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Article

Asymptotic Behavior for a Coupled Petrovsky–Petrovsky System with Infinite Memories

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Abstract: The main goal of this article is to obtain the existence of solutions for a nonlinear system of a coupled Petrovsky–Petrovsky system in the presence of infinite memories under minimal assumptions on the functions g_1, g_2 and φ_1, φ_2. Here, g_1, g_2 are relaxation functions and φ_1, φ_2 represent the sources. Also, a general decay rate for the associated energy is established. Our work is partly motivated by recent results, with a necessary modification imposed by the nature of our problem. In this work, we limit our results to studying the system in a bounded domain. The case of the entire domain $\mathbb{R}^n$ requires separate consideration. Of course, obtaining such a result will require not only serious technical work but also the use of new techniques and methods. In particular, one of the most significant points in achieving this goal is the use of the perturbed Lyapunov functionals combined with the multiplier method. To the best of our knowledge, there is no result addressing the linked Petrovsky–Petrovsky system in the presence of infinite memory, and we have overcome this lacune.

Keywords: Lyapunov functions; energy decay; infinite memories; source terms; partial differential equation

MSC: 35L05; 35L15; 35L70; 93D15

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1. Introduction

From a mathematical point of view, partial differential equations (in short, PDEs) are a very powerful instrument to describe real phenomena (e.g., explosion, boundedness, and stability) arising from biology, plasma physics, epidemiology, etc. In this context, we mention, for instance, refs. [1–3].

This study is concerned with the following viscoelastic system:

$$\begin{cases} |u_t|^{\ell} u_{tt} - \Delta_x^2 u + \int_0^{\infty} g_1(s) \Delta_x^2 u(x, t-s) ds + \varphi_1(u, v) = 0, & \text{in } \Omega_\infty, \\ |v_t|^{\ell} v_{tt} - \Delta_x^2 v + \int_0^{\infty} g_2(s) \Delta_x^2 v(x, t-s) ds + \varphi_2(u, v) = 0, & \text{in } \Omega_\infty, \\ u(x,t) = v(x,t) = 0 & \text{on } \Gamma_\infty, \\ u(x,-t) = u_0(x,t),\ v(x,-t) = v_0(x,t), & \text{in } \Omega_\infty, \\ u_t(x,t=0) = u_1(x),\ v_t(x,t=0) = v_1(x), & x \in \Omega, \\ u(x,t=0) = u_0(x),\ v(x,t=0) = v_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega_\infty = \Omega \times (0,\infty)$; $\Gamma_\infty = \partial\Omega \times (0,\infty)$; Ω is a regular and bounded domain in $\mathbb{R}^n (n \geq 1)$, with a smooth boundary $\partial\Omega$ of class $\mathcal{C}^4$; and ℓ is a real number such that

$$\begin{cases} 0 < \ell \leq \frac{2}{n-2}, & \text{if } n \geq 3, \\ \ell > 0, & \text{if } n \in \{1,2\}. \end{cases}$$

The functions u and v denote the transverse displacements of equations, and φ_1, φ_2 are source terms that define how the two equations interact with one another. The softening functions g_1 and g_2 represent the viscoelastic materials that have the property of keeping past memories. The initial data $(u_0,u_1),(v_0,v_1)$ belong to a suitable space. The interaction of two scalar fields is described in the theory of viscoelasticity by this problem (see [4–8]). For the single viscoelastic wave equation, there are many results concerning global well-posedness and stability; see, for example, [9–13]. To get us started, consider the wave equation presented; the authors of [14] investigated the conventional version of the following coupled system of quasilinear viscoelastic equations:

$$\begin{cases} |u_t|^\rho u_{tt} + \int_0^t g_1(t-s)\Delta_x u(s)ds + \varphi_1(x,u) = \Delta_x u + \gamma_1 \Delta_x u_{tt}, \\ |v_t|^\rho v_{tt} + \int_0^t g_2(t-s)\Delta_x v(s)ds + \varphi_2(x,u) = \Delta_x v + \gamma_2 \Delta_x v_{tt}. \end{cases}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$, with a smooth boundary $\partial\Omega$, $\gamma_i \geq 0, i = 1,2$ are constants, and ρ is a real number, such that

$$\begin{cases} 0 < \rho < \frac{2n}{(n-2)}, & \text{if } n \geq 3 \\ \rho > 0, & \text{if } n \in \{1,2\}, \end{cases}$$

and the initial data are given by the functions u_0, v_0, u_1, and v_1. The relaxation functions g_1 and g_2 are continuous, and the nonlinear terms are represented by $\varphi_1(u,v)$, $\varphi_2(u,v)$. The authors used the perturbed energy approach to demonstrate the energy decay finding.

Many authors thought about the very initial boundary value problem in the following coupled system:

$$\begin{cases} u_{tt} + \int_0^t g_1(t-s)\Delta_x u(s)ds + h_1(u_t) = \varphi_1(x,u) + \Delta_x u, \\ v_{tt} + \int_0^t g_2(t-s)\Delta_x v(s)ds + h_2(v_t) = \varphi_2(x,u) + \Delta_x v. \end{cases} \tag{2}$$

If the viscoelastic terms $g_i = 0, i = 1,2$ are not included in (2), several results concerning the local and global existence in the presence of a weak solution were found by Rammaha and Sakuntasathien [15]. Using the same method as in [16], the authors demonstrated that any weak solution with negative starting energy will blow up in finite time. In case of the presence of the memory, that is, $g_i \neq 0, i = 1,2$, there are various results concerning the asymptotic behavior and blow up of viscoelastic system solutions. For example, Liang and Gao [17] investigated the problem (2), with $h_1(u_t) = -\Delta_x u_t$, $h_2(v_t) = -\Delta_x u_t$. The authors showed that the decay rate of the energy functions is exponential under appropriate conditions on the functions $g_i, i = 1,2$, $\varphi_i, i = 1,2$, and for a specific initial data in the stable set. On the other hand, there are solutions with positive initial energy that blow up in finite time given certain specific initial data in the unstable set. Moreover, $h_1(u_t) = |u_t|^{m-1}u_t$ and $h_2(v_t) = |v_t|^{r-1}v_t$. Han and Wang [18] provided numerous results concerning local existence, global existence, and finite temporal blow-up (the initial energy $E(0) < 0$).

The generic version of the weakly damped viscoelastic wave equations is written as

$$\begin{cases} u_{tt} + \int_0^t g_1(t-s)\Delta_x u(s)ds + h_1(u_t) = \varphi_1(x,u) + \Delta_x u, \\ v_{tt} + \int_0^t g_2(t-s)\Delta_x v(s)ds + h_2(v_t) = \varphi_2(x,u) + \Delta_x v. \end{cases}$$

When the memory is infinite, the more general form of the wave equation can be given by

$$u_{tt} - \int_0^\infty \mu(s)\Delta_x u(t-s)ds + g(u_t) = f + \alpha\Delta_x u.$$

There have been so many results concerning the wave equation with respect to global well-posedness and stability up until now; see, for instance, [19]. For the coupled wave equations with infinite memories, Messaoudi and Al-Gharabli [20] considered the following system:

$$\begin{cases} u_{tt} + \int_0^\infty g(s)\Delta_x u(t-s) + \lambda|u_t|^{m-1}u_t = \varphi_1(u,v) + \Delta_x u, \\ v_{tt} + \int_0^\infty h(s)\Delta_x v(t-s) + \mu|v_t|^{r-1}v_t = \varphi_2(u,v) + \Delta_x v. \end{cases}$$

For coupled Petrovsky–Petrovsky equations, here we mention the work in [21] where the author considered the following coupled system:

$$\begin{cases} u_{tt} + \Delta_x^2 u + au_2 + g_1(u_t) = 0, \\ v_{tt} + \Delta_x^2 v + av_2 + g_2(v_t) = 0, \end{cases}$$

with Ω being a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ of class $\mathcal{C}^4$, and $a : \Omega \to \mathbb{R}$, $(g_i)_{i\in\{1,2\}} : \mathbb{R} \to \mathbb{R}$ are some given functions. Under suitable assumptions, he proved that this system is well-posed by using the nonlinear semi-groups theory, and dissipative by exploiting the multiplier method.

Motivated by prior research, the current study investigates the effect of infinite memory and source terms on the solutions to (1). Under suitable assumptions, we establish the decay properties of the solutions of (1). It is noted that our system is different from the one in Bahlil and Feng [1], making the methods used in our work different from theirs. In this research, we are able, essentially and mainly, to link the rate of decrease to the energy functional associated with the solution directly to that of the functions g_1, g_2 with an improvement in the conditions taken on these relaxation functions. We found that the two functions g_1, g_2 are responsible for the decay rate of the energy functional and then that of the existed solution. On the other hand, the functions φ_1, φ_2 obstruct the solution if they can overcome and dominate it.

This paper is structured as follows. In the next section, we provide some preliminaries and useful lemmas used to obtain our results. In the Section 3, we derive the decay properties and separately report the general results obtained for the most important case. The decaying results are obtained without the assumptions (A_1) in Section 4. Finally, in the Section 5 we give some examples on the relaxation functions to illustrate the energy decay rate given by Theorem 2.

2. Assumptions and Supporting Results

This part contains some material required for the statement and proof of our result. Set

$$H_0^1(\Omega) = \Big\{u \in H^1(\Omega) : u_0 = 0\Big\}.$$

Let λ_1 be the first eigenvalue of the spectral Dirichlet problem

$$\begin{cases} \Delta_x^2 u = \lambda_1 u, & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega, \\ \|\nabla_x u\|_2 \le \frac{1}{\sqrt{\lambda_1}}\|\Delta_x u\|_2. \end{cases}$$

We will employ embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, for $\frac{2n}{n-2} \ge q \ge 2$, if $0 \le n$ and $2 \le q$, if $n = 1,2$ and $L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $q < r$. Then, for some $c_s > 0$,

$$\|\nu\|_q \le c_s\|\nabla_x \nu\|_2, \quad \|\nu\|_q \le c_s\|\nu\|_r, \quad \text{for } \nu \in H_0^1(\Omega).$$

We will need the following assumptions:

(A_1) The relaxation functions $(g_i)_{i\in\{1,2\}}$ are differentiable functions such that,

$$g_i(s) \geq 0, \qquad \text{for } s \geq 0 \text{ and } i \in \{1,2\},$$

$$1 - \int_0^\infty g_i(s)ds = l_i > 0, \qquad \text{for } i \in \{1,2\},$$

and there are two differentiable positive nonincreasing functions $(\zeta_i)_{i\in\{1,2\}}$, such that

$$g_i'(s) \leq -\zeta_i(s)g_i(s), \quad \text{for } s \geq 0 \text{ and } i \in \{1,2\},$$

and

$$g_i'(s) \leq 0, \quad \text{for } s \geq 0 \text{ and } i \in \{1,2\}.$$

Our assumptions about the functions $g_i, i = 1,2$ are currently the most general. These assumptions are natural for systems arising in the study of time-PDEs.

(A_2) For $i \in \{1,2\}$, the functions $\varphi_i : \mathbb{R}^2 \to \mathbb{R}$ are $\mathcal{C}^1$, such that

$$\begin{cases} \varphi_1(u,v) = a|u+v|^{p-1}(u+v) + b|u|^{\frac{p-3}{2}}|v|^{\frac{p+1}{2}}u, \\ \varphi_2(u,v) = \varphi_1(v,u), \end{cases} \qquad \text{for all } (u,v) \in \mathbb{R}^2,$$

with $a,b > 0$, and a function Φ exists, such that

$$u\varphi_1(u,v) + v\varphi_2(u,v) = (p+1)\Phi(u,v), \text{ for all } (u,v) \in \mathbb{R}^2,$$

where

$$\Phi(u,v) = \frac{1}{(p+1)}(a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}), \quad \varphi_1(u,v) = \frac{\partial\Phi}{\partial u}, \quad \varphi_2(u,v) = \frac{\partial\Phi}{\partial v}.$$

(A_3) Two constants $c_0, c_1 > 0$ exist, such that

$$c_0(|u|^{p+1} + |v|^{p+1}) \leq \Phi(u,v) \leq c_1(|u|^{p+1} + |v|^{p+1}), \text{ for all } (u,v) \in \mathbb{R}^2,$$

and

$$\left|\frac{\partial\varphi_i}{\partial u}(u,v)\right| + \left|\frac{\partial\varphi_i}{\partial v}(u,v)\right| \leq C(|u|^{p-1} + |v|^{p-1}), \quad \text{for } i \in \{1,2\} \quad \text{where} \quad 1 \leq p < 6.$$

(A_4)

$$\begin{cases} p \geq 3, & \text{if } n = 1,2, \\ p = 3, & \text{if } n = 3. \end{cases}$$

3. Main Result for System

To demonstrate our solution for the problem (1), we follow the approach of Dafermos [22] by taking into account a new auxiliary variable, the relative history of u and v, as follows:

$$\eta_1 = \eta^{1t}(x,s) = u(x,t) - u(x,t-s) \quad \text{in } \Omega_\infty \times (0,\infty),$$

$$\eta_2 = \eta^{2t}(x,s) = v(x,t) - v(x,t-s) \quad \text{in} \quad \Omega_\infty \times (0,\infty),$$

and the weighted L^2-spaces

$$\begin{aligned}\mathcal{M}_i &= L^2_{g_i}(\mathbb{R}, H^4(\Omega)) \cap H^2_0(\Omega)) \\ &= \left\{\xi_i : \mathbb{R}^+ \to H^4(\Omega) \cap H^2_0(\Omega) : \int_0^\infty g_i(s)\|\Delta_x \zeta_i(s)\|_2^2 ds < \infty\right\}, \quad \text{for } i \in \{1,2\},\end{aligned}$$

which are a Hilbert spaces endowed with inner products and norms

$$\langle \xi_i, \zeta_i \rangle_{\mathcal{M}_i} = \int_0^\infty g_i(s)\left(\int_\Omega \Delta_x \xi_i(s)\Delta_x \zeta_i(s)dx\right)ds, \quad \text{for } i \in \{1,2\},$$

and

$$\|\xi_i\|^2_{\mathcal{M}_i} = \int_0^\infty g_i(s)\|\Delta_x \xi_i(s)\|_2^2 ds, \quad \text{for } i \in \{1,2\}.$$

Our analysis is given in phase space

$$\widetilde{\mathcal{H}} = H^2_0(\Omega) \cap H^4(\Omega) \times H^2_0(\Omega) \cap H^4(\Omega) \times H^2_0(\Omega) \times H^2_0(\Omega) \times \mathcal{M}_1 \times \mathcal{M}_2.$$

Therefore, problem (1) is equivalent to

$$\begin{cases} |u_t|^\ell u_{tt} - l_1\Delta_x^2 u - \Delta_x u_{tt} - \int_0^\infty g_1(s)\Delta_x^2 \eta^{1t}(x,s)ds + \varphi_1(u,v) = 0 & \text{in } \Omega_\infty, \\ |v_t|^\ell v_{tt} - l_2\Delta_x^2 v - \Delta_x v_{tt} - \int_0^\infty g_2(s)\Delta_x^2 \eta^{2t}(x,s)ds + \varphi_2(u,v) = 0 & \text{in } \Omega_\infty, \\ \eta^{1t}_t(x,t) + \eta^{1t}_s(x,s) = u_t(x,t) & \text{in } \Omega_\infty \times (0,\infty), \\ \eta^{2t}_t(x,t) + \eta^{2t}_s(x,s) = v_t(x,t) & \text{in } \Omega_\infty \times (0,\infty), \\ \eta^{10}(x,s) = \eta_{10}(x,s) & \text{in } \Omega_\infty, \\ \eta^{20}(x,s) = \eta_{20}(x,s) & \text{in } \Omega_\infty, \\ u(x,t) = v(x,t) = \eta^{1t}(x,t=0) = \eta^{2t}(x,t=0) = 0 & \text{on } \Gamma_\infty, \\ u(x,-t) = u_0(x,t),\ v(x,-t) = v_0(x,t), & \text{in } \Omega_\infty, \\ u_t(x,t=0) = u_1(x), v_t(x,t=0) = v_1(x), & x \in \Omega, \\ u(x,t=0) = u_0(x), v(x,t=0) = v_0(x), & x \in \Omega. \end{cases} \tag{3}$$

We define the energy function associated with the problem (3) by

$$\begin{aligned} E(t) \quad : \quad &= \frac{1}{\ell+2}\|u_t(t)\|^{\ell+2}_{\ell+2} + \frac{1}{\ell+2}\|v_t(t)\|^{\ell+2}_{\ell+2} + \frac{1}{2}\|\nabla_x u_t(t)\|_2^2 + \frac{1}{2}\|\nabla_x v_t(t)\|_2^2 + \\ &\frac{l_1}{2}\|\Delta_x u(t)\|_2^2 + \frac{l_2}{2}\|\Delta_x v(t)\|_2^2 + \int_0^\infty g_1(s)\|\Delta_x \eta^1(s)\|_2^2 ds \\ &+ \int_0^\infty g_2(s)\|\Delta_x \eta^2(s)\|_2^2 ds + \int_\Omega \Phi(u(t), v(t)dx. \end{aligned}$$

The following result can be proven by the Faedo–Galerkin procedure.

Theorem 1. *Suppose that* (A_1)–(A_4) *holds, and assume that* $(u_0, v_0, u_1, v_1, \eta_{10}, \eta_{20}) \in \widetilde{\mathcal{H}}$. *Then, a unique weak solution exists*

$$(u, v, u_t, v_t, \eta^1, \eta^2) \in \mathcal{C}([0,\infty) : \widetilde{\mathcal{H}}),$$

of (3) *satisfying*

$$u, v \in L^\infty([0,\infty) : H^4(\Omega) \cap H^2_0(\Omega)), \quad \eta^i \in L^\infty([0,\infty) : \mathcal{M}_i), \quad \textit{for } i \in \{1,2\},$$

$$u_t, v_t \in L^\infty([0,\infty) : H^2_0(\Omega)).$$

Proof. To generate an approximation solution, we employ the conventional Faedo–Galerkin approach. Let $\{w_j\}_{j=1}^\infty$ be the eigenfunctions of the operator $A = -\Delta_x$ with the zero Dirichlet boundary condition and $D(A) = H^4(\Omega) \cap H^2_0(\Omega)$. It is known that $\{w_j\}_{j=1}^\infty$ forms an orthonormal basis for $L^2(\Omega)$, $H^2_0(\Omega)$ and $H^4(\Omega) \cap H^2_0(\Omega)$. We consider two smooth orthonormal bases $\{\xi_j^1(x,s)\}_{j=1}^{\infty}$ and $\{\xi_j^2(x,s)\}_{j=1}^{\infty}$ for $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. For any integer $n \in \mathbb{N}$, we consider the finite-dimensional subspaces

$$W_n = \mathrm{Span}\{\omega_1, \ldots, \omega_n\} \subset V_2, \quad Q_1 n = \mathrm{Span}\{\xi_1^1, \ldots, \xi_n^1\} \subset \mathcal{M}_1,$$
$$Q_2 n = \mathrm{Span}\{\xi_1^2, \ldots, \xi_n^2\} \subset \mathcal{M}_2.$$

We will find an approximate solution in the following form:

$$u^n(t) = \sum_{j=1}^{j=n} a_{nj}(t)\omega_j(x) \;\;,\;\; \eta^{1,n}(s) = \sum_{j=1}^{j=n} b_{nj}(t)\xi_j^1(x,s),$$
$$v^n(t) = \sum_{j=1}^{j=n} d_{nj}(t)\omega_j(x) \;\;,\;\; \eta^{2,n}(s) = \sum_{j=1}^{j=n} h_{nj}(t)\xi_j^2(x,s),$$

satisfying the approximate problem:

$$\begin{aligned} \langle |u_t|^{\ell} u_{tt}^n, \omega_j\rangle_\Omega \quad &+ \quad \ell_1\langle \Delta_x u^n, \Delta_x \omega_j\rangle_\Omega + \langle \nabla_x u_{tt}^n, \nabla_x \omega_j\rangle_\Omega - \left\langle \int_0^\infty g_1(s)\Delta_x \eta^{1,n}(s)ds, \Delta_x\omega_j \right\rangle_\Omega \\ &+\langle \varphi_1(u^n, v^n), \omega_j\rangle_\Omega = 0, \end{aligned}$$

$$\begin{aligned} \langle |v_t|^{\ell} v_{tt}^n, \omega_j\rangle_\Omega \quad &+ \quad \ell_2\langle +\Delta_x v^n, \Delta_x \omega_j\rangle_\Omega + \langle \nabla_x u_{tt}^n, \nabla_x \omega_j\rangle_\Omega - \left\langle \int_0^\infty g_2(s)\Delta_x \eta^{2,n}(s)ds, \Delta_x\omega_j \right\rangle_\Omega \\ &+\langle \varphi_2(u^n, v^n), \omega_j\rangle_\Omega = 0, \end{aligned}$$

$$\begin{aligned} \left\langle \partial_t \eta^{1,n}, \xi_j^1 \right\rangle_{\mathcal{M}_1} &= -\left\langle \partial_s \eta^{1,n}, \xi_j^1 \right\rangle_{\mathcal{M}_1} + \left\langle u_t^n(t), \xi_j^1 \right\rangle_{\mathcal{M}_1}, \\ \left\langle \partial_t \eta^{2,n}, \xi_j^2 \right\rangle_{\mathcal{M}_2} &= -\left\langle \partial_s \eta^{2,n}, \xi_j^2 \right\rangle_{\mathcal{M}_2} + \left\langle v_t^n(t), \xi_j^2 \right\rangle_{\mathcal{M}_2}. \end{aligned}$$

□

Lemma 1. *The energy function satisfies the following inequality:*

$$E'(t) \leq \frac{1}{2}\int_0^\infty g_1'(s)\|\Delta_x \eta^1(s)\|^2 ds + \frac{1}{2}\int_0^\infty g_2'(s)\|\Delta_x \eta^2(s)\|^2 ds.$$

Proof. Multiply the first equation in (3) by $u_t(t)$ and the second one by $v_t(t)$; then, integrate the result over Ω to obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{\ell+2}\|u_t(t)\|_{\ell+2}^{\ell+2} + \frac{1}{\ell+2}\|v_t(t)\|_{\ell+2}^{\ell+2} + \tfrac{1}{2}\|\nabla_x u_t(t)\|_2^2 + \right. \\ &\left.\frac{1}{2}\|\nabla_x v_t(t)\|_2^2 + \ell_1\|\Delta_x u(t)\|_2^2 + l_2\|\Delta_x v(t)\|_2^2\right] + \\ &\frac{d}{dt}\int_\Omega \Phi(u(t),v(t))dx - \int_0^\infty g_1(s)\int_\Omega \Delta_x\eta^1(s)\Delta_x u_t(t)dxds - \\ &\int_0^\infty g_2(s)\int_\Omega \Delta_x\eta^2(s)\Delta_x v_t(t)dxds = 0\,. \end{aligned} \tag{4}$$

Since

$$\begin{cases} u_t(x,t) = \eta_t^1(x,s) + \eta_s^1(x,s), \\ v_t(x,t) = \eta_t^2(x,s) + \eta_s^2(x,s), \end{cases} \;,\quad \text{for } (x,s,t) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+,$$

we have

$$\begin{aligned}
&\int_0^\infty g_1(s)\int_\Omega \Delta_x\eta^1(s)\Delta_x u_t(t)dxds \\
=&\int_0^\infty g_1(s)\int_\Omega \Delta_x\eta^1(s)\Delta_x\eta_t^1(t)dxds \\
&+\int_0^\infty g_1(s)\int_\Omega \Delta_x\eta^1(s)\Delta_x\eta_s^1(t)dxds \\
=&\frac{1}{2}\int_0^\infty g_1(s)\frac{d}{dt}\|\Delta_x\eta^1(s)\|_2^2ds-\frac{1}{2}\int_0^\infty\left\|g_1'^1(s)\right\|_2^2ds, \qquad (5)
\end{aligned}$$

and

$$\int_0^\infty g_2(s)\int_\Omega \Delta_x\eta^2(s)\Delta_x v_t(t)dxds=\frac{1}{2}\int_0^\infty g_2(s)\frac{d}{dt}\|\Delta_x\eta^2(s)\|_2^2ds-\frac{1}{2}\int_0^\infty\left\|g_2'^2(s)\right\|_2^2ds. \quad (6)$$

By substituting (5) and (6) into (4), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Bigg[\frac{1}{\ell+2}\|u_t(t)\|_{\ell+2}^{\ell+2}+\frac{1}{\ell+2}\|v_t(t)\|_{\ell+2}^{\ell+2}+\frac{1}{2}\|\nabla_x u_t(t)\|_2^2+\frac{1}{2}\|\nabla_x v_t(t)\|_2^2 \\
&+\frac{l_1}{2}\|\Delta_x u(t)\|_2^2+\frac{l_2}{2}\|\Delta_x v(t)\|_2^2 \\
&+\int_\Omega \Phi(u(t),v(t))dx+\frac{1}{2}\frac{d}{dt}\int_0^\infty g_1(s)\|\nabla_x\eta^1(s)\|_2^2ds-\frac{1}{2}\int_0^\infty g_1'^1(s)\|_2^2ds \\
&+\frac{1}{2}\frac{d}{dt}\int_0^\infty g_2(s)\|\nabla_x\eta^2(s)\|_2^2ds-\frac{1}{2}\int_0^\infty\|g_2'^2(s)\|_2^2ds=0.
\end{aligned} \qquad (7)$$

Integrating (7) over $(0,t)$ yields

$$E(t)-\frac{1}{2}\int_0^t\int_0^\infty\left\|g_1'^1(s)\right\|_2^2ds-\frac{1}{2}\int_0^t\int_0^\infty\left\|g_2'^2(s)\right\|_2^2ds=E(0).$$

□

Lemma 2. *Under the assumptions of Theorem 1, the functional $\phi(t)$ defined by*

$$\phi(t)=\frac{1}{\ell+1}\int_\Omega |u_t|^\ell u_t u dx+\int_\Omega |v_t|^\ell v_t v dx+\int_\Omega \nabla_x u_t\nabla_x u dx+\int_\Omega \nabla_x v_t\nabla_x v dx,$$

satisfies, for some positive constants c', c'', c_1, c_2 and for any $t\geq 0$,

$$\begin{aligned}
\phi'(t) \leq\ & \frac{1}{\ell+1}\|u_t(t)\|_{\ell+2}^{\ell+2}+\frac{1}{\ell+1}\|v_t(t)\|_{\ell+2}^{\ell+2}-c'\|\Delta_x u(t)\|_2^2-c''\|\Delta_x v(t)\|_2^2 \\
&+c_1\int_0^\infty g_1(s)\|\Delta_x\eta^1(s)\|_2^2ds+c_2\int_0^\infty g_2(s)\|\Delta_x\eta^2(s)\|_2^2ds+ \\
&+\|\nabla_x u_t\|_2^2+\|\nabla_x v_t\|_2^2-(p+1)\int_\Omega \Phi(u,v)dx. \qquad (8)
\end{aligned}$$

Proof. Differentiating $\phi(t)$ with respect to t and using (3) gives

$$\begin{aligned}
\phi'(t) =\ & \|u_t\|_{\ell+2}^{\ell+2}+\|v_t(t)\|_{\ell+2}^{\ell+2}+\int_\Omega u(t)\left(l_1\Delta_x^2 u+\int_0^\infty g_1(s)\Delta_x^2\eta^1(s)ds-\varphi_1(u,v)\right)dx \\
&+\int_\Omega v(t)\left(l_2\Delta_x^2 v+\int_0^\infty g_2(s)\Delta_x^2\eta^2(s)ds-\varphi_2(u,v)\right)dx \\
&+\|\nabla_x u_t\|_2^2+\|\nabla_x v_t\|_2^2.
\end{aligned}$$

By using Young and Hölder's inequality, we can obtain for any $\delta > 0$,

$$\begin{aligned} -\int_\Omega \Delta_x u \int_0^\infty g_1(s)\Delta_x \eta^1(s)dsdx &\leq \delta\|\Delta_x u\|_2^2 + \frac{1}{4\delta}\int_\Omega \left(\int_0^\infty g_1(s)\Delta_x\eta^1(s)ds\right)^2 dx \\ &\leq \delta\|\Delta_x u\|^2 + \frac{1-l_1}{4\delta}\int_0^\infty g_1(s)\|\Delta_x \eta^1(s)\|_2^2 ds, \end{aligned} \tag{9}$$

$$-\int_\Omega \Delta_x v \int_0^\infty g_2(s)\Delta_x\eta^2(s)dsdx \leq \delta\|\Delta_x v\|_2^2 + \frac{1-l_2}{4\delta}\int_0^\infty g_2(s)\|\Delta_x\eta^2(s)\|_2^2 ds. \tag{10}$$

It follows from the assumptions on φ_1 and φ_2 that

$$-\int_\Omega (\varphi_1(u,v)u + \varphi_2(u,v)v)dx = -(p+1)\int_\Omega \Phi(u,v)dx. \tag{11}$$

By summing up (9)–(11), we obtain that for any $\delta > 0$,

$$\begin{aligned} \phi'(t) \leq\; & \frac{1}{\ell+1}\|u_t(t)\|_{\ell+2}^{\ell+2} + \frac{1}{\ell+1}\|v_t(t)\|_{\ell+2}^{\ell+2} - (l_1-\delta)\|\Delta_x u(t)\|_2^2 - (l_2-\delta)\|\Delta_x v(t)\|_2^2 \\ & + c_1\int_0^\infty g_1(s)\|\Delta_x\eta^1(s)\|_2^2 ds + c_2\int_0^\infty g_2(s)\|\Delta_x\eta^2(s)\|_2^2 ds \\ & + \|\nabla_x u_t\|_2^2 + \|\nabla_x v_t\|_2^2 - (p+1)\int_\Omega \Phi(u,v)dx. \end{aligned} \tag{12}$$

Now, by taking $\delta > 0$ so small so that

$$l_1 - \delta > \frac{l_1}{2}, \qquad l_2 - \delta > \frac{\ell_2}{2},$$

we can obtain (8) from (12), and hence the proof is completed. □

Lemma 3. *Under the assumptions of Theorem 1, some positive constants c_3, δ_1 exist such that, along the solution of system (3), the function $\psi_1(t)$ defined by*

$$\psi_1(t) = \int_\Omega \left(\Delta_x u_t(t) - \frac{1}{1+\ell}|u_t|^\ell u_t\right)\int_0^\infty g_1(s)\eta^1(s)dsdx,$$

satisfies

$$\begin{aligned} \psi_1'(t) \leq\; & \left(\frac{\delta_1 c}{\lambda_1}\left(\frac{2(\ell+2)}{\ell+1}E(0)\right)^{2\ell+1} + \delta_1\right)\|\Delta_x u\|_2^2 + \\ & + (1-l_1)\left(\frac{l_1^2}{4\delta_1} + 1 + \frac{c_s^2}{4\delta_1\lambda_1}\right)\int_0^\infty g_1(s)\Delta_x\eta^1(s)ds \\ & + \frac{\delta_1 c}{\lambda_1}\left(\frac{2(\ell+2)}{\ell+1}E(0)\right)^{\ell+1}\|\Delta_x v\|_2^2 + \frac{3(1-\ell_1)}{4}\|\Delta_x u_t\|_2^2 + \frac{3(1-l_1)}{4}\|u_t\|_{\ell+2}^{\ell+2} \\ & - \frac{2g_1(0)}{\lambda_1(1-l_1)}\int_0^\infty g_1'(s)\|\Delta_x\eta^1(s)\|_2^2 ds. \end{aligned} \tag{13}$$

Proof. From (3), we obtain

$$
\begin{aligned}
\psi_1'(t) &= \int_\Omega \left(-l_1\Delta_x^2 u - \int_0^\infty g_1(s)\Delta_x^2\eta^1(s)ds + \varphi_1(u,v)\right)\left(\int_0^\infty g_1(s)\eta^1(s)ds\right)dx \\
&\quad - \int_\Omega \left(\Delta_x u_t(t) - \frac{1}{1+\ell}|u_t|^\ell u_t\right)\int_0^\infty g_1(s)\eta_t^1(s)dsdx \\
&= \underbrace{l_1\int_\Omega \Delta_x u(t)\int_0^\infty g_1(s)\Delta_x\eta^1(s)dsdx}_{=I_1} + \underbrace{\int_\Omega\left(\int_0^\infty g_1(s)\Delta_x\eta^1(s)ds\right)^2 dx}_{=I_2} \\
&\quad + \underbrace{\int_\Omega \varphi_1(u,v)\int_0^\infty g_1(s)\eta^1(s)dsdx}_{=I_3} + \underbrace{\int_\Omega \nabla_x u_t\int_0^\infty g_1(s)\nabla_x\eta_t^1(s)dsdx}_{=I_4} \qquad (14) \\
&\quad + \underbrace{\int_\Omega |u_t|^\ell u_t\int_0^\infty g_1(s)\eta_t^1(s)dsdx}_{=I_5}.
\end{aligned}
$$

By using Young and Hölder's inequality, we conclude that for any $\delta_1 > 0$,

$$I_1 \le \delta_1\|\Delta_x u\|^2 + \frac{l_1^2(1-\ell_1)}{4\delta_1}\|\eta^1\|^2_{\mathcal{M}_1}, \qquad (15)$$

$$I_2 \le (1-l_1)\|\eta^1\|^2_{\mathcal{M}_1}, \qquad (16)$$

and

$$
\begin{aligned}
I_3 &\le \delta_1\int_\Omega |\varphi_1(u,v)|^2dx + \frac{1}{4\delta_1}\int_\Omega\left(\int_0^\infty g_1(s)\eta^1(s)ds\right)^2dx \qquad (17) \\
&\le C\delta_1(\|\nabla_x u\|^2 + \|\nabla_x v\|^2)^{2\ell+3} + \frac{(1-l_1)c_s^2}{4\delta_1}\int_0^\infty g_1(s)\|\nabla_x\eta^1(s)\|^2ds \\
&\le C\delta_1\left(\frac{2(\ell+2)}{\ell+1}E(0)\right)^{2\ell+1}(\|\nabla_x u\|^2 + \|\nabla_x v\|^2) + \frac{(1-\ell_1)c_s^2}{4\delta_1}\int_0^\infty g_1(s)\|\nabla_x\eta^1(s)\|^2ds \\
&\le \frac{\delta_1}{\lambda_1}C\left(\frac{2(\ell+2)}{\ell+1}E(0)\right)^{2\ell+1}(\|\Delta_x u\|_2^2 + \delta_1\|\Delta_x v\|_2^2) + \frac{(1-l_1)c_s^2}{4\delta_1\lambda_1}\int_0^\infty g_1(s)\|\Delta_x\eta^1(s)\|^2ds,
\end{aligned}
$$

where we used the fact

$$\int_\Omega |\varphi_1(u,v)|^2dx \le C(\|\nabla_x u\|^2 + \|\nabla_x v\|^2)^{2p+3}.$$

Noting that

$$
\begin{aligned}
\int_0^\infty g_1(s)\eta_t^1(s)ds &= -\int_0^\infty g_1(s)\eta_s^1(s)ds + \int_0^\infty u_t(t)g_1(s)ds \\
&= \int_0^\infty g_1'(s)\eta^1(s)ds + (1-l_1)u_t, \qquad (18)
\end{aligned}
$$

I_4 can be estimated as follows:

$$
\begin{aligned}
I_4 &= (1-l_1)\|\nabla_x u_t\|_2^2 + \int_\Omega \nabla_x u_t \int_0^\infty g_1'(s)\eta^1(s)dsdx \\
&\leq \frac{3(1-l_1)}{4}\|\nabla_x u_t\|_2^2 + \frac{1}{1-l_1}\int_\Omega \left(\int_0^\infty -g_1'(s)ds\right)\left(\int_0^\infty -g_1'(s)\nabla_x\eta^1(s)ds\right)dx \\
&\leq \frac{3(1-l_1)}{4}\|\nabla_x u_t\|_2^2 - \frac{g_1(0)}{1-l_1}\int_0^\infty g_1'(s)\|\nabla_x\eta^1(s)\|_2^2 ds \\
&\leq \frac{3(1-l_1)}{4}\|\nabla_x u_t\|_2^2 - \frac{g_1(0)}{\lambda_1(1-l_1)}\int_0^\infty g_1'(s)\|\Delta_x\eta^1(s)\|_2^2 ds,
\end{aligned}
\tag{19}
$$

by using (18), we obtain

$$
\begin{aligned}
I_5 &= (1-l_1)\|u_t\|_{\ell+2}^{\ell+2} + \int_\Omega |u_t|^\ell u_t \int_0^\infty g_1'(s)\eta^1(s)dsdx \\
&\leq \frac{3(1-l_1)}{4}\|u_t\|_{\ell+2}^{\ell+2} + \frac{1}{1-l_1}\int_\Omega \left(\int_0^\infty -g_1'(s)ds\right)\left(\int_0^\infty -g_1'(s)\eta^1(s)ds\right)dx \\
&\leq \frac{3(1-l_1)}{4}\|u_t\|_{\ell+2}^{\ell+2} - \frac{g_1(0)c_s^2}{1-l_1}\int_0^\infty g_1'(s)\|\nabla_x\eta^1(s)\|_2^2 ds \\
&\leq \frac{3(1-l_1)}{4}\|u_t\|_{\ell+2}^{\ell+2} - \frac{g_1(0)c_s^2}{\lambda_1(1-l_1)}\int_0^\infty g_1'(s)\|\Delta_x\eta^1(s)\|_2^2 ds.
\end{aligned}
\tag{20}
$$

Inserting (15), (16), (17), (19), and (20) into (14), we obtain (13). This completes the proof. □

We have the following lemma using the same argument as Lemma 3.

Lemma 4. *According to the assumptions of Theorem 1, the functional ψ_2 defined by*

$$
\psi_2(t) = \int_\Omega \left(\Delta_x v_t(t) - \frac{1}{1+\ell}|v_t|^\ell v_t\right)\int_0^\infty g_2(s)\eta^2(s)dsdx,
$$

satisfies, along the solution of system (3) for some positive constant c_3, δ_1,

$$
\begin{aligned}
\psi_2'(t) \leq\ & \delta_1\left[\frac{c}{\lambda_1}\left(\frac{2(\ell+2)}{\ell+1}E(0)\right)^{2\ell+1} + 1\right]\|\Delta_x v\|_2^2 \\
+\ & (1-l_2)\left[\frac{l_2^2}{4\delta_1} + 1 + \frac{c_s^2}{4\delta_1\lambda_1}\right]\int_0^\infty g_2(s)\Delta_x\eta^2(s)ds \\
& + \frac{\delta_1 c}{\lambda_1}\left(\frac{2(\ell+2)}{\ell+1}E(0)\right)^{\ell+1}\|\Delta_x u\|_2^2 + \frac{3(1-l_2)}{4}\|\Delta_x v_t\|_2^2 + \frac{3(1-l_1)}{4}\|v_t\|_{\ell+2}^{\ell+2} \\
& - \frac{2g_2(0)}{\lambda_1(1-l_2)}\int_0^\infty g_2'(s)\|\Delta_x\eta^2(s)\|_2^2 ds.
\end{aligned}
$$

In the sequel, we shall define the functional $\mathcal{L}(t)$ by

$$
\mathcal{L}(t) = E(t) + \varepsilon_1\phi(t) + \varepsilon_2(\psi_1(t) + \psi_2(t)),
$$

where ε_1 and ε_2 are positive constants that will be determined later.

Lemma 5. *For small enough $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$,we can obtain for any $t \geq 0$,*

$$
\frac{1}{2}E(t) \leq \mathcal{L}(t) \leq \frac{3}{2}E(t).
\tag{21}
$$

Proof. It is not difficult to see that a positive constant $\varepsilon > 0$ exists, such that

$$\begin{aligned} |\mathcal{L}(t) - E(t)| &\leq \frac{\varepsilon_1 + \varepsilon_2}{2}(\|u_t\|_{\ell+2}^{\ell+2} + \|v_t\|_{\ell+2}^{\ell+2} + \|\nabla_x u_t\|_2^2 + \|\nabla_x v_t\|_2^2) \\ &\quad + \varepsilon_1\|\Delta_x u\|^2 + \varepsilon_2\|\Delta_x v\|^2 + C\varepsilon_2\int_0^\infty g_1(s)\|\Delta_x \eta^1(s)\|^2 ds \\ &\quad + C\varepsilon_2\int_0^\infty g_2(s)\|\Delta_x\eta^2(s)\|^2 ds + \varepsilon_1\int_\Omega \Phi(u,v)dx \\ &\leq \varepsilon E(t). \end{aligned}$$

This implies that

$$(1-\varepsilon)E(t) \leq \mathcal{L}(t) \leq (1+\varepsilon)E(t).$$

Noting that $\varepsilon > 0$ is small enough if $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are small. Hence, we can obtain (21) if we choose small enough $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

This completes the proof. □

Lemma 6. *Two positive constants, k_0 and k_1, exist such that for any $t \geq 0$,*

$$\mathcal{L}'(t) \leq -k_0 E(t) + k_1\left(\int_0^\infty g_1(s)\|\Delta_x\eta^1(s)\|_2^2 ds + \int_0^\infty g_2(s)\|\Delta_x\eta^2(s)\|_2^2 ds\right). \tag{22}$$

Proof. It follows from Lemmata 1–4, that for any $t \geq 0$,

$$\begin{aligned} \mathcal{L}'(t) &\leq -\left(\varepsilon_1 - \frac{3(1-l_1)}{4}\varepsilon_2\right)\|u_t\|_{\ell+2}^{\ell+2} - \left(\varepsilon_1 - \frac{3(1-l_2)}{4}\varepsilon_2\right)\|v_t\|_{\ell+2}^{\ell+2} \\ &\quad - \left[\varepsilon_1 c' - \varepsilon_2\left\{\frac{2\delta_1 c}{\lambda_1}\left(\frac{2(\ell+2)}{(\ell+1)}E(0)\right)^{2\ell+1}\right\}\right]\|\Delta_x u\|_2^2 \\ &\quad - \left[\varepsilon_1 c'' - \varepsilon_2\left\{\frac{2\delta_1 c}{\lambda_1}\left(\frac{2(\ell+2)}{(\ell+1)}E(0)\right)^{2\ell+1}\right\}\right]\|\Delta_x v\|_2^2 \\ &\quad - \left[\varepsilon_1\left[1 - \delta_1 c_s^2\frac{\ell+2}{\ell+1}\left(\frac{2(\ell+2)}{(\ell+1)}E(0)\right)^{\ell+1}\right] - \varepsilon_2\frac{3(1-l_1)}{4}\right]\|\Delta_x u_t\|_2^2 \\ &\quad - \left[\varepsilon_1\left[1 - \delta_1 c_s^2\frac{\ell+2}{\ell+1}\left(\frac{2(\ell+2)}{(\ell+1)}E(0)\right)^{\ell+1}\right] - \varepsilon_2\frac{3(1-l_2)}{4}\right]\|\Delta_x v_t\|_2^2 \\ &\quad + \left[\varepsilon_1\left[\frac{(1-l_1)l_1^2}{4\delta_1} + 1 + \frac{c_s^2}{4\delta_1\lambda_1} + c_1\right]\right]\int_0^\infty g_1(s)\|\Delta_x\eta^1(s)\|_2^2 ds \\ &\quad + \left[\varepsilon_1\left(\frac{(1-l_1)l_2^2}{4\delta_1} + 1 + \frac{c_s^2}{4\delta_1\lambda_1} + c_1\right)\right]\int_0^\infty g_2(s)\|\Delta_x\eta^2(s)\|_2^2 ds \\ &\quad + \left(\frac{1}{2} - \frac{2\varepsilon_2 g_1(0)}{\lambda_1(1-\ell_1)}\right)\int_0^\infty g_1'^1(s)\|_2^2 ds \\ &\quad + \left[\frac{1}{2} - \frac{2\varepsilon_2 g_2(0)}{\lambda_1(1-\ell_2)}\right]\int_0^\infty g_2'^2(s)\|_2^2 ds - (p+1)\varepsilon_1\int_\Omega \Phi(u,v)dx. \end{aligned}$$

First, we take δ_1 satisfying

$$\delta_1 < \frac{1}{c_s^2\frac{(\ell+2)}{(\ell+1)}\left(\frac{2(\ell+2)}{(\ell+1)}E(0)\right)^{\ell+1}}.$$

Now, choose small enough $\varepsilon_2 > 0$ so that

$$\varepsilon_1 \left[1 - \delta_1 c_s^2 \frac{\ell+2}{\ell+1} \left(\frac{2(\ell+2)}{(\ell+1)} E(0) \right)^{\ell+1} \right] - \varepsilon_2 \frac{3(1-l_1)}{4} > 0,$$

$$\varepsilon_1 \left[1 - \delta_1 c_s^2 \frac{\ell+2}{\ell+1} \left(\frac{2(\ell+2)}{(\ell+1)} E(0) \right)^{\ell+1} \right] - \varepsilon_2 \frac{3(1-l_2)}{4} > 0,$$

$$\varepsilon_1 - \frac{3(1-l_1)}{4} \varepsilon_2 > 0, \quad \varepsilon_1 - \frac{3(1-l_2)}{4} \varepsilon_2 > 0.$$

In light of the above estimates, we can obtain (22). The proof is completed. □

Theorem 2. *Assume that* (A_1)–(A_4) *hold. Let* $(u_0, v_0, u_1, v_1, \eta^{10}, \eta^{20}) \in \widetilde{\mathcal{H}}$. *Then, two constants* $\mu \in (0,1)$ *and* $\delta_1 > 0$ *exist such that for any* $\delta_0 \in (0, \mu]$,

$$E(t) \leq \delta_1 \left(1 + \int_0^t h^{1-\delta_0}(s) \right) exp \left(-\delta_0 \int_0^t \zeta(s) ds \right) + \delta_1 \int_t^\infty h(s) ds, \tag{23}$$

where $\zeta(t) = \min\{\zeta_1(t), \zeta_2(t)\}$ *and* $h(t) = \max\{g_1(t), g_2(t)\}$.

In order to prove this theorem, the following lemma from [20] is needed.

Lemma 7 ([20]). *Under the assumptions of Theorem 2, two constants* $\beta_1 > 0$ *and* $\beta_2 > 0$ *exist such that for any* $t \geq 0$,

$$\zeta(t)\mathcal{L}'(t) + \beta_1 E'(t) \leq -k_0 \zeta(t) E(t) + \beta_2 \zeta(t) \int_t^\infty h(s) ds,$$

where $\zeta(t) = \min\{\zeta_1(t), \zeta_2(t)\}$ *and* $h(t) = \max\{g_1(t), g_2(t)\}$.

Proof of Theorem 2. Define the functional $\mathcal{E}(t)$ by

$$\mathcal{E}(t) = \zeta(t)\ell(t) + \beta_1 E(t). \tag{24}$$

It is not difficult to verify that $\mathcal{E}(t) \sim E(t)$. Let

$$R(t) = \zeta(t) \int_0^\infty h(s) ds.$$

Using (24) and the fact that $\zeta(t) > 0$ and $\zeta'(t) \leq 0$ a.e. $t \geq 0$, we deduce that for some $\gamma_0 > 0$,

$$\mathcal{E}'(t) \leq -\gamma_0 \zeta(t) \mathcal{E}(t) + \beta_2 R(t), \qquad a.e. \; t \geq 0.$$

In addition, the following inequality holds for any $\delta_0 \in (0, \gamma_0]$,

$$\mathcal{E}'(t) \leq -\delta_0 \zeta(t) \mathcal{E}(t) + \beta_2 R(t), \qquad \text{a.e. } t \geq 0. \tag{25}$$

Integrating (25) over $[0, T]$ leads to

$$\mathcal{E}(T) \leq e^{-\delta_0 \int_0^T \zeta(s) ds} \left(\mathcal{E}(0) + \beta_2 \int_0^T e^{\delta_0 \int_0^T \zeta(s) ds} R(t) dt \right),$$

which, together with the fact $\mathcal{E}(t) \sim E(t)$, yields

$$E(T) \leq \frac{1}{\beta_1} e^{-\delta_0 \int_0^T \zeta(s) ds} \left(\mathcal{E}(0) + \beta_2 \int_0^T e^{\delta_0 \int_0^T \zeta(s) ds} R(t) dt \right). \tag{26}$$

It follows that

$$\begin{aligned}\int_0^T e^{\delta_0\int_0^T\zeta(s)ds}R(t)dt &= \frac{1}{\delta_0}\int_0^T\left(\int_0^\infty h(s)ds\right)\frac{d}{dt}\left(e^{\delta_0\int_0^T\zeta(s)ds}\right)dt\\ &= \frac{1}{\delta_0}(e^{\delta_0\int_0^t\zeta(s)ds}\int_T^\infty h(s)ds-\int_0^\infty h(s)ds\\ &+\int_0^T e^{\delta_0\int_0^t\zeta(s)ds}h(t)dt).\end{aligned}\tag{27}$$

Inserting (27) into (26) gives

$$E(T)\le\frac{1}{\beta_1}\left(\mathcal{E}(0)+\frac{\beta_2}{\delta_0}\int_0^T e^{\delta_0\int_0^t\zeta(s)ds}h(t)dt\right)e^{-\delta_0\int_0^T\zeta(s)ds}+\frac{\beta_2}{\beta_1\delta_0}\int_T^\infty h(s)ds.\tag{28}$$

By using (A_1), we infer that for any $t\ge 0$,

$$\begin{aligned}&\frac{d}{dt}\left(e^{\int_0^t\zeta(s)ds}(g_1(t)+g_2(t))\right)\\ =\;&(g_1'(t)+g_2'(t))e^{\int_0^t\zeta(s)ds}+(g_1(t)+g_2(t))\zeta(t)e^{\int_0^t\zeta(s)ds}\\ \le\;&[-\zeta_1(t)g_1(t)-\zeta_2(t)g_2(t)]e^{\int_0^t\zeta(s)ds}+\zeta(t)(g_1(t)+g_2(t))e^{\int_0^t\zeta(s)ds}\\ \le\;&[(\zeta(t)-\zeta_1(t))g_1(t)+(\zeta(t)-\zeta_2(t))g_2(t)]e^{\int_0^t\zeta(s)ds}\le 0.\end{aligned}\tag{29}$$

It follows from (29) that

$$e^{\int_0^t\zeta(s)ds}h(t)\le e^{\int_0^t\zeta(s)ds}(g_1(t)+g_2(t))\le g_1(0)+g_2(0)\le 2h(0),$$

and

$$\int_0^T e^{\delta_0\int_0^t\zeta(s)ds}h(t)dt\le(2h(0))^{\delta_0}\int_0^T h^{1-\delta_0}(t)dt.$$

Therefore, (23) follows from (28) and (29), and thus demonstrating Theorem 2. □

Remark 1. *If $\varepsilon_0\in(0,1)$ exists, for which*

$$\int_0^{+\infty}(h(s))^{1-\varepsilon_0}ds<+\infty,\tag{30}$$

then we can choose $0<\delta_0\le\gamma_1$, $\gamma_1=\min\{\varepsilon_0,\gamma_0\}$, such that $\int_0^{+\infty}(h(s))^{1-\delta_0}ds<+\infty$, and, consequently, (23) takes the form

$$E(t)\le\delta_2\left(exp\left(-\delta_0\int_0^t\zeta(s)ds\right)+\int_t^\infty h(s)ds\right),\quad \delta_2>0.\tag{31}$$

4. Kernels with Exponential Decay

In this section, we investigate the cases of exponentially decaying kernels, and the results will be obtained without (A_1).

Theorem 3. *Assume that (A_2)–(A_4) hold true. Let $(u_0,v_0,u_1,v_1,\eta^{10},\eta^{20})\in\widetilde{\mathcal{H}}$, such that*

$$g_i'(t)\le-\xi_i g_i(t),\quad \textit{for } t\ge 0,\ i\in\{1,2\}.$$

Then, there are two constants $\mu>0$ and $\delta_1>0$; we have

$$E(t)\le\delta_1 e^{-\mu t}.\tag{32}$$

Proof. We multiply (23) by $\xi = \min\{\xi_1, \xi_2\}$ and use Lemma 1 to obtain

$$\xi\mathcal{L}'(t) \leq -k_0\xi E(t) + k_1\xi\left(\int_0^\infty g_1(s)\|\Delta_x\eta^1(s)\|^2 ds + \int_0^\infty g_2(s)\|\Delta_x\eta^2(s)\|^2 ds\right). \tag{33}$$

Now, using the fact that

$$\begin{aligned} k_1\left(\int_0^\infty g_1(s)\|\Delta_x\eta^1(s)\|^2 ds + \int_0^\infty g_2(s)\|\Delta_x\eta^2(s)\|^2 ds\right) &\leq -\frac{k_1}{\xi}\int_0^\infty g_1(s)'^1(s)\|^2 ds \\ &\quad -\frac{k_1}{\xi}\int_0^\infty g_2(s)'^2(s)\|^2 ds \\ &\leq -cE'(t). \end{aligned} \tag{34}$$

implies that

$$\xi\mathcal{L}'(t) \leq -k_0\xi E(t) - cE'(t).$$

The functional $\Phi = \xi\mathcal{L}(t) + cE(t)$ satisfies $\Phi \sim E$; we easily obtain

$$E(t) \leq \delta_1 e^{-\mu t}.$$

□

Remark 2. *It is worth mentioning here that our stability result was obtained without imposing the condition (A_4), which was imposed in [20].*

5. Examples

We illustrate the energy decay rate given by Theorem 2 throughout the following examples, which are introduced in [9].

Example 1. *Let $g_i(t) = a_i e^{-b_i(1+t)}$, with $b_i > 0$ and $a_i > 0$, for $i \in \{1,2\}$ small enough so that (A_1), with $(\zeta_i)_{i\in\{1,2\}} = (b_i)_{i\in\{1,2\}}$, holds. In this case, $\zeta(t) = \min\{b_1, b_2\} = b_0$ and $h(t) = A_0 e^{-b_0(1+t)}$, where $A_0 = \max\{a_1, a_2\}$. Then, (30) is satisfied and, consequently, (31) gives, for two positive constants c_1, c_2,*

$$E(t) \leq c_1 e^{-c_2(1+t)}, \quad \text{for all } t \in \mathbb{R}^+.$$

Example 2. *Let $g_i(t) = \frac{a_i}{(1+t)^{b_i}}$, with $b_i > 1$ and $a_i > 0$, for $i \in \{1,2\}$ small enough so that (A_1) with $(\zeta_i)_{i\in\{1,2\}} = \left(\frac{b_i}{1+t}\right)_{i\in\{1,2\}}$ holds. In this case, $\zeta(t) = \frac{b_0}{1+t}$ and $h(t) = \frac{A_0}{(1+t)^{b_0}}$, where $A_0 = \max\{a_1, a_2\}$ and $b_0 = \min\{b_1, b_2\}$. Then, (30) is satisfied, and, hence, (31) yields*

$$E(t) \leq c_1 e^{-c_2\ln(1+t)} = c_1(1+t)^{-c_2}, \quad \text{for all } t \in \mathbb{R}^+,$$

where $\zeta(t) = \min\{\zeta_1(t), \zeta_2(t)\}, \quad h(t) = \max\{g_1(t), g_2(t)\}$.

6. Concluding Remarks

The main purpose of this paper was to establish the solution of nonlinear systems in coupling Petrovsky–Petrovsky systems with infinite memory under minimum assumptions on the functions g_1, g_2 and φ_1, φ_2. Moreover, the general decay rate of the relevant energy is also established. The results are limited on the bounded domain Ω of $\mathbb{R}^n$. To conclude, we should mention that the original contributions in the present paper are:

1. We used classical methods to solve a non-trivial problem with useful new results to rival state-of-the-art work in Thorems 1–3.
2. It is shown that we are able to link the rate of decrease to the energy functional associated with the solution directly to that of the functions g_1, g_2, with an improvement in the conditions taken on these relaxation functions in (23).

3. We found that the two functions g_1, g_2 are responsible for the decay rate of the energy functional and then that of the existed solution. On the other hand, the functions φ_1, φ_2 obstruct the solution if they can overcome and dominate [23].
4. We give more cases to the kernel functions to discuss their impact on the decay rate.

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