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Existence of Fixed Points of Suzuki-Type Contractions of Quasi-Metric Spaces

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Abstract: In order to generalize classical Banach contraction principle in the setup of quasi-metric spaces, we introduce Suzuki-type contractions of quasi-metric spaces and prove some fixed point results. Further, we suggest a correction in the definition of another class of quasi-metrics known as Δ-symmetric quasi-metrics satisfying a weighted symmetry property. We discuss equivalence of various types of completeness of Δ-symmetric quasi-metric spaces. At the end, we consider the existence of fixed points of generalized Suzuki-type contractions of Δ-symmetric quasi-metric spaces. Some examples have been furnished to make sure that generalizations we obtain are the proper ones.

Keywords: quasi-metric spaces; suzuki-type contractions; fixed points; Δ-symmetric quasi-metric spaces

MSC: 47H10; 54H25; 54E50

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1. Introduction

Numerous disciplines of pure and applied mathematics, including nonlinear functional analysis, and the numerical solutions of differential equations make substantial use of fixed point theory ([1]). Banach contraction principle (BCP), which is a key finding in metric fixed point theory provides a useful method for solving functional equations. Due to its usefulness, this theorem has been expanded and generalized over time to encompass various spaces and mappings satisfying distinct contraction conditions.

Suzuki [2] and Kikkawa and Suzuki [3] generalized BCP by introducing Suzuki-type contractions of metric spaces. Note that Suzuki-type contractions characterize the completeness of underlying metric spaces. Moreover, Suzuki-type contractions are in the form of implication which allow many discontinuous mappings of metric spaces to be Suzuki-type contractions, while Banach contractions are necessarily continuous mappings of metric spaces. Suzuki-type contractions have a wide scope of applications, for instance, in delay differential equations (see [4]). Eroglu et al. [5] discussed fixed point results for rational type contractions in quasi-metric spaces. Romaguera [6] discussed fixed point results for generalized Ćirić's contractions of quasi-metric spaces. Ghasab et al. [7] presented some new fixed point results in $\mathscr{F}$-quasi-metric spaces and an application.

Kannan [8] presented a fixed point theorem for contractions known as Kannan-type contractions. Unlike Banach contractions, Kannan-type contractions are not necessarily continuous. Moreover, Kannan-type contractions characterize the completeness of underlying metric spaces (compare [9]). Note that Banach contractions do not characterize metric completeness (see [10]). Further, Banach and Kannan-type contractions are independent of each other. For more on Kannan-type contractions, we refer the reader to [11,12].

Another important class of contractions is due to Suzuki [2] known as Suzuki-type contractions. These contractions characterize metric completeness and are not necessarily continuous as well. Suzuki-type contractions properly generalize Banach contractions.

Quasi-metric spaces, by relaxing the symmetry property present in metric spaces, give rise to asymmetric functional analysis. The topology of quasi-metric space, in general, is T_0. Further, these spaces have various applications, for instance, in fractal theory [13], software engineering [14], denotational semantics and complexity analysis [15], and many more in the literature. Dağ et al. [16] have generalized the classical Banach contraction principle in the setup of quasi-metric spaces with some illustrative examples. In [17], they have obtained Kannan-type fixed point theorems in quasi-metric spaces along with quasi-metric completeness characterization.

Recently, Romaguera's investigations revealed that unlike the Banach contraction principle and Kannan-type fixed point theorem, a basic Suzuki-type fixed point theorem cannot be transported as it is in the setup of quasi-metric spaces (see for details [18] (Example 3)). Romaguera proposed a new type of Suzuki-type contraction by tweaking the original Suzuki type contraction a little bit and obtained a variant of Suzuki type fixed point theorems for mappings of quasi-metric spaces. Romaguera introduced a $2-$basic contraction of the Suzuki type and proved fixed point results in the setup of Smyth's complete quasi-metric space. Note that $2-$basic contraction of the Suzuki type does not reduce to the original Suzuki type contraction whenever quasi-metric is replaced by metric.

In this manuscript, in the context of quasi-metric spaces, we further investigate the Suzuki-type fixed point theorems. We proposed a new type of Suzuki-type contraction, which is reduced to the original Suzuki-type contraction when quasi-metric is replaced with metric.

As we know from [18] (Example 3), Suzuki-type contractions cannot be transported in quasi-metric spaces, so the transportation of generalization of such contractions is out of the question. There is a special type of quasi-metric space known as Δ-symmetric quasi-metric spaces (see [19]). These spaces are Hausdorff quasi-metric spaces. In this paper, we prove the existence and uniqueness of fixed points of generalized Suzuki-type contractions of Δ-symmetric quasi-metric spaces. We present a remark about the possible range for Δ that is being used in Δ-symmetric quasi-metric spaces and suggest a correction in the definition of these spaces given in [19]. Moreover, we discuss, bicompleteness, Smyth, and $d-$sequential completeness of these spaces as well. We present a situation (an example) where the mapping is generalized Suzuki type contraction, but it is neither Suzuki type contraction nor $2-$basic contraction of Suzuki type.

For the purposes of this paper, the symbols $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}$ will represent the sets of real numbers, natural numbers, and integer numbers, respectively. Also, $(\Im, d)$ represents the quasi-metric space, and $(\Im, d^s)$ represents the metric space induced by quasi-metric d otherwise mentioned.

Definition 1. *Let $d : \Im \times \Im \to \mathbb{R}$ be a function on a non-empty set $\Im$. Consider for all $\ell, \hbar, \wp \in \Im$, the following conditions:*

(d_1) $d(\ell, \hbar) \geq 0$,
(d_2) $d(\ell, \hbar) = 0$ *iff* $\ell = \hbar$,
(d_3) $d(\ell, \hbar) = d(\hbar, \ell)$,
(d_4) $d(\ell, \hbar) \leq d(\ell, \wp) + d(\wp, \hbar)$,
(d_5) $d(\ell, \hbar) = d(\hbar, \ell) = 0$ *iff* $\ell = \hbar$.

The function d is a metric on $\Im$ if it satisfies the conditions (d_1) through (d_4), and a quasi-metric on $\Im$ if d satisfies (d_1), (d_4), and (d_5).

For a positive real number $\Delta > 0$, then quasi-metric space $(\Im, d)$ is said to be Δ-symmetric if

$$d(\ell, \hbar) \leq \Delta d(\hbar, \ell),$$

for all $\ell, \hbar \in \Im$. It is obvious that the Δ-symmetric quasi-metric space $(\Im, d)$ constitutes a metric space if $\Delta = 1$. See [19] for more details and examples of Δ-symmetric quasi-metric spaces.

Remark 1. *Let $(\Im, d)$ be a Δ-symmetric quasi-metric space with at least two distinct points. Recently, in [19] Δ-symmetric quasi-metric is defined for $\Delta > 0$ but*

$$d(\ell, \hbar) \leq \Delta d(\hbar, \ell) \leq \Delta^2 d(\ell, \hbar)$$

for all $\ell, \hbar \in \Im$ such that $\ell \neq \hbar$ implies $1 - \Delta^2 \leq 0$ which means that $1 \leq \Delta^2$, then we have either $\Delta \leq -1$ or $\Delta \geq 1$. Hence $\Delta \geq 1$ (as $\Delta > 0$) for Δ-symmetric quasi-metric spaces with at least two distinct points.

If $(\Im, d)$ is a quasi-metric space, then the mapping d_{-1} is defined as

$$d_{-1}(\ell, \hbar) = d(\hbar, \ell),$$

is another quasi-metric on $\Im$. It is named a conjugate of d, and the function

$$d^s(\ell, \hbar) = \max\{d(\ell, \hbar), d_{-1}(\ell, \hbar)\}$$

is a metric on $\Im$. Every quasi-metric d on $\Im$ produces a topology τ_d, which is T_0 (in general) with the class of open balls

$$\{B_d(\ell, \varepsilon) : \ell \in \Im, \varepsilon > 0\}$$

as its basis, where

$$B_d(\ell, \varepsilon) = \{\hbar \in \Im : d(\ell, \hbar) < \varepsilon\}$$

for all $\ell \in \Im$ and $\varepsilon > 0$. If the topology τ_d on $\Im$ is T_1 (or T_2), then quasi-metric space $(\Im, d)$ is said to be T_1 (or T_2). Note that $(\Im, d)$ is T_1 iff

$$d(\ell, \hbar) = 0 \Rightarrow \ell = \hbar$$

for all $\ell, \hbar \in \Im$. A sequence $(\ell_k)_{k \in \mathbb{N}}$ in $(\Im, d)$ is convergent to $z_0 \in \Im$ in τ_d ($\tau_{d_{-1}}$) if for every $\varepsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that

$$d(z_0, \ell_k) < \varepsilon \, (d(\ell_k, z_0) < \varepsilon)$$

for all $k \geq n_0$. A sequence $(\ell_k)_{k \in \mathbb{N}}$ in $(\Im, d)$ is said to be a left (right) $\tilde{K}$-Cauchy sequence if for any $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$d(\ell_k, \ell_j) < \varepsilon \, (d(\ell_j, \ell_k) < \varepsilon),$$

for any $j \geq k \geq n_0$. A quasi-metric space $(\Im, d)$ is

- left (right) $\tilde{K}$-sequentially complete if each left (right) $\tilde{K}$-Cauchy sequence converges in the topology τ_d,
- d-sequentially complete if any Cauchy sequence in $(\Im, d^s)$ converges in topology τ_d,
- bicomplete if the metric space $(\Im, d^s)$ is complete, and
- Smyth is complete if every left $\tilde{K}$-Cauchy sequence in $(\Im, d)$ converges in the topology τ_{d^s}.

Note that in Δ-symmetric quasi-metric spaces $(\Im, d)$, every left (right) $\tilde{K}$-Cauchy sequence is a right (left) K-Cauchy sequence. If a sequence $(\ell_k)_{k \in \mathbb{N}}$ in Δ-symmetric quasi-metric space $(\Im, d)$ is left (right) $\tilde{K}$-Cauchy, then we refer to it as Δ-Cauchy in $(\Im, d)$. Smyth completeness implies bicompleteness, but the converse does not hold in general. Both bicompleteness and left (right) $\tilde{K}$-sequentially completeness imply d-sequentially completeness but converses do not hold in general. For more details on above mentioned notions, we refer the readers to [16,18], and the references therein.

Throughout this paper, otherwise stated, λ denotes a decreasing function $\lambda : [0,1) \to \left(\frac{1}{2},1\right]$ defined as

$$\lambda(\alpha) = \frac{1}{1+\alpha}.$$

2. Fixed Points of Suzuki-Type d (d_{-1})-Contractions of Quasi-Metric Spaces

In this section, we start with the following definition.

Definition 2. *For a quasi-metric space $(\Im, d)$ and a mapping $\Psi : \Im \to \Im$,*

1. *if there is a $\alpha \in [0,1)$ such that*

$$\begin{aligned} d(\Psi\ell, \Psi\hbar) &\leq \alpha d(\ell,\hbar) \text{ and} \\ d(\Psi\ell, \Psi\hbar) &\leq \alpha d(\hbar,\ell) \end{aligned}$$

for all $\ell, \hbar \in \Im$, then Ψ is called d -contraction and d_{-1}-contraction on $(\Im, d)$, respectively (see [16]),

2. *if there is a $\alpha \in [0,1)$ such that*

$$\lambda(\alpha) d(\ell, \Psi\ell) \leq d^s(\ell,\hbar) \Rightarrow d^s(\Psi\ell, \Psi\hbar) \leq \alpha d(\ell,\hbar) \tag{1}$$

for all $\ell, \hbar \in \Im$, then Ψ is called Suzuki-type d-contraction on $(\Im, d)$,

3. *if there is a $\alpha \in [0,1)$ such that*

$$\lambda(\alpha) d(\ell, \Psi\ell) \leq d^s(\ell,\hbar) \Rightarrow d^s(\Psi\ell, \Psi\hbar) \leq \alpha d(\hbar,\ell) \tag{2}$$

for all $\ell, \hbar \in \Im$, then Ψ is called Suzuki-type d_{-1}-contraction on $(\Im, d)$,

4. *if there is a $\alpha \in [0,1)$ such that*

$$\lambda(\alpha) d(\ell, \Psi\ell) \leq d(\ell,\hbar) \Rightarrow d(\Psi\ell, \Psi\hbar) \leq \alpha d(\ell,\hbar) \tag{3}$$

for all $\ell, \hbar \in \Im$, then Ψ is called Suzuki type contraction on $(\Im, d)$,

5. *if there is a $\alpha \in (0,1)$ such that*

$$\min\{d(\ell, \Psi\ell), d(\hbar, \Psi\hbar)\} \leq 2d(\ell,\hbar) \Rightarrow d(\Psi\ell, \Psi\hbar) \leq \alpha d(\ell,\hbar) \tag{4}$$

for all $\ell, \hbar \in \Im$, then Ψ is called $2-$basic contraction of Suzuki type (see [18]).

The following theorem is a direct consequence of [2] (Theorem 2).

Theorem 1. *Let $(\Im, d)$ be a complete metric space and $\Psi : \Im \to \Im$. Assume that there exists $\alpha \in [0,1)$ such that*

$$d(\ell, \Psi\ell) \leq 2d(\ell,\hbar) \Rightarrow d(\Psi\ell, \Psi\hbar) \leq \alpha d(\ell,\hbar)$$

for all $\ell, \hbar \in \Im$. Then there exists a unique fixed point z of Ψ. Moreover

$$\lim_{k\to\infty} \Psi^k \ell = z$$

for all $\ell \in \Im$.

Remark 2. *Theorem 1 does not hold for quasi-metric spaces (see [18] (Example 3)).*

The following Theorem is a direct consequence of [3] (Theorem 2).

Theorem 2. *Let $(\Im, d)$ be a complete metric space and $\Psi : \Im \to \Im$ a Suzuki-type contraction. Then Ψ has a unique fixed point.*

The following example (compare with [18] (Example 3)) shows that the above Theorem cannot be transported as it is in quasi-metric space.

Example 1. *Let $\Im = \mathbb{N} \cup \{\infty\}$ and d be a quasi-metric on $\Im$ given as*

$$\begin{aligned} &d(\ell, \ell) = 0 \text{ for all } \ell \in \Im, \\ &d(p, \infty) = 0 \text{ for all } p \in \mathbb{N}, \\ &d(\infty, p) = \tfrac{1}{p} \text{ for all } p \in \mathbb{N}, \\ &d(p, q) = \tfrac{1}{q} \text{ for all } p, q \in \mathbb{N} \text{ with } p \neq q. \end{aligned}$$

We conclude that $(\Im, d)$ is Smyth complete since any non-eventually constant left-$\tilde{K}$ Cauchy sequence in $(\Im, d)$ converges to ∞ in $(\Im, d^s)$. Let $\Psi : \Im \to \Im$ be a mapping defined as

$$\Psi\ell = \begin{cases} 1, \text{ if } \ell = \infty, \\ 2\ell, \text{ if } \ell \in \mathbb{N}. \end{cases}$$

Clearly Ψ has no fixed point. But it fulfills the condition (3) for $\alpha = \dfrac{1}{2}$.

- *If $\ell = p$ and $\hbar = q$ with $p \neq q$, we have*

$$d(\Psi\ell, \Psi\hbar) = d(2p, 2q) = \frac{1}{2q} = \frac{1}{2}d(p, q) = \frac{1}{2}d(\ell, \hbar).$$

- *If $\ell = \infty$ and $\hbar = p$, $p \in \mathbb{N}$, we have*

$$d(\Psi\ell, \Psi\hbar) = d(1, 2p) = \frac{1}{2p} = \frac{1}{2}d(\infty, p) = \frac{1}{2}d(\ell, \hbar).$$

- *If $\ell = p$, $p \in \mathbb{N}$, and $\hbar = \infty$, $p \in \mathbb{N}$, we have*

$$\lambda(\alpha)d(\ell, \Psi\ell) \leq d(\ell, \Psi\ell) = d(p, 2p) = \frac{1}{2p} > 0 = d(p, \infty) = d(\ell, \hbar).$$

Thus we have shown that Ψ is a Suzuki-type contraction of quasi-metric space $(\Im, d)$ without a fixed point.

Romaguera [18] presented the following theorem.

Theorem 3. *[18] Every 2−basic contraction $\Psi : \Im \to \Im$ of a Smyth's complete T_1 quasi-metric space $(\Im, d)$ has a unique fixed point.*

We present the following proposition, which we will use to prove some fixed point theorems for Suzuki-type d (d_{-1})-contractions.

Proposition 1. *Let $(\Im, d)$ be a quasi-metric space. If Ψ is a Suzuki-type $d(d_{-1})$-contraction, then the following holds:*

(a) Ψ is a Suzuki-type contraction on $(\Im, d^s)$.

(b) For any $\ell_0 \in \Im$, the sequence $\left(\Psi^k \ell_0\right)_{k \in \mathbb{N}}$ is a Cauchy sequence in $(\Im, d^s)$.

Proof. (a) Assuming that Ψ is a Suzuki-type d-contraction on $(\Im, d)$, there exists $\alpha \in (0, 1)$ such that

$$\lambda(\alpha)d(\ell, \Psi\ell) \leq d^s(\ell, \hbar) \Rightarrow d^s(\Psi\ell, \Psi\hbar) \leq \alpha d(\ell, \hbar), \tag{5}$$

for all $\ell, \hbar \in \Im$. Since

$$\lambda(\alpha)d^s(\ell, \Psi\ell) \leq d^s(\ell, \hbar) \Rightarrow \lambda(\alpha)d(\ell, \Psi\ell) \leq d^s(\ell, \hbar),$$

therefore by (5), we get

$$d^s(\Psi\ell, \Psi\hbar) \leq \alpha d(\ell, \hbar) \leq \alpha d^s(\ell, \hbar). \tag{6}$$

Hence,

$$\lambda(\alpha)d^s(\ell, \Psi\ell) \leq d^s(\ell, \hbar) \Rightarrow d^s(\Psi\ell, \Psi\hbar) \leq \alpha d^s(\ell, \hbar)$$

for all $\ell, \hbar \in \Im$. In a similar way, we prove that if Ψ is a Suzuki-type d_{-1}-contraction on $(\Im, d)$, then Ψ is a Suzuki-type contraction on $(\Im, d^s)$.

(b) Since Ψ is a Suzuki-type contraction on $(\Im, d^s)$ from (a), so from [3], $\left(\Psi^k \ell_0\right)$ for $k \in \mathbb{N}$ is a Cauchy sequence in $(\Im, d^s)$. □

Theorem 4. *Suppose $(\Im, d)$ is bicomplete then any Suzuki-type $d(d_{-1})$-contraction on $(\Im, d)$ has a unique fixed point.*

Proof. From Proposition 1 (a), if Ψ is a Suzuki-type d (d_{-1})-contraction on $(\Im, d)$, then Ψ is a Suzuki-type contraction on $(\Im, d^s)$. As $(\Im, d)$ is bicomplete, $(\Im, d^s)$ is complete. Hence, from [2,3], we obtain that Ψ has a unique fixed point. □

3. Fixed Points and Δ-Symmetric Quasi-Metric Spaces

We start this section with a lemma showing that bicompleteness implies Smyth completeness in Δ-symmetric quasi-metric spaces.

Lemma 1. *Every Δ-symmetric quasi-metric space $(\Im, d)$ is bicomplete if it is Smyth complete.*

Proof. Let $(\Im, d)$ be bicomplete Δ-symmetric quasi-metric space. Let $(\ell_k)_{k\in\mathbb{N}}$ be a left $\tilde{K}$-Cauchy sequence in $(\Im, d)$. As d is Δ-symmetric, so $(\ell_k)_{k\in\mathbb{N}}$ is the right $\tilde{K}$-Cauchy sequence in $(\Im, d)$. Hence $(\ell_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $(\Im, d^s)$ which is complete. Consequently $(\ell_k)_{k\in\mathbb{N}}$ converges in $(\Im, d^s)$. This implies that $(\Im, d)$ is Smyth's complete quasi-metric. The Converse part is trivial and left for the reader. □

The following proposition shows that Δ-symmetric quasi-metric spaces are T_1. In fact, they are more than that.

Proposition 2. *Every Δ-symmetric quasi-metric space $(\Im, d)$ is T_1 quasi-metric space.*

Proof. Suppose that for $\ell, \hbar \in \Im$,

$$d(\ell, \hbar) = 0$$

implies that

$$d(\hbar, \ell) \leq \Delta d(\ell, \hbar) = 0.$$

Hence $\ell = \hbar$, that is, $(\Im, d)$ is T_1 quasi-metric space. □

Proposition 3. *The topology of Δ-symmetric quasi-metric spaces $(\Im, d)$ is Hausdorff.*

Proof. As Δ-symmetric quasi-metric space is T_1 and τ_d convergence implies $\tau_{d_{-1}}$ convergence, so from [13] (Proposition 2.1), the topology of Δ-symmetric quasi-metric space is Hausdorff. □

Proposition 4. *A Δ-symmetric quasi-metric space $(\Im, d)$ is d (d_{-1})-sequentially complete if $(\Im, d)$ is bicomplete.*

Proof. Suppose $(\Im, d)$ is a d-sequentially complete Δ-symmetric quasi-metric space, then the Cauchy sequence $(\ell_k)_{k\in\mathbb{N}}$ in metric space $(\Im, d^s)$ converges in the topology τ_d. By Δ-symmetric condition, $(\ell_k)_{k\in\mathbb{N}}$ is also converged in the topology $\tau_{d_{-1}}$. This implies $(\ell_k)_{k\in\mathbb{N}}$ converges in $(\Im, d^s)$. Hence, $(\Im, d)$ is bicomplete. Converse follows directly from the

definition of bicompleteness. On similar lines, the proof follows for d_{-1}-sequentially complete Δ-symmetric quasi-metric space. □

Theorem 5. *Every Suzuki-type d (d_{-1})-contraction on a d-sequentially complete Δ-symmetric quasi-metric space $(\Im, d)$ has a unique fixed point.*

Proof. Suppose $(\Im, d)$ is d-sequentially complete Δ-symmetric quasi-metric space and Ψ is a Suzuki-type d-contraction. As $(\Im, d)$ is Δ-symmetric quasi-metric space, by Proposition 4, $(\Im, d)$ is bicomplete. So by Theorem 4, $(\Im, d)$ has a unique fixed point. □

Consider the following definition.

Definition 3. *Let $(\Im, d)$ be a quasi-metric space and $\Psi : \Im \to \Im$ a mapping. If there is a $\alpha \in [0, 1)$ such that*

$$\lambda(\alpha) d(\ell, \Psi\ell) \leq d^s(\ell, \hbar) \Rightarrow d(\Psi\ell, \Psi\hbar) \leq \alpha U_\Psi(\ell, \hbar), \tag{7}$$

where,

$$U_\Psi(\ell, \hbar) = \max\left\{ d(\ell, \hbar), d(\ell, \Psi\ell), d(\hbar, \Psi\hbar), d\left(\Psi\ell, \Psi^2\ell\right), \frac{d\left(\Psi^2\ell, \Psi\hbar\right)}{2}, \frac{d(\hbar, \Psi\ell) + d(\ell, \Psi\hbar)}{2} \right\}. \tag{8}$$

for all $\ell, \hbar \in \Im$, then Ψ is called generalized Suzuki-type (GST) contraction on $(\Im, d)$.

Now we present fixed point results for GST contractions of Δ-symmetric quasi-metric spaces.

Theorem 6. *Let $(\Im, d)$ be a d-sequentially complete Δ-symmetric quasi-metric space. Then every GST contraction on $(\Im, d)$ has a unique fixed point.*

Proof. Let Ψ be a GST contraction on $(\Im, d)$ and α_1 be a real number such that $0 \leq \alpha < \alpha_1 < 1$. For a fixed $\varrho_1 \in \Im$, define an iterative sequence as $\varrho_{k+1} = \Psi\varrho_k$ for all $k \in \mathbb{N}$. If there exists a $k \in \mathbb{N}$ such that

$$\varrho_k = \varrho_{k+1} = \Psi\varrho_k,$$

then ϱ_k is a fixed point of Ψ. So assume that $\varrho_k \neq \varrho_{k+1}$ for all $k \in \mathbb{N}$. Since $\lambda(\alpha) \leq 1$, therefore

$$\lambda(\alpha) d(\varrho_{k+1}, \Psi\ell_k) \leq d(\varrho_k, \Psi\varrho_k) = d(\varrho_k, \varrho_{k+1}),$$

holds for all $k \in \mathbb{N}$. By the given hypothesis,

$$\begin{aligned}
&d(\varrho_{k+1}, \varrho_{k+2}) = d(\Psi\varrho_k, \Psi\varrho_{k+1}) \\
&\leq \alpha U_\Psi(\varrho_k, \varrho_{k+1}) \\
&= \alpha_1 \max\left\{ \begin{array}{l} d(\varrho_k, \varrho_{k+1}), d(\varrho_k, \Psi\varrho_k), d(\varrho_{k+1}, \Psi\varrho_{k+1}), d\left(\Psi\varrho_k, \Psi^2\varrho_k\right), \\ \dfrac{d\left(\Psi^2\varrho_k, \Psi\varrho_{k+1}\right)}{2}, \dfrac{d(\varrho_{k+1}, \Psi\varrho_k) + d(\varrho_k, \Psi\varrho_{k+1})}{2} \end{array} \right\} \\
&\leq \alpha_1 \max\left\{ d(\varrho_k, \varrho_{k+1}), d(\varrho_{k+1}, \varrho_{k+2}), \frac{d(\varrho_k, \varrho_{k+2})}{2} \right\} \\
&\leq \alpha_1 \max\{d(\varrho_k, \varrho_{k+1}), d(\varrho_{k+1}, \varrho_{k+2})\}.
\end{aligned}$$

Thus

$$d(\varrho_{k+1}, \varrho_{k+2}) \leq \alpha_1 \max\{d(\varrho_k, \varrho_{k+1}), d(\varrho_{k+1}, \varrho_{k+2})\}.$$

If

$$\max\{d(\varrho_k, \varrho_{k+1}), d(\varrho_{k+1}, \varrho_{k+2})\} = d(\varrho_{k+1}, \varrho_{k+2}),$$

then

$$d(\varrho_{k+1}, \varrho_{k+2}) \leq \alpha_1 d(\varrho_{k+1}, \varrho_{k+2}) < d(\varrho_{k+1}, \varrho_{k+2}),$$

a contradiction. Consequently, we obtain $d(\varrho_{k+1}, \varrho_{k+2}) \leq \alpha_1 d(\varrho_k, \varrho_{k+1})$.
That is, $d(\varrho_{k+1}, \varrho_{k+2}) \leq \alpha_1^k d(\varrho_1, \varrho_2)$ for all $k \in \mathbb{N}$. This implies $\{\varrho_k\}$ is a Δ-Cauchy sequence in $(\Im, d)$. Since $(\Im, d)$ is d-sequential complete, there exists a $z_0 \in \Im$ such that

$$\lim_{k\to\infty} d(\varrho_k, z_0) = 0. \tag{9}$$

Now, we claim that

$$d(z_0, \Psi\ell) \leq \alpha \max\{d(z_0, \ell), d(\ell, z_0), d(\ell, \Psi\ell)\}, \tag{10}$$

for all $\ell \neq z_0$. By (9), there exists an $n_0 \in \mathbb{N}$ such that

$$d(z_0, \varrho_k) < \frac{1}{3\Delta} d(z_0, \ell)$$

for all $k \geq n_0$, which implies

$$\begin{aligned}
\lambda(\alpha) d(\varrho_k, \Psi\varrho_k) &\leq d(\varrho_k, \Psi\varrho_k) = d(\varrho_k, \varrho_{k+1}) \\
&\leq d(\varrho_k, z_0) + d(z_0, \varrho_{k+1}) \\
&\leq \Delta d(z_0, \varrho_k) + d(z_0, \varrho_{k+1}) \\
&\leq \Delta \frac{1}{3\Delta} d(z_0, \ell) + \frac{1}{3\Delta} d(z_0, \ell) \\
&= \frac{2}{3} d(z_0, \ell) \\
&= d(z_0, \ell) - \frac{1}{3} d(z_0, \ell) \\
&\leq d(z_0, \ell) - \Delta d(z_0, \varrho_k) \\
&\leq d(z_0, \ell) - d(z_0, \varrho_k) \leq d(\varrho_k, \ell).
\end{aligned}$$

Thus,

$$\lambda(\alpha) d(\varrho_k, \Psi\varrho_k) \leq d^s(\varrho_k, \ell)$$

holds for all $k \geq n_0$. From (8), we have

$$\begin{aligned}
d(\varrho_{k+1}, \Psi\ell) &= d(\Psi\varrho_k, \Psi\ell) \leq \alpha U_\Psi(\varrho_k, \ell) \\
&\leq \alpha \max\left\{ \begin{array}{l} d(\varrho_k, \ell), d(\varrho_k, \Psi\varrho_k), d(\ell, \Psi\ell), d\left(\Psi\varrho_k, \Psi^2\varrho_k\right), \\ \dfrac{d\left(\Psi^2\varrho_k, \Psi\ell\right)}{2}, \dfrac{d(\ell, \Psi\varrho_k) + d(\varrho_k, \Psi\ell)}{2} \end{array} \right\} \\
&= \alpha \max\left\{ \begin{array}{l} d(\varrho_k, \ell), d(\varrho_k, \varrho_{k+1}), d(\ell, \Psi\ell), d(\varrho_{k+1}, \varrho_{k+2}), \\ \dfrac{d(\varrho_{k+2}, \Psi\ell)}{2}, \dfrac{d(\ell, \varrho_{k+1}) + d(\varrho_k, \Psi\ell)}{2} \end{array} \right\}.
\end{aligned}$$

On taking limit as $k \to \infty$, we have

$$\begin{aligned}
d(z_0, \Psi\ell) &\leq \alpha \max\left\{ d(z_0, \ell), d(\ell, \Psi\ell), \frac{d(z_0, \Psi\ell)}{2}, \frac{d(\ell, z_0) + d(z_0, \Psi\ell)}{2} \right\} \\
&\leq \alpha \max\left\{ d(z_0, \ell), d(\ell, \Psi\ell), \frac{d(z_0, \ell) + d(\ell, \Psi\ell)}{2}, \frac{d(\ell, z_0) + d(z_0, \Psi\ell)}{2} \right\} \\
&= \alpha \max\left\{ d(z_0, \ell), d(\ell, \Psi\ell), \frac{d(\ell, z_0) + d(z_0, \Psi\ell)}{2} \right\}
\end{aligned}$$

holds for all $\ell \neq z_0$. If

$$\max\left\{ d(z_0, \ell), d(\ell, \Psi\ell), \frac{d(z_0, \Psi\ell) + d(\ell, z_0)}{2} \right\} = \frac{d(z_0, \Psi\ell) + d(\ell, z_0)}{2},$$

then

$$d(z_0, \Psi\ell) \leq \alpha\left(\frac{d(z_0, \Psi\ell) + d(\ell, z_0)}{2}\right)$$

implies that

$$d(z_0, \Psi\ell) \leq \left(\frac{\alpha}{2-\alpha}\right)d(\ell, z_0) < \alpha d(\ell, z_0) \leq \alpha \max\{d(z_0, \ell), d(\ell, z_0), d(\ell, \Psi\ell)\}$$

for all $\ell \neq z_0$. Hence, (10) holds. Now, we prove $z_0 = \Psi z_0$. Assume on the contrary that

$$x \neq \Psi x \tag{11}$$

for all $x \in \Im$. Set $\ell = \Psi z_0$. In view of (11), $\ell \neq z_0$. Since

$$\lambda(\alpha)d(z_0, \Psi z_0) \leq d(z_0, \Psi z_0) = d(z_0, \ell),$$

then, by (8), we have

$$\begin{aligned}
d(\ell, \Psi\ell) &= d(\Psi z_0, \Psi\ell) \\
&\leq \alpha \max\left\{ \begin{array}{l} d(z_0, \ell), d(z_0, \Psi z_0), d(\ell, \Psi\ell), d\left(\Psi z_0, \Psi^2 z_0\right), \\ \dfrac{d\left(\Psi^2 z_0, \Psi\ell\right)}{2}, \dfrac{d(\ell, \Psi z_0) + d(z_0, \Psi\ell)}{2} \end{array} \right\} \\
&\leq \alpha \max\left\{ d(z_0, \ell), d(\ell, \Psi\ell), \frac{d(z_0, \Psi\ell)}{2} \right\} \\
&= \alpha \max\left\{ d(z_0, \ell), d(\ell, \Psi\ell), \frac{d(z_0, \ell) + d(\ell, \Psi\ell)}{2} \right\} \\
&\leq \alpha \max\{d(z_0, \ell), d(\ell, \Psi\ell)\},
\end{aligned}$$

that is,

$$d(\ell, \Psi\ell) \leq \alpha \max\{d(z_0, \ell), d(\ell, \Psi\ell)\}.$$

If

$$\max\{d(z_0, \ell), d(\ell, \Psi\ell)\} = d(\ell, \Psi\ell),$$

then

$$d(\ell, \Psi\ell) = d(\Psi z_0, \Psi\ell) \leq \alpha d(\ell, \Psi\ell) < d(\ell, \Psi\ell),$$

which is a contradiction. Hence, we get

$$d(\ell, \Psi\ell) \leq \alpha d(z_0, \ell)$$

along with (10) implies

$$d(z_0, \Psi\ell) \leq \alpha \max\{d(z_0, \ell), d(\ell, z_0)\}.$$

Therefore, we obtain

$$\begin{aligned}
d(\ell, \Psi\ell) &\leq d(\ell, z_0) + d(z_0, \Psi\ell) \\
&\leq d(\ell, z_0) + \alpha \max\{d(z_0, \ell), d(\ell, z_0)\} \\
&\leq d^s(\ell, z_0) + \alpha d^s(\ell, z_0) \\
&= (1+\alpha)d^s(\ell, z_0).
\end{aligned}$$

Thus,

$$\frac{1}{(1+\alpha)}d(\ell, \Psi\ell) \leq d^s(\ell, z_0).$$

By (8), we have

$$\begin{aligned} d(\Psi\ell,\Psi z_0) &\leq \alpha \max U_\Psi(\ell,z_0) \\ &\leq \alpha \max\left\{ \begin{array}{l} d(\ell,z_0), d(\ell,\Psi\ell), d(z_0,\Psi z_0), d\big(\Psi\ell,\Psi^2\ell\big), \\ \dfrac{d\big(\Psi^2\ell,\Psi z_0\big)}{2}, \dfrac{d(z_0,\Psi\ell)+d(\ell,\Psi z_0)}{2} \end{array} \right\}, \end{aligned}$$

holds for all $\ell \neq z_0 \in \Im$. Put $\ell = \varrho_k$ in above inequality

$$\begin{aligned} d(\varrho_{k+1},\Psi z_0) &= d(\Psi\varrho_k,\Psi z_0) \\ &\leq \alpha \max\left\{ \begin{array}{l} d(\varrho_k,z_0), d(\varrho_k,\Psi\varrho_k), d(z_0,\Psi z_0), d\big(\Psi\varrho_k,\Psi^2\varrho_k\big), \\ \dfrac{d\big(\Psi^2\varrho_k,\Psi z_0\big)}{2}, \dfrac{d(z_0,\Psi\varrho_k)+d(\varrho_k,\Psi z_0)}{2} \end{array} \right\} \\ &\leq \alpha \max\left\{ \begin{array}{l} d(\varrho_k,z_0), d(\varrho_k,\varrho_{k+1}), d(z_0,\Psi z_0), d(\varrho_{k+1},\varrho_{k+2}), \\ \dfrac{d(\varrho_{k+2},\Psi z_0)}{2}, \dfrac{d(z_0,\varrho_{k+1})+d(\varrho_k,\Psi z_0)}{2} \end{array} \right\} \end{aligned}$$

holds for all $k \in \mathbb{N}$. Taking the limit as $k \to \infty$ on both sides of the above inequality, we have

$$d(z_0,\Psi z_0) \leq \alpha d(z_0,\Psi z_0) < d(z_0,\Psi z_0),$$

a contradiction. Hence, $z_0 = \Psi z_0$. That is, Ψ has a fixed point. Now, we prove the uniqueness. For the proof by contradiction suppose that Ψ has two distinct fixed points say z_0, and z_1. Since

$$\lambda(\alpha) d(z_0,\Psi z_0) \leq d(z_0,\Psi z_0) = d(z_0,z_0) = 0 \leq d(z_0,z_1),$$

therefore we get

$$\begin{aligned} d(z_0,z_1) &= d(\Psi z_0,\Psi z_1) \\ &\leq k \max\left\{ \begin{array}{l} d(z_0,z_1), d(z_0,\Psi z_0), d(z_1,\Psi z_1), d\big(\Psi z_0,\Psi^2 z_0\big), \\ \dfrac{d\big(\Psi^2 z_0,\Psi z_1\big)}{2}, \dfrac{d(z_1,\Psi z_0)+d(z_0,\Psi z_1)}{2} \end{array} \right\} \\ &= k d(z_0,z_1) < d(z_0,z_1), \end{aligned}$$

a contradiction. This proves the uniqueness of fixed points and completes the proof as well. □

Now we give an example to show that (GST) contractions are proper generalization of d-contractions and Suzuki-type d-contractions in the context of Δ-symmetric quasi-metric space.

Example 2. *Let* $\Im = \{\ell, \hbar, \wp\}$ *and mapping* $d : \Im \times \Im \to [0, +\infty)$ *defined as*

$$\begin{aligned} &d(\ell,\hbar) = 1, \quad d(\hbar,\ell) = 1.1, \quad d(\wp,\ell) = 1.4, \quad d(\ell,\wp) = 2, \\ &d(\hbar,\wp) = 1.3, \quad d(\wp,\hbar) = 1.25, \quad d(\ell,\ell) = 0 \end{aligned}$$

for all $\ell \in \Im$. *Note that* $(\Im, d)$ *is a* Δ*-symmetric quasi-metric space for any* $\Delta \geq \dfrac{1.4}{1.1}$. *Consider a mapping* $\Psi : \Im \to \Im$ *defined as*

$$\Psi x = \left\{ \begin{array}{ll} \hbar, & \text{if } x \in \{\hbar, \wp\}, \\ \wp, & \text{if } x = \ell. \end{array} \right.$$

As

$$\lambda(\alpha) d(\hbar,\Psi\hbar) = \lambda(\alpha) d(\hbar,\hbar) = 0 \leq 1.1 = d^s(\hbar,\ell).$$

Note that

$$d(\Psi\hbar, \Psi\ell) = d^s(\Psi\hbar, \Psi\ell) = d^s(\hbar, \wp) = d(\hbar, \wp) = 1.3 \nleq 1.1\alpha = \alpha d(\hbar, \ell)$$

for any $\alpha \in [0, 1)$. *Hence* Ψ *is neither a d-contraction nor a Suzuki-type d-contraction. Moreover*

$$\min\{d(\ell, \Psi\ell), d(\hbar, \Psi\hbar)\} = \min\{d(\ell, \wp), d(\hbar, \hbar)\} = 0 \leq 2d(\ell, \hbar)$$

and

$$d(\Psi\ell, \Psi\hbar) = d(\wp, \hbar) = 1.25 \nleq \alpha = \alpha d(\ell, \hbar)$$

for any $\alpha \in (0, 1)$, *implies that* Ψ *is not a* $2-$*basic contraction on* $(\Im, d)$. *Now, we show that* Ψ *is a GST contraction of* $(\Im, d)$. *Now,*

$$d(\Psi\hbar, \Psi\ell) = d(\hbar, \wp) = 1.3$$

and

$$\begin{aligned} U_\Psi(\hbar, \ell) &= \max\left\{ \begin{array}{l} d(\hbar, \ell), d(\hbar, \Psi\hbar), d(\ell, \Psi\ell), d\left(\Psi\hbar, \Psi^2\hbar\right), \\ \dfrac{d\left(\Psi^2\hbar, \Psi\ell\right)}{2}, \dfrac{d(\ell, \Psi\hbar) + d(\hbar, \Psi\ell)}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} d(\hbar, \ell), d(\hbar, \hbar), d(\ell, \wp), d(\hbar, \hbar), \\ \dfrac{d(\hbar, \wp)}{2}, \dfrac{d(\ell, \hbar) + d(\hbar, \wp)}{2} \end{array} \right\} \\ &= \max\left\{1.1, 0, 2, 0, \frac{1.3}{2}, \frac{1 + 1.3}{2}\right\} = 2. \end{aligned}$$

Thus

$$d(\Psi\hbar, \Psi\ell) \leq \alpha U_\Psi(\hbar, \ell)$$

holds for $\alpha = 0.7$. *Note that*

$$\lambda(\alpha)d(\ell, \Psi\ell) = \lambda(\alpha)d(\ell, \wp) = 1.17 \nleq 1.1 = d^s(\ell, \hbar)$$

for $\alpha = 0.7$. *Note that*

$$d(\Psi\ell, \Psi\wp) = d(\wp, \hbar) = 1.25$$

and

$$\begin{aligned} U_\Psi(\ell, \wp) &= \max\left\{ \begin{array}{l} d(\ell, \wp), d(\ell, \Psi\ell), d(\wp, \Psi\wp), d\left(\Psi\ell, \Psi^2\ell\right), \\ \dfrac{d\left(\Psi^2\ell, \Psi\wp\right)}{2}, \dfrac{d(\wp, \Psi\ell) + d(\ell, \Psi\wp)}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} d(\ell, \wp), d(\ell, \wp), d(\wp, \hbar), d(\wp, \hbar), \\ \dfrac{d(\hbar, \hbar)}{2}, \dfrac{d(\wp, \wp) + d(\ell, \hbar)}{2} \end{array} \right\} \\ &= \max\left\{2, 2, 1.25, 1.25, 0, \frac{1}{2}\right\} = 2. \end{aligned}$$

Thus

$$d(\Psi\ell, \Psi\wp) \leq \alpha U_\Psi(\ell, \wp)$$

holds for $\alpha = 0.7$. *Now*

$$d(\Psi\wp, \Psi\ell) = d(\hbar, \wp) = 1.3$$

and

$$\begin{aligned} U_{\Psi}(\wp,\ell) &= \max\left\{ \begin{array}{l} d(\wp,\ell), d(\wp,\Psi\wp), d(\ell,\Psi\ell), d\left(\Psi\wp,\Psi^2\wp\right), \\ \dfrac{d\left(\Psi^2\wp,\Psi\ell\right)}{2}, \dfrac{d(\ell,\Psi\wp)+d(\wp,\Psi\ell)}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} d(\wp,\ell), d(\wp,\hbar), d(\ell,\wp), d(\hbar,\hbar), \\ \dfrac{d(\hbar,\wp)}{2}, \dfrac{d(\ell,\hbar)+d(\wp,\wp)}{2} \end{array} \right\} \\ &= \max\left\{1.4, 1.25, 2, 0, \frac{1.3}{2}, \frac{1}{2}\right\} = 2. \end{aligned}$$

Thus

$$d(\Psi\wp,\Psi\ell) \leq \alpha U_{\Psi}(\wp,\ell)$$

holds for $\alpha = 0.7$. *As*

$$d(\Psi\hbar,\Psi\wp) = d(\hbar,\hbar) = 0 \text{ and } d(\Psi\wp,\Psi\hbar) = d(\hbar,\hbar) = 0$$

so

$$d(\Psi\hbar,\Psi\wp) \leq \alpha U_{\Psi}(\hbar,\wp) \text{ and } d(\Psi\wp,\Psi\hbar) \leq \alpha U_{\Psi}(\wp,\hbar)$$

holds for $\alpha = 0.7$. *Hence,* Ψ *is a (GST) contraction for* $\alpha = 0.7$.

Corollary 1. *Let* $(\Im, d)$ *be a d-sequential complete* Δ*-symmetric quasi-metric space and* $\Psi : \Im \to \Im$ *a mapping. If there is an* $\alpha \in [0,1)$ *such that*

$$\begin{aligned} &\lambda(\alpha) d(\ell,\Psi\ell) \leq d^s(\ell,\hbar) \\ &\text{implies} \\ &d(\Psi\ell,\Psi\hbar) \leq \alpha \max\left\{ d(\ell,\hbar), d(\ell,\Psi\ell), d(\hbar,\Psi\hbar), \frac{d(\hbar,\Psi\ell)+d(\ell,\Psi\hbar)}{2} \right\} \end{aligned}$$

for all $\ell, \hbar \in \Im$. *Then* Ψ *has a unique fixed point.*

The following corollary generalizes Theorems 1, 2 and [16] (Theorem 5).

Corollary 2. *Let* $(\Im, d)$ *be a d-sequential complete* Δ*-symmetric quasi-metric space and* $\Psi : \Im \to \Im$ *a mapping. If there is an* $\alpha \in [0,1)$ *such that*

$$\lambda(\alpha) d(\ell,\Psi\ell) \leq d^s(\ell,\hbar) \Rightarrow d(\Psi\ell,\Psi\hbar) \leq \alpha d(\ell,\hbar)$$

or all $\ell, \hbar \in \Im$. *Then* Ψ *has a unique fixed point.*

4. Conclusions

In this paper, we obtained fixed point results for Suzuki-type contractions of quasi-metric spaces. We discussed the equivalence of various types of completeness notions of quasi-metric spaces, including Smyth completeness. Note that Smyth completeness has various applications in computer science and engineering, for instance, in denotational semantics and complexity analysis (see [15]); further quasi metrics have applications in software engineering [14] and fractal theory [13]. Results obtained here in this paper can further be explored in connection with these applications.

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