2}) < 1$, it follows Darbo's theorem; A is a contraction and has at least one fixed point in Q_{r_2}. Then, there exists at least one nondecreasing solution $\psi \in L_1(I)$ to (1). □

Corollary 2. *From Theorems 2 and 3, problems (1) and (2) have at least one nondecreasing solution* $\psi \in L_1(I)$.

2.3. Uniqueness of the Solution

Now, consider the following assumptions:

$(i)^*$ $f : I \times R^3 \to R$, $g_i : I \times R \to R$ and $f_i : I \times R^3 \to R^+$ are measurable in $t \in I$ $\forall \xi, \vartheta, \omega \in R$
and satisfy the Lipschitz condition such that

$$|f(t,\xi_1,\vartheta_1,\omega_1) - f(t,\xi_2,\vartheta_2,\omega_2)| \leq b(|\xi_1-\xi_2| + |\vartheta_1-\vartheta_2| + |\omega_1-\omega_2|), \quad (4)$$
$$t \in I,\ \xi_i, \vartheta_i, \omega_i \in R,\ \ i = 1,2,$$

$$|g_i(t,\vartheta_1) - g_i(t,\vartheta_2)| \leq b_i|\vartheta_1 - \vartheta_2|,\ t \in I,\ \ \vartheta_i \in R,\ \ i = 1,2 \quad (5)$$

and

$$\int_0^T |f_i(t,\vartheta_1) - f_i(t,\vartheta_2)|dt \leq c_i\|\vartheta_1 - \vartheta_2\|,\ t \in I,\ \ \vartheta_i \in R,\ \ i = 1,2. \tag{6}$$

$(v)^*$ $\hbar_1 : I \times R \to R$ and $\hbar_2 : I \times R \to R \setminus \{0\}$ are measurable in $t \in I\ \forall \xi \in R$ and satisfy the Lipschitz condition such that

$$|\hbar_i(t,\xi_1) - \hbar_i(t,\xi_2)| \leq a_i|\xi_1 - \xi_2|,\ t \in I,\ \ \xi_i \in R,\ \ i = 1,2. \tag{7}$$

Moreover, h_i, i = 1,2 are nondecreasing for every nondecreasing argument.
From Equation (4), we have

$$|f(t,\xi_1,\vartheta_1,\omega_1)| - |f(t,0,0,0)| \leq |f(t,\xi_1,\vartheta_1,\omega_1) - f(t,0,0,0)| \leq b(|\xi_1| + |\vartheta_1| + |\omega_1|),$$

$$|f(t,\xi_1,\vartheta_1,\omega_1)| \leq |f(t,0,0,0)| + b(|\xi_1| + |\vartheta_1| + |\omega_1|)$$

and

$$|f(t,\xi_1,\vartheta_1,\omega_1)| \leq \|m\| + b(|\xi_1| + |\vartheta_1| + |\omega_1|).$$

Additionally, from Equations (5)–(7), we obtain

$$|g_i(t,\vartheta_1)| \leq |g_i(t,0)| + b_i|\vartheta_1|,\ \ |f_i(t,\vartheta_1)| \leq |f_i(t,0)| + c_i|\vartheta_1|\ \ \textit{and}$$
$$|\hbar_i(t,\xi_1)| \leq |\hbar_i(t,0)| + a_i|\xi_1|.$$

So, we have proved the following Lemma.

Lemma 1. *Assumptions $(i)^*$ and $(v)^*$ imply assumptions (i) and (v), respectively.*

Theorem 4. *Let assumptions $(i)^*$, (ii), $(v)^*$, and (vi) be satisfied. If*

$$bb_1 + bc_1 \diagup T + bb_2(\|k_2\| + c_2r_1) + bc_2(\|m_2\| + b_2r_1) < 1,$$

then the solution to problems (1) and (2) is unique.

Proof. Firstly, let Υ_1, Υ_2 be two solutions in Q_{r_1} to (2). Then,

$$\begin{aligned}
& |\Upsilon_2(\mathfrak{r}) - \Upsilon_1(\mathfrak{r})| = \\
& \left|f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_2(\mathfrak{r})),\ \diagup\int_0^{\mathfrak{r}} f_1(s,\Upsilon_2(s))ds,\ g_2(\mathfrak{r},\Upsilon_2(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_2(s))ds\right)\right. \\
- \quad & \left. f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_1(\mathfrak{r})),\ \diagup\int_0^{\mathfrak{r}} f_1(s,\Upsilon_1(s))ds,\ g_2(\mathfrak{r},\Upsilon_1(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_1(s))ds\right)\right| \\
\leq \quad & b\left|g_1(\mathfrak{r},\Upsilon_2(\mathfrak{r})) - g_1(\mathfrak{r},\Upsilon_1(\mathfrak{r}))\right| + b \diagup\int_0^{\mathfrak{r}} |f_1(s,\Upsilon_2(s)) - f_1(s,\Upsilon_1(s))|ds \\
+ \quad & b\left|g_2(\mathfrak{r},\Upsilon_2(\mathfrak{r})) - g_2(\mathfrak{r},\Upsilon_1(\mathfrak{r}))\right|.\int_0^{\mathfrak{r}} |f_2(s,\Upsilon_2(s))|ds \\
+ \quad & b|g_2(\mathfrak{r},\Upsilon_1(\mathfrak{r}))|.\int_0^{\mathfrak{r}} |f_2(s,\Upsilon_2(s)) - f_2(s,\Upsilon_1(s))|ds \\
\leq \quad & bb_1|\Upsilon_2(\mathfrak{r}) - \Upsilon_1(\mathfrak{r})| + b \diagup c_1\|\Upsilon_2 - \Upsilon_1\| + bb_2|\Upsilon_2(\mathfrak{r}) - \Upsilon_1(\mathfrak{r})|.(\|k_2\| + c_2r_1) \\
+ \quad & bc_2\|\Upsilon_2 - \Upsilon_1\|(|m_2(\mathfrak{r})| + b_2|\Upsilon(\mathfrak{r})|).
\end{aligned}$$

Then,

$$\begin{aligned}\int_0^T |\Upsilon_2(\mathfrak{r})-\Upsilon_1(\mathfrak{r})|d\mathfrak{r} &\leq bb_1\int_0^T |\Upsilon_2(\mathfrak{r})-\Upsilon_1(\mathfrak{r})|d\mathfrak{r}+b\nleftrightarrow c_1\|\Upsilon_2-\Upsilon_1\|\int_0^T d\mathfrak{r}\\ &+ bb_2(\|k_2\|+c_2r_1)\int_0^T |\Upsilon_2(\mathfrak{r})-\Upsilon_1(\mathfrak{r})|d\mathfrak{r}\\ &+ bc_2\|\Upsilon_2-\Upsilon_1\|.\left(\int_0^T |m_2(\mathfrak{r})|d\mathfrak{r}+b_2\int_0^T |\Upsilon(\mathfrak{r})|d\mathfrak{r}\right)\\ \|\Upsilon_2-\Upsilon_1\| &\leq bb_1\|\Upsilon_2-\Upsilon_1\|+bc_1\nleftrightarrow\|\Upsilon_2-\Upsilon_1\|T\\ &+ bb_2(\|k_2\|+c_2r_1)\|\Upsilon_2-\Upsilon_1\|+bc_2\|\Upsilon_2-\Upsilon_1\|.(\|m_2\|+b_2r_1).\end{aligned}$$

Hence,

$$\|\Upsilon_2-\Upsilon_1\|\left(1-(bb_1+bc_1\nleftrightarrow T+bb_2(\|k_2\|+c_2r_1)+bc_2(\|m_2\|+b_2r_1))\right)\leq 0,$$

then $\Upsilon_1=\Upsilon_2$ and the solution of (2) is unique.

Next, let ψ_1,ψ_2 be two solutions in Q_{r_2} of (1), then

$$\begin{aligned}&|\psi_2(\mathfrak{r})-\psi_1(\mathfrak{r})|=\\ &\left|\hbar_1(\mathfrak{r},\psi_2(\phi_1(\mathfrak{r})))+\Upsilon(\mathfrak{r})\hbar_2(\mathfrak{r},\psi_2(\phi_2(\mathfrak{r})))-\hbar_1(\mathfrak{r},\psi_1(\phi_1(\mathfrak{r})))-\Upsilon(\mathfrak{r})\hbar_2(\mathfrak{r},\psi_1(\phi_2(\mathfrak{r})))\right|\\ \leq\ &\left|\hbar_1(\mathfrak{r},\psi_2(\phi_1(\mathfrak{r})))-\hbar_1(\mathfrak{r},\psi_1(\phi_1(\mathfrak{r})))\right|+|\Upsilon(\mathfrak{r})|\left|\hbar_2(\mathfrak{r},\psi_2(\phi_2(\mathfrak{r})))-\hbar_2(\mathfrak{r},\psi_1(\phi_2(\mathfrak{r})))\right|\\ \leq\ &a_1|\psi_2(\phi_1(\mathfrak{r}))-\psi_1(\phi_1(\mathfrak{r}))|+Ma_2|\psi_2(\phi_2(\mathfrak{r}))-\psi_1(\phi_2(\mathfrak{r}))|,\end{aligned}$$

Then,

$$\int_0^T |\psi_2(\mathfrak{r})-\psi_1(\mathfrak{r})|d\mathfrak{r}\leq\frac{a_1}{N_1}\int_0^T |\psi_2(\theta_1)-\psi_1(\theta_1)|d\theta_1+\frac{Ma_2}{N_2}\int_0^T |\psi_2(\theta_2)-\psi_1(\theta_2)|d\theta_2.$$

Hence

$$\|\psi_2-\psi_1\|\left(1-(\frac{a_1}{N_1}+\frac{Ma_2}{N_2})\right)\leq 0.$$

Then, $\psi_1=\psi_2$, and the solution to (1) is unique. □

2.4. Hyers–Ulam Stability

Now, replace the assumption (v) by $(v)^{**}$ as follows:

$(v)^{**}$ $\hbar_1: I\times R\to R$ is measurable in $t\in I$ for any $\xi\in R$ and continuous in $\xi\in R$ for all $t\in I$, and $\hbar_2: I\times R\to R\setminus\{0\}$ is measurable in $t\in I$ for any $\xi\in R$ and continuous in $\xi\in R$ for all $t\in I$. Moreover, there exist a bounded and measurable $\nu(t)$ such that $|\nu(t)|\leq\nu$, and they satisfy the Lipschitz condition such that

$$|\hbar_i(t,\xi_1)-\hbar_i(t,\xi_2)|\leq a_i|\xi_1-\xi_2|,\ t\in I,\ \ \xi_i\in R,\ \ i=1,2.$$

Moreover, $\hbar_i$ is nondecreasing for every nondecreasing argument.

Definition 1. *The functional Equation (1) is Hyers–Ulam stable if* $\forall\epsilon>0,\ \exists\ \delta(\epsilon)$ *such that, for any* δ*-approximate solution* ψ_s *to (1), it satisfies*

$$\left\|\psi_s(\mathfrak{r})-\left(\hbar_1(\mathfrak{r},\psi_s(\phi_1(\mathfrak{r})))+\Upsilon_s(\mathfrak{r})\hbar_2(\mathfrak{r},\psi_s(\phi_2(\mathfrak{r})))\right)\right\|<\delta.\ \ \textit{Then},\ \ \|\psi-\psi_s\|<\epsilon$$

where Υ_s *is an approximate solution to (2) such that*

$$\left\| \Upsilon_s(\mathfrak{r}) - f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_s(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon_s(s))ds,\ g_2(\mathfrak{r},\Upsilon_s(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_s(s))ds\right)\right\| < \delta.$$

Then, $\| \Upsilon - \Upsilon_s \| < \epsilon_1$.

Theorem 5. *Let the assumptions of Theorem 4 be satisfied. Assume that* $(v)^{**}$ *is met; then, problems (1) and (2) are Hyers–Ulam stable.*

Proof. Firstly,

$$\begin{aligned}
& | \Upsilon(\mathfrak{r}) - \Upsilon_s(\mathfrak{r})| = \\
& \left|f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds,\ g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\right) - \Upsilon_s(\mathfrak{r})\right| \\
\leq\ & \left|f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds,\ g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\right)\right. \\
-\ & f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_s(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon_s(s))ds,\ g_2(\mathfrak{r},\Upsilon_s(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_s(s))ds\right) \\
-\ & \left.\Upsilon_s(\mathfrak{r}) + f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_s(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon_s(s))ds,\ g_2(\mathfrak{r},\Upsilon_s(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_s(s))ds\right)\right| \\
\leq\ & b\left|g_1(\mathfrak{r},\Upsilon(\mathfrak{r})) - g_1(\mathfrak{r},\Upsilon_s(\mathfrak{r}))\right| + b\succ\int_0^{\mathfrak{r}} |f_1(s,\Upsilon(s)) - f_1(s,\Upsilon_s(s))|ds \\
+\ & b\left|g_2(\mathfrak{r},\Upsilon(\mathfrak{r})) - g_2(\mathfrak{r},\Upsilon_s(\mathfrak{r}))\right|.\int_0^{\mathfrak{r}} |f_2(s,\Upsilon(s))|ds \\
+\ & b|g_2(\mathfrak{r},\Upsilon_s(\mathfrak{r}))|.\int_0^{\mathfrak{r}} |f_2(s,\Upsilon(s)) - f_2(s,\Upsilon_s(s))|ds \\
+\ & \left|\Upsilon_s(\mathfrak{r}) - f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_s(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon_s(s))ds,\ g_2(\mathfrak{r},\Upsilon_s(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_s(s))ds\right)\right| \\
\leq\ & bb_1|\Upsilon(\mathfrak{r}) - \Upsilon_s(\mathfrak{r})| + b\succ c_1\|\Upsilon - \Upsilon_s\| + bb_2|\Upsilon(\mathfrak{r}) - \Upsilon_s(\mathfrak{r})|.(\|k_2\| + c_2r_1) \\
+\ & b(|m_2(\mathfrak{r})| + b_2|\Upsilon_s(\mathfrak{r})|)c_2\|\Upsilon - \Upsilon_s\| \\
+\ & \left|\Upsilon_s(\mathfrak{r}) - f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon_s(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon_s(s))ds,\ g_2(\mathfrak{r},\Upsilon_s(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon_s(s))ds\right)\right|,
\end{aligned}$$

Then,

$$\begin{aligned}
\int_0^T |\Upsilon(\mathfrak{r}) - \Upsilon_s(\mathfrak{r})|d\mathfrak{r} \quad &\leq \quad bb_1\int_0^T |\Upsilon(\mathfrak{r}) - \Upsilon_s(\mathfrak{r})|d\mathfrak{r} + b\succ c_1\|\Upsilon - \Upsilon_s\|\int_0^T d\mathfrak{r} \\
&+ \quad bb_2(\|k_2\| + c_2r_1)\int_0^T |\Upsilon(\mathfrak{r}) - \Upsilon_s(\mathfrak{r})|d\mathfrak{r} \\
&+ \quad bc_2\|\Upsilon - \Upsilon_s\|(\int_0^T |m_2(\mathfrak{r})|d\mathfrak{r} + b_2\int_0^T |\Upsilon_s(\mathfrak{r})|d\mathfrak{r}) + \delta \\
\|\Upsilon - \Upsilon_s\| \quad &\leq \quad bb_1\|\Upsilon - \Upsilon_s\| + b\succ c_1\|\Upsilon - \Upsilon_s\|T \\
&+ \quad bb_2(\|k_2\| + c_2r_1)\|\Upsilon - \Upsilon_s\| + bc_2\|\Upsilon - \Upsilon_s\|(\|m_2\| + b_2r_1) + \delta.
\end{aligned}$$

Hence,

$$\|\Upsilon - \Upsilon_s\| \quad \leq \quad \frac{\delta}{1 - \left(bb_1 + bc_1\succ T + bb_2(\|k_2\| + c_2r_1) + bc_2(\|m_2\| + b_2r_1)\right)} = \epsilon_1. \quad (8)$$

Next,

$$\begin{aligned}
& |\psi(\mathfrak{r}) - \psi_s(\mathfrak{r})| = \\
& \left|\hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \Upsilon(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) - \psi_s(\mathfrak{r})\right| \\
\leq \quad & \Big|\hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \Upsilon(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) - \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r}))) - \Upsilon_s(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))\\
- \quad & \psi_s(\mathfrak{r}) + \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r}))) + \Upsilon_s(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))\Big| \\
\leq \quad & \left|\hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) - \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r})))\right| + |\ \Upsilon\ (\mathfrak{r}) - \Upsilon_s(\mathfrak{r})||\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))| \\
+ \quad & |\ \Upsilon_s\ (\mathfrak{r})|\left|\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) - \hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))\right| \\
+ \quad & |\psi_s(\mathfrak{r}) - (\hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r}))) + \Upsilon_s(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r}))))| \\
\leq \quad & a_1|\psi(\phi_1(\mathfrak{r})) - \psi_s(\phi_1(\mathfrak{r}))| + \nu|\ \Upsilon\ (\mathfrak{r}) - \Upsilon_s(\mathfrak{r})| + Ma_2|\psi(\phi_2(\mathfrak{r})) - \psi_s(\phi_2(\mathfrak{r}))| \\
+ \quad & |\psi_s(\mathfrak{r}) - (\hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r}))) + \Upsilon_s(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r}))))|,
\end{aligned}$$

Then,

$$\begin{aligned}
\int_0^T |\psi(\mathfrak{r}) - \psi_s(\mathfrak{r})|d\mathfrak{r} \quad \leq \quad & \frac{a_1}{N_1}\int_0^T |\psi(\theta_1) - \psi_s(\theta_1)|d\theta_1 + \nu\int_0^T |\ \Upsilon\ (\mathfrak{r}) - \Upsilon_s(\mathfrak{r})|d\mathfrak{r} \\
+ \quad & \frac{Ma_2}{N_2}\int_0^T |\psi(\theta_2) - \psi_s(\theta_2)|d\theta_2 + \delta.
\end{aligned}$$

Hence,

$$\|\psi - \psi_s\| \leq \frac{\nu\|\ \Upsilon - \Upsilon_s\ \| + \delta}{1 - (\frac{a_1}{N_1} + \frac{Ma_2}{N_2})}.$$

From (8), we have

$$\|\ \Upsilon - \Upsilon_s\ \| \ \leq \ \epsilon_1.$$

Then,

$$\|\psi - \psi_s\| \leq \frac{\nu\epsilon_1 + \delta}{1 - (\frac{a_1}{N_1} + \frac{Ma_2}{N_2})} = \epsilon.$$

□

2.5. Continuous Dependence on Constraint

Theorem 6. *Let the assumptions of Theorem 4 be satisfied for* $\nwarrow, \nwarrow^*, g_i$ *and* g_i^*, $i = 1, 2$. *Then the unique solution* $\Upsilon \in L_1(I)$ *depends continuously on* $\nwarrow, g_1$, *and* g_2 *in the sense that*

$$\forall \epsilon > 0,\ \exists\ \delta(\epsilon)\ \ \textit{such that}$$

$$max\{|\nwarrow - \nwarrow^*|,\ |g_1(\mathfrak{r}, \Upsilon) - g_1^*(\mathfrak{r}, \Upsilon)|,\ |g_2(\mathfrak{r}, \Upsilon) - g_2^*(\mathfrak{r}, \Upsilon)|\} < \delta, then\ \ \|\ \Upsilon - \Upsilon^*\ \| < \epsilon^*.$$

where Υ^* *is a solution to*

$$\Upsilon^*(\mathfrak{r}) = f\left(\mathfrak{r},\ \ g_1^*(\mathfrak{r}, \Upsilon^*(\mathfrak{r})),\ \ \nwarrow^* \int_0^{\mathfrak{r}} f_1(s, \Upsilon^*(s))ds,\ \ g_2^*(\mathfrak{r}, \Upsilon^*(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s, \Upsilon^*(s))ds\right).$$

Proof.

$$\begin{aligned}
& |\Upsilon(\mathfrak{r}) - \Upsilon^*(\mathfrak{r})| = \\
& \left| f\left(\mathfrak{r},\ g_1(\mathfrak{r}, \Upsilon(\mathfrak{r})),\ \succ \int_0^{\mathfrak{r}} f_1(s, \Upsilon(s))ds,\ g_2(\mathfrak{r}, \Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s, \Upsilon(s))ds\right)\right. \\
- \quad & \left. f\left(\mathfrak{r},\ g_1^*(\mathfrak{r}, \Upsilon^*(\mathfrak{r})), \succ^* \int_0^{\mathfrak{r}} f_1(s, \Upsilon^*(s))ds,\ g_2^*(\mathfrak{r}, \Upsilon^*(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s, \Upsilon^*(s))ds\right)\right| \\
\leq \quad & b\left|g_1(\mathfrak{r}, \Upsilon(\mathfrak{r})) - g_1^*(\mathfrak{r}, \Upsilon(\mathfrak{r}))\right| + b\left|g_1^*(\mathfrak{r}, \Upsilon(\mathfrak{r})) - g_1^*(\mathfrak{r}, \Upsilon^*(\mathfrak{r}))\right| \\
+ \quad & b|\succ - \succ^*|.\int_0^{\mathfrak{r}} |f_1(s, \Upsilon(s))|ds + b\succ^* \int_0^{\mathfrak{r}} |f_1(s, \Upsilon(s))ds - f_1(s, \Upsilon^*(s))|ds \\
+ \quad & b\left|g_2(\mathfrak{r}, \Upsilon(\mathfrak{r})) - g_2^*(\mathfrak{r}, \Upsilon(\mathfrak{r}))\right|.\int_0^{\mathfrak{r}} |f_2(s, \Upsilon(s))|ds \\
+ \quad & b\left|g_2^*(\mathfrak{r}, \Upsilon(\mathfrak{r})) - g_2^*(\mathfrak{r}, \Upsilon^*(\mathfrak{r}))\right|.\int_0^{\mathfrak{r}} |f_2(s, \Upsilon(s))|ds \\
+ \quad & b|g_2^*(\mathfrak{r}, \Upsilon^*(\mathfrak{r}))|.\int_0^{\mathfrak{r}} |f_2(s, \Upsilon(s)) - f_2(s, \Upsilon^*(s))|ds \\
\leq \quad & b\delta + bb_1|\Upsilon(\mathfrak{r}) - \Upsilon^*(\mathfrak{r})| + b\delta(\|k_1\| + c_1r_1) + bc_1\succ^* \|\Upsilon - \Upsilon^*\| + b\delta(\|k_2\| + c_2r_1) \\
+ \quad & bb_2|\Upsilon(\mathfrak{r}) - \Upsilon^*(\mathfrak{r})|.(\|k_2\| + c_2r_1) + bc_2(|m_2(\mathfrak{r})| + b_2|\Upsilon^*(\mathfrak{r})|).\|\Upsilon - \Upsilon^*\|,
\end{aligned}$$

Then,

$$\begin{aligned}
\int_0^T |\Upsilon(\mathfrak{r}) - \Upsilon^*(\mathfrak{r})|d\mathfrak{r} \quad = \quad & b\delta\int_0^T d\mathfrak{r} + bb_1\int_0^T |\Upsilon(\mathfrak{r}) - \Upsilon^*(\mathfrak{r})|d\mathfrak{r} + b\delta(\|k_1\| + c_1r_1)\int_0^T d\mathfrak{r} \\
+ \quad & bc_1\succ^* \|\Upsilon - \Upsilon^*\|\int_0^T d\mathfrak{r} + b\delta(\|k_2\| + c_2r_1)\int_0^T d\mathfrak{r} \\
+ \quad & bb_2(\|k_2\| + c_2r_1)\int_0^T |\Upsilon(\mathfrak{r}) - \Upsilon^*(\mathfrak{r})|d\mathfrak{r} \\
+ \quad & bc_2\|\Upsilon - \Upsilon^*\|(\int_0^T |m_2(\mathfrak{r})|d\mathfrak{r} + b_2\int_0^T |\Upsilon^*(\mathfrak{r})|d\mathfrak{r})
\end{aligned}$$

and

$$\begin{aligned}
\|\Upsilon - \Upsilon^*\| \quad \leq \quad & b\delta T + bb_1\|\Upsilon - \Upsilon^*\| + b\delta(\|k_1\| + c_1r_1)T + bc_1\succ^* \|\Upsilon - \Upsilon^*\|T + b\delta(\|k_2\| + c_2r_1)T \\
+ \quad & bb_2(\|k_2\| + c_2r_1)\|\Upsilon - \Upsilon^*\| + bc_2\|\Upsilon - \Upsilon^*\|(\|m_2\| + b_2r_1).
\end{aligned}$$

Hence,

$$\|\Upsilon - \Upsilon^*\| \quad \leq \quad \frac{\Big(bT + bT(\|k_1\| + c_1r_1) + bT(\|k_2\| + c_2r_1)\Big)\delta}{1 - (bb_1 + bc_1\succ^* T + bb_2(\|k_2\| + c_2r_1) + bc_2(\|m_2\| + b_2r_1))} = \epsilon^*.$$

□

2.6. Dependence of ψ on Υ

Theorem 7. *Let the assumptions of Theorem 5 be satisfied. Then, the unique solution $\psi \in L_1(I)$ depends continuously on Υ in the sense that*

$$\forall\epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad \textit{such that if} \quad |\Upsilon(\mathfrak{r}) - \Upsilon^*(\mathfrak{r})| < \delta, \quad \textit{then} \quad \|\psi - \psi^*\| < \epsilon$$

where ψ^* is the solution to

$$\psi^*(\mathfrak{r}) = \hbar_1(\mathfrak{r}, \psi^*(\phi_1(\mathfrak{r}))) + \Upsilon^*(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi^*(\phi_2(\mathfrak{r}))) \tag{9}$$

and Υ^* *is the solution to*

$$\Upsilon^*(\mathfrak{r}) = f\left(\mathfrak{r},\ g_1(\mathfrak{r}, \Upsilon^*(\mathfrak{r})), \lambda \int_0^{\mathfrak{r}} f_1(s, \Upsilon^*(s))ds,\ g_2(\mathfrak{r}, \Upsilon^*(\mathfrak{r})). \int_0^{\mathfrak{r}} f_2(s, \Upsilon^*(s))ds\right).$$

Proof. Let $\delta > 0$ be given, and let ψ^* be the solution to (9). Then,

$$\begin{aligned}
& |\psi(\mathfrak{r}) - \psi^*(\mathfrak{r})| = \\
& \left|\hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \Upsilon(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) - \hbar_1(\mathfrak{r}, \psi^*(\phi_1(\mathfrak{r}))) - \Upsilon^*(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi^*(\phi_2(\mathfrak{r})))\right| \\
\leq\ & \left|\hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) - \hbar_1(\mathfrak{r}, \psi^*(\phi_1(\mathfrak{r})))\right| + |\ \Upsilon\ (\mathfrak{r}) - \Upsilon^*(\mathfrak{r})||\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))| \\
+\ & |\ \Upsilon^*\ (\mathfrak{r})|\left|\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) - \hbar_2(\mathfrak{r}, \psi^*(\phi_2(\mathfrak{r})))\right| \\
\leq\ & a_1|\psi(\phi_1(\mathfrak{r})) - \psi^*(\phi_1(\mathfrak{r}))| + \nu|\ \Upsilon\ (\mathfrak{r}) - \Upsilon^*(\mathfrak{r})| + Ma_2|\psi(\phi_2(\mathfrak{r})) - \psi^*(\phi_2(\mathfrak{r}))|,
\end{aligned}$$

Then,

$$\begin{aligned}
\int_0^T |\psi(\mathfrak{r}) - \psi^*(\mathfrak{r})|d\mathfrak{r} \quad \leq\quad & \frac{a_1}{N_1}\int_0^T |\psi(\theta_1) - \psi^*(\theta_1)|d\theta_1 + \nu\int_0^T |\ \Upsilon\ (\mathfrak{r}) - \Upsilon^*(\mathfrak{r})|d\mathfrak{r} \\
+\quad & \frac{Ma_2}{N_2}\int_0^T |\psi(\theta_2) - \psi^*(\theta_2)|d\theta_2
\end{aligned}$$

and

$$\|\psi_2 - \psi_1\| \leq \frac{\nu\|\ \Upsilon - \Upsilon^*\ \|}{1 - (\frac{a_1}{N_1} + \frac{Ma_2}{N_2})},$$

Then,

$$\|\psi_2 - \psi_1\| \leq \frac{\nu\epsilon^*}{1 - (\frac{a_1}{N_1} + \frac{Ma_2}{N_2})} = \epsilon.$$

□

Corollary 3. *The solution* $\psi \in L_1(I)$ *to problems (1) and (2) depends continuously on the functions* g_i *and* $i = 1, 2$ *and on the parameter* λ.

Proof. From Theorem 6, the results follow. □

3. Hybrid Functional Integral Equation

Here, we examine the equivalence between problem (1) and (2) and the IHFIE (3).
(i) Let $\psi \in L_1(I)$ be the solution to (3).
Putting

$$\Upsilon(\mathfrak{r}) = \frac{\psi(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))}, \tag{10}$$

then the solution $\psi \in L_1(I)$ to (3) is given by (1) where $\Upsilon \in L_1(I)$ is the solution to (2).
(ii) Let $\psi \in L_1(I)$ be the solution to (1); then, we obtain

$$\Upsilon(\mathfrak{r}) = \frac{\psi(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))},$$

By substituting this into (2), we obtain the IHFIE (3).
Then, we have proved the following.

Corollary 4. *Problems (1) and (2) and the IHFIE (3) are equivalences.*

Corollary 5. *Let the assumptions of Theorems 2–6 be satisfied. Then,*

(i) The IHFIE (3) has a unique nondecreasing solution $\psi \in L_1(I)$.

(ii) The solution $\psi \in L_1(I)$ to (3) depends continuously on the parameter $\succ$ and on the two functions g_1 and g_2.

Hyers–Ulam Stability

Definition 2. *The IHFIE (3) is Hyers–Ulam stable if $\forall \epsilon > 0, \exists\ \delta(\epsilon)$ such that, for any δ-approximate solution ψ_s to (3), it satisfies*

$$\begin{aligned} &\left\| \frac{\psi_s(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))} - \right. \\ &f\left(\mathfrak{r},\ g_1\left(\mathfrak{r}, \frac{\psi_s(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))}, \succ \int_0^{\mathfrak{r}} f_1\left(s, \frac{\psi_s(s) - \hbar_1(s, \psi_s(\phi_1(s)))}{\hbar_2(s, \psi_s(\phi_2(s)))}\right) ds, \right. \\ &\left. \left. g_2\left(\mathfrak{r}, \frac{\psi_s(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))}\right) . \int_0^{\mathfrak{r}} f_2\left(s, \frac{\psi_s(s) - \hbar_1(s, \psi_s(\phi_1(s)))}{\hbar_2(s, \psi_s(\phi_2(s)))}\right) ds \right) \right\| < \delta, \end{aligned} \tag{11}$$

then $\|\psi - \psi_s\| < \epsilon$.

Theorem 8. *Let the assumptions of Theorem 5 be satisfied; then, the IHFIE (3) is Hyers–Ulam stable.*

Proof. From (10) and (11), we can deduce that

$$\left\| \Upsilon_s(\mathfrak{r}) - f\left(\mathfrak{r},\ g_1(\mathfrak{r}, \Upsilon_s(\mathfrak{r})),\ \succ \int_0^{\mathfrak{r}} f_1(s, \Upsilon_s(s)) ds,\ g_2(\mathfrak{r}, \Upsilon_s(\mathfrak{r})) . \int_0^{\mathfrak{r}} f_2(s, \Upsilon_s(s)) ds \right) \right\| < \delta,$$

Then, from the proof of Theorem 5, we can obtain (8)

$$\| \Upsilon - \Upsilon_s \| \leq \frac{\delta}{1 - \left(bb_1 + bc_1 \succ T + bb_2(\|k_2\| + c_2 r_1) + bc_2(\|m_2\| + b_2 r_1) \right)} = \epsilon_1.$$

Now,

$$\begin{aligned} & |\psi(\mathfrak{r}) - \psi_s(\mathfrak{r})| = \\ & \left| \frac{\psi(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))} . \hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) + \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) \right. \\ - & \left. \left(\frac{\psi_s(\mathfrak{r}) - \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r})))}{\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))} . h_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r}))) \right) - \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r}))) \right| \\ \leq & \left| \Upsilon(\mathfrak{r}) \hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) - \Upsilon_s(\mathfrak{r}) \hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r}))) \right| + \left| \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) - \hbar_1(\mathfrak{r}, \psi_s(\phi_1(\mathfrak{r}))) \right| \\ \leq & \left| \Upsilon(\mathfrak{r}) \hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) - \Upsilon(\mathfrak{r}) \hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r}))) \right| \\ + & \left| \Upsilon(\mathfrak{r}) \hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r}))) - \Upsilon_s(\mathfrak{r}) \hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r}))) \right| \\ + & a_1 |\psi(\phi_1(\mathfrak{r})) - \psi_s(\phi_1(\mathfrak{r}))| \\ \leq & |\Upsilon(\mathfrak{r})| |\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))) - \hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))| + |\Upsilon(\mathfrak{r}) - \Upsilon_s(\mathfrak{r})| |\hbar_2(\mathfrak{r}, \psi_s(\phi_2(\mathfrak{r})))| \\ + & a_1 |\psi(\phi_1(\mathfrak{r})) - \psi_s(\phi_1(\mathfrak{r}))| \\ \leq & M a_2 |\psi(\phi_2(\mathfrak{r})) - \psi_s(\phi_2(\mathfrak{r}))| + \nu |\Upsilon(\mathfrak{r}) - \Upsilon_s(\mathfrak{r})| + a_1 |\psi(\phi_1(\mathfrak{r})) - \psi_s(\phi_1(\mathfrak{r}))| \end{aligned}$$

Then,

$$\begin{aligned}\int_0^T |\psi(\mathfrak{r}) - \psi_s(\mathfrak{r})|d\mathfrak{r} \quad &\leq \quad \frac{Ma_2}{N_2}\int_0^T |\psi(\theta_2) - \psi_s(\theta_2)|d\theta_2 + \nu\int_0^T |\ \Upsilon\ (\mathfrak{r}) - \Upsilon_s(\mathfrak{r})|d\mathfrak{r} \\ &+ \quad \frac{a_1}{N_1}\int_0^T |\psi(\theta_1) - \psi_s(\theta_1)|d\theta_1.\end{aligned}$$

Hence,

$$\|\psi - \psi_s\| \leq \frac{\nu\|\ \Upsilon - \Upsilon_s\ \|}{1 - (\frac{a_1}{N_1} + \frac{Ma_2}{N_2})},$$

From (8), we have

$$\|\ \Upsilon - \Upsilon_s\ \| \ \leq \ \epsilon_1,$$

Then,

$$\|\psi - \psi_s\| \leq \frac{\nu\epsilon_1}{1 - (\frac{a_1}{N_1} + \frac{Ma_2}{N_2})} = \epsilon.$$

□

4. Solvability in Unbounded Interval $L_1(R_+)$

Here, we examine the existence of solutions, $\Upsilon \in L_1(R_+)$, to (2). Take into account the following assumptions:

(vii) $f : R_+ \times R^3 \to R$ and $g_i : R_+ \times R \to R$, i=1,2 are Carathéodory functions [21], and there exist bounded and integrable functions $m,\ m_i, q \in L_1(R_+)$ where $|m(t)| \leq m$, $|m_i(t)| \leq m_i$, $|q(t)| \leq q$, and $b, b_i \in R_+$ such that

$$|f(t,\xi,\vartheta,\omega)| \leq |m(t)| + b|\xi| + q(t)|\vartheta| + b|\omega| \ \ and \ \ |g_i(t,\vartheta)| \leq |m_i(t)| + b_i|\vartheta|$$

and

$$\lim_{t\to\infty} q(t) = 0, \quad \sup_{t\in R_+}\int_0^t |q(s)|ds = q^*.$$

Moreover, f is nonincreasing for every nondecreasing argument such that $t_1 < t_2$, and, for all $\xi_1 < \xi_2$, $\vartheta_1 < \vartheta_2$, $\omega_1 < \omega_2$ implies

$$\mathbf{f}(t_1,\xi_1,\vartheta_1,\omega_1) \geq \mathbf{f}(t_2,\xi_2,\vartheta_2,\omega_2)$$

and g_i is nondecreasing for every nondecreasing argument.

(viii) $f_i : R_+ \times R \to R_+$, i=1,2 are Carathéodory functions [21], and there exist bounded and integrable functions $k_i \in L_1(R_+)$ and $c_i \geq 0$ such that

$$|f_i(t,\vartheta)| \leq |k_i(t)| + c_i|\vartheta|.$$

(ix) There exists a positive root ρ_1 of the algebraic equation

$$bb_2c_2\rho_1^2 + (bb_1 + q^* \diagdown c_1 + bc_2m_2^* + bb_2k_2^* - 1)\rho_1 + m^* + bm_1^* + q^* \diagdown k_1^* + bk_2^*m_2^* \ = \ 0.$$

Theorem 9. *Let assumptions (vii)–(ix) hold if* $2(bb_1 + bb_2(k_2^* + c_2\rho_1)) < 1$. *Then, the functional integral Equation (2) has at least one nonincreasing solution* $\Upsilon \in L_1(R_+)$ *to (2).*

Proof. Let B_{ρ_1} be a closed ball of all nonincreasing functions

$$B_{\rho_1} = \{\Upsilon \in L_1(R_+) : \| \Upsilon \| \leq \rho_1\}.$$

Associate the operator

$$K\Upsilon(\mathfrak{r}) = f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds,\ g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\right),$$

Then, we deduce that K transforms nonincreasing functions into other nonincreasing functions, and $K\Upsilon$ is nonincreasing on R_+.

Let $\Upsilon \in B_{\rho_1}$. Then,

$$\begin{aligned}
|K\Upsilon(\mathfrak{r})| &= \left|f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds,\ g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\right)\right| \\
&\leq |m(\mathfrak{r})| + b\Big(|m_1(\mathfrak{r})| + b_1|\Upsilon(\mathfrak{r})|\Big) + |q(\mathfrak{r})| \succ \left(\int_0^{\mathfrak{r}} (k_1(s) + c_1|\Upsilon(s)|)ds\right) \\
&+ b\Big(|m_2(\mathfrak{r})| + b_2|\Upsilon(t)|\Big).\int_0^{\mathfrak{r}} (k_2(s) + c_2|\Upsilon(s)|)ds \\
&\leq |m(\mathfrak{r})| + b(|m_1(\mathfrak{r})| + b_1|\Upsilon(\mathfrak{r})|) + |q(\mathfrak{r})| \succ (k_1^* + c_1\rho_1) \\
&+ b(|m_2(\mathfrak{r})| + b_2|\Upsilon(\mathfrak{r})|).(k_2^* + c_2\rho_1),
\end{aligned} \tag{12}$$

Then,

$$\begin{aligned}
\int_0^{\infty} |K\Upsilon(\mathfrak{r})|d\mathfrak{r} &\leq \int_0^{\infty} |m(\mathfrak{r})|d\mathfrak{r} + b(\int_0^{\infty} |m_1(\mathfrak{r})|d\mathfrak{r} + b_1\int_0^{\infty} |\Upsilon(\mathfrak{r})|d\mathfrak{r}) + \succ (k_1^* + c_1\rho_1)\int_0^{\infty} |q(\mathfrak{r})|d\mathfrak{r} \\
&+ b(k_2^* + c_2\rho_1).(\int_0^{\infty} |m_2(\mathfrak{r})|d\mathfrak{r} + b_2\int_0^{\infty} |\Upsilon(\mathfrak{r})|d\mathfrak{r}) \\
\|K\Upsilon\| &\leq m^* + bm_1^* + bb_1\rho_1 + q^* \succ (k_1^* + c_1\rho_1) + b(k_2^* + c_2\rho_1).(m_2^* + b_2\rho_1) = \rho_1.
\end{aligned}$$

Thus, the operator K maps the ball B_{ρ_1} into itself. Moreover, our assumptions imply that the operator K is continuous on B_{ρ_1}.

Now, let Ω be nonempty subset of B_{ρ_1}. Fix $\epsilon > 0$ and take a measurable set $D \subset R_+$ such that $D \leq \epsilon$. Then, for any $\Upsilon \in \Omega$, we obtain

$$\begin{aligned}
\|K\Upsilon\|_{L_1(D)} &\leq \int_D |K\Upsilon(\mathfrak{r})|d\mathfrak{r} \\
&\leq \int_D |m(\mathfrak{r})|d\mathfrak{r} + b(\int_D |m_1(\mathfrak{r})|d\mathfrak{r} + b_1\int_D |\Upsilon(\mathfrak{r})|d\mathfrak{r}) \\
&+ \succ (k_1^* + c_1\rho_1)\int_D |q(\mathfrak{r})|d\mathfrak{r} + b(k_2^* + c_2\rho_1).(\int_D |m_2(\mathfrak{r})|d\mathfrak{r} + b_2\int_D |\Upsilon(\mathfrak{r})|d\mathfrak{r}),
\end{aligned}$$

Then,

$$\begin{aligned}
\beta(K\Upsilon(\mathfrak{r})) &= \lim_{\epsilon\to 0} \|K\Upsilon\|_{L_1(D)} \\
\beta(K\Upsilon(\mathfrak{r})) &\leq bb_1\beta(\Upsilon(\mathfrak{r})) + bb_2(k_2^* + c_2\rho_1)\beta(\Upsilon(\mathfrak{r}))
\end{aligned}$$

and

$$\beta(K\Omega) \leq (bb_1 + bb_2(k_2^* + c_2\rho_1))\beta(\Omega). \tag{13}$$

Next, fixing $T > 0$, we obtain

$$\begin{aligned}
\int_T^{\infty} |K\Upsilon(\mathfrak{r})|d\mathfrak{r} &\leq \int_T^{\infty} |m(\mathfrak{r})|d\mathfrak{r} + b(\int_T^{\infty} |m_1(\mathfrak{r})|d\mathfrak{r} + b_1\int_T^{\infty} |\Upsilon(\mathfrak{r})|d\mathfrak{r}) + \succ (k_1^* + c_1\rho_1)\int_T^{\infty} |q(\mathfrak{r})|d\mathfrak{r} \\
&+ b(k_2^* + c_2\rho_1).(\int_T^{\infty} |m_2(\mathfrak{r})|d\mathfrak{r} + b_2\int_T^{\infty} |\Upsilon(\mathfrak{r})|d\mathfrak{r}).
\end{aligned}$$

However,

$$d(\Omega) = \lim_{T\to 0}\left(sup\left(\int_T^{\infty} |\Upsilon(\mathfrak{r})|d\mathfrak{r} : \Upsilon \in \Omega\right)\right),$$

Then,

$$d(K\Upsilon(\mathfrak{r})) \leq (bb_1 + bb_2(k_2^* + c_2\rho_1))d(\Upsilon(\mathfrak{r})). \tag{14}$$

According to (13) and (14), we obtain

$$\gamma(K\Omega) \leq (bb_1 + bb_2(k_2^* + c_2\rho_1))\gamma(\Omega).$$

Finally, we obtain

$$\begin{aligned}\chi(K\Omega) &\leq \gamma(K\Omega) \leq (bb_1 + bb_2(k_2^* + c_2\rho_1))\gamma(\Omega)\\ \chi(K\Omega) &\leq 2(bb_1 + bb_2(k_2^* + c_2\rho_1))\chi(\Omega).\end{aligned}$$

where χ is the Hausdorff MNC [6,9,22,23].

Since $(bb_1 + bb_2(k_2^* + c_2\rho_1)) < 1$, it follows from Darbo's theorem that K is a contraction and that it has at least one fixed point in B_{ρ_1}. Then, there exists at least one nonincreasing solution $\Upsilon \in L_1(R_+)$ to (2). □

Corollary 6. *Let the assumptions of Theorem 9 hold. Then, from (12), we can deduce that*

$$|\Upsilon(\mathfrak{r})| \leq \frac{m + bm_1 + q \nearrow (k_1^* + c_1\rho_1) + bm_2(k_2^* + c_2\rho_1)}{1 - (bb_1 + bb_2(k_2^* + c_2\rho_1))} \leq \alpha.$$

Now, consider the functional Equation (1) under these assumptions:

(x) $\phi_i : R_+ \to R_+$, $\phi_i(t) \leq t$, $i = 1, 2$ are increasing and absolutely continuous, and there exist two constants $N_i > 0$, $i = 1,\ 2$ such that $\phi_i'(t) \geq N_i$ a.e on R_+.

(xi) $\hbar_1 : R_+ \times R \to R$ and $\hbar_2 : R_+ \times R \to R \setminus \{0\}$ are Carathéodory functions [21], and there exists a bounded and integrable function $u \in L_1(R_+)$ and a bounded and measurable $\nu(t)$, $|\nu(t)| < \nu$, and $a \in R_+$, such that

$$|\hbar_1(t,\xi)| \leq |u(t)| + a|\xi| \ \ and \ \ |\hbar_2(t,\xi)| \leq \nu.$$

Moreover, $\hbar_i$, $i = 1, 2$ are nonincreasing for every nondecreasing argument.

(xii) $\frac{a}{N_1} < 1$.

Theorem 10. *Let the assumptions of Theorem 9 be satisfied. Assume that (x)–(xii) are met. Then, there exists at least one nonincreasing solution $\psi \in L_1(R_+)$ to (1).*

Proof. Let B_{ρ_2} be a closed ball of all of the nonincreasing functions

$$B_{\rho_2} = \{\psi \in L_1(R_+) : \|\psi\| \leq \rho_2\}.$$

Associate the operator

$$H\psi(\mathfrak{r}) = \hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \ \Upsilon(\mathfrak{r})\ \hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r}))),$$

then we deduce that H transforms nonincreasing functions into other nonincreasing functions, and $H\psi$ is nonincreasing on R_+.

Let $\psi \in B_{\rho_2}$. Then,

$$\begin{aligned} |H\psi(\mathfrak{r})| &= \left|\hbar_1(\mathfrak{r}, \psi(\phi_1(\mathfrak{r}))) + \Upsilon(\mathfrak{r})\hbar_2(\mathfrak{r}, \psi(\phi_2(\mathfrak{r})))\right| \\ &\leq \left(|u(\mathfrak{r})| + a|\psi(\phi_1(\mathfrak{r}))|\right) + |\Upsilon(\mathfrak{r})|\nu, \end{aligned}$$

Then,

$$\int_0^\infty |H\psi(\mathfrak{r})|d\mathfrak{r} \leq \int_0^\infty |u(\mathfrak{r})|d\mathfrak{r} + \frac{a}{N_1}\int_0^\infty |\psi(\theta_1)|d\theta_1 + \nu\int_0^\infty |\Upsilon(\mathfrak{r})|d\mathfrak{r}$$

and

$$\|H\psi\| \leq u^* + \frac{a}{N_1}\rho_2 + \nu\rho_1 = \rho_2, \ \rho_2 = \frac{u^* + \nu\rho_1}{1 - \frac{a}{N_1}}.$$

Hence, the operator H maps the ball B_{ρ_2} into itself. Moreoover, our assumptions imply that the operator H is continuous on B_{ρ_2}.

Now, let Y be a nonempty subset of B_{ρ_2}. Fix $\epsilon > 0$ and take a measurable set $D \subset R_+$ such that $D \leq \epsilon$. Then, for any $\psi \in$ Y, we obtain

$$\begin{aligned} \|H\psi\|_{L_1(D)} \leq \int_D |H\psi(\mathfrak{r})|d\mathfrak{r} &\leq \int_D |u(\mathfrak{r})|d\mathfrak{r} + \frac{a}{N_1}\int_D |\psi(\phi_1(\mathfrak{r}))|\phi_1\grave{}(\mathfrak{r})d\mathfrak{r} + \nu\int_D |\Upsilon(\mathfrak{r})|d\mathfrak{r} \\ &\leq \int_D |u(\mathfrak{r})|d\mathfrak{r} + \frac{a}{N_1}\int_{\phi_1(D)} |\psi(\mathfrak{r})|d\mathfrak{r} + \nu\int_D |\Upsilon(\mathfrak{r})|d\mathfrak{r}, \end{aligned}$$

Then,

$$\begin{aligned} \beta(H\psi(\mathfrak{r})) &= \lim_{\epsilon\to 0}\|H\psi\|_{L_1(D)} \\ \beta(H\psi(\mathfrak{r})) &\leq \frac{a}{N_1}\beta(\psi(\mathfrak{r})) \end{aligned}$$

and

$$\beta(H\mathrm{Y}) \leq \frac{a}{N_1}\beta(\mathrm{Y}). \tag{15}$$

Next, fixing $T > 0$, we obtain

$$\int_T^\infty |H\psi(\mathfrak{r})|d\mathfrak{r} \leq \int_T^\infty |u(\mathfrak{r})|d\mathfrak{r} + \frac{a}{N_1}\int_{\phi(T)}^\infty |\psi(\mathfrak{r})|d\mathfrak{r} + \nu\int_T^\infty |\Upsilon(\mathfrak{r})|d\mathfrak{r}.$$

However,

$$d(\mathrm{Y}) = \lim_{T\to 0}\left(sup\left(\int_T^\infty |\psi(\mathfrak{r})|d\mathfrak{r} \ : \psi \in \mathrm{Y}\right)\right),$$

and, since $\lim_{T\to\infty}\phi(T) = \infty$, then

$$d(H\mathrm{Y}) \leq \frac{a}{N_1}d(\mathrm{Y}). \tag{16}$$

According to (15) and (16), we obtain

$$\gamma(H\mathrm{Y}) \leq \frac{a}{N_1}\gamma(\mathrm{Y}).$$

Finally, we obtain

$$\begin{aligned}\chi(HY) &\leq \gamma(HY) \leq \frac{a}{N_1}\gamma(Y)\\ \chi(HY) &\leq 2\frac{a}{N_1}\chi(Y).\end{aligned}$$

where χ is the Hausdorff MNC [6,9,22,23].

Since $2\frac{a}{N_1} < 1$, it follows from Darbo's theorem that H is a contraction and that it has at least one fixed point in B_{ρ_2}. Then, there exists at least one nonincreasing solution $\psi \in L_1(R_+)$ to (1). □

Corollary 7. *From Theorems 9 and 10, problems (1) and (2) have at least one nonincreasing solution $\psi \in L_1(R_+)$.*

4.1. Asymptotic Stability

Now, replace assumptions (vii), (xi), and (xii) with $(vii)^*$, $(xi)^*$, and $(xii)^*$, as follows:

$(vii)^*$ $f : R_+ \times R^3 \to R$ and $g_i : R_+ \times R \to R$, $i = 1,2$ are measurable in $t \in R_+$ $\forall \xi, \vartheta, \omega \in R$ and satisfy the Lipschitz condition,

$$\begin{aligned}&|f(t,\xi,\vartheta,\omega) - f(t,\xi_1,\vartheta_1,\omega_1)| \leq b|\xi-\xi_1| + q(t)|\vartheta-\vartheta_1| + b|\omega-\omega_1| \text{ and}\\ &|g_i(t,\vartheta) - g_i(t,\vartheta_1)| \leq b_i|\vartheta-\vartheta_1| \qquad (17)\\ &\forall (t,\xi,\vartheta,\omega),(t,\xi_1,\vartheta_1,\omega_1) \in R_+ \times R \times R,\ (t,\vartheta),(t,\vartheta_1) \in R_+ \times R.\end{aligned}$$

Moreover, f is nonincreasing for every nondecreasing argument.

$(xi)^*$ $\hbar_1 : R_+ \times R \to R$ and $\hbar_2 : R_+ \times R \to R \setminus \{0\}$ are measurable in $t \in R_+$ $\forall \xi \in R$ and satisfies Lipschitz condition,

$$|\hbar_i(t,\xi) - \hbar_i(t,\xi_1)| \leq a_i|\xi-\xi_1| \quad \forall (t,\xi),(t,\xi_1) \in R_+ \times R. \qquad (18)$$

Moreover, $\hbar_i$ is nonincreasing for every nondecreasing arguments.

$(xii)^*$ $a_1 + \alpha\, a_2 < 1$.

From Equation (17) and (18), we have

$$\begin{aligned}&|f(t,\xi,\vartheta,\omega)| \leq |m(t)| + b|\xi| + q(t)|\vartheta| + b|\omega|,\ \ |g_i(t,\vartheta)| \leq |m_1(t)| + b_i|\vartheta| \text{ and}\\ &|\hbar_1(t,\xi)| \leq |u(t)| + a|\xi|.\end{aligned}$$

So, we have proved the following Lemma.

Lemma 2. *Assumptions (vii) and (xi) imply assumptions $(vii)^*$ and $(xi)^*$, respectively.*

Theorem 11. *Suppose that $(vii)^*$, $(viii)$, $(xi)^*$, and $(xii)^*$ hold. Then, the solution to problems (1) and (2) is asymptotically stable in the sense that for any $\epsilon > 0$, there exist $T(\epsilon) > 0$ and $\rho_2 > 0$. Moreover, for $\psi, \bar{\psi} \in B_{\rho_2}$ any two solutions, then $|\psi(\mathfrak{r}) - \bar{\psi}(\mathfrak{r})| \leq \epsilon$ for $\mathfrak{r} \geq T(\epsilon)$.*

Proof. Let ψ and $\bar{\psi}$ be two solutions to problems (1) and (2) and $\phi_1 = \phi_2 = 1$. Then,

$$\begin{aligned}|\psi(\mathfrak{r}) - \bar{\psi}(\mathfrak{r})| &= \left|\hbar_1(\mathfrak{r},\psi(\mathfrak{r})) + \Upsilon(\mathfrak{r})\hbar_2(\mathfrak{r},\psi(\mathfrak{r})) - \hbar_1(\mathfrak{r},\bar{\psi}(\mathfrak{r})) - \bar{\Upsilon}(\mathfrak{r})h_2(\mathfrak{r},\bar{\psi}(\mathfrak{r}))\right|\\ &\leq |\hbar_1(\mathfrak{r},\psi(\mathfrak{r})) - \hbar_1(\mathfrak{r},\bar{\psi}(\mathfrak{r}))| + |\,\Upsilon(\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})||\hbar_2(\mathfrak{r},\psi(\mathfrak{r}))|\\ &+ |\bar{\Upsilon}(\mathfrak{r})||\hbar_2(\mathfrak{r},\psi(\mathfrak{r})) - \hbar_2(\mathfrak{r},\bar{\psi}(\mathfrak{r}))|\\ &\leq a_1|\psi(\mathfrak{r}) - \bar{\psi}(\mathfrak{r})| + \nu|\,\Upsilon(\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})| + |\bar{\Upsilon}(\mathfrak{r})|a_2|\psi(\mathfrak{r}) - \bar{\psi}(\mathfrak{r})|,\end{aligned}$$

Then,

$$|\psi(\mathfrak{r}) - \bar{\psi}(\mathfrak{r})| \quad \leq \quad \frac{\nu|\ \Upsilon\ (\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})|}{1-(a_1+\alpha a_2)}.$$

where

$$\begin{aligned}
& |\ \Upsilon\ (\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})| = \\
& \left|f\left(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds, g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\right)\right. \\
- \quad & \left. f\left(\mathfrak{r},\ g_1(\mathfrak{r},\bar{\Upsilon}(\mathfrak{r})),\ \succ\int_0^{\mathfrak{r}} f_1(s,\bar{\Upsilon}(s))ds,\ g_2(\mathfrak{r},\bar{\Upsilon}(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\bar{\Upsilon}(s))ds\right)\right| \\
\leq \quad & b\left|g_1(\mathfrak{r},\Upsilon(\mathfrak{r})) - g_1(\mathfrak{r},\bar{\Upsilon}(\mathfrak{r}))\right| + \ \succ|q(\mathfrak{r})|\int_0^{\mathfrak{r}}|f_1(s,\Upsilon(s)) - f_1(s,\bar{\Upsilon}(s))|ds \\
+ \quad & b|g_2(\mathfrak{r},\Upsilon(\mathfrak{r})) - g_2(\mathfrak{r},\bar{\Upsilon}(\mathfrak{r}))|.\int_0^{\mathfrak{r}}|f_2(s,\Upsilon(s))|ds \\
+ \quad & b|g_2(\mathfrak{r},\bar{\Upsilon}(\mathfrak{r}))|.\int_0^{\mathfrak{r}}|f_2(s,\Upsilon(s)) - f_2(s,\bar{\Upsilon}(s))|ds \\
\leq \quad & bb_1|\ \Upsilon\ (\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})| + 2\succ|q(\mathfrak{r})|\int_0^{\mathfrak{r}}|k_1(s)|ds + bb_2|\ \Upsilon\ (\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})|.(k_2^* + c_2\rho_1) \\
+ \quad & b(|m_2(\mathfrak{r})| + b_2|\bar{\Upsilon}(\mathfrak{r})|)(2\int_0^t |k_2(s)|ds) \\
\leq \quad & bb_1|\ \Upsilon\ (\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})| + 2\succ|q(\mathfrak{r})|\epsilon_2 + bb_2|\ \Upsilon\ (\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})|.(k_2^* + c_2\rho_1) + 2b\epsilon_3(|m_2(\mathfrak{r})| + b_2|\bar{\Upsilon}(\mathfrak{r})|),
\end{aligned}$$

Then,

$$|\ \Upsilon\ (\mathfrak{r}) - \bar{\Upsilon}(\mathfrak{r})| \quad \leq \quad \frac{2\succ q\epsilon_2 + 2b\epsilon_3(m_2+\alpha b_2)}{1-(bb_1+bb_2(k_2^*+c_2\rho_1))} = \bar{\epsilon}.$$

Hence,

$$|\psi(\mathfrak{r}) - \bar{\psi}(\mathfrak{r})| \quad \leq \quad \frac{\nu\bar{\epsilon}}{1-(a_1+\alpha a_2)} = \epsilon.$$

□

4.2. Continuous Dependence on Some Results

Now, replace assumption $(viii)$ with $(viii)^*$, as follows:

$(viii)^* f_i : R_+ \times R \to R^+$, $i = 1,\ 2$ are Carathéodory functions [21] and satisfy the Lipschitz condition

$$\int_0^{\infty} |f_i(t,\vartheta) - f_i(t,\vartheta_1)|dt \leq c_i\|\vartheta - \vartheta_1\|,\ \ i = 1,\ 2,\ \ \forall (t,\vartheta),(t,\vartheta_1) \in R_+ \times R.$$

Theorem 12. *Let (vii)*–(viii)* and (xi)*–(xii)* occur. Then, the solution to problems (1) and (2) exhibits asymptotic dependency on the parameter $\succ$ and on the functions g_1 and g_2 such that $\forall \epsilon > 0,\ \exists \delta(\epsilon)$, that is,*

$$max\{|\succ - \succ^*|,\ \|g_1(\mathfrak{r},\Upsilon) - g_1^*(\mathfrak{r},\Upsilon)\|,\ \|g_2(\mathfrak{r},\Upsilon) - g_2^*(\mathfrak{r},\Upsilon)\|\} < \delta \Rightarrow \|\psi - \psi^*\| < \epsilon,\ \ \mathfrak{r} > T(\epsilon)$$

where ψ^* is the solution to

$$\psi^*(\mathfrak{r}) = \hbar_1(\mathfrak{r},\psi^*(\phi_1(\mathfrak{r}))) + \Upsilon^*(\mathfrak{r})\hbar_2(\mathfrak{r},\psi^*(\phi_2(\mathfrak{r}))),\ \ \mathfrak{r} \geq 0$$

and Υ^* is the solution to

$$\Upsilon^*(\mathfrak{r}) = f\left(\mathfrak{r},\ g_1^*(\mathfrak{r},\Upsilon^*(\mathfrak{r})),\succ^*\int_0^{\mathfrak{r}} f_1(s,\Upsilon^*(s))ds,\ g_2^*(\mathfrak{r},\Upsilon^*(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon^*(s))ds\right),\ \ \mathfrak{r} \geq 0.$$

Proof. Firstly,

$$\begin{aligned}
|\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})| &= \Big|f\Big(\mathfrak{r},\ g_1(\mathfrak{r},\Upsilon(\mathfrak{r})),\ \lambda\int_0^{\mathfrak{r}} f_1(s,\Upsilon(s))ds,\ g_2(\mathfrak{r},\Upsilon(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon(s))ds\Big)\\
&- f\Big(\mathfrak{r},\ g_1^*(\mathfrak{r},\Upsilon^*(\mathfrak{r})),\ \lambda^*\int_0^{\mathfrak{r}} f_1(s,\Upsilon^*(s))ds,\ g_2^*(\mathfrak{r},\Upsilon^*(\mathfrak{r})).\int_0^{\mathfrak{r}} f_2(s,\Upsilon^*(s))ds\Big)\Big|\\
&\le b\Big|g_1(\mathfrak{r},\Upsilon(\mathfrak{r}))-g_1^*(\mathfrak{r},\Upsilon(\mathfrak{r}))\Big|+b\Big|g_1^*(\mathfrak{r},\Upsilon(\mathfrak{r}))-g_1^*(\mathfrak{r},\Upsilon^*(\mathfrak{r}))\Big|\\
&+ |q(\mathfrak{r})||\lambda-\lambda^*|\int_0^{\mathfrak{r}}|f_1(s,\Upsilon(s))|ds+|q(\mathfrak{r})|\lambda^*\int_0^{\mathfrak{r}}|f_1(s,\Upsilon(s))-f_1(s,\Upsilon^*(s))|ds\\
&+ b|g_2(\mathfrak{r},\Upsilon(\mathfrak{r}))-g_2^*(\mathfrak{r},\Upsilon(\mathfrak{r}))|.\int_0^{\mathfrak{r}}|f_2(s,\Upsilon(s))|ds\\
&+ b|g_2^*(\mathfrak{r},\Upsilon(\mathfrak{r}))-g_2^*(\mathfrak{r},\Upsilon^*(\mathfrak{r}))|.\int_0^{\mathfrak{r}}|f_2(s,\Upsilon(s))|ds\\
&+ b|g_2(\mathfrak{r},\Upsilon^*(\mathfrak{r}))|.\int_0^{\mathfrak{r}}|f_2(s,\Upsilon(s))-f_2(s,\Upsilon^*(s))|ds\\
&\le b\Big|g_1(\mathfrak{r},\Upsilon(\mathfrak{r}))-g_1^*(\mathfrak{r},\Upsilon(\mathfrak{r}))\Big|+bb_1|\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})|+|q(\mathfrak{r})|\delta(k_1^*+c_1\rho_1)\\
&+ |q(\mathfrak{r})|\lambda^*c_1\|\Upsilon-\Upsilon^*\|+b|g_2(\mathfrak{r},\Upsilon(\mathfrak{r}))-g_2^*(\mathfrak{r},\Upsilon(\mathfrak{r}))|.(k_2^*+c_2\rho_1)\\
&+ bb_2|\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})|.(k_2^*+c_2\rho_1)+b(|m_2(\mathfrak{r})|+b_2|\Upsilon^*(\mathfrak{r})|).c_2\|\Upsilon-\Upsilon^*\|,
\end{aligned}$$

Then,

$$\begin{aligned}
\int_0^\infty|\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})|d\mathfrak{r} &\le b\int_0^\infty\Big|g_1(\mathfrak{r},\Upsilon(\mathfrak{r}))-g_1^*(\mathfrak{r},\Upsilon(\mathfrak{r}))\Big|d\mathfrak{r}+bb_1\int_0^\infty|\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})|d\mathfrak{r}\\
&+ \delta(k_1^*+c_1\rho_1)\int_0^\infty|q(\mathfrak{r})|d\mathfrak{r}+\lambda^*c_1\|\Upsilon-\Upsilon^*\|\int_0^\infty|q(\mathfrak{r})|d\mathfrak{r}\\
&+ b(k_2^*+c_2\rho_1).\int_0^\infty|g_2(\mathfrak{r},\Upsilon(\mathfrak{r}))-g_2^*(\mathfrak{r},\Upsilon(\mathfrak{r}))|d\mathfrak{r}\\
&+ bb_2(k_2^*+c_2\rho_1)\int_0^\infty|\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})|d\mathfrak{r}\\
&+ bc_2\|\Upsilon-\Upsilon^*\|(\int_0^\infty|m_2(\mathfrak{r})|d\mathfrak{r}+b_2\int_0^\infty|\Upsilon^*(\mathfrak{r})|d\mathfrak{r})
\end{aligned}$$

and

$$\begin{aligned}
\|\Upsilon-\Upsilon^*\| &\le b\delta+bb_1\|\Upsilon-\Upsilon^*\|+\delta(k_1^*+c_1\rho_1)q^*+\lambda^*c_1\|\Upsilon-\Upsilon^*\|q^*\\
&+ b(k_2^*+c_2\rho_1)\delta+bb_2(k_2^*+c_2\rho_1)\|\Upsilon-\Upsilon^*\|+bc_2\|\Upsilon-\Upsilon^*\|(m_2^*+b_2\rho_1).
\end{aligned}$$

Hence,

$$\|\Upsilon-\Upsilon^*\| \le \frac{\Big(b+(k_1^*+c_1\rho_1)q^*+b(k_2^*+c_2\rho_1)\Big)\delta}{1-\Big(bb_1+\lambda^*c_1q^*+bb_2(k_2^*+c_2\rho_1)+bc_2(m_2^*+b_2\rho_1)\Big)}=\epsilon_4. \quad (19)$$

Now, let ψ and ψ^* be two solutions to problems (1) and (2) and $\phi_1=\phi_2=1$. Then,

$$\begin{aligned}
|\psi(\mathfrak{r})-\psi^*(\mathfrak{r})| &= \Big|\hbar_1(\mathfrak{r},\psi(\mathfrak{r}))+\Upsilon(\mathfrak{r})\hbar_2(\mathfrak{r},\psi(\mathfrak{r}))-\hbar_1(\mathfrak{r},\psi^*(\mathfrak{r}))-\Upsilon^*(\mathfrak{r})\hbar_2(\mathfrak{r},\psi^*(\mathfrak{r}))\Big|\\
&\le |\hbar_1(\mathfrak{r},\psi(\mathfrak{r}))-\hbar_1(\mathfrak{r},\psi^*(\mathfrak{r}))|+|\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})||\hbar_2(\mathfrak{r},\psi(\mathfrak{r}))|\\
&+ |\Upsilon^*(\mathfrak{r})||\hbar_2(\mathfrak{r},\psi(\mathfrak{r}))-\hbar_2(\mathfrak{r},\psi^*(\mathfrak{r}))|\\
&\le a_1|\psi(\mathfrak{r})-\psi^*(\mathfrak{r})|+\nu|\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})|+|\Upsilon^*(\mathfrak{r})|a_2|\psi(\mathfrak{r})-\psi^*(\mathfrak{r})|,
\end{aligned}$$

Then,

$$\begin{aligned}\int_0^\infty |\psi(\mathfrak{r})-\psi^*(\mathfrak{r})|d\mathfrak{r} &\leq a_1\int_0^\infty |\psi(\mathfrak{r})-\psi^*(\mathfrak{r})|d\mathfrak{r}+\nu\int_0^\infty |\Upsilon(\mathfrak{r})-\Upsilon^*(\mathfrak{r})|d\mathfrak{r}\\ &+ |\Upsilon^*(\mathfrak{r})|a_2\int_0^\infty |\psi(\mathfrak{r})-\psi^*(\mathfrak{r})|d\mathfrak{r}\end{aligned}$$

and

$$\|\psi-\psi^*\| \leq a_1\|\psi-\psi^*\|+\nu\|\Upsilon-\Upsilon^*\|+|\Upsilon^*(\mathfrak{r})|a_2\|\psi-\psi^*\|$$

From (19), we have

$$\|\Upsilon-\Upsilon^*\| \leq \epsilon_4,$$

Then,

$$\|\psi-\psi^*\| \leq \frac{\nu\epsilon_4}{1-(a_1+\alpha^*a_2)}=\epsilon.$$

□

Now, from Theorems 9–12, we have the following corollaries.

Corollary 8. *The IHFIE (3) has at least one nonincreasing solution $\psi \in L_1(R_+)$.*

Corollary 9. *The IHFIE (3) is asymptotically stable.*

Corollary 10. *The IHFIE (3) is asymptotically dependent on the parameter $\diagdown$ and on the functions g_1 and g_2.*

Proof. From the equivalence of problems (1)–(3), the results follow. □

5. Examples

Example 1. *Consider the delay functional equation*

$$\psi(t)=e^{-ln(t+1)}+\frac{|\psi(t)|}{3}+\frac{7.5}{e}\left(e^t+\frac{|\psi(t)|}{8}\right),\ t\in[0,1]. \tag{20}$$

subject to the quadratic functional integral constraint

$$\begin{aligned}\Upsilon(t) &= e^{-ln(t+1)}+\frac{1}{3}(e^t+\frac{|\Upsilon(t)|}{2})+\frac{1}{12}\int_0^t(e^s+\frac{|\Upsilon(s)|}{8})ds\\ &+ \frac{1}{3}\left(\frac{1}{e^t}+\frac{|\Upsilon(t)|}{4}\right)\int_0^t(e^s+\frac{|\Upsilon(s)|}{6})ds.\ t\in[0,1]\end{aligned} \tag{21}$$

Set

$$\begin{aligned}\hbar_1(t,\psi) &= e^{-ln(t+1)}+\frac{|\psi(t)|}{3},\quad h_2(t,\psi)=e^t+\frac{|\psi(t)|}{8},\\ \mathbf{g_1}(t,\Upsilon) &= e^t+\frac{|\Upsilon(t)|}{2},\quad \mathbf{g_2}(t,\Upsilon)=\frac{1}{e^t}+\frac{|\Upsilon(t)|}{4},\\ \mathbf{f_1}(t,\Upsilon) &= e^t+\frac{|\Upsilon(t)|}{8},\quad \mathbf{f_2}(t,\Upsilon)=e^t+\frac{|\Upsilon(t)|}{6}.\end{aligned}$$

Putting

$$\|m\|=ln(2),\ \|m_1\|=e-1,\ \|m_2\|=\frac{-1}{e}+1,\ \|k_1\|=\|k_2\|=e-1,$$

$$\nwarrow = \frac{1}{4},\ b = \frac{1}{3}\ b_1 = \frac{1}{2},\ b_2 = \frac{1}{4},\ c_1 = \frac{1}{8},\ c_2 = \frac{1}{6},\ T = 1,$$

$$\|u_1\| = ln(2),\ \|u_2\| = e-1,\ a_1 = \frac{1}{3},\ a_2 = \frac{1}{8},\ \phi_1 = \phi_2 = 1,\ and\ M = 2.8685734 \approx \frac{7.5}{e}.$$

We can find that

$$\frac{a_1}{N_1} + \frac{Ma_2}{N_2} = 0.691908 < 1\ \ , bb_1 + bb_2(\|k_2\| + c_2r_1) = 0.34379020 < 1\ \ and$$

$$bb_1 + b \nwarrow c_1T + bb_2(\|k_2\| + c_2r_1) + bc_2(\|m_2\| + b_2r_1) = 0.42325805 < 1.$$

Then, problems (20) and (21) have at least one solution.

Now, we find an implicit hybrid functional integral equation that is equivalent to problems (20) and (21).

Example 2. *Consider the following an implicit hybrid functional integral equation*

$$\begin{aligned}
&\frac{\psi(t) - (e^{-ln(t+1)} + \frac{|\psi(t)|}{3})}{e^t + \frac{|\psi(t)|}{8}} = e^{-ln(t+1)} + \frac{1}{3}\left(e^t + \frac{1}{2}(\frac{\psi(t) - (e^{-ln(t+1)} + \frac{|\psi(t)|}{3})}{e^t + \frac{|\psi(t)|}{8}})\right) \\
&+\frac{1}{12}\int_0^t (e^s + \frac{1}{8}(\frac{\psi(s) - (e^{-ln(s+1)} + \frac{|\psi(s)|}{3})}{e^s + \frac{|\psi(s)|}{8}})ds \qquad (22)\\
&+\frac{1}{3}\left(\frac{1}{e^t} + \frac{1}{4}(\frac{\psi(t) - (e^{-ln(t+1)} + \frac{|\psi(t)|}{3})}{e^t + \frac{|\psi(t)|}{8}})\right)\int_0^t (e^s + \frac{1}{6}(\frac{\psi(s) - (e^{-ln(s+1)} + \frac{|\psi(s)|}{3})}{e^s + \frac{|\psi(s)|}{8}}))ds.
\end{aligned}$$

Set

$$\Upsilon(t) = \frac{\psi(t) - (e^{-ln(t+1)} + \frac{|\psi(t)|}{3})}{e^t + \frac{|\psi(t)|}{8}}$$

Then, (22) has at least one solution $\psi \in L_1(I)$.

Example 3. *Taking into account the next constrained problem of the delay functional equation*

$$\psi(t) = \frac{1}{e^t} + \frac{|\psi(t)|}{4} + \frac{|\Upsilon(t)|}{6},\ \ t \geq 0. \qquad (23)$$

subject to the quadratic functional integral constraint

$$\begin{aligned}
\Upsilon(t) &= e^{-t} + \frac{1}{3}(te^{-t} + \frac{|\Upsilon(t)|}{2}) + \frac{e^{-t}}{4}\int_0^t (e^{-s} + \frac{|\Upsilon(s)|}{8})ds \\
&+ \frac{1}{3}\left(\frac{1}{e^t} + \frac{|\Upsilon(t)|}{4}\right)\int_0^t (e^{-s} + \frac{|\Upsilon(s)|}{6})ds.\ t \geq 0. \qquad (24)
\end{aligned}$$

Set

$$\begin{aligned}
\hbar_1(t,\psi) &= \frac{1}{e^t} + \frac{|\psi(t)|}{4},\quad \hbar_2(t,\psi) = \frac{1}{6},\\
f(t,\eta(t),\omega(t),\vartheta(t)) &= e^{-t} + \frac{1}{3}(\eta(t)) + \frac{e^{-t}}{4}(\omega(t)) + \frac{1}{3}(\vartheta(t)),\\
\eta(t) &= te^{-t} + \frac{|\Upsilon(t)|}{2},\quad \omega(t) = \frac{e^{-t}}{4}\int_0^t (e^{-s} + \frac{|\Upsilon(s)|}{8})ds,\\
\vartheta(t) &= \left(\frac{1}{e^t} + \frac{|\Upsilon(t)|}{4}\right)\int_0^t (e^{-s} + \frac{|\Upsilon(s)|}{6})ds.
\end{aligned}$$

Putting

$$m^* = m_1^* = m_2^* = k_1^* = k_2^* = q^* = 1,\ \gtrdot = \frac{1}{4},\ b = \frac{1}{3}\ b_1 = \frac{1}{2},\ b_2 = \frac{1}{4},\ c_1 = \frac{1}{8},\ c_2 = \frac{1}{6},$$

$$u = \frac{1}{e^t},\ a = \frac{1}{4},\ \nu = \frac{1}{6},\ \phi_1 = \phi_2 = 1,\ \textit{and}\ \alpha = 3.517491075.$$

We can find that

$$\frac{a}{N_1} = 0.25 < 1\ \ , bb_1 + bb_2(k_2^* + c_2\rho_1) = 0.2988540427 < 1.$$

Then, problems (23) and (24) have at least one solution.

Example 4. *Consider the following an implicit hybrid functional integral equation*

$$\begin{aligned}\frac{\psi(t) - (\frac{1}{e^t} + \frac{|\psi(t)|}{4})}{\frac{1}{6}} &= e^{-t} + \frac{1}{3}(te^{-t} + \frac{\frac{\psi(t) - (\frac{1}{e^t} + \frac{|\psi(t)|}{4})}{\frac{1}{6}}}{2})\\ &+ \frac{e^{-t}}{4}\int_0^t (e^{-s} + \frac{\frac{\psi(s) - (\frac{1}{e^s} + \frac{|\psi(s)|}{4})}{\frac{1}{6}}}{8})ds\\ &+ \frac{1}{3}\left(\frac{1}{e^t} + \frac{\frac{\psi(t) - (\frac{1}{e^t} + \frac{|\psi(t)|}{4})}{\frac{1}{6}}}{4}\right)\int_0^t (e^{-s} + \frac{\frac{\psi(s) - (\frac{1}{e^s} + \frac{|\psi(s)|}{4})}{\frac{1}{6}}}{6})ds.\ t \geq 0.\end{aligned} \tag{25}$$

Set

$$\Upsilon(t) = \frac{\psi(t) - (\frac{1}{e^t} + \frac{|\psi(t)|}{4})}{\frac{1}{6}}$$

Then, (25) has at least one solution $\psi \in L_1(R_+)$.

6. Conclusions

In this study, we have conducted a thorough examination of the constrained problem involving the nonlinear delay functional Equation (1) subject to the quadratic functional integral constraint (2). To address this problem, we employed the technique associated with MNC.

In this investigation, we discussed two cases: In the first case, we studied the existence of nondecreasing solutions, Υ, on a bounded domain for constraint (2). Then, we studied the existence of nondecreasing solutions, $\psi \in L_1(I)$, for (1). Moreover, we established some sufficient conditions to guarantee the uniqueness of the solution and its dependence on the parameter $\gtrdot$ as well as the functions g_1 and g_2. We also studied the equivalence between problems (1) and (2) and the IHFIE (3). Furthermore, we thoroughly investigated the Hyers–Ulam stability of problems (1) and (2) and IHFIE (8). In the second case, we established the solvability and asymptotic stability and dependency of the nonincreasing solution $\psi \in L_1(R_+)$ to problems (1) and (2) on the parameter $\gtrdot \geq 0$ and the functions g_1 and g_2. Finally, we provided some illustrative examples to demonstrate the practical application and validity of our obtained results.

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