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On Solving Stochastic Optimization Problems

Ovidiu Blăjină and Ionuţ Gabriel Ghionea *

Manufacturing Engineering Department, Faculty of Industrial Engineering and Robotics, National University of Science and Technology Politehnica Bucharest, Splaiul Independenţei 313, Dist. 6, 060042 Bucharest, Romania; ovidiu.blajina@upb.ro

* Correspondence: gabriel.ghionea@upb.ro

Abstract: Many optimization mathematical models, associated with the technical-economic processes of real-world problems, have elements of uncertainty in their structure, which places them in stochastic optimization programming. Their diversity and complexity, due to the large uncertainty space, require special methods of solving, because there is no general solution method. Within this context, in this paper we consider the category of optimization models that can contain random variable type coefficients and/or imposed probability levels on the constraints. The purpose of the paper is to propose a methodology dedicated to these studied models. Applying the methodology leads to developing a deterministic linear programming model, associated with the initial stochastic model. In fact, the proposed methodology reduces the stochastic formulation to a deterministic formulation. The methodology is illustrated with a numerical case study based on a manufacturing problem. Solving the obtained deterministic model is carried out in the version assisted by a specialized software product (*WinQSB* Version 2.0). It allows for the performing of a sensitivity analysis of the optimal solution, and/or a parametric analysis relative to certain model coefficients, both also presented in the paper. The main result of the study in this paper is the proposed methodology, which is applicable on a large scale, for any mathematical model of stochastic optimization of the mentioned type, regardless of complexity, dimensions and the domain of the process to which it is associated. The numerical results obtained when applying this methodology indicate its efficiency and effectiveness in finding the solution for the studied models. The approach to this issue in the present paper is determined by the wide range of stochastic optimization problems in the various studied real-life processes and by the imperative need to adopt the best decisions in conditions of uncertainty.

Keywords: linear programming; stochastic programming; sensitivity analysis; parametric analysis; WinQSB

MSC: 60H35 computational methods for stochastic equations (aspects of stochastic analysis)

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Citation: Blăjină, O.; Ghionea, I.G. On Solving Stochastic Optimization Problems. *Mathematics* **2023**, *11*, 4451. https://doi.org/10.3390/math11214451

Academic Editor: Konstantin Kozlov

Received: 25 September 2023
Revised: 22 October 2023
Accepted: 25 October 2023
Published: 27 October 2023

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1. Introduction

The act of obtaining the best decision under given circumstances is known as optimization. For deterministic optimization problems, no randomness or uncertainty is considered, and so the solutions are relatively simple. However, many real-world optimization problems involve some sort of uncertainty in the form of randomness [1]. Randomness usually enters the problem through the objective function and/or the constraints set.

Linear programming is a branch of mathematical programming, considered a tool for optimizing a result through a deterministic optimization model. It is a quantitative analysis technique to achieve the desired solution, and it is considered the most important method of optimization in different fields of research. It is used to obtain the optimal solution to a problem within some constraints [2]. Linear programming is concerned with the optimization (maximization or minimization) of a linear objective function in many variables subject to linear equality and inequality constraints [3].

Stochastic programming is involved in modeling optimization problems that invoke uncertainty. Real, everyday problems often include coefficients that are unknown at the time a decision should be made. A stochastic programming model is formulated with random requirements when the optimal decision is uncertain relative to the state of the future events. The size of such model grows proportionally to the number of possible use of uncertain coefficients. Solution approaches to stochastic programming models are driven by the type of probability distributions that set up the random coefficients.

Stochastic optimization has been studied in a broad set of communities that each developed methods to solve problems that were important for them.

Dantzig was the first researcher who introduced stochastic programming problems [4]. A procedure for solving this problem, considering both randomness and interval parameters in the problem formulation, was established by Barik et al. [5]. Ahmed [6] proposed the formulation technique of two-stage stochastic programming with fixed recourse. An extensive analysis of the different categories of stochastic programming models with the indication of computational methods is carried out by Birge. Much of stochastic programming research has considered various theoretical properties. Important convexity, continuity, and stability properties appear for recourse models [7]. For chance-constrained models, Prékopa derived key convexity conditions [8]. A procedure to find a solution for a stochastic programming problem where the coefficients in objective function follow uniform distribution and where some of the parameters follow varying continuous probability distributions with known mean and variance is proposed by Doshi and Trivedi [9]. Stochastic integer programming models arise when the decision variables are required to take on integer values. In most practical situations, this involves a loss of convexity and makes the application of decomposition methods problematic. Techniques for solving such models is an active research area [10]. A multistage stochastic optimization problem with quasi-variation inequality constraints is analyzed by Scopelliti [11] by using the Rockafellar and Wets multistage stochastic approach. Powell [12] treats the diversity of optimization problems under uncertainty, a policy search based on look-ahead approximations and risk in such problems.

Stochastic programming is largely used in different optimization application domains like finance, technology, ecology, maintenance, telecommunications, transportation, manufacturing, etc. [13]. Recently, there has been a strong orientation of researchers towards the development of specific methods dedicated to certain processes in particular fields. The paper by [14] is centered on an optimal control problem with applications in car suspension systems and the accumulation of pollution caused by the consumption of gas, oil, etc. Shao et al. [15] present a study of deregulated electricity using fuzzy variables and robust optimization so that the expectation of a risk-averse distribution system operator is fulfilled. Rigoberto and Lopez-Barrientos [16] treat the insuring of the extraction tasks of non-renewable resources using stochastic dynamic programming.

The purpose of our paper is to propose a methodology for solving the optimization mathematical models associated with technical-economic processes, which contain random variable type coefficients and/or imposed probability levels (different from unity) on the constraints. Starting from the initial stochastic model, a deterministic linear programming model will be developed, solvable with specific methods and which, subsequently, makes possible a sensitivity analysis of the optimal solution and/or a parametric analysis relative to certain coefficients.

The methodology is applicable to any stochastic optimization model of the mentioned type, no matter its size, number of variables, or restrictions. Larger dimensions of the initial model do not increase the degree of difficulty of setting up the deterministic model. From a dimensional point of view, the resulting deterministic model has the same number of variables as the initial stochastic model, but a greater number of restrictions due to the possible combinations between the multiple values that the discrete random variables can take. The proposed set-up of the deterministic model does not require a sophisticated mathematical apparatus. The results obtained by solving this last model are then attributed to the initial

stochastic model. The spectrum of real-world applications that can be formalized with the type of stochastic model studied in this paper is diverse; consequently, the proposed methodology is applicable on a large scale, without any limitations of its use in practice.

The paper is organized as follows. Section 2 gives a brief overview of some theoretical aspects of classical linear programming. In Section 3, the characteristics of the stochastic programming model type studied in this paper are described. It also deals with the ways this model is converted into a deterministic linear programming model. Section 4 is dedicated to the proposed methodology for solving and analyzing the considered type of stochastic model. In Section 5, a numerical case study is presented to exemplify the practical application of the proposed methodology. Section 6 contains the main conclusions of this paper.

2. Theoretical Aspects of Linear Programming

In this section, certain essential theoretical aspects regarding linear programming problems are presented, since the methodology proposed in Section 4 aims precisely at the construction of a deterministic mathematical model of linear programming associated with the initial stochastic model.

A linear programming problem may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints. The constraints may be equalities or inequalities.

A linear programming problem in general form has the following mathematical model:

$$\begin{array}{ll} \text{Maximize or minimize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\ \text{Subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq (\geq, =)\, b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq (\geq, =)\, b_2 \\ & \cdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq (\geq, =)\, b_m \end{array} \quad (1)$$

where the x_j (j = 1,..., n) values are decision variables, and the c_j, a_{ij}, b_i (i = 1,..., m; j = 1,..., n) values are constants, called coefficients, which are given or specified by a problem's assumptions.

In linear programming, the expression being optimized (maximized or minimized) z is called the objective function. The other relations in the model (1) constitute the system of problem constraints. In each relation, only one of the symbols "$\leq$", "$\geq$", or "=" appears. The left hand side of each relation is a linear function in the variables x_1, x_2,..., x_n.

A linear programming problem is in standard form if in its mathematical model all the constraints have the symbol "=" and, in addition, the non-negativity conditions of the variables also appear. This model can be rewritten in the reduced format:

$$\begin{array}{ll} \text{Maximize or minimize } z = \sum_{j=1}^{n} c_jx_j & \\ \text{Subject to} & \sum_{j=1}^{n} a_{ij}x_j = b_i,\ i = 1, \ldots, m \\ & x_j \geq 0,\ j = 1, \ldots, n \end{array} \quad (2)$$

The mathematical model can have two canonical forms:

$$\begin{array}{ll} \text{Maximize (minimize) } z = \sum_{j=1}^{n} c_jx_j & \\ \text{Subject to} & \sum_{j=1}^{n} a_{ij}x_j \leq (\geq)\, b_i,\ i = 1, \ldots, m \\ & x_j \geq 0,\ j = 1, \ldots, n \end{array} \quad (3)$$

All these forms of linear programming models are equivalent; with elementary linear transformations, they can go from one form to another. The most convenient way to solve a model is its standard form.

Solving a linear programming problem consists in determining those values of the decision variables x_1, x_2,. . ., x_n, which satisfy the problem's constraints and conditions and which optimize the value of the objective function z. To solve a classical linear programming problem, the simplex method or interior-point methods are used. As it is well known, in a such model, all the constraints must be necessarily satisfied, and the variables are non-negative. If the mathematical model contains only two variables, the graphic method can also be applied.

3. Studied Stochastic Optimization Model

Let us assume the following mathematical model:

$$\begin{aligned} &\text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ &\text{Subject to } \mathrm{P}\left(\sum_{j=1}^{n} a_{lj}x_j \le b_l\right) \ge p_l, l = 1, \ldots, q \\ &\qquad \sum_{j=1}^{n} a_{sj}x_j \le b_s, s = q+1, \ldots, m \\ &\qquad x_1, \ldots, x_n \ge 0 \end{aligned} \tag{4}$$

where some or all the coefficients in the objective function c_j $(j = 1, \ldots, n)$ and/or some of the coefficients in the constraints a_{sj} $(s = q+1, \ldots, m, j = 1, \ldots, n)$ are random variables, and/or the first q constraints are set to satisfy a given probability level p_l $(l = 1, \ldots, q)$. It is a stochastic programming model to solve.

Such a model can no longer be solved with the simplex algorithm. It is necessary to develop, in advance, an equivalent deterministic model of linear programming.

Let us assume that in the first q constraints of the model

$$\mathrm{P}\left(\sum_{j=1}^{n} a_{lj}x_j \le b_l\right) \ge p_l, l = 1, \ldots, q \tag{5}$$

each coefficient b_l $(l = 1, \ldots, q)$ is an independent random continuous variable, which follows a normal distribution, with mean μ_l and dispersion (variance) σ_l^2. The inequality (5) shows that the l-th constraint would not be satisfied with the probability $1 - p_l$ for any admissible choice of the x_j (j = 1,. . ., n) values.

However,

$$\mathrm{P}\left(\sum_{j=1}^{n} a_{lj}x_j \le b_l\right) = \mathrm{P}\left(\frac{\sum_{j=1}^{n} a_{lj}x_j - \mu_l}{\sigma_l} \le \frac{b_l - \mu_l}{\sigma_l}\right) \tag{6}$$

where σ_l is the standard deviation.

Since the variable b_l is normally distributed, based on the transformation relation of a normal distribution into a standard normal distribution, it results that $\frac{b_l - \mu_l}{\sigma_l}$ is the standardized normal variable, with mean zero and unit variance [12].

Based on Gauss–Laplace's function:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{u^2}{2}} \mathrm{d}u \tag{7}$$

relation (5) becomes

$$-\Phi\left(\frac{\sum_{j=1}^{n} a_{lj}x_j - \mu_l}{\sigma_l}\right) \geq p_l \tag{8}$$

or

$$\frac{\sum_{j=1}^{n} a_{lj}x_j - \mu_l}{\sigma_l} \leq -\Phi^{-1}(p_l) \tag{9}$$

The value $\Phi^{-1}(p_l)$ can be evaluated from a standard normal table. We note $\Phi^{-1}(p_l) = k_l$. Then the relation (9) is rewritten as follows:

$$\sum_{j=1}^{n} a_{lj}x_j \leq \mu_l - k_l\sigma_l \tag{10}$$

We assume that the mathematical model contains a number of t coefficients of the type of discrete random variables, independent two by two from a statistical point of view, denoted by $rv^{(k)}$ ($k = 1,\ldots, t$). Each random variable $rv^{(k)}$ has a finite number n_k of possible values in its distribution table:

$$rv^{(k)} : \begin{pmatrix} v_1^{(k)} & v_2^{(k)} & \cdots & v_{n_k}^{(k)} \\ p_1^{(k)} & p_2^{(k)} & \cdots & p_{n_k}^{(k)} \end{pmatrix}, k = 1, \ldots, t \tag{11}$$

The coefficients of the mathematical model of the discrete random variable type can take any of their values within the model. As a result, several numerical forms of the initial mathematical model are possible. Their number N is equal to

$$N = \prod_{k=1}^{t} n_k \tag{12}$$

Any of the N numerical forms of the model can be expressed synthetically by grouping the concrete values of the coefficients in a matrix of the following form:

$$M_i = \begin{bmatrix} c_1 & c_2 & \cdots & c_n & \\ a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}, i = 1, \ldots, N \tag{13}$$

Among the matrix elements, t are random variables, and the rest are deterministic values.

Since the t random variables are independent two by two, it follows that the probability of existence of each matrix M_i, denoted by p_i ($i = 1, \ldots, N$), is equal to the product of the probabilities of the values taken by the random variables in the respective matrix:

$$p_i = \prod_{k=1}^{k} p_{j,i}^{(k)}, j = 1, \ldots, n_k, i = 1, \ldots, N \tag{14}$$

where $p_{j,i}^{(k)}$ is the probability of the value $v_j^{(k)}$, $j = 1, \ldots, n_k$, of the random coefficient $rv^{(k)}$, $k = 1, \ldots, t$, in the matrix M_i, $i = 1, \ldots, N$.

Each coefficient of the objective function, c_j, $j = 1, \ldots, n$, which is of discrete random variable type, will be replaced in the deterministic mathematical model with its average

value, calculated as the weighted sum of the possible values of that coefficient, multiplied by the value of the probability of the matrices' existence, of which they are a part:

$$\widetilde{c}_j = \sum_{i=1}^{N} p_i \cdot c_j^{(i)}, j = 1, \ldots, n \tag{15}$$

where $c_j^{(i)}$ is the value of the random variable c_j in the matrix M_i, $i = 1, \ldots, N$.

The final deterministic model will contain all possible, distinct constraints that can be formed by combining different values of the random variables in the stochastic model's constraints structure.

Thus, the optimal solution of this obtained deterministic model will satisfy any numerical variant of the mathematical model among the N possible ones, resulting from the initial stochastic model.

The deterministic mathematical model, thus obtained, has its uniqueness ensured by the application of the methodology proposed in Section 4. Such model includes all possible values of the random variables and satisfies the restrictions with the imposed probabilities.

4. Methodology for Solving the Studied Stochastic Models

In formulating an optimization model for a specific technical-economic problem, four stages are applied, namely: (1) determining decision variables; (2) formulating the objective function; (3) determining and modeling the constraint functions; (4) formulating the model of the linear programming statement subject to the given constraints.

The diversity as well as the complexity of stochastic programming models mean that there is no general solution method applicable to them, like the simplex algorithm in classical linear programming.

In order to solve and analyze a stochastic programming model with configuration (4), we propose the following methodology through the application of which a deterministic linear programming model is built, and afterward it can be solved and analyzed:

Step 1. It is analyzed if the initial mathematical model is of stochastic type. If yes, step 2 follows; otherwise, the model is solved with the simplex algorithm.
Step 2. If the mathematical model contains preassigned probability levels on the constraints (see relation 5), then these are replaced in the model with relations of form (10), then go to step 3.
Step 3. If the mathematical model has coefficients of a discrete random variable type, then step 4 follows; otherwise, the model is solved with the simplex algorithm.
Step 4. All possible numerical matrices M_i, $i = 1, \ldots, N$, of form (13) are built.
Step 5. The probability p_i, $i = 1, \ldots, N$, of occurrence of each numerical matrix M_i from step 4 is calculated with relation (14).
Step 6. The mean value $\widetilde{c}_j$ of each random coefficient of the objective function, c_j, $j = 1, \ldots, n$, is inserted in the model at the position of the respective coefficient.
Step 7. All the possible, distinct constraints that can be formed by combining different values of the random variables in the stochastic model's constraint structure are inserted.
Step 8. The obtained deterministic linear programming model is solved with the simplex algorithm. The optimal solution of this model and the corresponding objective function value are assigned to the initial stochastic mathematical model.
Step 9. In this step, the sensitivity analysis of the optimal solution of the deterministic mathematical model is performed.
Step 10. The parametric analysis of the objective function is performed in relation to the coefficients of the objective function or to the free terms of the constraints from the deterministic mathematical model.

Remarks:

1. The calculations volume that is required to obtain the deterministic linear programming model increases with the number of random (discrete) variables in the stochastic model, as well as with the number of their possible values.
2. The calculations required in steps 8–10 are recommended to be performed using a specialized software product, such as WinQSB 2.0, POM-QM.
3. When solving the deterministic model, three situations from classical linear programming are possible: to admit a finite solution (single or multiple); not to admit a solution (situation generated by a conflict between constraints); to admit an infinite optimum (a situation that cannot occur in the case of the numerical data of the model of a real technical-economic problem).

The advantages of applying the methodology proposed above for stochastic models of type (4) consist of the following aspects: use in a wide variety of practical applications that involve stochastic optimization; the relatively simple construction of the deterministic model; ensuring the uniqueness of the final deterministic model, practically, of the convergence of the proposed algorithm; solving the built linear programming model using the simplex algorithm.

The assisted realization of the last three steps of the methodology with the help of a specialized software product has the following main advantages: guaranteeing the correctness of the numerical results obtained when applying the simplex solving algorithm; elimination of the analyst's routine activities; an important saving of time. The possible small approximation errors that may interfere with the accuracy of the obtained results are due exclusively to the numerical calculation errors inherent in the current calculation systems.

5. Numerical Case Study

Let us consider the following manufacturing problem.

A company makes two types of products, P_1 and P_2, by processing them successively on two machines U_1 and U_2. The unit processing times (in minutes) of the products are indicated in Table 1.

Table 1. The unit processing times (minutes).

	P_1	P_2
U_1	1	2
U_2	0.5	1.5

The processing times available for the two machines are random variables independently and normally distributed with means $\mu_1 = 60$, $\mu_2 = 50$ and dispersions $\sigma_1^2 = 4.84$, $\sigma_2^2 = 2.25$.

For the manufacture of the two products, the company uses two raw materials M_1 and M_2, available in limited quantities. The technological coefficients (unit consumption of raw materials), the unit profits obtained by selling products, and the available quantities of raw materials are indicated in Table 2.

Table 2. Available quantities of raw materials.

	P_1	P_2	**Available Raw Material**
M_1	a_{31}	1	60
M_2	1.2	a_{42}	85
Unit profit	12	C_2	-

From the company's available data, it appears that the three coefficients a_{31}, a_{42}, c_2 are of the discrete random variable type with the following distributions:

- due to the quality of the raw materials:

$$a_{31} = \begin{pmatrix} 2 & 3 \\ 0.6 & 0.4 \end{pmatrix}; \; a_{42} = \begin{pmatrix} 3 & 3.4 \\ 0.5 & 0.5 \end{pmatrix} \tag{16}$$

- due to market demand fluctuations:

$$c_2 = \begin{pmatrix} 13 & 14 \\ 0.7 & 0.3 \end{pmatrix} \tag{17}$$

These three random variables are independent two by two from a statistical point of view.

We propose determining a manufacturing plan relative to the two types of products P_1 and P_2, so that the total average profit is maximum, and the working time on the two machines does not exceed the available time with the minimum probabilities of 95% and 90%, respectively.

We will denote with $x_1,\ x_2$, the decision variables representing the quantities to be manufactured for products P_1 and P_2, respectively, and with $b_1,\ b_2$, the quantities available from the two raw materials. The mathematical model of the considered problem has the following form:

$$\begin{array}{ll} [\text{Maximize}] & z = 12x_1 + c_2x_2 \\ \text{Subject to} & \mathrm{P}(x_1 + 2x_2 \leq\ b_1) \geq 0.95 \\ & \mathrm{P}(0.5x_1 + 1.5x_2 \leq b_2) \geq 0.90 \\ & a_{31}x_1 + x_2 \leq 60 \\ & 1.2x_1 + a_{42}x_2 \leq 85 \\ & x_1,\ x_2 \geq 0 \end{array} \tag{18}$$

According to the proposed methodology, the first two constraints in model (18), which have preassigned probability levels, are replaced by the following two normal constraints:

$$\begin{array}{c} x_1 + 2x_2 \leq 60 - 1.65 \cdot 2.2 \\ 0.5x_1 + 1.5x_2 \leq 50 - 1.28 \cdot 1.5 \end{array} \tag{19}$$

The stochastic model becomes

$$\begin{array}{ll} [\text{Maximize}] & z = 12x_1 + c_2x_2 \\ \text{Subject to} & x_1 + 2x_2 \leq\ 56.37 \\ & 0.5x_1 + 1.5x_2 \leq 48.08 \\ & a_{31}x_1 + x_2 \leq 60 \\ & 1.2x_1 + a_{42}x_2 \leq 85 \\ & x_1,\ x_2 \geq 0 \end{array} \tag{20}$$

The three random coefficients in the model (20), $a_{31},\ a_{42},\ c_2$, each having two possible values, lead to eight possible numerical forms of it. Their coefficients are grouped in the following eight numerical matrices (21):

$$M_1 = \begin{bmatrix} 12 & 13 & - \\ 1 & 2 & 56.37 \\ 0.5 & 1.5 & 48.08 \\ 2 & 1 & 60 \\ 1.2 & 3 & 85 \end{bmatrix}; M_2 = \begin{bmatrix} 12 & 13 & - \\ 1 & 2 & 56.37 \\ 0.5 & 1.5 & 48.08 \\ 2 & 1 & 60 \\ 1.2 & 3.4 & 85 \end{bmatrix}; M_3 = \begin{bmatrix} 12 & 14 & - \\ 1 & 2 & 56.37 \\ 0.5 & 1.5 & 48.08 \\ 2 & 1 & 60 \\ 1.2 & 3.4 & 85 \end{bmatrix};$$
$$M_4 = \begin{bmatrix} 12 & 14 & - \\ 1 & 2 & 56.37 \\ 0.5 & 1.5 & 48.08 \\ 2 & 1 & 60 \\ 1.2 & 3 & 85 \end{bmatrix}; M_5 = \begin{bmatrix} 12 & 13 & - \\ 1 & 2 & 56.37 \\ 0.5 & 1.5 & 48.08 \\ 3 & 1 & 60 \\ 1.2 & 3 & 85 \end{bmatrix}; M_6 = \begin{bmatrix} 12 & 13 & - \\ 1 & 2 & 56.37 \\ 0.5 & 1.5 & 48.08 \\ 3 & 1 & 60 \\ 1.2 & 3.4 & 85 \end{bmatrix}; \quad (21)$$
$$M_7 = \begin{bmatrix} 12 & 14 & - \\ 1 & 2 & 56.37 \\ 0.5 & 1.5 & 48.08 \\ 3 & 1 & 60 \\ 1.2 & 3 & 85 \end{bmatrix}; M_8 = \begin{bmatrix} 12 & 14 & - \\ 1 & 2 & 56.37 \\ 0.5 & 1.5 & 48.08 \\ 3 & 1 & 60 \\ 1.2 & 3.4 & 85 \end{bmatrix}$$

The probabilities of occurrence of these matrices are calculated as follows:

$$\begin{gathered} p_1 = 0.7 \cdot 0.6 \cdot 0.5 = 0.21;\ p_2 = 0.7 \cdot 0.6 \cdot 0.5 = 0.21;\ p_3 = 0.3 \cdot 0.6 \cdot 0.5 = 0.09; \\ p_4 = 0.3 \cdot 0.6 \cdot 0.5 = 0.09;\ p_5 = 0.7 \cdot 0.4 \cdot 0.5 = 0.14;\ p_6 = 0.7 \cdot 0.4 \cdot 0.5 = 0.14; \\ p_7 = 0.3 \cdot 0.4 \cdot 0.5 = 0.06;\ p_8 = 0.3 \cdot 0.4 \cdot 0.5 = 0.06 \end{gathered} \quad (22)$$

The mean value of the random coefficient c_2 is determined as follows:

$$\widetilde{c_2} = 13 \cdot 0.21 + 13 \cdot 0.21 + 14 \cdot 0.09 + 14 \cdot 0.09 + 13 \cdot 0.14 + 13 \cdot 0.14 + 14 \cdot 0.06 + 14 \cdot 0.06 = 13.3 \quad (23)$$

The obtained deterministic linear programming model is the following:

$$\begin{array}{ll} [\text{Maximize}]\ \widetilde{z} = 12x_1 + 13.3x_2 \\ \text{Subject to}\ \ x_1 + 2x_2 \leq\ 56.37 \\ \qquad\qquad 0.5x_1 + 1.9x_2 \leq 48.08 \\ \qquad\qquad 2x_1 + x_2 \leq 60 \\ \qquad\qquad 3x_1 + x_2 \leq 60 \\ \qquad\qquad 1.2x_1 + 3x_2 \leq 85 \\ \qquad\qquad 1.2x_1 + 3.4x_2 \leq 85 \\ \qquad\qquad x_1,\ x_2 \geq 0 \end{array} \quad (24)$$

We will solve model (24) with the module *Linear and Integer Programming* of the specialized software product WinQSB.

In the initial working window of the module, we entered the coefficients of the model in the *Spreadsheet Matrix Form* format (Figure 1). With the *Switch to Normal Model Form* command, the inserted model can be converted to the *Normal Model Form* format, presented in Figure 2.

The *Solve the Problem* command from the *Solve and Analyze* menu analytically solves the model with the simplex algorithm. In the results window, the combined report of the model solution is obtained (shown in Figure 3). The upper half of the report refers to the optimal solution and the corresponding optimal value of the objective function. The lower half refers to the constraints of the model.

Since model (24) contains only two variables, the graphic method can also be used to solve it. The *Graphic Method* command, from the *Solve and Analyze* menu, returns the graphic solution of the model in the *Graphic Solution* window (Figure 4). The solution of the problem is represented in the figure by the green-colored point, being a peak of the hatched domain.

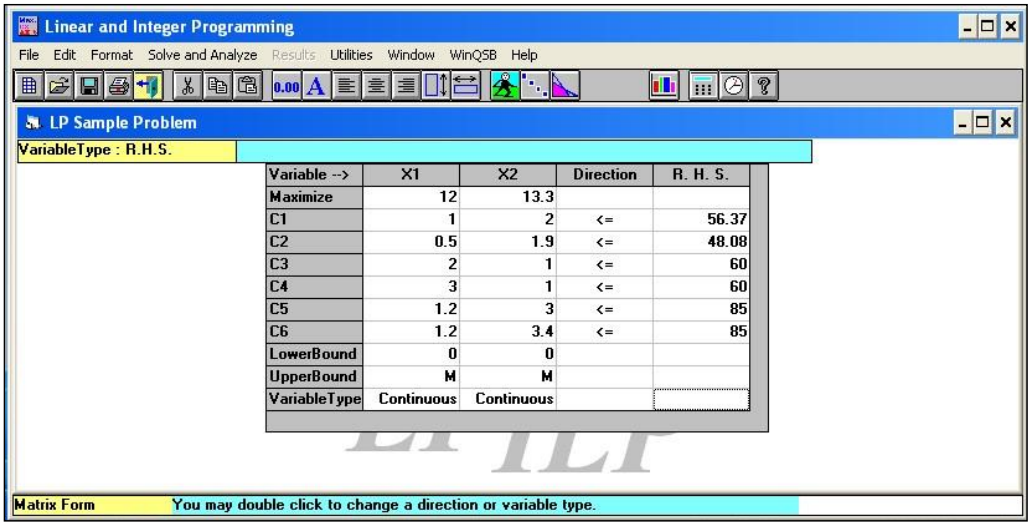


Figure 1. Mathematical model in *Spreadsheet Matrix Form*.

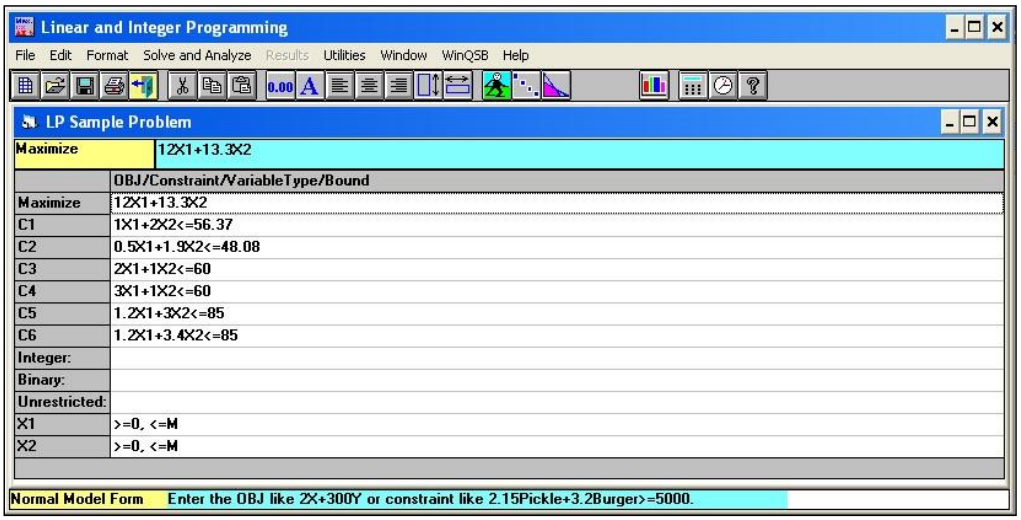


Figure 2. Mathematical model in *Normal Model Form*.

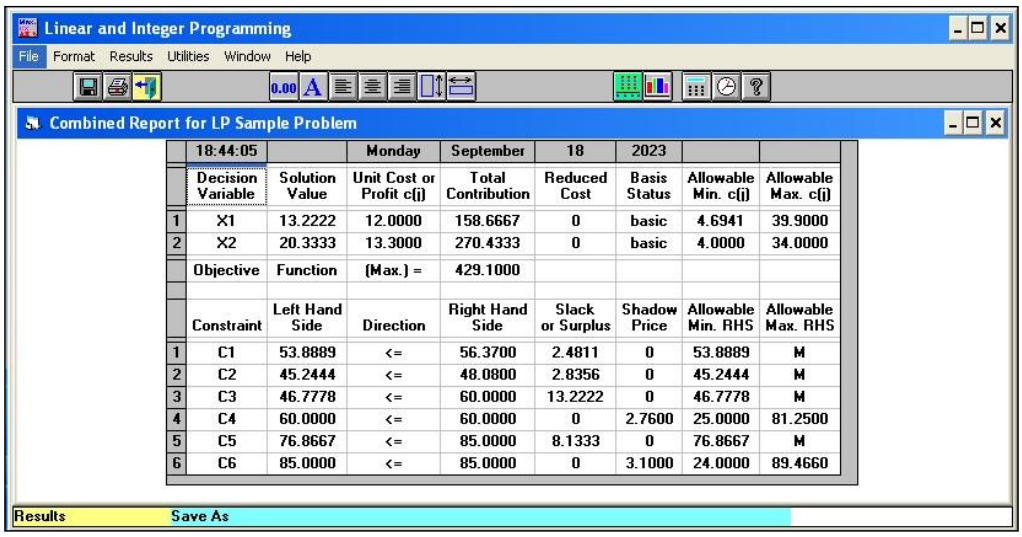


Figure 3. Combined report of the model solution.

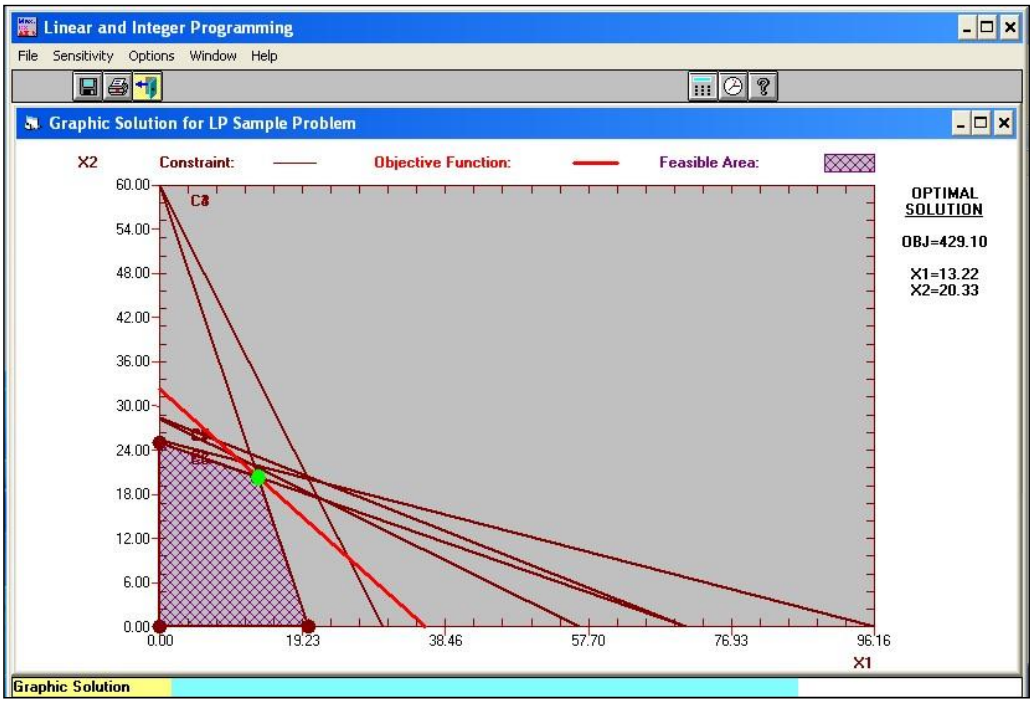


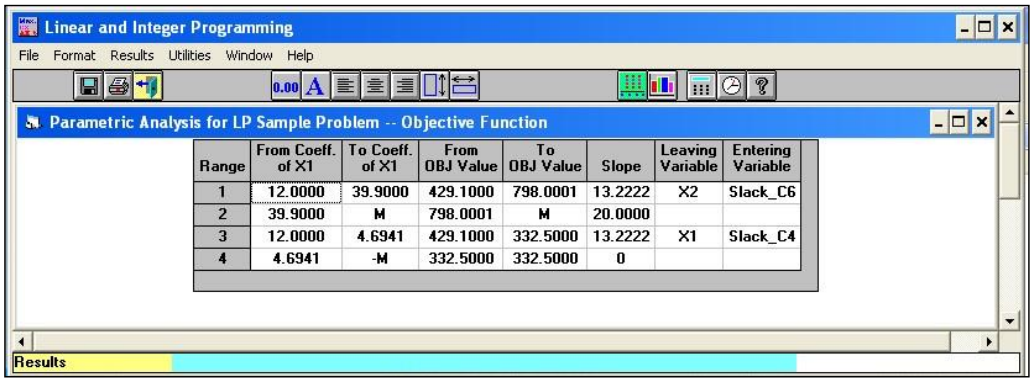
Figure 4. Graphic solution of the model.

The optimal solution of the deterministic model is $x_1 = 13.22$, $x_2 = 20.33$ (the coordinates of the point in the above Figure 4). The corresponding maximum value of the objective function $\widetilde{z}_{\max} = 429.10$. The optimal solution ($x_1$ and x_2) and the value $\widetilde{z}_{\max}$ of the deterministic linear programming model (24) are assigned as the solution and the value of the objective function z, respectively, of the initial stochastic model (18).

In the economic context of the original problem stated, the quantities of products that will have to be made are 13.22 units of the product P_1 and 20.33 units of the product P_2, with an average total profit equal to 429.10 monetary units.

The sensitivity analysis of the optimal solution of the deterministic model (24) uses the columns *Allowable Min c(j)* and *Allowable Max c(j)* from the combined report (see Figure 3). They contain the limits of the admissibility intervals of the coefficients of the objective function $\widetilde{z}$, for which the optimal values of the variables, $x_1 = 13.22$ and $x_2 = 20.33$, do not change. Thus: $c_1 \in [4.69,\ 39.90]$ and $c_2 \in [4,\ 34]$.

The parametric analysis uses the *Perform Parametric Analysis* command, chosen from the *Solve and Analyze* menu. For the parametric analysis relative to the values of a coefficient in the objective function, for example, c_1 of the variable x_1, the *Parametric Analysis* dialog box is completed accordingly and then, by pressing the *OK* command button, displays the table with the parametric analysis (Figure 5) in the results window. The columns *From Coeff. of X1* and *To Coeff. of X1* contain the limits of the variation intervals of the coefficient c_1. The columns *From OBJ Value* and *To OBJ Value* contain the extreme values of the corresponding variation intervals of the objective function $\widetilde{z}$. For example, the value $\widetilde{z} = 798.0001$ corresponds to the value of the parameter $c_1 = 39.9$. The letter M is the conventional notation in the program for $\propto$. In the *Slope* column, for each interval, the slope is indicated; it represents the ratio between the value of the objective function variation and the coefficient variation c_1, and it is equal to the value of x_1, which does not change in the range specified for its coefficient c_1. For $c_1 \in (-\infty,\ 4.6941)$, it obtains $x_1 = 0$, and $\widetilde{z}$ has the constant value 332.5. For $c_1 \in (\ 4.6941, \infty)$, the dependence between the objective $\widetilde{z}$ and the parameter c_1 is directly proportional to the different variation sections of the parameter.



Range	From Coeff. of X1	To Coeff. of X1	From OBJ Value	To OBJ Value	Slope	Leaving Variable	Entering Variable
1	12.0000	39.9000	429.1000	798.0001	13.2222	X2	Slack_C6
2	39.9000	M	798.0001	M	20.0000		
3	12.0000	4.6941	429.1000	332.5000	13.2222	X1	Slack_C4
4	4.6941	-M	332.5000	332.5000	0		

Figure 5. The table of the parametric analysis relative to the coefficient c_1.

With the *Graphic Parametric Analysis* command from the *Results* menu, the graph that shows the variation of the objective function depending on the values of the coefficient c_1 is obtained (Figure 6).

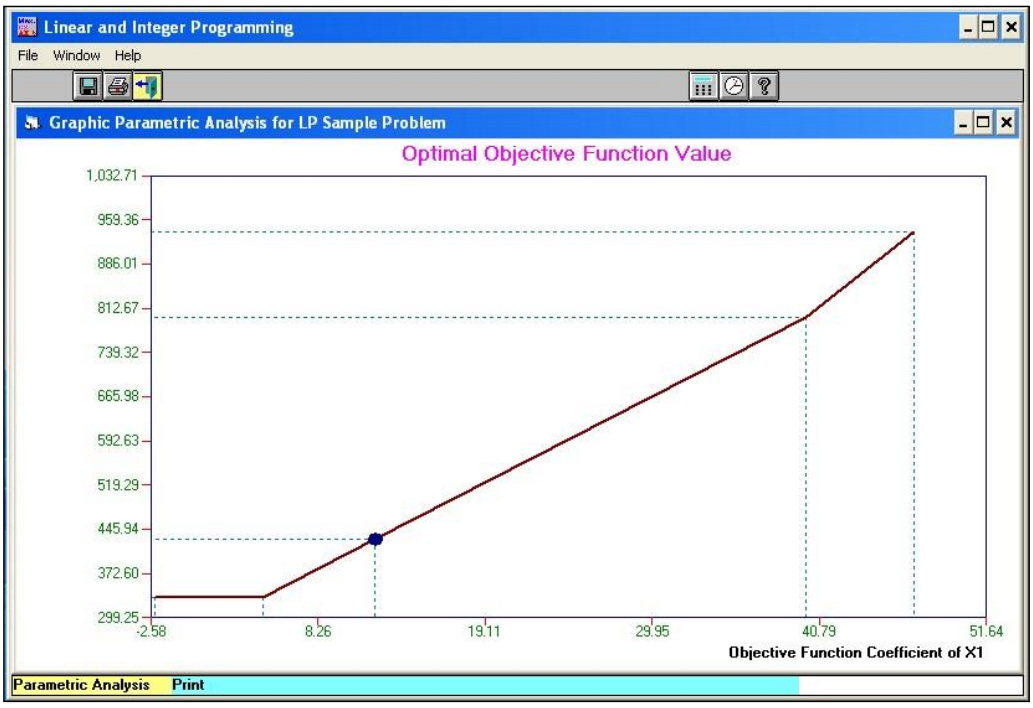
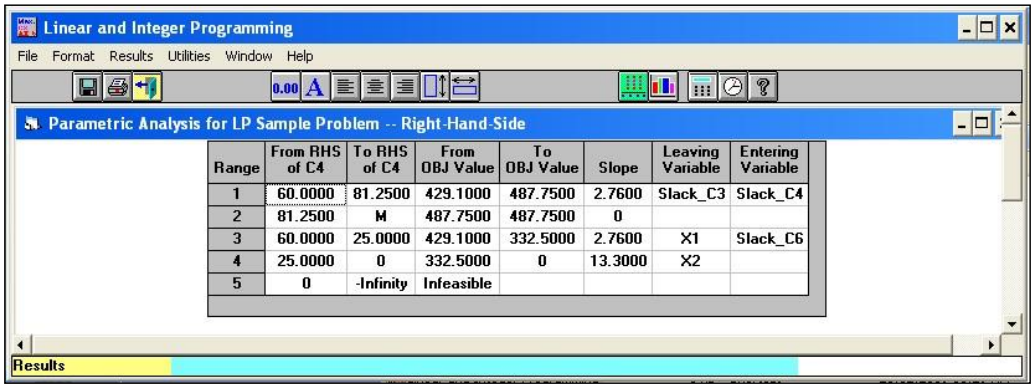


Figure 6. The graph of the parametric analysis relative to the coefficient c_1.

In a similar way, the parametric analysis is carried out in relation to the other coefficient c_2 of the objective function of the mathematical model (24).

For the parametric analysis relative to the values of the right hand side of a constraint, for example, the coefficient b_4, the table with the resulting parametric analysis, displayed in the results window, is presented in Figure 7. The columns *From RHS of C4* and *To RHS of C4* contain the limits of the variation intervals of the coefficient b_4. The columns *From OBJ Value* and *To OBJ Value* contain the extreme values of the corresponding variation intervals of the objective function $\widetilde{z}$. For example, the value $\widetilde{z} = 332.5$ corresponds to the value of the parameter $b_4 = 25$. The columns *From RHS of C4* and *To RHS of C4* contain the limits of the variation intervals of the coefficient b_4.



Range	From RHS of C4	To RHS of C4	From OBJ Value	To OBJ Value	Slope	Leaving Variable	Entering Variable
1	60.0000	81.2500	429.1000	487.7500	2.7600	Slack_C3	Slack_C4
2	81.2500	M	487.7500	487.7500	0		
3	60.0000	25.0000	429.1000	332.5000	2.7600	X1	Slack_C6
4	25.0000	0	332.5000	0	13.3000	X2	
5	0	-Infinity	Infeasible				

Figure 7. The table of the parametric analysis relative to the coefficient b_3.

The columns *From OBJ Value* and *To OBJ Value* contain the extreme values of the corresponding variation intervals of the objective function $\widetilde{z}$. For example, the value $\widetilde{z} = 332.5$ corresponds to the value of the parameter $b_4 = 25$. For $b_4(-\infty,\ 0)$, $\widetilde{z}$ is infeasible. For $b_4(\ 0,\ 81\ .25)$, the dependence between the objective $\widetilde{z}$ and the parameter b_4 is directly proportional to the different variation sections of the parameter. For $b_4(\ 81.25, \infty)$, $\widetilde{z}$ has the constant value 487.75, due to the fixed and limited values of the coefficients b_1, b_2, b_3, b_5, b_6 of the other five constraints.

The graphic representation corresponding to the parametric analysis relative to the coefficient b_4 is presented in Figure 8.

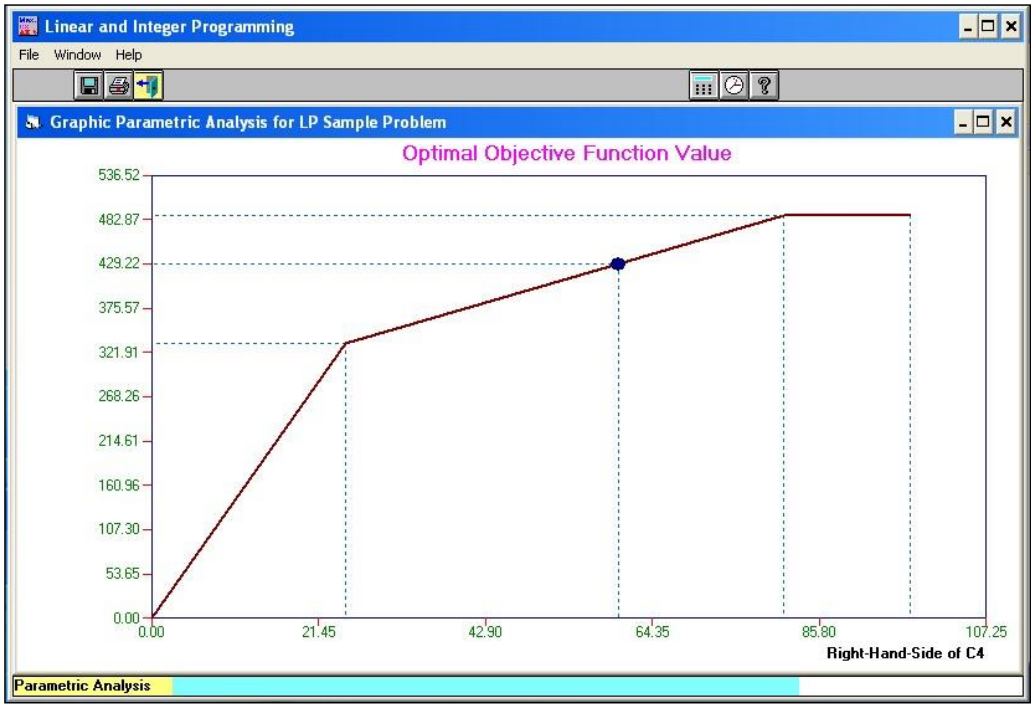


Figure 8. The graph of the parametric analysis relative to the coefficient b_3.

In a similar way, the parametric analysis is carried out in relation to each right hand side coefficient b_1, b_2, b_3, b_5 or b_6 of the constraints in the deterministic mathematical model (24).

Mentioning the commands from the *WinQSB* program used to solve the problem aimed to familiarize the reader with the main operating commands of this efficient software product, specialized in solving optimization problems.

The initial stochastic model (18) proposed in this section is simple in terms of dimensional complexity. It contains only two variables, so that when solving the obtained deterministic model (24), the graphic method, available in the *WinQSB* software, Version 2.0, can be used, along with the analytical method. At the same time, understanding the practical application of the methodology developed in the paper regarding the construction of the deterministic model is facilitated by the reduced dimensions of the studied model.

The subsequent approach to solving some larger models does not raise special, additional problems, compared to those encountered in the considered numerical example.

If the number of decision variables in the deterministic mathematical model is at least equal to 3, its solution is possible only with the analytical method: the simplex algorithm.

For a concrete technical-economic optimization problem to be solved, first it must develop the associated mathematical model. Obviously, the accuracy of the obtained numerical results is dependent on the quality of this initial mathematical model. Finally, all obtained numerical results must be interpreted, in the context of the practical development of the concrete problem.

The deterministic model, obtained after applying the methodology, presents the following advantages: uniqueness; solving with the simplex algorithm; performing the sensitivity analysis of the obtained solution and the parametric analyses.

The proposed methodology is applicable to the category described by stochastic optimization models. The optimal solution (if it exists) of the deterministic model is considered "just" a solution for the initial stochastic model.

6. Conclusions

In this paper, we present a study for solving and analyzing stochastic programming models which contain coefficients of the random variable type and preassigned probability levels on the constraints. This study is motivated by the fact that in socio-economic activities, as well as in a major number of engineering applications, determining the best/optimal solution for a concrete numerical optimization problem is a necessity to achieve the best performance.

The ever-increasing dynamics of the competitive market economy causes changes or updates in the input data of the problems. This implies the performance of the two types of post-optimization analyses, namely sensitivity and/or parametric. This is why the authors have included these two types of analysis in the methodology, as a requirement for competence and managerial responsibility.

From a scientific point of view, with the help of the methodology proposed in this paper, starting from a complex stochastic model of type (4), a deterministic linear programming model can be developed, which can be solved with the specific method. There is no dimensional limit of the initial model that influences the obtaining of the final model. Increasing the number of constraints in this last model does not increase the difficulty of obtaining the solution if computer-aided solving is used.

Therefore, finding the solution for the studied stochastic optimization models is relatively simple using the proposed methodology. If the deterministic model does not admit a solution, the reason is the existence of some contradictions between the constraints of the stochastic model, taken over by the construction of the associated deterministic model.

From an applicative point of view, the methodology provides the ability to effectively solve the problem, as well as to perform, as the case may be, an analysis of the solution sensitivity and/or the parametric analysis of the objective function (in relation to the coefficients of the objective function or to the free terms of the constraints).

Unlike the studies of other authors, the methodology proposed in this paper presents the following characteristics: it does not require the use of a complicated mathematical apparatus; it has wide applicability for any stochastic optimization process in real life, having the mathematical model included in the category studied in this paper; there are no

limit conditions for the development of the deterministic model; it guarantees obtaining the deterministic model associated with the stochastic one; it ensures the uniqueness of the constructed deterministic model; it also offers the possibility of solving and computer-assisted analysis of the final model.

The assisted realization of the last three steps of the methodology with the help of a specialized software product (such as *WinQSB*) has the main advantages of guaranteeing the correctness of the numerical results obtained when applying the simplex algorithm; elimination of the analyst's routine activities; an important saving of time.

The use of computers in the calculation is also important when it is necessary to re-optimize the initial stochastic mathematical model due to changes in the values of some of its coefficients. These changes are caused by the dynamics of the real-time evolution of the studied process.

This paper can be a starting point for future developments. It could be interesting to extend the analysis, both from a theoretical and computational point of view, to other categories of stochastic programming, inspired by real-world applications.

Author Contributions: Conceptualization, O.B.; methodology, O.B.; software use, O.B.; validation, O.B. and I.G.G.; resources, O.B. and I.G.G.; data curation, O.B.; writing—review and editing, O.B. and I.G.G.; supervision, I.G.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: There are no additional data available.

Acknowledgments: The authors wish to thank the reviewers for numerous suggestions that have greatly improved the quality of this article. Also, we wish to thank National University of Science and Technology Politehnica of Bucharest for the support given to the publication of this paper and the creators of *WinQSB* software for the version 2.0 used in this research.

Conflicts of Interest: The authors declare no conflict of interest.

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