mathematics

Article

Periodic Flows in a Viscous Stratified Fluid in a Homogeneous Gravitational Field

Yuli D. Chashechkin * and Artem A. Ochirov

Ishlinsky Institute for Problems in Mechanics RAS, 119526 Moscow, Russia; ochir@ipmnet.ru
* Correspondence: chakin@ipmnet.ru; Tel.: +7-(495)-434-01-92

Abstract: The density of a fluid or gas, which depends on the temperature, pressure and concentration of dissolved substances or suspended particles, changes under the influence of a large number of physical factors. We assume that an undisturbed liquid is heterogeneous. The propagation of periodic flows in viscous, uniformly stratified fluids is considered. The analysis is based on a system of fundamental equations for the transfer of energy, momentum and matter in periodic flows. Taking into account the compatibility condition, dispersion relations are constructed for two-dimensional internal, acoustic and surface linear periodic flows with a positive definite frequency and complex wave number in a compressible viscous fluid exponentially stratified by density. The temperature conductivity and diffusion effects are neglected. The obtained regularly perturbed solutions of the dispersion equations describe the conventional weakly damped waves. The families of singular solutions, specific for every kind of periodic flow, characterize the before unknown thin ligaments that accompany each type of wave. In limited cases, the constructed regular solutions transform into well-known expressions for a viscous homogeneous and an ideal fluid. Singular solutions are degenerated in a viscous homogeneous fluid or disappear in an ideal fluid. The developing method of the fundamental equation system analysis is directed to describe the dynamics and spatial structure of periodic flows in heterogeneous fluids in linear and non-linear approximations.

Keywords: heterogeneous fluid; stratification; viscosity; compressibility; linear models; complete description; dispersion relations

MSC: 76A02; 76Q05; 76M45

check for updates

Citation: Chashechkin, Y.D.; Ochirov, A.A. Periodic Flows in a Viscous Stratified Fluid in a Homogeneous Gravitational Field. *Mathematics* **2023**, *11*, 4443. https://doi.org/10.3390/math11214443

Academic Editor: Ramoshweu Solomon Lebelo

Received: 28 September 2023
Revised: 23 October 2023
Accepted: 24 October 2023
Published: 26 October 2023

Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

In natural, laboratory and industrial conditions, the density of a liquid or gas depends on the temperature, pressure, concentration of dissolved substances or suspended particles. It is not constant and changes under the influence of a large number of physical factors. An oscillating source forms waves that propagate over long distances in a medium with a weak dissipation. Historically, it is customary to distinguish acoustic waves, the existence of which is provided by the compressibility of the medium and gravitational waves associated with the action of the gravity field. Inertial waves propagate in a globally rotating medium and capillary waves run at the interface between the media. The existence of a large group of hybrid waves is provided by the combined action of a number of factors [1,2].

In the mass forces (gravity and inertia) field, the fluid medium is separated. Heavy particles sink, light particles float up, and the medium is naturally stratified. Compressibility under the action of hydrostatic pressure has an additional impact on density. The choice of the coordinate system depends on the overall geometry of the problem. The consideration of flows with scales much smaller than the Earth's radius is carried out in a Cartesian coordinate system with an axis z pointing vertically upwards. The acceleration of gravity g is directed downwards.

The density distribution in the direction of gravity $\rho(z)$ is characterized by the scale $\Lambda = |d \ln \rho(z)/dz|^{-1}$, frequency $N = \sqrt{g/\Lambda}$ and buoyancy period $T_b = 2\pi/N$. In the atmosphere and ocean, the average buoyancy period lies in the range of $3 < T_b < 10$ min [3,4]. In the "instantaneous" density profiles of the atmosphere and ocean, thin, highly gradient interfaces are expressed. They separate thick, more homogeneous layers, thereby forming a "fine structure" of the medium [4,5].

In practice, several characteristic types of average density distributions have been identified. Further, the models of continuous (linear or exponential), two-layer or multilayer stratification (the last two with a persistent density gap) will be used. In a large group of flows, density variations are much less than the average value.

At the end of the 18th century, B. Franklin observed sea fluctuations in the free surface and the interface between water and olive oil in a ship lighting lamp, which was later mounted on a swing. He noted the need to analyze the influence of fluid density heterogeneity in mathematical research [6]. Initially, the effects of stratification began to be taken into account in calculations of the internal wave propagation in the atmosphere and ocean, which were carried out by famous English scientists G.G. Stokes [7], Lord Rayleigh [8], H. Lamb [9] and others.

A systematic study of the influence of stratification on the pattern of flows in the atmosphere and ocean, including navigation (the "dead water" effect), which was noticed in ancient times, gained interest after the publication of the scientific results from F. Nansen's polar expeditions [10]. V. Ekman developed the methodology and planned the experiments. In order to conduct laboratory studies of the phenomenon of the "dead water", he used a review of the first publications on the theory of internal waves in the treatise [11]. In a series of thorough experiments, V. Ekman determined the conditions for the generation of large waves using a moving model of a ship at a smoothed interface between fresh and salt (sea) water and determined the influence of the movement mode on the position of the model's hull and drag [10]. However, in general, the work on the consideration of the equations of internal waves and the "exotic" phenomenon of "dead water" fell out of scientific circulation for more than half a century and did not affect the development of the general theory of fluid flows.

At least two of the reasons have to be noted: the smallness of the density variations compared to its average value, limiting the effect on inertial properties, and the insufficient development of the mathematical apparatus. G.G. Stokes noted in a fundamental article [12], written several years before a thorough study of wave propagation in homogeneous and layer-by-layer stratified media [13], "As it is quite useless to consider cases of the utmost degree of generality, I shall suppose the fluid to be homogeneous..." However, a few years later, he also emphasized the limitations of the approximation used: "The three equations of which (l) is the type are not the general equations of motion which apply to a heterogeneous fluid when internal friction is taken into account, which are those numbered (10) in my former paper, but are applicable to a homogeneous incompressible fluid, or to a homogeneous elastic fluid subject to small variations of density, such as those which accompany sonorous vibrations" [7].

Accordingly, when studying the waves of other types—acoustic [14] or gravitational-capillary—at the interface between the atmosphere and the hydrosphere [1,15], the unperturbed density was assumed to be homogeneous. Here and further, general rotation effects and associated inertial waves [1,16] are not considered.

The interest in the mathematical study of the stratification influence started to form in the middle of the last century. During this period, the precision instruments identified the thin, highly gradient structure of the Baltic Seawaters [5]. Next, the flows induced by diffusion on an inclined wall in a continuously stratified atmosphere were discovered [17]. The development of interest in studying the influence of stratification was facilitated by the papers [18,19], which showed the important role of diffusion-induced flows on topography not only in the atmosphere, where they manifested themselves not only in the form of mountain and valley winds but also in the ocean. At the same time, experimental [20] and theoretical studies of internal waves in continuously stratified media [21] were developing. Numerous expeditions have shown the existence of fine structure and its influence on the dynamics of the atmosphere and ocean in various regions of the Earth.

The number of original articles and reviews describing the influence of stratification on individual phenomena (internal waves, currents and vortices) was increasing rapidly. The propagation of acoustic vibrations in a continuously stratified medium was considered [22]. The influence of viscosity was initially taken into account only in terms of the exponential attenuation of wave amplitudes [1,21]. It was analyzed in more detail when describing the propagation of gravitational surface [23–29], internal [30] and acoustic waves [31], considering the boundary layers formed simultaneously with the waves.

From the general content of papers and monographs [1,2,11,15,16], it follows that the basis of a rational mathematical description of inhomogeneous fluid flows is a system of fundamental equations-differential analogues of the momentum, energy and matter conservation laws with physically justified initial and boundary conditions. All the equations that were originally presented in the first edition of the treatise [1], published in 1944, were quite complex for general analysis. In practice, the reduced forms of the general system of equations are usually used, which makes it possible to study the properties of individual flow components, such as waves, vortices, jets, and wakes with the required degree of completeness. In this work, the main attention is paid to the analysis of periodic flows, the temporal variability of which is proportional to a function of the form $f \propto \exp(-i\omega t)$.

In the experiment, as in the early stage of the development of the analytical theory of waves [11], it was emphasized that the measured physical quantities-parameters of periodic flows, such as the period T_w (frequency ω,), length λ, group c_g and phase c_{ph} velocity of the wave, are characterized by real numbers. From the very beginning of the theoretical study, periodic flows began to be described using complex numbers, introduced to reduce the notation and convenience of calculations. The immersion of problems in the algebra of complex numbers leads to the expansion of the dimension of the problem space and the emergence of additional "physically unrealizable" solutions. Accordingly, there is a need to select a part of the solutions corresponding to the initial formulation, with the introduction of criteria explaining the procedure.

The physical interpretation of the solutions depends on the choice of the algorithm for the rules for immersing the problem in the algebra of complex numbers. Traditionally, starting with the works of scientists in the 19th century, the frequency ω of a waveform $f \propto \exp(ikx - i\omega t)$ has been chosen as a complex value. Its real part determines the dispersion relation, the functional relationship between frequency ω and wave vector k, and the imaginary part determines the stability condition and the wave attenuation coefficient [1]. An innumerable number of works, including popular monographs, are devoted to the study of flow and wave stability [32,33]. The history of the development of flow stability studies is traced in detail in [34]. Researchers investigated the problem of finding the liquid surface shape and the criteria for the development of instability under the action of various destabilizing factors, such as surface electric charge (Tonks–Frenkel instability) [35,36], Rayleigh-Taylor and Marangoni thermal convective instabilities [37], etc.

However, the amplitude and wavelength change within the distance of the source, but the frequency of periodic motion remains constant, as it follows from the consideration of the experimental patterns of non-dispersive waves, propagation in a medium at rest. In this regard, it is natural to maintain the frequency, which is a measure of the wave energy, as a positive definite real quantity in calculations, and take the wave number to be complex [38]. Substituting expansions of this type into a linearized system of fundamental equations, the solution of which is found using methods of singular perturbation theory [39], allows for a new classification of the structural components of periodic flows based on the properties of complete solutions.

This part of the solutions of the fundamental equation system, which includes regularly perturbed functions, characterizes waves slowly decaying in the direction of propagation in weakly dissipative media. Singularly perturbed components of the solution describe ligaments—thin flows that determine the structure of the medium in both linear and weakly nonlinear approximations [40–42].

In the hydrosphere and atmosphere, there are types of waves that differ significantly in frequency (in particular, acoustic and internal waves in the thickness of a stratified liquid [1,11,21,22,31]) or in the distribution of displacement amplitudes in depth (surface and internal waves) [1,11,21]. This makes it possible to study their properties within the framework of individual specialized equations-acoustics [1,22,31], internal [21] surface gravity or capillary waves [11,43]. Modern researchers often consider the problem of acoustic wave propagation in compressible media with complex structures using numerical and analytical methods [44,45]. At the same time, an important part of the periodic flow, which determines the fine structure of the flow, remains without attention.

The patterns of propagation of a set of two-dimensional periodic disturbances—waves and ligaments—in an incompressible fluid, when the reduced continuity equation allows us to introduce a stream function convenient for analysis, are considered in the thickness [38] and on the surface of a viscous stratified fluid [46]. This paper is the first to consider the problem of propagating a complete set of two-dimensional infinitesimal periodic disturbances in a continuously stratified compressible fluid.

2. System of Fundamental Equations of Periodic Flows in the Atmosphere and Ocean

2.1. The Complete System of Equations Determining the Flow of the Liquid

Periodic wave processes occurring in a viscous liquid are considered. The liquids existing in nature are heterogeneous. The inhomogeneous distribution of density ρ is determined using the equation of state:

$$\rho = \rho(P, S, s_n, T). \tag{1}$$

The symbol P denotes pressure, S stands for entropy, s_n denotes salinity of the n-th impurity and T stands for temperature

Far from the conditions of phase transitions, the values of the temperature gradient and the impurity content are limited, and it is permissible to use a linearized equation of state:

$$\begin{gathered} \rho = \rho_0\left(1 - \alpha_T(T - T_0) + \alpha_P(P - P_0) + \sum_n \alpha_{s_n}(s_n - s_{n0})\right). \\ \alpha_T = -\frac{1}{\rho}\frac{\partial\rho}{\partial T},\ \alpha_P = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial P}\right)_S,\ \alpha_{s_n} = \frac{1}{\rho}\frac{\partial\rho}{\partial s_n} \end{gathered} \tag{2}$$

Here, α_T denotes the coefficient of thermal expansion of the liquid; α_P stands for the coefficient of adiabatic compressibility of the liquid; α_{s_n} denotes the coefficient of contraction of the n-th impurity; and T_0, P_0, s_{n0} refer to the reference level of temperature, pressure and salinity, respectively.

The fundamental system of equations in addition to the equation of state consists of the equations for describing the matter transfer, the concentration of impurity, temperature and momentum. Taking into account that the neglect of thermophores is the Ludwig–Soret effect [47,48] and Dufour effect [49], the system of equations is written as follows [38,46]:

$$G = G(P, S, s_n, T), \rho = \rho(P, S, s_n, T) \tag{3}$$

$$\partial_t\rho + \nabla_j\left(p^j\right) = Q_\rho, \tag{4}$$

$$\partial_t\left(p^i\right) + \nabla_j\Pi^{ij} = \rho g^i + 2\varepsilon^{ijk}p_j\Omega_k + Q^i, \tag{5}$$

$$\partial_t(\rho T) + \nabla_j\left(p^j T\right) = \Delta(\kappa_T\rho T) + Q_T, \tag{6}$$

$$\partial_t(\rho s_n) + \nabla_j \left(p^j s_n \right) = \Delta(\kappa_s \rho s_n) + Q_{s_n}. \tag{7}$$

Here, G is Gibbs potential; $Q_\rho, Q^i, Q_T, Q_{s_n}$ represent the source of mass, momentum, temperature and the salinity concentration, respectively; $\mathbf{p}$ denotes momentum; $\Pi^{ij} = \rho u^i u^j + P\delta^{ij} - \sigma^{ij}$ stands for the momentum flux density tensor; u^i is the component of the fluid velocity $\mathbf{u} = \mathbf{p}/\rho$; δ^{ij} is the Kronecker delta; $\sigma^{ij} = \mu \left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} - \frac{2}{3}\delta^{ij}\frac{\partial u^k}{\partial x^k} \right) + \zeta\delta^{ij}\frac{\partial u^k}{\partial x^k}$ denotes the viscous stress tensor; μ, ζ are dynamic and bulk viscosities, respectively; $\mathbf{g}$ is the gravity acceleration; ε^{ijk} is the Levi-Civita symbol; $\boldsymbol{\Omega}$ is the global rotation angular velocity; and κ_T, κ_{s_n} stand for thermal and mass diffusivity, respectively.

Equations (1), (3)–(7) form a fundamental system of equations that determine the fluid flow. The complete solution of the system of Equations (3)–(7) defines all components of flow in liquids—waves: acoustic, gravitational (internal and surface), capillary, hybrid and ligaments—accompanying components that identify the fine flow structure. Usually, researchers ignore the fine structure, limiting themselves to a partial solution of a system of equations. In this work, we construct a theory that takes into account all flow components.

To complete the formulation, it is necessary to add initial and boundary conditions of the problem. The initial conditions depend on the shape and type of the oscillation source. Often, when studying the properties of periodic flows, instead of initial conditions, researchers specify the type of solution and look for steady-state solutions of a given type. No-slip, no-flux boundary and initial conditions on the surface of a solid impermeable body Σ are written as follows:

$$\mathbf{u}|_{\Sigma} = 0,\ \mathbf{u}|_{t \le 0} = 0,\ P|_{t \le 0} = P_0,\ s_n|_{t \le 0} = s_{n0},\ T|_{t \le 0} = T_0, \tag{8}$$

If the distance to the boundaries greatly exceeds the characteristic dimensions of the observed phenomena, then a model of an unbounded medium is often used. In this case, the boundary conditions are transformed into the conditions of physical implementation–attenuation with removal:

$$\mathbf{u}|_{\mathbf{r} \to \infty} \to 0, \tag{9}$$

If the model under consideration contains a free surface or interface between layers of immiscible liquids, then it is necessary to add standard hydrodynamic boundary conditions: kinematic and dynamic boundary conditions. The kinematic boundary condition is written for both contacting layers (or for one medium in the case of a free surface): the substantial derivative of the function F defining the shape of the free surface is equal to zero at the boundary:

$$\frac{DF}{Dt} \equiv \frac{\partial F}{\partial t} + (\mathbf{u} \cdot \nabla)F = 0, \tag{10}$$

Dynamic boundary conditions are determined using the balance of forces at the interface (free surface of the liquid):

$$n^k_{(1)}\sigma^{ik}_{(1)} + n^k_{(2)}\sigma^{ik}_{(2)} = 0, \tag{11}$$

Here, $\mathbf{n}$ is the unit normal vector, and the subscript (1) and (2) refer to the two contacting media. If the model takes into account the effects of surface tension, then on the right side of (11), it is necessary to take into account Laplace forces as well.

2.2. The Reduced System of Equations

The fundamental system of equations is complete and allows one to determine the patterns of changes in basic physical quantities during the propagation of periodic disturbances in continuous media. Since the complete system of equations is of a high order and very complex to analyze, it is simplified to study the properties of individual processes. An extremely simplified model in which it is possible to track the dynamics and evolution of the structure of periodic flows takes into account the uneven distribution of density, without indicating the physical nature of the heterogeneity formation.

The system of Equations (3)–(7) is noticeably reduced in the constant temperature model in the absence of impurities in a weakly compressible fluid. The consideration is carried out in a Cartesian coordinate system $Oxyz$ in which the Oz axis is directed against the direction of the gravity acceleration $\mathbf{g}$. The Oxy plane determines the position of the reference level. In a weakly compressible viscous fluid, bulk viscosity takes on a zero value. In the absence of mass sources, $Q_\rho = 0$ and under the assumptions made, the reduced system of equations will take the following form:

$$\rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}) = \rho\nu\Delta\mathbf{u} - \nabla P + \rho\mathbf{g} \tag{12}$$

$$\partial_t \rho + \mathbf{u} \cdot \nabla\rho + \rho \mathrm{div}\mathbf{u} = 0 \tag{13}$$

$$\rho = \rho_0(z)(1 + \widetilde{\rho}(x, y, z, t)) \tag{14}$$

The initial stratification $\rho_0(z)$ when describing models is often defined as a linear $\rho_0(z) = \rho_{00}(1 - z/\Lambda)$ or exponential $\rho_0(z) = \rho_{00}\exp(-z/\Lambda)$ function. The symbol ρ_{00} indicates the density value at the reference level $z = 0$, and the symbol $\Lambda = |d\ln\rho/dz|^{-1}$ characterizes the stratification scale. In nature, liquids are usually weakly stratified and the scale of stratification is on the order of tens or hundreds of kilometers. When considering phenomena with characteristic dimensions much smaller than the scale of stratification, the density value in the linear and exponential stratification models turns out to be practically the same. In this case, the researchers select the more user-friendly mathematical model. Real measurements show that in an atmosphere with good accuracy, stratification can be considered linear or exponential [3].

A stably stratified liquid is characterized by the limiting frequency of its own mechanical vibrations [8]—buoyancy frequency—the square of which is given by

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz} \tag{15}$$

The equation of state (2) under the assumptions made is simplified as follows:

$$\rho = \rho_0(z)(1 - \alpha_P(P - P_0)) \tag{16}$$

Fluid pressure is represented as the sum of reference level pressure P, hydrostatic pressure and perturbation pressure $\widetilde{P}$:

$$P = P_0 + \int_z^0 \rho(x, y, \xi, t) g d\xi + \widetilde{P}(x, y, z, t) \tag{17}$$

Taking into account the equation of state (16), the definition of the velocity of sound $c^2 = (\partial P/\partial\rho)_S$ and the definition of pressure (17), the relation (15) for the buoyancy frequency takes the following form:

$$N^2 = \frac{g^2}{c^2}\left(\frac{c_p}{c_V} - 1\right), \tag{18}$$

Here, c_P, c_V is the heat capacity at constant pressure and at constant volume, respectively.

The resulting system of equations, despite significant simplifications, qualitatively completely describes periodic flows in viscous inhomogeneous continuous media. The boundary and initial conditions will not change.

In the model under consideration, there are intrinsic parameters. These parameters determine the characteristic scales of the flow components and the characteristic times of their observation. A set of kinetic coefficients allows one to form their own parameters.

Intrinsic parameters for liquids with the parameters of water and air are presented in Tables 1 and 2, respectively.

Table 1. Intrinsic parameters of hydrosphere.

Parameter	Fluid			
	Stratified		Homogeneous	
	Strongly	Weakly	Potentially	Actually
Buoyancy frequency N, s^{-1}	1	0.01	0.00001	0.0
Buoyancy period T_b	10 s	10 min	10 days	∞
Scale of stratification Λ	10 m	100 km	10^8 km	∞
Viscous wave scale $\delta_N^{g\nu} = (g\nu)^{1/3}N^{-1}$, cm	2	200	$2 \cdot 10^5$	∞
Stokes microscale $\delta_N^{\nu} = \sqrt{\nu/N}$, cm	0.1	1	30	∞

Table 2. Intrinsic parameters of the atmosphere.

Parameter	Fluid			
	Stratified		Homogeneous	
	Strongly	Weakly	Potentially	Actually
Buoyancy frequency N, s^{-1}	1	0.01	0.00001	0.0
Buoyancy period T_b	10 s	10 min	10 days	∞
Scale of stratification Λ	10 m	100 km	10^8 km	∞
Viscous wave scale $\delta_N^{g\nu} = (g\nu)^{1/3}N^{-1}$, cm	5	500	$5 \cdot 10^5$	∞
Stokes microscale $\delta_N^{\nu} = \sqrt{\nu/N}$, cm	0.4	4	120	∞

The natural parameters presented in the table have to be supplemented with temporal and spatial scales that do not depend on the level of fluid stratification. Taking into account compressibility, a time scale $\tau_c^{\nu} = \nu/c^2$ is added. It takes values for water $\tau_c^{\nu} \simeq 4 \cdot 10^{-13}$ s and for air $\tau_c^{\nu} \simeq 10^{-10}$ s. Spatial scale $\delta_c^{\nu} = \nu/c$ is added. It takes values for water $\delta_c^{\nu} \simeq 7 \cdot 10^{-10}$ m, and for air $\delta_c^{\nu} \simeq 5 \cdot 10^{-8}$ m. In viscous liquids (homogeneous and heterogeneous), a capillary-viscous time scale appears $\tau_{\nu g}^{\gamma} = \gamma/\nu g$. The symbol $\gamma = \sigma/\rho_{00}$ denotes the surface tension coefficient of the liquid σ normalized to the equilibrium density value ρ_{00}. For water, the capillary-viscous time scale takes on values $\tau_{\nu g}^{\gamma} \simeq 7$ s, and for air $\tau_{\nu g}^{\gamma} \simeq 400$ s. The spatial scale in viscous liquids $\delta_g^{\nu} = \sqrt[3]{\nu^2/g}$ has the value $\delta_g^{\nu} \simeq 5 \cdot 10^{-5}$ m in water and $\delta_g^{\nu} \simeq 3 \cdot 10^{-4}$ m in air. The capillary length $\delta_g^{\gamma} = \sqrt{\gamma/g}$ is in both the viscous liquid model and in the inviscid liquid one. For water capillary length, it takes the value $\delta_g^{\gamma} \simeq 3 \cdot 10^{-3}$ m, and for air it is $\delta_g^{\gamma} \simeq 8 \cdot 10^{-2}$ m.

Small disturbances of physical quantities (pressure, density, velocity) often occur in nature. Let us solve the problem using the decomposition method for a small parameter that plays the role of the amplitude of periodic movements.

3. Periodic Flows in the Thickness of a Uniformly Stratified Liquid

3.1. Linearization of the Equation System

The perturbations of the target values (velocity, density and pressure) are considered small. To obtain dispersion relations, we linearize the system of Equations (12)–(14), (16). If we assume that the fluid is exponentially stratified, then in a linear approximation in terms of the amplitude of periodic motion, the reduced system of fundamental equations is written as follows:

$$\begin{cases} \partial_t\widetilde{\rho} - \frac{w}{\Lambda} + \partial_x u + \partial_y v + \partial_z w = 0 \\ \partial_t u - \nu\Delta u + \frac{1}{\rho_{00}}\partial_x\widetilde{P} = 0 \\ \partial_t v - \nu\Delta v + \frac{1}{\rho_{00}}\partial_y\widetilde{P} = 0 \\ \partial_t w - \nu\Delta w + \frac{1}{\rho_{00}}\partial_z\widetilde{P} + g\widetilde{\rho} = 0 \\ \frac{1}{\rho_{00}c^2}\partial_t\widetilde{P} - \frac{wg}{c^2} + \partial_x u + \partial_y v + \partial_z w = 0 \end{cases} \tag{19}$$

Here u, v, w are the components of the velocity field $\mathbf{u} = (u, v, w)$. We look for the solution of the equation system (19) in the form of periodic flows $\propto \exp(i\omega t)$:

$$\begin{pmatrix} u \\ v \\ w \\ \widetilde{P} \\ \widetilde{\rho} \end{pmatrix} = \begin{pmatrix} U_m \\ V_m \\ W_m \\ P_m \\ P_m \end{pmatrix} \exp(i\mathbf{kr} - i\omega t) = \begin{pmatrix} U_m \\ V_m \\ W_m \\ P_m \\ P_m \end{pmatrix} \exp\left(ik_x x + ik_y y + ik_z z - i\omega t\right) \tag{20}$$

Here, U_m, V_m, W_m, P_m, P_m are the amplitudes of the corresponding quantities; $\mathbf{k}$ is the wave vector, the components of which have the right to be complex values k_x, k_y, k_z; and the frequency of periodic motion ω is considered positive definite.

3.2. Dispersion Relation: Classification of Flow Components

By substituting the type of solution (20) into the system of Equation (19), we obtain a system of algebraic equations. The compatibility condition of the algebraic equations system determines the dispersion relations between the components of the wave vector and the frequency of periodic motion:

$$\begin{gathered} D_\nu(k)\left(\omega^2 D_\nu^2(k) - \omega N^2 D_\nu(k) + c^2 k_\perp^2 N_c^2 - c^2\omega k^2 D_\nu(k)\right) = 0, \\ D_\nu(k) = \omega + i\nu k^2,\ k^2 = k_x^2 + k_y^2 + k_z^2,\ k_\perp^2 = k_x^2 + k_y^2,\ N^2 = \frac{g}{\Lambda},\ N_c^2 = N^2 - \frac{g^2}{c^2} \end{gathered} \tag{21}$$

Dispersion relation (21) coincides with the relation obtained earlier [50] in which the limiting transition to a non-rotating weakly compressible fluid (second viscosity $\zeta \to 0$) has been made. It is convenient to find and analyze the regular and singular components of the solution to the dispersion relation (21) in dimensionless variables if one chooses the scales of the problem as non-dimensional parameters. Intrinsic scales characterize the spatial and temporal dimensions of the observed phenomena (see Tables 1 and 2). We choose the inverse buoyancy frequency $\tau_b = N^{-1}$ as the time scale, and the viscous wave scale $\delta_N^{g\nu} = (g\nu)^{1/3}N^{-1}$ as the spatial scale. With the selected non-dimensional parameters, the dispersion relation (21) is written as follows:

$$\begin{gathered} \left(ik_*^2\varepsilon + \omega_*\right)\left(k_{\perp *}^2\left(\frac{\varepsilon}{\eta} - \frac{1}{\varepsilon^2}\right) + \omega_*^2\left(ik_*^2\varepsilon + \omega_*\right)^2 - \omega_*\left(ik_*^2\varepsilon + \omega_*\right) - k_*^2\omega_*\frac{\varepsilon}{\eta}\left(ik_*^2\varepsilon + \omega_*\right)\right) = 0, \\ \varepsilon = \frac{\delta_g^\nu}{\delta_N^{g\nu}} = \frac{\sqrt{\nu/N}}{(g\nu)^{1/3}N^{-1}} = \frac{N\nu^{1/3}}{g^{2/3}},\ \eta = \frac{\tau_c^\nu}{\tau_b} = \frac{N\nu}{c^2} \end{gathered} \tag{22}$$

The ratio of the natural parameters of the medium-viscous scale δ_g^ν and viscous wave scale $\delta_N^{g\nu}$ and the time scale ratio η characterize the small parameters of the problem. The dimensionless components of the wave vector and the dimensionless frequency are indicated by the subscript «*». Since at the highest degree of Equation (22), there is a small parameter, the equation is singularly perturbed with respect to k_{*z}. Consequently, the solution of the form $k_{*z} = k_{*z}\left(k_{*x}, k_{*y}, \omega_*\right)$ contains regular and singular components. The solutions of Equation (22) are written as follows:

$$k_{*z} = \pm\sqrt{-k_{*\perp}^2 + \frac{i\omega_*}{\varepsilon}} \tag{23}$$

$$k_{*z} = \pm\sqrt{\frac{-\varepsilon\omega_*\left(\omega_* + 2k_{*\perp}^2\varepsilon(i+\eta\omega_*) - i\eta(2\omega_*^2 - 1)\right) - \sqrt{-\varepsilon^2\omega_*^2(\eta + i\omega_*) + 4\omega_* k_{*\perp}^2(\varepsilon^3 - \eta)(i+\eta\omega_*)}}{2\varepsilon^2\omega_*(i+\eta\omega_*)}} \tag{24}$$

$$k_{*z} = \pm\sqrt{\frac{-\varepsilon\omega_*\left(\omega_* + 2k_{*\perp}^2\varepsilon(i+\eta\omega_*) - i\eta(2\omega_*^2 - 1)\right) + \sqrt{-\varepsilon^2\omega_*^2(\eta + i\omega_*) + 4\omega_* k_{*\perp}^2(\varepsilon^3 - \eta)(i+\eta\omega_*)}}{2\varepsilon^2\omega_*(i+\eta\omega_*)}} \tag{25}$$

In dimensional form, the roots (23)–(25) are written as follows:

$$k_z = \pm\sqrt{-k_\perp^2 + \frac{i\omega}{\nu}} \tag{26}$$

$$k_{*z} = \pm\sqrt{\frac{-\left(i\nu N^2 + 2\nu\omega(\nu k_\perp^2 - i\omega) + c^2(2i\nu k_\perp^2 + \omega)\right) - \sqrt{-\nu^2N^4 + 2\nu c^2(2\nu k_\perp^2 N_c^2 - i\omega N^2) + c^4\left(\omega^2 + \frac{4i\nu N_c^2 k_\perp^2}{\omega}\right)}}{2\nu(ic^2 + \nu\omega)}} \tag{27}$$

$$k_{*z} = \pm\sqrt{\frac{-\left(i\nu N^2 + 2\nu\omega(\nu k_\perp^2 - i\omega) + c^2(2i\nu k_\perp^2 + \omega)\right) + \sqrt{-\nu^2N^4 + 2\nu c^2(2\nu k_\perp^2 N_c^2 - i\omega N^2) + c^4\left(\omega^2 + \frac{4i\nu N_c^2 k_\perp^2}{\omega}\right)}}{2\nu(ic^2 + \nu\omega)}} \tag{28}$$

The choice of a sign in solutions (23)–(25) or (26)–(28) is determined by the boundary conditions for the decay of periodic motion with distance from the source of disturbances. The solutions (25) and (28) describe the regular component and determine the wave motion, the solutions (23)–(24) and (26)–(27) determine the singular component of the solution and define two types of ligaments. Wave roots (25) or (28) can be obtained approximately using regular decomposition. Ligament roots (23)–(24) or (26)–(27) can be obtained approximately using singular value decomposition [39]. The presented expressions are exact solutions of the dispersion relation. By substituting numerical values of kinetic coefficients, the components of the wave vector corresponding to both ligament and wave solutions can be calculated.

To verify the solutions obtained, we consider some limiting cases.

4. High-Frequency Acoustic Waves

Let us consider the limit of high-frequency oscillations corresponding to acoustic oscillations if their oscillation frequency significantly exceeds the buoyancy frequency of the medium $\omega \gg N$ [22,50]. In this approximation, the dispersion Equation (21) is rewritten as follows:

$$D_\nu(k)\left(D_\nu(k)\omega\left(D_\nu(k)\omega - c^2k^2\right) - g^2k_\perp^2\right) = 0, \tag{29}$$

The solution of dispersion relation (29) is written as follows:

$$\begin{aligned} k_z &= \pm\sqrt{-k_\perp^2 + \tfrac{i\omega}{\nu}}; \\ k_z &= \pm\sqrt{-k_\perp^2 - \frac{c^2\omega - 2i\nu\omega^2 + \sqrt{c^4\omega^2 - \frac{4g^2\nu k_\perp^2}{\omega}(ic^2 + \nu\omega)}}{2\nu(ic^2 + \nu\omega)}}; \\ k_z &= \pm\sqrt{-k_\perp^2 - \frac{c^2\omega - 2i\nu\omega^2 - \sqrt{c^4\omega^2 - \frac{4g^2\nu k_\perp^2}{\omega}(ic^2 + \nu\omega)}}{2\nu(ic^2 + \nu\omega)}}; \end{aligned} \tag{30}$$

The roots (30) describe the wave motion and two attached ligaments. The sign-in solution (30) is chosen based on the need for attenuation of periodic motion $\mathrm{Im}(k_z) > 0$ when moving in the positive direction of the axis Oz. For oppositely directed motion, the solutions are symmetrical.

When moving to a 2D formulation (if we consider the movement to be independent of the horizontal coordinate y), one of the ligaments degenerates and the solution contains one wave and one ligament component:

$$k_z = \pm\sqrt{-k_x^2 - \frac{c^2\omega - 2i\nu\omega^2 + \sqrt{c^4\omega^2 - \frac{4g^2\nu k_x^2}{\omega}(ic^2+\nu\omega)}}{2\nu(ic^2+\nu\omega)}};$$
$$k_z = \pm\sqrt{-k_x^2 - \frac{c^2\omega - 2i\nu\omega^2 - \sqrt{c^4\omega^2 - \frac{4g^2\nu k_x^2}{\omega}(ic^2+\nu\omega)}}{2\nu(ic^2+\nu\omega)}}; \quad (31)$$

In the limit of an inviscid fluid, the dispersion relation (29) is simplified even further and written in the following form:

$$\omega^2\left(\omega^2 - k^2c^2\right) - g^2k_\perp^2 = 0, \quad (32)$$

The ligament components of the solution to relation (32) degenerate and only the wave component remains:

$$k_z = \pm\sqrt{-k_\perp^2 - \frac{g^2k_\perp^2}{c^2\omega^2} + \frac{\omega^2}{c^2}} \quad (33)$$

5. Low-Frequency Gravity Waves

In the limit of low-frequency oscillations $\omega \ll N$, the dispersion relation (21) takes the following form:

$$D_\nu(k)\left(c^2\omega i k^4\nu - c^2N^2k_\perp^2 + c^2k^2\omega^2 + N^2\omega D_\nu(k) + g^2k_\perp^2\right) = 0, \quad (34)$$

The relation (34) also contains a solution in the form of a wave disturbance and two attached ligaments:

$$k_z = \pm\sqrt{-k_\perp^2 + \frac{i\omega}{\nu}};$$
$$k_z = \pm\sqrt{-k_\perp^2 - \frac{ic^2\omega^2 - N^2\nu\omega + \sqrt{N^4\nu^2\omega^2 + 4ic^2k_\perp^2\omega\nu(g^2 - c^2N^2) - c^4\omega^4}}{2c^2\nu\omega}};$$
$$k_z = \pm\sqrt{-k_\perp^2 - \frac{ic^2\omega^2 - N^2\nu\omega - \sqrt{N^4\nu^2\omega^2 + 4ic^2k_\perp^2\omega\nu(g^2 - c^2N^2) - c^4\omega^4}}{2c^2\nu\omega}}; \quad (35)$$

When transitioning to a flat formulation, one of the ligaments degenerates.

Relation (34) also contains a solution in the form of a wave disturbance and two attached ligaments:

$$k_z = \pm\sqrt{-k_x^2 - \frac{ic^2\omega^2 - N^2\nu\omega + \sqrt{N^4\nu^2\omega^2 + 4ic^2k_x^2\omega\nu(g^2 - c^2N^2) - c^4\omega^4}}{2c^2\nu\omega}};$$
$$k_z = \pm\sqrt{-k_x^2 - \frac{ic^2\omega^2 - N^2\nu\omega - \sqrt{N^4\nu^2\omega^2 + 4ic^2k_x^2\omega\nu(g^2 - c^2N^2) - c^4\omega^4}}{2c^2\nu\omega}}; \quad (36)$$

In an ideal liquid, the dispersion relation (34) is simplified as follows:

$$c^2k^2\omega^2 - c^2k_\perp^2N^2 + N^2\omega^2 + g^2k_\perp^2 = 0, \quad (37)$$

The solution (37), which is represented only by a wave component, in an ideal liquid transforms into a well-known expression that does not include the wavelength $\omega^2 = N^2\sin^2\theta$, which describes the geometry of the wave packet in the shape of a "St. Andrew's cross" (θ is the angle of inclination of the wave vector to the horizontal) [20,21]. Ligaments in the ideal fluid model degenerate:

$$k_z = \pm\sqrt{-k_\perp^2 + k_\perp^2\frac{N^2c^2 - g^2}{c^2\omega^2} - \frac{N^2}{c^2}}; \quad (38)$$

The limiting cases discussed in paragraphs 4 and 5 show that ligaments are observed in the entire frequency range from infra-low-frequency mechanical vibrations to high-frequency sound vibrations. The fine structure of the flow accompanies wave motion and requires attention when analyzing phenomena.

6. Periodic Flows in a Two-Layer System of Stratified Liquids

In a two-layer system, which consists of a stratified weakly compressible ocean and a stratified compressible atmosphere, it is necessary to write down the boundary conditions at the interface. In a two-layer system, the pressure in both media is written in the form of the sum of hydrostatic pressure and perturbation pressure, so in a 2D formulation (if we consider the movement independent of the horizontal coordinate y) it is written as follows:

$$P^{o,a} = \int_z^\zeta \rho^{o,a}(x,\xi,t) g d\xi + \widetilde{P}^{o,a}(x,z,t) \tag{39}$$

Here and further, the superscripts "o" and "a" denote quantities related to the ocean (the lower denser liquid) and the atmosphere (the upper less dense liquid), respectively. The symbol $\zeta = \zeta(x,t)$ denotes the function that determines the deviation of the interface between media from the equilibrium position $z = 0$. The system of equations of motion, taking into account Expression (39), is written as follows:

$$z < \zeta: \ \partial_t \mathbf{u}^o - \nu^o \Delta \mathbf{u}^o + \frac{1}{\rho_{00}^o} \nabla P^o - \rho^o \mathbf{g} = 0 \tag{40}$$

$$\partial_t \rho^o + \mathbf{u}^o \cdot \nabla \rho^o + \rho^o \mathrm{div} \mathbf{u}^o = 0 \tag{41}$$

$$\rho^o = \rho_0^o(z)(1 - \alpha_P^o(P^o - P_0^o)) \tag{42}$$

$$z > \zeta: \ \partial_t \mathbf{u}^a - \nu^a \Delta \mathbf{u}^a + \frac{1}{\rho_{00}^a} \nabla P^a - \rho^a \mathbf{g} = 0 \tag{43}$$

$$\partial_t \rho^a + \mathbf{u}^a \cdot \nabla \rho^a + \rho^a \mathrm{div} \mathbf{u}^a = 0 \tag{44}$$

$$\rho^a = \rho_0^a(z)(1 - \alpha_P^a(P^a - P_0^a)) \tag{45}$$

The system of Equations (40)–(45) is supplemented with the boundary conditions at the interface: $z = \xi$

$$z = \zeta: \ \partial_t \zeta + u^o \partial_x \zeta = w^o \tag{46}$$

$$\partial_t \zeta + u^a \partial_x \zeta = w^a \tag{47}$$

$$P^o - 2\rho^o \nu^o \mathbf{n} \cdot ((\mathbf{n} \cdot \nabla) \mathbf{u}^o) = P^a - 2\rho^a \nu^a \mathbf{n} \cdot ((\mathbf{n} \cdot \nabla) \mathbf{u}^a) - \sigma \mathrm{div} \mathbf{n} \tag{48}$$

$$\mathbf{u}^o \cdot \boldsymbol{\tau} = \mathbf{u}^a \cdot \boldsymbol{\tau} \tag{49}$$

$$\rho^o \nu^o (\boldsymbol{\tau} \cdot ((\mathbf{n} \cdot \nabla) \mathbf{u}^o) + \mathbf{n} \cdot ((\boldsymbol{\tau} \cdot \nabla) \mathbf{u}^o)) = \rho^a \nu^a (\boldsymbol{\tau} \cdot ((\mathbf{n} \cdot \nabla) \mathbf{u}^a) + \mathbf{n} \cdot ((\boldsymbol{\tau} \cdot \nabla) \mathbf{u}^a)) \tag{50}$$

$$\mathbf{n} = \frac{\nabla(z-\zeta)}{|\nabla(z-\zeta)|} = \left(\frac{-\partial_x \zeta}{\sqrt{1 + (\partial_x \zeta)^2}}, \frac{1}{\sqrt{1 + (\partial_x \zeta)^2}} \right), \ \boldsymbol{\tau} = \left(\frac{1}{\sqrt{1 + (\partial_x \zeta)^2}}, \frac{\partial_x \zeta}{\sqrt{1 + (\partial_x \zeta)^2}} \right)$$

Here, σ is the coefficient of surface tension at the interface between contacting media, and $\mathbf{n}, \boldsymbol{\tau}$ are the normal and tangent vectors to the interface, respectively. After carrying out the linearization procedure and transferring the boundary conditions to the equilibrium surface $z = 0$ [51], the mathematical formulation in a linear approximation takes the following form:

$$
\begin{gathered}
z<0: \int_z^\zeta \mathrm{e}^{-\frac{\xi}{\Lambda^o}} g \partial_x \widetilde{\rho}^o(x, \xi, t) + \mathrm{e}^{-\frac{\zeta}{\Lambda^o}} g \partial_x \zeta + \partial_t u^o - \nu^o \Delta u^o + \partial_x \widetilde{P}^o = 0 \\
\mathrm{e}^{-\frac{z}{\Lambda^o}} \partial_t w^o - \nu^o \mathrm{e}^{-\frac{z}{\Lambda^o}} \Delta w^o + \frac{\partial_z \widetilde{P}^o}{\rho_{00}^o} = 0 \\
\partial_t \widetilde{\rho}^o - \frac{w^o}{\Lambda^o} + \partial_x u^o + \partial_z w^o = 0 \\
\frac{1}{\rho_{00}^o c^{o2}} \partial_t \widetilde{P}^o - \frac{w^o g}{c^{o2}} + \partial_x u^o + \partial_z w^o = 0
\end{gathered} \tag{51}
$$

$$
\begin{gathered}
z>0: \int_z^\zeta \mathrm{e}^{-\frac{\xi}{\Lambda^a}} g \partial_x \widetilde{\rho}^a(x, \xi, t) + \mathrm{e}^{-\frac{\zeta}{\Lambda^a}} g \partial_x \zeta + \partial_t u^a - \nu^a \Delta u^a + \partial_x \widetilde{P}^a = 0 \\
\mathrm{e}^{-\frac{z}{\Lambda^a}} \partial_t w^a - \nu^a \mathrm{e}^{-\frac{z}{\Lambda^a}} \Delta w^a + \frac{\partial_z \widetilde{P}^a}{\rho_{00}^a} = 0 \\
\partial_t \widetilde{\rho}^a - \frac{w^a}{\Lambda^a} + \partial_x u^a + \partial_z w^a = 0 \\
\frac{1}{\rho_{00}^a c^{a2}} \partial_t \widetilde{P}^a - \frac{w^a g}{c^{a2}} + \partial_x u^a + \partial_z w^a = 0
\end{gathered} \tag{52}
$$

$$
\begin{gathered}
z=0: \partial_t \zeta - w^o = 0,\ \partial_t \zeta - w^a = 0,\ u^o - u^a = 0, \\
\widetilde{P}^o - \widetilde{P}^a + 2\rho_{00}^a \nu^a \partial_z w^a - 2\rho_{00}^o \nu^o \partial_z w^o + \sigma \partial_{xx} \zeta = 0, \\
\rho^o \nu^o (\partial_z u^o + \partial_x w^o) - \rho^a \nu^a (\partial_z u^a + \partial_x w^a) = 0
\end{gathered} \tag{53}
$$

We look for a solution to the system of Equations (51)–(53) in the form of periodic flows $\propto \exp(i\omega t)$:

$$
\begin{pmatrix} u^{o,a} \\ w^{o,a} \\ \widetilde{P}^{o,a} \\ \widetilde{\rho}^{o,a} \\ \zeta \end{pmatrix} = \begin{pmatrix} U_m^{o,a} \\ W_m^{o,a} \\ P_m^{o,a} \\ P_m^{o,a} \\ A_m \end{pmatrix} \exp(i\mathbf{k}^{o,a}\mathbf{r} - i\omega t) = \begin{pmatrix} U_m^{o,a} \exp(ik_z^{o,a} z) \\ W_m^{o,a} \exp(ik_z^{o,a} z) \\ P_m^{o,a} \exp(ik_z^{o,a} z) \\ P_m^{o,a} \exp(ik_z^{o,a} z) \\ A_m \end{pmatrix} \exp(ik_x x - i\omega t) \tag{54}
$$

Substituting the type of solution (54) into the main Equations (51) and (52) leads to a system of algebraic equations connecting the components of wave vectors $k_x,\ k_z^{o,a}$ and the frequency of periodic disturbances ω:

$$
\begin{pmatrix}
-gk_x^2 + \frac{\omega(i+k_z^o\Lambda^o)\left(\nu^o\left(k_x^2+k_z^{o2}\right)-i\omega\right)}{\Lambda^o} & -k_x\left(N^{o2}(i+k_z^o\Lambda^o) + \omega\left(\nu^o\left(k_x^2+k_z^{o2}\right)-i\omega\right)\right) & 0 & 0 \\
0 & \nu^o\left(k_x^2+k_z^{o2}\right) - i\omega & \frac{ik_z^o \mathrm{e}^{\frac{z}{\Lambda^o}}}{\rho_{00}^o} & 0 \\
ik_x & ik_z^o - \frac{1}{\Lambda^o} & 0 & -i\omega \\
ik_x & -\frac{g}{c^{o2}} + ik_z^o & -\frac{i\omega}{\rho_{00}^o c^{o2}} & 0 \\
-gk_x^2 + \frac{\omega(i+k_z^a\Lambda^a)\left(\nu^a\left(k_x^2+k_z^{a2}\right)-i\omega\right)}{\Lambda^a} & -k_x\left(N^{a2}(i+k_z^a\Lambda^a) + \omega\left(\nu^a\left(k_x^2+k_z^{a2}\right)-i\omega\right)\right) & 0 & 0 \\
0 & \nu^a\left(k_x^2+k_z^{a2}\right) - i\omega & \frac{ik_z^a \mathrm{e}^{\frac{z}{\Lambda^a}}}{\rho_{00}^a} & 0 \\
ik_x & ik_z^a - \frac{1}{\Lambda^a} & 0 & -i\omega \\
ik_x & -\frac{g}{c^{a2}} + ik_z^a & -\frac{i\omega}{\rho_{00}^a c^{a2}} & 0
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{55}
$$

The resulting system is divided into two independent systems of equations that describe the relationships between the upper and lower media. The compatibility condition for each of the systems leads to dispersion relations for the lower one:

$$
\begin{gathered}
\frac{\omega}{c^{o2}\Lambda^{o2}} \left[\omega\left(\nu^o\left(k_x^2 + k_z^{o2}\right) - i\omega\right)\left(-gk_x^2\Lambda^o + \omega\left(i + k_z^0\Lambda^o\right)\left(\nu^o\left(k_x^2 + k_z^{o2}\right) - i\omega\right)\right) + \right. \\
+ \mathrm{e}^{\frac{z}{\Lambda^o}} k_z^o \left(\left(g + ic^{o2}k_z^{o2}\right)\left(-gk_x^2\Lambda^o + \omega\left(i + k_z^0\Lambda^o\right)\left(\nu^o\left(k_x^2 + k_z^{o2}\right) - i\omega\right)\right) + \right. \\
\left.\left. + c^{o2}k_x^2\Lambda^o\left(N^{o2}(ik_z^o\Lambda^o - 1) + \omega\left(i\nu\left(k_x^2 + k_z^{o2}\right) + \omega\right)\right)\right)\right] = 0
\end{gathered} \tag{56}
$$

and top liquid:

$$\begin{aligned}\frac{\omega}{c^{a2}\Lambda^{a2}}&\Big[\omega\big(\nu^a\big(k_x^2+k_z^{a2}\big)-i\omega\big)\big(-gk_x^2\Lambda^a+\omega(i+k_z^a\Lambda^a)\big(\nu^a\big(k_x^2+k_z^{a2}\big)-i\omega\big)\big)+\\&+e^{\frac{z}{\Lambda^a}}k_z^a\big(\big(g+ic^{a2}k_z^{a2}\big)\big(-gk_x^2\Lambda^a+\omega(i+k_z^a\Lambda^a)\big(\nu^a\big(k_x^2+k_z^{a2}\big)-i\omega\big)\big)+\\&+c^{a2}k_x^2\Lambda^a\big(N^{a2}(ik_z^a\Lambda^a-1)+\omega\big(i\nu\big(k_x^2+k_z^{a2}\big)+\omega\big)\big)\big)\Big]=0\end{aligned}\tag{57}$$

Let us consider expressions (56) and (57) in a dimensionless form. We choose the natural parameters of each medium as non-dimensional scales: as the time scale, we take the inverse buoyancy frequency $\tau_b = N^{-1}$, and as the spatial scale, we select the viscous wave scale $\delta_N^{g\nu} = (g\nu)^{1/3}N^{-1}$:

$$\begin{aligned}\frac{\omega_*}{\varepsilon^{o2}}&\big[\varepsilon^o\eta^o\omega_*\big(\varepsilon^o\omega_*(k_{z*}^o+i\varepsilon^o)\big(\varepsilon^o k_{z*}^{o2}-i\omega_*\big)+\varepsilon^o k_{x*}^4\big(-1+\varepsilon^{o2}\omega_* k_{z*}^o+i\varepsilon^{o3}\omega_*\big)+\\&+k_{x*}^2\big(2\varepsilon^{o3}\omega_* k_{z*}^{o3}-2i\varepsilon^{o2}\omega_*^2 k_{z*}^o+\omega_*\big(i+2\varepsilon^{o3}\omega_*\big)+k_{z*}^{o2}\big(-\varepsilon^o+2i\varepsilon^{o4}\omega_*\big)\big)\big)+\\&+e^{\frac{z}{\Lambda^o}}k_{z*}^o\big(i\varepsilon^{o4}\omega_* k_{x*}^4+\varepsilon^o\omega_*(k_{z*}^o+i\varepsilon^o)\big(\varepsilon^{o2}k_{z*}^o-i\eta^o\big)\big(i\varepsilon^o k_{z*}^{o2}+\omega_*\big)-\\&-k_{x*}^2\big(\eta^o-2i\varepsilon^{o4}\omega_* k_{z*}^{o2}+\varepsilon^{o5}\omega_* k_{z*}^o-\varepsilon^{o2}\eta^o\omega_* k_{z*}^o+\varepsilon^{o3}\big(1-i\eta^o\omega_*-\omega_*^2\big)\big)\big)\big]=0\end{aligned}\tag{58}$$

$$\begin{aligned}\frac{\omega_*}{\varepsilon^{a2}}&\big[\varepsilon^a\eta^a\omega_*\big(\varepsilon^a\omega_*(k_{z*}^a+i\varepsilon^a)\big(\varepsilon^a k_{z*}^{a2}-i\omega_*\big)+\varepsilon^a k_{x*}^4\big(-1+\varepsilon^{a2}\omega_* k_{z*}^a+i\varepsilon^{a3}\omega_*\big)+\\&+k_{x*}^2\big(2\varepsilon^{a3}\omega_* k_{z*}^{a3}-2i\varepsilon^{a2}\omega_*^2 k_{z*}^a+\omega_*\big(i+2\varepsilon^{a3}\omega_*\big)+k_{z*}^{a2}\big(-\varepsilon^a+2i\varepsilon^{a4}\omega_*\big)\big)\big)+\\&+e^{\frac{z}{\Lambda^a}}k_{z*}^a\big(i\varepsilon^{a4}\omega_* k_{x*}^4+\varepsilon^a\omega_*(k_{z*}^a+i\varepsilon^a)\big(\varepsilon^{a2}k_{z*}^a-i\eta^a\big)\big(i\varepsilon^a k_{z*}^{a2}+\omega_*\big)-\\&-k_{x*}^2\big(\eta^a-2i\varepsilon^{a4}\omega_* k_{z*}^{a2}+\varepsilon^{a5}\omega_* k_{z*}^a-\varepsilon^{a2}\eta^a\omega_* k_{z*}^a+\varepsilon^{a3}\big(1-i\eta^a\omega_*-\omega_*^2\big)\big)\big)\big]=0\end{aligned}\tag{59}$$

$$\varepsilon^a = N^a\sqrt[3]{\frac{\nu^a}{g^2}},\ \varepsilon^o = N^o\sqrt[3]{\frac{\nu^o}{g^2}},\ \eta^a=\frac{N^a\nu^a}{c^{a2}},\ \eta^o=\frac{N^o\nu^o}{c^{o2}}.$$

Expressions (58) and (59) are reduced to the dispersion relations in an incompressible fluid when passing to the limit $c^{o,a}\to\infty\,(\eta^{o,a}\to 0)$:

$$\omega_*\Big(i\varepsilon^o\Big(k_{x*}^2+k_{z*}^{o2}\Big)+\omega_*\Big)\Big(\varepsilon^o\omega_*(k_{z*}^o+i\varepsilon^o)\Big(\varepsilon^o k_{z*}^{o2}-i\omega_*\Big)+k_{x*}^2\Big(-1+\varepsilon^{o2}\omega_* k_{z*}^o+i\varepsilon^{o3}\omega_*\Big)\Big)=0\tag{60}$$

$$\omega_*\Big(i\varepsilon^a\Big(k_{x*}^2+k_{z*}^{a2}\Big)+\omega_*\Big)\Big(\varepsilon^a\omega_*(k_{z*}^a+i\varepsilon^a)\Big(\varepsilon^a k_{z*}^{a2}-i\omega_*\Big)+k_{x*}^2\Big(-1+\varepsilon^{a2}\omega_* k_{z*}^a+i\varepsilon^{a3}\omega_*\Big)\Big)=0\tag{61}$$

The small parameter $\eta^{o,a}$ for liquids with the parameters of water and air turns out to be significantly smaller than the small parameter $\varepsilon^{o,a}$. The approximate solutions of dispersion relations (58) and (59) $k_{*z}^{o,a}$ have the following form:

$$k_{*z}^{o,a}=k_{0*z}^{o,a}+\eta k_{1*z}^{o,a}\tag{62}$$

In solution $k_{0*z}^{o,a}$ (62) takes one of the following values:

$$k_{0*z}^{o,a}=0;\tag{63}$$

$$k_{0*z}^{o,a}=-\frac{i\varepsilon^{o,a}}{4}-\frac{1}{2}\sqrt{-\frac{\varepsilon^{o,a}}{4}-\frac{2\varepsilon^{o,a}k_{x*}^2-i\omega_*}{\varepsilon^{o,a}}+\theta}\pm\frac{1}{2}\sqrt{-\frac{\varepsilon^{o,a2}}{2}-\frac{2\varepsilon^{o,a}k_{x*}^2-i\omega_*}{\varepsilon^{o,a}}-\theta-\frac{i\varepsilon^{o,a3}-8i(\varepsilon^{o,a}k_{x*}^2-i\omega_*)+4i(2\varepsilon^{o,a}k_{x*}^2-i\omega_*)}{4\sqrt{-\frac{\varepsilon^{o,a2}}{4}+\theta-\frac{2\varepsilon^{o,a}k_{x*}^2-i\omega_*}{\varepsilon^{o,a}}}}}\tag{64}$$

$$k_{0*z}^{o,a}=-\frac{i\varepsilon^{o,a}}{4}+\frac{1}{2}\sqrt{-\frac{\varepsilon^{o,a}}{4}-\frac{2\varepsilon^{o,a}k_{x*}^2-i\omega_*}{\varepsilon^{o,a}}+\theta}\pm\frac{1}{2}\sqrt{-\frac{\varepsilon^{o,a2}}{2}-\frac{2\varepsilon^{o,a}k_{x*}^2-i\omega_*}{\varepsilon^{o,a}}-\theta+\frac{i\varepsilon^{o,a3}-8i(\varepsilon^{o,a}k_{x*}^2-i\omega_*)+4i(2\varepsilon^{o,a}k_{x*}^2-i\omega_*)}{4\sqrt{-\frac{\varepsilon^{o,a2}}{4}-\frac{2\varepsilon^{o,a}k_{x*}^2-i\omega_*}{\varepsilon^{o,a}}+\theta}}};\tag{65}$$

$$\theta=\frac{\big(i+\sqrt{3}\big)\Big(\alpha+\sqrt{\alpha^2-4\beta^3}\Big)^{1/3}}{6\cdot 2^{1/3}\varepsilon^{o,a}\omega_*}+\frac{2\varepsilon^{o,a}k_{x*}^2-i\omega_*}{3\varepsilon^{o,a}}+\frac{\big(i-\sqrt{3}\big)2^{1/3}\beta}{3\varepsilon^{o,a}\Big(\alpha+\sqrt{\alpha^2-4\beta^3}\Big)^{1/3}}$$

$$\beta=\Big(-16\varepsilon^{o,a2}k_{x*}^4\omega_*+\omega_*^2(3i\varepsilon^{o,a}+\omega_*)+k_{x*}^2\Big(-3\varepsilon^{o,a4}\omega_*+4i\varepsilon^{o,a}\Big(-3+4\omega_*^2\Big)\Big)\Big)$$

$$\alpha = \omega_*^2\Big(128i\varepsilon^{o,a3}\omega_* k_{x*}^6 + 2\omega_*^3\Big(-9i\varepsilon^{o,a3}+\omega_*\Big) + 12\varepsilon^{o,a2}k_{x*}^4\Big(-12+3i\varepsilon^{o,a3}\omega_* + 16\omega_*^2\Big) + 3k_{x*}^2\Big(9\varepsilon^{o,a4}\Big(-1+2\omega_*^2\Big) - 4i\varepsilon^{o,a}\omega_*\Big(-6+5\omega_*^2\Big)\Big)\Big)$$

and $k_{1*z}^{o,a}$ takes the corresponding (63)–(65) values:

$$k_{1*z}^{o,a} = \frac{\Big(k_{0z*}^{o,a} + \varepsilon^{o,a}\mathrm{e}^{-\frac{z}{\Lambda^{o,a}}}\Big(\varepsilon^{o,a}\omega_*\Big(k_{x*}^2 + k_{0z*}^{o,a2} - i\omega_*\Big)\Big)\Big)\Big(\varepsilon^{o,a}\omega_*\big(k_{0z*}^{o,a}+i\varepsilon^{o,a}\big)\Big(\varepsilon^{o,a}k_{0z*}^{o,a2} - i\omega_*\Big) + k_{x*}^2\big(-1+\varepsilon^{o,a2}k_{0z*}^{o,a}\omega_* + i\varepsilon^{o,a3}\omega_*\big)\Big)}{\varepsilon^{o,a3}\Big(-i\varepsilon^{o,a}\omega_* k_{x*}^4 + \omega_* k_{0z*}^{o,a}\Big(-5i\varepsilon^{o,a}k_{0z*}^{o,a3} + 4\varepsilon^{o,a2}k_{0z*}^2 - 3\omega_* k_{0z*}^{o,a} - 2i\varepsilon^{o,a}\omega_*\Big) + k_{x*}^2\Big(1 - 6i\varepsilon^{o,a}\omega_* k_{0z*}^{o,a2} + 2\varepsilon^{o,a2}\omega_* k_{0z*}^{o,a} - \omega_*^2\Big)\Big)} \tag{66}$$

Additional conditions for physical implementation are imposed on solutions (63)–(66):

$$\mathrm{Im}(k_{*x}) > 0,\ \mathrm{Im}(k_{*z}^o) < 0,\ \mathrm{Im}(k_{*z}^a) > 0 \tag{67}$$

Taking (67) into account, solution (63) turns out to be physically unrealizable in both media. Solution (64) describes a regular solution with respect to a small parameter $\varepsilon^{o,a}$ and the corresponding wave component of a periodic flow. Solution (65) describes a singular solution with respect to a small parameter $\varepsilon^{o,a}$ and corresponds to the ligament component of the periodic flow. To distinguish the roots, we introduce a redesignation for singular solutions $k_{*l}^{o,a}$. Mathematically, the solutions corresponding to the wave component are determined by the following condition:

$$\left|\mathrm{Re}(k_{*z}^{o,a})\right| \gg \left|\mathrm{Im}(k_{*z}^{o,a})\right| \tag{68}$$

and the solutions corresponding to the ligament component are determined by the following mathematical condition:

$$\left|\mathrm{Re}\big(k_{*l}^{o,a}\big)\right| \sim \left|\mathrm{Im}\big(k_{*l}^{o,a}\big)\right| \tag{69}$$

Taking into account the ligament components, the form of the complete solution (54) is rewritten as follows:

$$\begin{pmatrix} u^{o,a} \\ w^{o,a} \\ \widetilde{P}^{o,a} \\ \widetilde{\rho}^{o,a} \\ \zeta \end{pmatrix} = \begin{pmatrix} U_m^{o,a}\big(\exp(ik_z^{o,a}z) + \Theta\exp\big(ik_l^{o,a}z\big)\big) \\ W_m^{o,a}\big(\exp(ik_z^{o,a}z) + \Theta\exp\big(ik_l^{o,a}z\big)\big) \\ P_m^{o,a}\big(\exp(ik_z^{o,a}z) + \Theta\exp\big(ik_l^{o,a}z\big)\big) \\ P_m^{o,a}\big(\exp(ik_z^{o,a}z) + \Theta\exp\big(ik_l^{o,a}z\big)\big) \\ A_m \end{pmatrix} \exp(ik_x x - i\omega t) \tag{70}$$

Substituting the form of solution (70) for the boundary conditions (53), we obtain the dispersion relations connecting the components of the wave vector k_x with the frequency of wave motion ω. Substituting the approximate solutions (64), (66), (65) and (66) into the resulting relation, we obtain a dispersion equation. Restrictions (67) are imposed on the solution, and thus physically realizable roots are selected. The resulting expressions are cumbersome and difficult to analyze. Let us consider some limiting cases.

We consider the behavior of oscillations far from the interface between the media. In this case, we assume that $|z| \gg 1$ for the lower liquid and for the upper liquid. Thus, for the ocean in the dispersion relation (56), we can neglect the second term and use the following:

$$\frac{\omega^2\big(\nu^o\big(k_x^2+k_z^{o2}\big)-i\omega\big)\big(-gk_x^2\Lambda^o + \omega\big(i+k_z^0\Lambda^o\big)\big(\nu^o\big(k_x^2+k_z^{o2}\big)-i\omega\big)\big)}{c^{o2}\Lambda^{o2}} = 0 \tag{71}$$

or in a dimensionless form:

$$\omega_*^2\eta^o\Big(\omega_*\big(k_{z*}^o + i\varepsilon^o\big)\Big(\varepsilon^o k_{z*}^{o2} - i\omega_*\Big) + k_{x*}^4\Big(-1+\varepsilon^{o2}\omega_* k_{z*}^o + i\varepsilon^{o3}\omega_*\Big) = 0 \tag{72}$$

The solutions to expression (71) (or (72)) are found exactly, but due to their cumbersomeness, they are not given here.

For the atmosphere in the dispersion relation (57), based on similar reasoning, we neglect the first term and obtain the dispersion relation, which is far from the interface:

$$\frac{\omega k_z^a\big(\big(g+ic^{a2}k_z^{a2}\big)\big(-gk_x^2\Lambda^a + \omega\big(i+k_z^a\Lambda^a\big)\big(\nu^a\big(k_x^2+k_z^{a2}\big)-i\omega\big)\big) + c^{a2}k_x^2\Lambda^a\big(N^{a2}(ik_z^a\Lambda^a-1) + \omega\big(i\nu\big(k_x^2+k_z^{a2}\big)+\omega\big)\big)\big)}{c^{a2}\Lambda^{a2}} = 0 \tag{73}$$

or in a dimensionless form:

$$\begin{gathered}\frac{\omega_*^2}{\varepsilon^a}\eta^a\left(\varepsilon^a\omega_*(k_{z*}^a+i\varepsilon^a)\left(\varepsilon^a k_{z*}^{a2}-i\omega_*\right)+\varepsilon^a k_{x*}^4\left(-1+\varepsilon^{a2}\omega_* k_{z*}^a+i\varepsilon^{a3}\omega_*\right)+\right.\\ \left.+k_{x*}^2\left(2\varepsilon^{a3}\omega_* k_{z*}^{a3}-2i\varepsilon^{a2}\omega_*^2 k_{z*}^a+\omega_*\left(i+2\varepsilon^{a3}\omega_*\right)+k_{z*}^{a2}\left(-\varepsilon^a+2i\varepsilon^{a4}\omega_*\right)\right)\right)=0\end{gathered} \tag{74}$$

The solutions of expression (73) (or (74)) are not given here due to their cumbersomeness. For waves near the surface, we can assume that the dispersion relations (56) and (57) are simplified:

$$\begin{gathered}\frac{\omega}{c^{o2}\Lambda^{o2}}\left[\omega\left(\nu^o\left(k_x^2+k_z^{o2}\right)-i\omega\right)\left(-gk_x^2\Lambda^o+\omega\left(i+k_z^0\Lambda^o\right)\left(\nu^o\left(k_x^2+k_z^{o2}\right)-i\omega\right)\right)+\right.\\ +k_z^o\left(\left(g+ic^{o2}k_z^{o2}\right)\left(-gk_x^2\Lambda^o+\omega\left(i+k_z^0\Lambda^o\right)\left(\nu^o\left(k_x^2+k_z^{o2}\right)-i\omega\right)\right)+\right.\\ \left.\left.+\,c^{o2}k_x^2\Lambda^o\left(N^{o2}(ik_z^o\Lambda^o-1)+\omega\left(i\nu\left(k_x^2+k_z^{o2}\right)+\omega\right)\right)\right)\right]=0\end{gathered} \tag{75}$$

$$\begin{gathered}\frac{\omega}{c^{a2}\Lambda^{a2}}\left[\omega\left(\nu^a\left(k_x^2+k_z^{a2}\right)-i\omega\right)\left(-gk_x^2\Lambda^a+\omega(i+k_z^a\Lambda^a)\left(\nu^a\left(k_x^2+k_z^{a2}\right)-i\omega\right)\right)+\right.\\ +k_z^a\left(\left(g+ic^{a2}k_z^{a2}\right)\left(-gk_x^2\Lambda^a+\omega(i+k_z^a\Lambda^a)\left(\nu^a\left(k_x^2+k_z^{a2}\right)-i\omega\right)\right)+\right.\\ \left.\left.+\,c^{a2}k_x^2\Lambda^a\left(N^{a2}(ik_z^a\Lambda^a-1)+\omega\left(i\nu\left(k_x^2+k_z^{a2}\right)+\omega\right)\right)\right)\right]=0\end{gathered} \tag{76}$$

or in a dimensionless form:

$$\begin{gathered}\frac{\omega_*}{\varepsilon^{o2}}\left[\varepsilon^o\eta^o\omega_*\left(\varepsilon^o\omega_*(k_{z*}^o+i\varepsilon^o)\left(\varepsilon^o k_{z*}^{o2}-i\omega_*\right)+\varepsilon^o k_{x*}^4\left(-1+\varepsilon^{o2}\omega_* k_{z*}^o+i\varepsilon^{o3}\omega_*\right)+\right.\right.\\ \left.+k_{x*}^2\left(2\varepsilon^{o3}\omega_* k_{z*}^{o3}-2i\varepsilon^{o2}\omega_*^2 k_{z*}^o+\omega_*\left(i+2\varepsilon^{o3}\omega_*\right)+k_{z*}^{o2}\left(-\varepsilon^o+2i\varepsilon^{o4}\omega_*\right)\right)\right)+\\ +k_{z*}^o\left(i\varepsilon^{o4}\omega_* k_{x*}^4+\varepsilon^o\omega_*(k_{z*}^o+i\varepsilon^o)\left(\varepsilon^{o2}k_{z*}^o-i\eta^o\right)\left(i\varepsilon^o k_{z*}^{o2}+\omega_*\right)-\right.\\ \left.\left.-\,k_{x*}^2\left(\eta^o-2i\varepsilon^{o4}\omega_* k_{z*}^{o2}+\varepsilon^{o5}\omega_* k_{z*}^o-\varepsilon^{o2}\eta^o\omega_* k_{z*}^o+\varepsilon^{o3}\left(1-i\eta^o\omega_*-\omega_*^2\right)\right)\right)\right]=0\end{gathered} \tag{77}$$

$$\begin{gathered}\frac{\omega_*}{\varepsilon^{a2}}\left[\varepsilon^a\eta^a\omega_*\left(\varepsilon^a\omega_*(k_{z*}^a+i\varepsilon^a)\left(\varepsilon^a k_{z*}^{a2}-i\omega_*\right)+\varepsilon^a k_{x*}^4\left(-1+\varepsilon^{a2}\omega_* k_{z*}^a+i\varepsilon^{a3}\omega_*\right)+\right.\right.\\ \left.+k_{x*}^2\left(2\varepsilon^{a3}\omega_* k_{z*}^{a3}-2i\varepsilon^{a2}\omega_*^2 k_{z*}^a+\omega_*\left(i+2\varepsilon^{a3}\omega_*\right)+k_{z*}^{a2}\left(-\varepsilon^a+2i\varepsilon^{a4}\omega_*\right)\right)\right)+\\ +k_{z*}^a\left(i\varepsilon^{a4}\omega_* k_{x*}^4+\varepsilon^a\omega_*(k_{z*}^a+i\varepsilon^a)\left(\varepsilon^{a2}k_{z*}^a-i\eta^a\right)\left(i\varepsilon^a k_{z*}^{a2}+\omega_*\right)-\right.\\ \left.\left.-\,k_{x*}^2\left(\eta^a-2i\varepsilon^{a4}\omega_* k_{z*}^{a2}+\varepsilon^{a5}\omega_* k_{z*}^a-\varepsilon^{a2}\eta^a\omega_* k_{z*}^a+\varepsilon^{a3}\left(1-i\eta^a\omega_*-\omega_*^2\right)\right)\right)\right]=0\end{gathered} \tag{78}$$

Nevertheless, despite their simpler appearance, the roots of expressions (77) and (78), as well as complete expressions, can only be found asymptotically or numerically.

7. Discussion

The expressive properties of periodic flows in fluids—the regularity of wave displacements of the liquid-free surface, the high speed of sound vibrations propagation and the clarity of the pattern of periodic internal waves beams—formed the basis for the generally accepted classification of waves and predetermined the rules for constructing mathematical models of the phenomenon. To describe each wave process in a linear [1,2,16,21] or nonlinear approximation [41], its own system of equations was developed based on the system of fundamental equations of mechanics of fluids and gases [1,2,4,16], and general physical considerations [31,52].

Under natural conditions, sharp disturbances lead to the formation of several types of waves, which propagate with their own phase and group velocities and differ in attenuation laws. The parameters of wave processes—periods, wavelength, group and phase propagation velocities—are described by real numbers. The mathematical description of periodic flows is carried out in the algebra of complex numbers. The use of wave representations by exponential functions of complex frequency and complex wave vector allows us to construct the dispersion relations [1,2] and evaluate the stability of the flows under study [32,33].

Taking into account the special physical properties of the wave frequency—the measure of the energy of periodic motion—in this work, as in [38,40,42,46], the wave frequency ω is assumed to be real, and the wave number $\mathbf{k}$ is taken to be complex. In this approximation, the degree of the dispersion relation corresponds to the order of the system of differential equations. The solutions of the system of governing equations, constructed using methods of singular perturbation theory and by taking into account the type of small parameter of the process under study, contain two types of solutions. The real part of some wave numbers is large, and the imaginary part is small. The other types have real and imaginary parts of the same order. Accordingly, some of the solutions, including solutions with small values of the imaginary part of the wave vectors, contain functions that are regular in the small parameter and describe waves. For each type of wave, its own dispersion equation is constructed.

Another part of the solutions with large values of the wave vector imaginary parts determines the ligaments, which correspond to thin high-gradient fibers and interfaces in the thickness of a stratified liquid [30,38]. From the given analysis, it follows that specific ligaments accompany all types of waves—surface, internal and acoustic ones. The consideration of the ligaments' influence made it possible to pre-calculate the parameters of reflected and leaking waves, which occur when the reflecting beams of the internal waves of the critical level separate the medium with a high buoyancy frequency from a low-frequency layer not exceeding the wave frequency [5]. It is consistent with the data of later experiments [53].

From the theoretical point of view, the number of ligaments accompanying the wave is determined by the completeness degree taking into account the factors influencing the density and the dimension of the problem space. The minimum number—two ligaments—accompany two-dimensional waves in a medium with one dissipative parameter (kinematic viscosity). Their thickness is determined by the scale of the periodic Stokes flow $\delta_\omega^\nu = \sqrt{\nu/\omega}$ [13]. Considering the three-dimensionality of space, the effects of thermal diffusivity and diffusion lead to an increase in the number of ligaments with different properties [40]. The effects of nonlinear interaction between ligaments can increase the mutual influence of waves of different types [42].

The developed methodology for constructing complete solutions makes it possible to describe not only the wave component of a periodic flow but also the fine structure, manifested in the form of ligaments—thin jets accompanying the wave motion. The parameters of the observed phenomena in the process of propagation of periodic disturbances in liquids and gases, which are determined using the properties of the medium, define the requirements for the experimental methodology and the resolution (spatial and temporal) of the equipment for observing the complete picture of flows.

8. Conclusions

For the first time in a unified formulation, the propagation of infinitesimal periodic disturbances in the thickness and on the surface of a viscous compressible exponentially stratified fluid has been studied based on a system of fundamental equations. The analysis of linearized equations has been carried out using the methods of singular perturbation theory, taking into account the compatibility condition. The dispersion relations for periodic flows with a real positive definite frequency and complex wave number are calculated and analyzed. Complete solutions of the dispersion relations containing regular and singular roots are found. Regular roots, which determine the wave components of periodic flows, are regularly reduced to known dispersion relations for waves in a homogeneous viscous or ideal fluid. Singular roots define the ligament components of periodic flows. Ligaments describe the fine structure of periodic flows and characterize thin high-gradient jets and interfaces.

The general properties of solutions are that acoustic or internal waves propagating in the thickness, as well as gravitational waves at the interface of infinitely deep media, are accompanied by ligaments forming a fine structure of the medium. In extreme cases, the obtained relationships transform into known expressions for waves a viscous incompressible and an ideal homogeneous fluid.

The further application of the obtained expressions in studying the physical properties of periodic flows in configuration space and in comparison with experimental data using high-resolution instruments is of scientific and practical interest.

Author Contributions: Conceptualization, Y.D.C. and A.A.O.; methodology, Y.D.C. and A.A.O.; validation, Y.D.C. and A.A.O.; formal analysis, Y.D.C.; investigation, Y.D.C.; resources, Y.D.C.; data curation, Y.D.C.; writing—original draft preparation, Y.D.C. and A.A.O.; writing—review and editing, Y.D.C.; visualization, A.A.O.; supervision, Y.D.C.; project administration, Y.D.C.; funding acquisition, Y.D.C. All authors have read and agreed to the published version of the manuscript.

Funding: The work was supported by the Russian Science Foundation (project 19-19-00598-P "Hydrodynamics and energetics of drops and droplet jets: formation, motion, break-up, interaction with the contact surface" https://rscf.ru/en/project/19-19-00598/ (accessed on 23 October 2023)).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Landau, L.D.; Lifshitz, E.M. *Fluid Mechanics*; Course of Theoretical Physics; Pergamon Press: Oxford, UK, 1987; Volume 6, 560p.
2. Müller, P. *The Equations of Oceanic Motions*; CUP: Cambridge, UK, 2006; 302p.
3. NASA. *US Standard Atmosphere 1976—NOAA-S/T-76-1562. NASA-TM-X-74335*; Accession Number 77N16482; NASA: Washington, DC, USA. Available online: https://ntrs.nasa.gov/citations/19770009539 (accessed on 23 October 2023).
4. Matveev, L.T. *Fizika Atmosfery*; Gidrometeoizdat: St. Petersburg, Russia, 2000; 777p. (In Russian)
5. Fedorov, K.N. *The Thermohaline Finestructure of the Ocean*; Pergamon Marine Series; Elsevier: Amsterdam, The Netherlands, 2013; 180p.
6. Franklin, B. Behavior of oil on water. Letter to J. Pringle. In *Experiments and Observations on Electricity*; R. Cole: London, UK, 1769; pp. 142–144.
7. Stokes, G.G. On the theory of oscillatory waves. In *Mathematical and Physical Papers (Cambridge Library Collection—Mathematics)*; Cambridge University Press: Cambridge, UK, 2010; pp. 197–229. [CrossRef]
8. Rayleigh, L. Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density. *Proc. Lond. Math. Soc.* **1882**, *s114*, 170–177. [CrossRef]
9. Lamb, H. On atmospheric oscillations. *Proc. Roy. Soc.* **1911**, *84*, 551–574. [CrossRef]
10. Ekman, V.W. On Dead Water. In *The Norwegian North Polar Expedition 1893–1896. Scientific Results*; Nansen, F., Ed.; Yakov Dyewad: Christiania, Norway, 1906; Volume 15, 152p.
11. Lamb, H. *A Treatise on the Mathematical Theory of the Motion of Fluids*; Cambridge University Press: Cambridge, UK, 1879; 258p.
12. Stokes, G.G. On the theories of the internal friction of fluids in motion and of the equilibrium and motion of elastic solids. In *Mathematical and Physical Papers (Cambridge Library Collection—Mathematics)*; Cambridge University Press: Cambridge, UK, 2010; pp. 75–129. [CrossRef]
13. Stokes, G.G. On the Effect of Internal Friction of Fluids on the Motion of Pendulums. In *Mathematical and Physical Papers (Cambridge Library Collection—Mathematics)*; Cambridge University Press: Cambridge, UK, 2010; pp. 1–10. [CrossRef]
14. Rayleigh, J.W.S. *Theory of Sound*, 2nd ed.; Dover: New York, NY, USA, 1945.
15. Kochin, N.E.; Kibel, I.A.; Roze, N.V. *Theoretical Hydromechanics*; John Wiley & Sons Ltd.: Chichester, NH, USA, 1964; 560p.
16. Vallis, G.K. *Atmospheric and Oceanic Fluid Dynamics*; CUP: Cambridge, UK, 2017; 745p.
17. Prandtl, L. *Führer Durch die Strömungslehre*; Vieweg, Vieweg und Sohn: Braunschweig, Germany, 1942.
18. Phillips, O. On flows induced by diffusion in a stably stratified fluid. *Deep Sea Res. Oceanogr. Abstr.* **1970**, *17*, 435–443. [CrossRef]
19. Wunsch, C. On oceanic boundary mixing. *Deep Sea Res. Oceanogr. Abstr.* **1970**, *17*, 293–301. [CrossRef]
20. Turner, J.S. *Buoyancy Effects in Fluids*; Cambridge Monographs on Mechanics; Cambridge University Press: Cambridge, UK, 1980; 412p.
21. Lighthill, J.M. *Waves in Fluids*; Cambridge Mathematical Library; Cambridge University Press: Cambridge, UK, 1978; 594p.
22. Tolstoy, I.; Clay, C.S. *Ocean Acoustics: Theory and Experiment in Underwater Sound*; McGraw-Hill: New York, NY, USA, 1966; 293p.
23. Longuet-Higgins, M.S. Mass transport in water waves. *Phil. Trans. R. Soc. London Ser. A Math. Phys. Sci.* **1953**, *245*, 535–581. [CrossRef]
24. Longuet-Higgins, M.S. Mass transport in the boundary layer at a free oscillating surface. *J. Fluid Mech.* **1960**, *8*, 293–306. [CrossRef]
25. Liu, A.; Davis, S. Viscous attenuation of mean drift in water waves. *J. Fluid Mech.* **1977**, *81*, 63–84. [CrossRef]
26. Robertson, S.; Rousseaux, G. Viscous dissipation of surface waves and its relevance to analogue gravity experiments. In *Fluid Dynamics*; Cornell University Press: New York, NY, USA, 2018. [CrossRef]
27. Dore, B. Mass transport in layered fluid systems. *J. Fluid Mech.* **1970**, *40*, 113–126. [CrossRef]
28. Zhang, W.; Jin, H. Nonlinear Stability of the Monotone Traveling Wave for the Isothermal Fluid Equations with Viscous and Capillary Terms. *Mathematics* **2023**, *11*, 1734. [CrossRef]
29. Pei, F.; Wu, G.; Guo, Y. Construction of Infinite Series Exact Solitary Wave Solution of the KPI Equation via an Auxiliary Equation Method. *Mathematics* **2023**, *11*, 1560. [CrossRef]
30. Kistovich, Y.V.; Chashechkin, Y.D. Linear theory of beams of internal wave propagation an arbitrarily stratified liquid. *J. Appl. Mech. Tech. Phys.* **1998**, *39*, 302–309. [CrossRef]

31. Krasil'nikov, V.A.; Krylov, V.V. *Vvedeniye v Fizicheskuyu Akustiku*; Nauka: Moskva, Russia, 1984; 400p. (In Russian)
32. Lin, C.C. *The Theory of Hydrodynamic Stability*; Cambridge University Press: Cambridge, UK, 1955; 155p.
33. Chandrasekhar, S. *Hydrodynamic and Hydromagnetic Stability, International Series of Monographs on Physics*; Clarendon Press: Oxford, UK, 1961; 654p.
34. Darrigol, O. *Worlds of Flow: A History of Hydrodynamics from the Bernoullis to Prandt*; Oxford University Press: Oxford, UK, 2005; 356p.
35. Suvorov, V.G.; Zubarev, N.M. Formation of the Taylor cone on the surface of liquid metal in the presence of an electric field. *J. Phys. D Appl. Phys.* **2003**, *37*, 289. [CrossRef]
36. Zubarev, N.M. Exact solutions of the equations of motion of liquid helium with a charged free surface. *J. Exp. Theor. Phys.* **2002**, *94*, 534–544. [CrossRef]
37. Zeytounian, R.K. The Benard–Marangoni thermocapillary-instability problem. *Phys. Uspekhi* **1998**, *41*, 241. [CrossRef]
38. Chashechkin, Y.D. Foundations of engineering mathematics applied for fluid flows. *Axioms* **2021**, *10*, 286. [CrossRef]
39. Nayfeh, A.H. *Introduction to Perturbation Technique*; John Wiley & Sons: New York, NY, USA, 1993; 536p.
40. Chashechkin, Y.D. Singularly perturbed components of flows—Linear precursors of shock waves. *Math. Model. Nat. Phenom.* **2018**, *13*, 17. [CrossRef]
41. Whitham, G.B. *Linear and Nonlinear Waves*; Wiley Interscience: New York, NY, USA, 1999; 660p.
42. Chashechkin, Y.D. Conventional partial and new complete solutions of the fundamental equations of fluid mechanics in the problem of periodic internal waves with accompanying ligaments generation. *Mathematics* **2021**, *9*, 586. [CrossRef]
43. Thomson, W. Hydrokinetic solutions and observations. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **1871**, *42*, 362–377. [CrossRef]
44. Shishlenin, M.; Savchenko, N.; Novikov, N.; Klyuchinskiy, D. Modeling of 2D Acoustic Radiation Patterns as a Control Problem. *Mathematics* **2022**, *10*, 1116. [CrossRef]
45. Fellah, Z.E.A.; Fellah, M.; Ongwen, N.O.; Ogam, E.; Depollier, C. Acoustics of Fractal Porous Material and Fractional Calculus. *Mathematics* **2021**, *9*, 1774. [CrossRef]
46. Chashechkin, Y.D.; Ochirov, A.A. Periodic waves and ligaments on the surface of a viscous exponentially stratified fluid in a uniform gravity field. *Axioms* **2022**, *11*, 402. [CrossRef]
47. Soret, C. Sur l'etat d'équilibre que prend au point de vue de sa concentration une dissolution saline primitivement homogène dont deux parties sont portées a des températures difféntes. *Arch. Sci. Phys. Nat.* **1879**, *2*, 48.
48. Mortimer, R.G.; Eyring, H. Elementary transition state theory of the Soret and Dufour effects. *Proc. Natl. Acad. Sci. USA* **1980**, *77*, 1728–1731. [CrossRef] [PubMed]
49. Dufour, L. The Diffusion Thermoeffect. *Arch. Sci. Phys. Nat.* **1872**, *45*, 9–12.
50. Kistovich, A.V.; Chashechkin, Y.D. Regular and singular components of periodic flows in the fluid interior. *J. Appl. Math. Mech.* **2007**, *71*, 762. [CrossRef]
51. Joseph, D.D. Domain perturbations: The higher order theory of infinitesimal water waves. *Arch. Ration. Mech. Anal.* **1973**, *51*, 295. [CrossRef]
52. Rudenko, O.V. Giant nonlinearities in structurally inhomogeneous media and the fundamentals of nonlinear acoustic diagnostic techniques. *Phys. Uspekhi* **2006**, *176*, 1011–1036. [CrossRef]
53. Paoletti, M.S.; Swinney, H.L.; Paoletti, M.S.; Swinney, H.L. Propagating and evanescent internal waves in a deep ocean model. *J. Fluid Mech.* **2012**, *706*, 571–583. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.