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A Note on the Geometry of Certain Classes of Lichnerowicz Laplacians and Their Applications

Vladimir Rovenski [1], Sergey Stepanov [2] and Josef Mikeš [3,*]

[1] Department of Mathematics, University of Haifa, Mount Carmel, Haifa 31905, Israel; vrovenski@univ.haifa.ac.il
[2] Department of Mathematics, Finance University, Leningradsky Prospect, 49–55, 125468 Moscow, Russia; s.e.stepanov@mail.ru
[3] Department of Algebra and Geometry, Faculty of Science, Palacky University, 771 46 Olomouc, Czech Republic
* Correspondence: josef.mikes@upol.cz

Abstract: In the present paper, we prove vanishing theorems for the null space of the Lichnerowicz Laplacian acting on symmetric two tensors on complete and closed Riemannian manifolds and further estimate its lowest eigenvalue on closed Riemannian manifolds. In addition, we give an application of the obtained results to the theory of infinitesimal Einstein deformations.

Keywords: Lichnerowicz Laplacian; symmetric two-tensors; vanishing theorem; eigenvalue; infinitesimal Einstein deformations

MSC: 53B05; 35R01

1. Introduction

Let (M, g) be an n-dimensional $(n \geq 2)$ connected and complete Riemannian manifold with the Levi–Civita connection ∇. The metric g and connection of (M, g) naturally extend to tensor products over M, and the extensions are denoted by the same symbols. That is, if E is an arbitrary tensor bundle over (M, g), then it is equipped with an inner product, denoted by g_x in the fiber E_x for each $x \in M$ and a compatible connection, denoted by ∇.

The Lichnerowicz Laplacian Δ_L: $C^\infty(E) \to C^\infty(E)$ acting on C^∞-sections of E differs from the usual Laplacian Δ_0: $C^\infty(M) \to C^\infty(M)$ defined on C^∞-functions of M via the Weitzenböck decomposition formula $\Delta = \nabla^* \circ \nabla + \mathfrak{R}$, where ∇^* is the formal-adjoint operator of ∇ and $\mathfrak{R}$ is the Weitzenböck curvature operator linearly dependent on the Riemann curvature tensor R and Ricci tensor Ric of (M, g) (see [1], pp. 27–28).

On the other hand, in modern Riemannian geometry, there is one of the most important methods, the Bochner technique (see [2]; [3] (pp. 333–364); [4,5], etc.). In general, the Bochner technique is well known as a method of proving vanishing theorems for null space of a Laplace operator admitting a Weitzenbock decomposition and further estimating its lowest eigenvalue (see [6], p. 53). The Bochner technique will be the main research method of the certain classes of Lichnerowicz Laplacians in our paper.

2. On the Kernel and Estimate for the Eigenvalues of the Lichnerowicz Laplacian Acting on Symmetric Two-Tensors

1. In what follows, we denote by $S^qM = Sim(\otimes^q T^*M)$ the point-wise natural projection Sim: $\otimes^q T^*M \to S^qM$ called *symmetrization*. Throughout this paper, we consider the vector spaces of C^∞-sections of S^qM denoted by $C^\infty(S^qM)$. The Riemannian metric g induces a metric on the fibers of each of these spaces. If (M, g) is a compact (without

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boundary) connected Riemannian manifold, then all these spaces are also endowed with the *global scalar product* $\langle \cdot, \cdot \rangle$. In particular, the formula

$$\langle \varphi, \varphi' \rangle = \int_M \frac{1}{p!}\, g(\varphi, \varphi')\, dv_g$$

where $\varphi, \varphi' \in C^\infty(S^pM)$ and dv_g is the volume element of (M, g), determines the *global scalar product* or, in other words, L^2*-scalar product on* $C^\infty(S^pM)$.

Next, if D is a differential operator between some tensor bundles over M, its formal adjoint D^* is uniquely defined by the formula $\langle D, \cdot \rangle = \langle \cdot, D^* \rangle$ (see [6], p. 460). For example, the covariant derivative ∇: $C^\infty(\otimes^q T^*M) \to C^\infty(\otimes^{q+1} T^*M)$ has the formal adjoint operator ∇^* such that ∇^*: $C^\infty(\otimes^{q+1} T^*M) \to C^\infty(\otimes^q T^*M)$ (see [6], p. 54).

Let δ^*: $C^\infty(S^qM) \to C^\infty(S^{q+1}M)$ denote the symmetrized covariant operator on symmetric q-tensor fields (see [6], p. 356). Then, there exists its formal adjoint operator δ: $C^\infty(S^{q+1}M) \to C^\infty(S^qM)$ with respect to the L^2-scalar product, which is called the *divergence operator* (see [6], p. 356). Notice that δ is nothing but the $\otimes^{q+1} T^*M$ restriction of ∇^* to $S^{q+1}M$. Using the operators δ^* and δ, Sampson defined in ([7], p. 147) the second-order differential operator Δ_S: $C^\infty(S^qM) \to C^\infty(S^qM)$ via the formula $\Delta_S = \delta \circ \delta^* - \delta^* \circ \delta$ (see also [6], p. 356). We have proved that the Sampson operator Δ_S is the *Laplace operator* and its kernel is a finite-dimensional vector space on a closed manifold (M, g) (see [8]). Based on the information above, we called Δ_S the *Sampson Laplacian* in articles [8–10]. It has natural applications in the study of Ricci flow (see [11]) and in other geometrical problems (see [8–10]).

Let us compare the Sampson Laplacian Δ_S with the *Bochner Laplacian* $\bar{\Delta} = \nabla^* \circ \nabla$, see ([6], p. 54). First, it is easy to see that these two operators coincide if (M, g) is a locally Euclidean space. Second, the operator $\Delta_S - \bar{\Delta}$ has the order zero and can be defined using symmetric endomorphisms of the bundle S^pM. This means that we have the Weitzenböck decomposition formula $\Delta_S = \bar{\Delta} - \mathfrak{R}_q$, where $\mathfrak{R}_q$: $S^qM \to S^qM$ is an algebraic symmetric operator that depends linearly in a known way on the curvature tensor R and the Ricci tensor Ric of the metric g (see [8,9] and [6], p. 356). At the same time, we deduce from $\Delta_S = \bar{\Delta} - \mathfrak{R}_q$ the following equation (see also [12]):

$$\Delta_L\, \varphi = \Delta_S\, \varphi + 2\, \mathfrak{R}_q(\varphi) = \bar{\Delta}\varphi + \mathfrak{R}_q(\varphi)$$

of another Laplacian. This Laplacian was defined by Lichnerowicz Δ_L: $C^\infty(\otimes^q T^*M) \to C^\infty(\otimes^q T^*M)$ in his famous article ([1], pp. 315–316).

Consider a covering $\{U, x^1, \ldots, x^n\}$ of (M, g) with a system of coordinate neighborhoods, where U denotes a neighborhood and $x^1, \ldots, x^n$ denote local coordinates in U. Then, we can define the natural frame $\{X_1 = \partial/\partial x^1, \ldots, X_n = \partial/\partial x^n\}$ in an arbitrary coordinate neighborhood $\{U, x^1, \ldots, x^n\}$. In this case, $g_{ij} = g(X_i, X_j)$ are local components of the metric tensor g with the indices $i, j, k, l, \cdots \in \{1, 2, \ldots, n\}$. We will use the same definition of the curvature tensor as in [13]. That is, we denote by R_{ik} and R_{ikjl} the local components the Ricci Ric and curvature R tensors, respectively. These components are defined by the equations (see [13], pp. 145, 203, 249)

$$\nabla_i \nabla_j \xi^k - \nabla_j \nabla_i \xi^k = \xi^m R^k_{mij}$$

for a C^∞-vector field $\xi = \xi^k X_k$, $R_{ijkl} = g_{im} R^m_{jkl}$ and $R_{kl} = R^i_{kil}$, where we use the Einstein summation convention. Then, using direct calculation, we obtain (see also [8,12])

$$\Delta_S \varphi_{ij} = (\delta \circ \delta^* - \delta^* \circ \delta)\varphi_{ij} = \bar{\Delta}\varphi_{ij} - \underbrace{(R_{ik}\varphi^k_j + R_{jk}\varphi^k_i - 2R_{ikjl}\varphi^{kl})}_{\mathfrak{R}_2(\varphi)}$$

for the local components $\varphi_{ij} = \varphi(X_i, X_j)$ of an arbitrary symmetric two-tensor field $\varphi \in C^\infty(S^2M)$ and $\varphi^k_j = g^{ki}\varphi_{ij}$, where g^{ki} values are the contravariant components of the metric tensor g. At the same time, we deduce from (1) given in Section 2 the following equation:

$$\Delta_L \varphi = \Delta_S \varphi + 2\mathfrak{R}_2(\varphi) = \bar{\Delta}\varphi + \mathfrak{R}_2(\varphi)$$

of the Lichnerowicz Laplacian Δ_L: $C^\infty(S^2M) \to C^\infty(S^2M)$, considered in ([5], pp. 387–388).

2. In the paper, we consider the Lichnerowicz Laplacian Δ_L: $C^\infty(S^2M) \to C^\infty(S^2M)$ acting on C^∞-sections of the bundle of covariant symmetric two-tensor fields S^2M on (M, g). In general, this operator was introduced by Lichnerovich with the equality (see [1], pp. 315–316; [6], p. 133), which, for the case $q = 2$, can be rewritten in the form (see [5], pp. 387–388)

$$\Delta_L \varphi = \bar{\Delta}\varphi + \mathfrak{R}_2(\varphi), \tag{1}$$

where

$$\mathfrak{R}_2(\varphi)_{ij} = R_{ik}\varphi^k_j + R_{jk}\varphi^k_i - 2R_{ikjl}\varphi^{kl} \tag{2}$$

for an arbitrary $\varphi \in C^\infty(S^2M)$. By direct calculations, from (1) and (2), we obtain the formula

$$trace_g(\Delta_L \varphi) = \bar{\Delta}(trace_g \varphi) \tag{3}$$

for an arbitrary $\varphi \in C^\infty(S^2M)$. Therefore, if $trace_g\varphi = \Delta_L \varphi = 0$, then Equation (3) becomes an identity.

Remark 1. *If (M, g) is a locally reducible connected Riemannian manifold, then there exists a nonzero $\varphi \in C^\infty(S^2M)$ such that $\Delta_L \varphi = 0$. To prove this, we recall that if the holonomy of (M, g) is locally reducible, then (M, g) is locally isometric to a Riemannian product $(M_1, g_1) \times (M_2, g_2) := (M_1 \times M_2, g_1 + g_2)$. In addition, it is well known that if (M, g) is a connected Riemannian manifold, then there exists a nonzero $\varphi \in C^\infty(S^2M)$ such that $\nabla\varphi = 0$ if and only if the holonomy of (M, g) is locally reducible. In this case, the equation $\Delta_L \varphi = 0$ holds. Note that the above result is a local result.*

A smooth two-tensor field is called Δ_L*-harmonic* if it belongs to the null space of the Lichnerowicz Laplacian Δ_L (see [4], p. 104). Next, the following statement holds.

Theorem 1. *Let Δ_L: $C^\infty(S^2M) \to C^\infty(S^2M)$ be the Lichnerowicz Laplacian acting on C^∞-sections of the bundle of covariant symmetric two-tensor fields S^2M on a Riemannian manifold (M, g). Then, $trace_g\varphi$ is a harmonic function for an arbitrary Δ_L-harmonic section $\varphi \in C^\infty(S^2M)$. Therefore, if (M, g) is closed, then $trace_g\varphi$ is necessarily constant. On the other hand, if (M, g) is noncompact complete with non-negative Ricci curvature and φ lies in L^p for some $p \in (0, \infty]$, then $\varphi \in C^\infty(S^2_0M)$.*

Proof. Let $\varphi \in C^\infty(S^2M) \cap Ker\Delta_L$; then, we obtain from (3) that $\bar{\Delta}(trace_g\varphi) = 0$. If (M, g) is closed. Then, $trace_g\varphi$ is a constant via the *strong maximum principle* (see [3], p. 75). The Yau theorem from [14] shows that on any complete Riemannian manifold, a harmonic function, which lies in L^p for some $p \in \{1, \infty\}$, is necessarily constant. On the other hand, according to the well-known *Hopf maximum principle* (see [15], p. 30), any harmonic function on a closed Riemannian manifold is also constant. Furthermore, if (M, g) is complete with non-negative Ricci curvature and φ lies in L^p for some $p \in (1, \infty)$, then $\varphi = C$ for some constant C (see [16]). At the same time, a complete noncompact (M, g) with a non-negative Ricci curvature has infinite volume. In this case, C is necessarily zero. Via direct calculations from the Weitzenböck decomposition Formula (1), we obtain the following formula:

$$\frac{1}{2}\Delta\|\varphi\|^2 = -g(\Delta_L \varphi, \varphi) + \|\nabla\varphi\|^2 + g(\mathfrak{R}_2(\varphi), \varphi), \tag{4}$$

where $\Delta = div \circ grad$ is the Beltrami Laplacian on function. In addition, for any point $x \in M$, there exists an orthonormal eigenframe $\{e_1, \ldots, e_n\}$ of T_xM such that $\varphi_x(e_i, e_j) = \varepsilon_i \delta_{ij}$ for the Kronecker delta δ_{ij}. Then, we have (see [5], p. 388)

$$g(\mathfrak{R}_2(\varphi_x), \varphi_x) = \sum_{i \neq j} sec(e_i, e_j)(\varepsilon_i - \varepsilon_j)^2, \tag{5}$$

where $sec(e_i, e_j) = R(e_i, e_j, e_i, e_j)$ is the sectional curvature $\sigma_x = span\{e_i, e_j\}$ of (M, g) at an arbitrary point $x \in M$. In particular, if φ is a Δ_L-harmonic 2-tensor, then from (4), we obtain

$$\frac{1}{2}\,\Delta\|\varphi\|^2 = \|\nabla\varphi\|^2 + \sum_{i \neq j} sec(e_i, e_j)(\varepsilon_i - \varepsilon_j)^2. \tag{6}$$

Proceeding from (6), we can conclude that if the section curvature of (M, g) is positive semidefinite at any point of M, then $\Delta\|\varphi\|^2 \geq 0$. In this case, if φ lies in L^p for some $p \in (0, \infty)$, then $\|\varphi\|^2$ = constant. At the same time, if the sectional curvature of a complete noncompact Riemannian manifold (M, g) is positive semidefinite, then(M, g) has infinite volume; see [17]. Then $\|\varphi\|^2 = 0$. The theorem has been proven. □

Proceeding from (6) and using the *weak maximum principle* (see [3], p. 75), we can conclude that if the section curvature of (M, g) is positive semidefinite at any point of a connected open domain $U \subset M$ and is also positive (in all directions σ_x) at some point $x \in U$, then $\|\varphi\|^2$ is constant and $\nabla\varphi = 0$ in U. If $C > 0$, then φ is nowhere zero. Now, at a point $x \in U$, where the section curvature $sec(e_i, e_j)$ is positive, the left side of (6) is zero while the right side is non-negative. This contradiction shows $\varepsilon_1 = \cdots = \varepsilon_n = \varepsilon$ and hence $\varphi = \varepsilon \cdot g$ for some constant λ everywhere in U. On the other hand, the fact that $\nabla\varphi = 0$ means that φ is invariant under parallel translation. In this case, if the holonomy of (M, g) is locally irreducible, then the tensor φ has one eigenvalue, i.e., $\varphi = \lambda \cdot g$ for some constant λ at each point of U. As a result, we have the following theorem.

Theorem 2. *Let U be a connected open domain of a Riemannian manifold (M, g) and φ be a Δ_L-harmonic symmetric two-tensor field defined on U. If the section curvature of (M, g) is positive semidefinite at any point of U and the scalar function $\|\varphi\|^2$ has a local maximum at some point of U, then $\|\varphi\|^2$ is a constant function and φ is invariant under parallel translation in U. Moreover, if $sec > 0$ at some point of U or if the holonomy of (M, g) is locally irreducible, then φ is a constant multiple of g at all points of U.*

The following corollary is obvious.

Corollary 1. *Let (M, g) be a connected closed Riemannian manifold and φ be a Δ_L-harmonic symmetric two-tensor field defined on (M, g). If the section curvature of (M, g) is positive semidefinite at any of its points, then $\|\varphi\|^2$ is a constant function and φ is invariant under parallel translation in (M, g). Moreover, if $sec > 0$ at some point of (M, g) or if (M, g) is locally irreducible, then φ is constant multiple of g at all points of (M, g).*

Let us consider a complete manifold (M, g) with non-negative sectional curvature. For this case, we can prove the following theorem.

Theorem 3. *Let (M, g) be a connected noncompact and complete Riemannian manifold with non-negative sectional curvature. Then, there is no nonzero Δ_L-harmonic symmetric two-tensor such that it lies in L^p for some $1 < p < \infty$.*

Proof. By direct calculation, we find the following

$$\frac{1}{2}\,\Delta\|\varphi\|^2 = \|\varphi\|\,\Delta\|\varphi\| + \|d\|\varphi\|\|^2$$

for $\varphi \in C^\infty(S^2M)$. Then, Equation (4) can be rewritten in the form

$$\|\varphi\|\,\Delta\|\varphi\| = -g(\Delta_L\,\varphi,\varphi) + \|\nabla\varphi\|^2 - \|d\|\varphi\|\|^2 + g(\mathfrak{R}_2(\varphi),\varphi). \tag{7}$$

By using the first Kato inequality (see [18])

$$\|\nabla\varphi\|^2 \geq \|d\|\varphi\|\|^2$$

and Formula (7), we can write

$$\|\varphi\|\,\Delta\|\varphi\| \geq -g(\Delta_L\,\varphi,\varphi) + g(\mathfrak{R}_2(\varphi),\varphi). \tag{8}$$

Let (M,g) be a complete Riemannian manifold with non-negative sectional curvature and $\varphi \in C^\infty(S^2M) \cap Ker\Delta_L$, then we obtain from (8) that $\|\varphi\|\,\Delta\|\varphi\| \geq 0$. In this case, the Yau theorem from [14] shows the following: if u is a smooth non-negative subharmonic L^p-function for $1< p < \infty$ defined on a complete Riemannian manifold (M,g) such that $u\Delta u \geq 0$, then u is constant. At the same time, if (M,g) is a connected noncompact and complete Riemannian manifold with a non-negative sectional curvature, then (M,g) has infinite volume (see also [17]). In this case, this constant is equal to zero. □

3. Let Δ_L: $C^\infty(S^2M) \to C^\infty(S^2M)$ be the Lichnerowicz Laplacian acting on C^∞-sections of the bundle of covariant symmetric two-tensor fields S^2M on a closed manifold (M,g). This is a self-adjoint elliptic operator, and via the compactness of (M,g), it has a discrete spectrum $Spec^{(2)}\Delta_L = \{\lambda_1^{(2)}, \lambda_2^{(2)}, \dots\}$. Let $\lambda_a^{(2)} \in Spec^{(2)}\Delta_L$ for some $a = 1,2,\dots$ corresponds to the nonzero eigentensor such that $\varphi \neq \mu \cdot g$, then from (4) and (5), we obtain the integral formula

$$\lambda_a^{(2)} \int_M \|\varphi\|^2\, dv_g = \int_M \big(\|\nabla\varphi\|^2 + \textstyle\sum_{i\neq j} sec(e_i,e_j)(\varepsilon_i - \varepsilon_j)^2\big)\, dv_g.$$

Next, we suppose that the section curvature sec of a closed (M,g) is positive and let $K_{\min}$ denote its minimum. In this case, the inequality $Ric \geq (n-1)K_{\min} > 0$ holds. At the same time, from the above formula we deduce inequalities

$$\lambda_a^{(2)} \int_M \|\varphi\|^2\, dv_g \geq K_{\min} \int_M \textstyle\sum_{i\neq j} (\varepsilon_i - \varepsilon_j)^2\, dv_g > 0. \tag{9}$$

In this case, we can conclude that $\lambda_a^{(2)} > 0$.

On the other hand, the Laplacian $\Delta_0 = -div \circ grad$: $C^\infty(M) \to C^\infty(M)$ has the discrete spectrum $Spec\,\Delta_0 = \{0 = \bar\lambda_0^{(0)} < \bar\lambda_1^{(0)} \leq \bar\lambda_2^{(0)} \leq \dots \to \infty\}$. For simplicity, we will assume that (M,g) is connected; this will, for example, imply that the smallest eigenvalue, $\bar\lambda_0^{(0)}$, occurs with multiplicity 1. Here, we will focus on bounds on the first nonzero eigenvalue $\bar\lambda_1^{(0)}$ imposed by the geometry. The lower bound is due to Yang [19]: if (M,g) is a closed Riemannian manifold of dimension $n \geq 2$ and $Ric \geq (n-1)K_{\min}$ for some constant $K > 0$, then $\bar\lambda_1^{(0)} \geq \frac{n-1}{4}\, K_{\min} + \frac{\pi}{d^2}$, where d is the diameter of a closed (M,g).

Next, let $\lambda_a^{(a)}$ be the nonzero eigenvalue of Δ_L corresponding to an eigentensor $\varphi \in C^\infty(S^2M)$ such that $trace_g\varphi \neq 0$; then $\bar\Delta(trace_g\varphi) = \lambda_a^{(2)}(trace_g\varphi) \neq 0$. Therefore, $\lambda_a^{(2)} \in Spec(\bar\Delta)$ and hence $\lambda_a^{(2)} = \bar\lambda_b^{(0)}$ for some b. Thus, in view of the foregoing, we have the following theorem.

Theorem 4. *Let (M,g) be an n-dimensional closed Riemannian manifold with positive sectional curvature and Δ_L: $C^\infty(S^2M) \to C^\infty(S^2M)$ be the Lichnerowicz Laplacian. Then, the Lichnerowicz Laplacian Δ_L has the discrete spectrum $Spec^{(2)}\Delta_L = \{0 \leq \lambda_1^{(2)} \leq \lambda_2^{(a)} \leq \dots \to +\infty\}$ and the eigenvalue $\lambda_a^{(2)}$ corresponding to an arbitrary eigentensor $\varphi \in C^\infty(S^2M)$ with a nonzero trace*

that satisfies the inequality $\lambda_a^{(2)} \geq \frac{n-1}{4} K_{\min} + \frac{\pi}{d^2}$, *where* $K_{\min}$ *is the minimum of the sectional curvature and d is the diameter of* (M, g).

4. We denote by S_0^2M the sub-bundle of the bundle S^2M over a Riemannian manifold defined using the condition $trace_g\varphi = 0$ for any $\varphi \in S_0^2M$. In this case, there exists an orthonormal eigenframe $\{e_1, \ldots, e_n\}$ at any point $x \in M$ such that $\varphi_x(e_i, e_j) = \varepsilon_i\delta_{ij}$ for the Kronecker delta δ_{ij} and

$$\|\varphi\|^2 = \varepsilon_1^2 + \cdots + \varepsilon_n^2 = \frac{1}{n} \sum_{i<j}(\varepsilon_i - \varepsilon_j)^2 \tag{10}$$

since $trace_g\varphi = \varepsilon_1 + \cdots + \varepsilon_n = 0$.

On the other hand, we obtain from (3) that $trace_g(\Delta_L \varphi) = \Delta_L(trace_g\varphi) = 0$ for an arbitrary $\varphi \in C^\infty(S_0^2M)$. Then, we can conclude that Δ_L: $C^\infty(S_0^2M) \to C^\infty(S_0^2M)$ since $trace_g\varphi = 0$. Next, we will consider the Lichnerowicz Laplacian Δ_L: $C^\infty(S_0^2M) \to C^\infty(S_0^2M)$ acting on C^∞-sections of the bundle S_0^2M of covariant symmetric trace-less two-tensor fields on a closed manifold (M, g). Moreover, we suppose that the section curvature sec of a closed (M, g) is positive and that $K_{\min}$ denotes its minimum. In this case, we deduce from (9) and (10) the following inequality:

$$\lambda_a^{(2)} \int_M \|\varphi\|^2 \, dv_g \geq 2nK_{\min} \int_M \|\varphi\|^2 \, dv_g.$$

Then, from the previous inequality, we conclude that $\lambda_1^{(2)} \geq 2n\, K_{\min}$ for the first nonzero eigenvalues $\lambda_1^{(2)}$ of the Laplacian Δ_L: $C^\infty(S_0^2M) \to C^\infty(S_0^2M)$ that act on the bundle S_0^2M of traceless symmetric two-tensors on a Riemannian manifold with sectional curvature $sec \geq K > 0$. Thus, we have proved the following theorem.

Theorem 5. *Let* (M, g) *be an n-dimensional closed Riemannian manifold with sectional curvature* $sec \geq K_{\min} > 0$, *and let* Δ_L: $C^\infty(S_0^2M) \to C^\infty(S_0^2M)$ *be the Lichnerowicz Laplacian that acts on the bundle* S_0^2M *of the traceless symmetric two-tensor fields over* (M, g). *Then, the first nonzero eigenvalue* $\lambda_1^{(2)}$ *of* Δ_L *satisfies the inequality* $\lambda_1^{(2)} \geq 2n\, K_{\min}$.

Remark 2. *The sectional curvature of the Euclidean n-sphere* $\mathbb{S}^n$ *of radius* $R = 1$ *equals* $K_{\min} = 1$. *Then, the first nonzero eigenvalue* $\lambda_1^{(2)}$ *of the Lichnerowicz Laplacian* Δ_L, *which is defined on the Euclidean sphere* $\mathbb{S}^n$ *and the acting traceless symmetric two-tensors, satisfies the inequality* $\lambda_1^{(2)} \geq 2n$ *(compare with the results of [20]).*

3. Infinitesimal Einstein Deformations

1.Finally, consider the Lichnerowitz Laplacian Δ_L acting on the vector bundle S_0^2M of symmetric traceless two-tensor fields, which can be regarded as infinitesimal deformations of the metric g. Therefore, it arises in the analysis of the *stability of the Einstein metrics* (see [6], chapter 12).

Namely, let g be an Einstein metric on a closed manifold M, i.e., $Ric = s/n\, g$ for the scalar curvature s of (M, g). An infinitesimal Einstein deformation of an Einstein metric g is a symmetric two-tensor field φ such that (see [6], p. 347)

$$\Delta_L \varphi = \nabla d\,(trace_g\varphi) + 2s/n\, \varphi; \quad \delta\varphi = 0; \quad \int_M (trace_g\varphi)\, dv_g = 0,$$

where $(\delta\varphi)X := -(trace_g\nabla\varphi)X$ for any vector field X on M. Furthermore, a symmetric two-tensor field φ is an infinitesimal Einstein deformation of g if and only if it satisfies the following equation (see [6], p. 347):

$$\Delta_L \varphi = 2s/n\, \varphi; \quad \delta\varphi = 0; \quad trace_g\varphi = 0. \tag{11}$$

If $\varphi \in trace_g^{-1}(0) \cap \delta^{-1}(0)$, then it is called *transverse traceless tensor* or *TT-tensor*. Therefore, if φ is an infinitesimal Einstein deformation of g, then it is a TT-tensor and an eigenform of the Lichnerowicz Laplacian Δ_L , and $2\,s/n$ is its eigenvalue. On the other hand, if $2\,s/n$ is not an eigenvalue of Δ_L, then g is not deformable, i.e., Einstein deformations do not exist. We also recall that if an Einstein metric g does not have infinitesimal Einstein deformations, then it is called *rigid* (see [6], p. 347).

Remark 3. *An Einstein metric g is called stable if the smallest eigenvalue of Δ_L is greater than $2\,s/n$. It is obvious that the stability of g implies the rigidity and non-deformability of g (see [6], p. 132). In particular, the standard sphere and compact Einstein manifolds with strictly negative sectional curvature have stable metrics (see also [6], p. 132).*

We recall that $R_{iklj} = -R_{ikjl}$; then, the Lichnerowicz Laplacian can be rewritten in the form

$$\Delta_L\, \varphi_{ij} = \bar{\Delta}\varphi_{ij} + R_{ik}\varphi^k_j + R_{jk}\varphi^k_i + 2\,\overset{\circ}{R}(\varphi)_{ij}, \tag{12}$$

where $\overset{\circ}{R}$: $S^2_0M \to S^2_0M$ is the *curvature operator of the second kind* (see [21]), defined by $\overset{\circ}{R}(\varphi)_{ij} = R_{iklj}\varphi^{kl}$.

Remark 4. *In the monograph ([6], p. 133), the Lichnerowicz Laplacian was rewritten in the form $\Delta_L\, \varphi_{ij} = \bar{\Delta}\varphi_{ij} + R_{ik}\varphi^k_j + R_{jk}\varphi^k_i - 2\overset{\circ}{R}(\varphi)_{ij}$, where the curvature operator of the second kind was defined in ([6], p. 52) by the formula $\overset{\circ}{R}(\varphi)_{ij} = R_{ikjl}\varphi^{kl}$, since the local components R_{jkli} of the Riemannian curvature tensor R were defined in the monograph by the identities $R_{jkli} = g_{im}R_{jkl}{}^m$, where $R_{jkli} = R_{kjli}$.*

Then, using (11) and (12), we can rewrite Equation (4) in the form

$$1/2\,\Delta\|\varphi\|^2 = \|\nabla\varphi\|^2 + 2g(\overset{\circ}{R}(\varphi), \varphi). \tag{13}$$

Next, from (13), we deduce the integral formula

$$\int_M (\|\nabla\varphi\|^2 + 2g(\overset{\circ}{R}(\varphi), \varphi)\, dv_g = 0.$$

It is obvious that the inequality $\overset{\circ}{R} > 0$ conflicts with the above integral equality. Therefore, if $\overset{\circ}{R} > 0$ on S^2_0M, then (M, g) is not infinitesimal deformable (see also [5], p. 390). In particular, if (M, g) is a manifold with constant sectional curvature $\varepsilon > 0$ (see [13], p. 203), then its Riemann curvature tensor is given via the identities $R_{ijkl} = \varepsilon(g_{ik}g_{jl} - g_{il}g_{jk})$. For any traceless symmetric two-tensor φ with local components φ_{ij}, we have

$$g(\overset{\circ}{R}(\varphi), \varphi) = R_{ijkl}\varphi^{il}\varphi^{jk} = \varepsilon(g_{ik}g_{jl} - g_{il}g_{jk})\varphi^{il}\varphi^{jk} = \varepsilon(\|\varphi\|^2 - (trace_{g_0})^2) = \varepsilon\|\varphi\|^2 \geq 0,$$

where equality is possible in the case that $\varphi = 0$.

Remark 5. *In general, we say that $\overset{\circ}{R} \geq 0$ (respectively, $\overset{\circ}{R} > 0$) if the eigenvalues of $\overset{\circ}{R}$ as a bilinear form on S^2_0M are non-negative (respectively, strictly positive). At an arbitrary point $x \in M$, we choose orthogonal unit vectors $X, Y \in T_xM$ at an arbitrary point $x \in M$ and define the symmetric two-tensor field $\theta = X \otimes Y + Y \otimes X$; then, via direct calculation, we obtain $g(\overset{\circ}{R}(\theta), \theta) = 2\,sec(X, Y)$. Therefore, the sectional curvature of (M, g) is everywhere non-negative (respectively, positive) if the operator $\overset{\circ}{R}$ is non-negative (respectively, strictly positive) defined on any section of the bundle S^2_0M (see [21], p. 196).*

2. Let (M, g) be a closed connected Einstein manifold with a non-negative curvature operator of the second kind; then, from the above integral equality, we obtain $\nabla\varphi = 0$, i.e., φ is invariant under parallel translations. On the other hand, if (M, g) is a locally irreducible manifold, then $\varphi = \varepsilon g$ for some constant ε. At the same time, from (11), we obtain $n\,\varepsilon = trace_g\varphi = 0$, and then $\varphi = 0$. Therefore, the following statement holds.

Lemma 1. *Let (M, g) be a closed connected Einstein manifold with the positive-definite curvature operator of the second kind; then, it is not infinitesimal deformable. On the other hand, if the curvature operator of the second kind is non-negative and φ is an infinitesimal Einstein deformation of g, then φ is invariant under parallel translations. Furthermore, if (M, g) is also a locally irreducible manifold, then g is rigid.*

Remark 6. *Firstly, we recall that a closed Riemannian manifold (M, g) with a positive curvature operator of the second kind is diffeomorphic to a spherical spatial form (see [21]) and hence M has the metric $\bar{g}$ of constant positive sectional curvature. In addition, $(M, \bar{g})$ is not infinitesimal deformable (see also [6], p. 132). Second, it is well known that a non-negative curvature operator preserves product manifolds, in the sense that if (M, g) is isometric to $(M_1 \times M_2, g_1 + g_2)$, then (M, g) has a non-negative curvature operator if and only if both (M_1, g_1) and (M_2, g_2) have a non-negative curvature operator.*

We proved in [10] that if $sec \geq \varepsilon$ for a positive constant ε, then

$$g(\overset{\circ}{R}(\varphi), \varphi) \geq \frac{1}{n}\,(n^2\varepsilon - s)\,\|\varphi\|^2$$

for $\varphi \in C^\infty(S_0^2M)$. In this case, we can formulate the following corollary.

Corollary 2. *Let (M, g) be a closed Einstein manifold with $sec \geq \varepsilon$ for a positive constant ε such that $s < n^2\varepsilon$; then, it is not infinitesimally Einstein-deformable. On the other hand, if the inequality $s \leq n^2\varepsilon$ holds and φ is an infinitesimal Einstein deformation of g, then φ is invariant under parallel translations. If, at the same time, (M, g) is also a locally irreducible manifold, then g is rigid.*

Remark 7. *In particular, if (M, g) has constant sectional curvature $\varepsilon > 0$, then $s = n(n-1)\varepsilon > 0$. In this case, the inequality $s < n^2\varepsilon$ holds. Therefore, if a Riemannian manifold has constant sectional curvature $\varepsilon > 0$, then it is not infinitesimally deformable (see also [6], p. 132).*

3. In conclusion, we define the operator Δ_E using the Weitzenböck decomposition formula

$$\Delta_E = \bar{\Delta} + 2\overset{\circ}{R}. \tag{14}$$

This is a self-adjoint elliptic operator (see [6], p. 347), and from the compactness of (M, g), it has a discrete spectrum. In particular, if Δ_E is a positive definite on $\ker(\Delta_{E|TT}) := \delta^{-1}(0) \cap trace_g^{-1}$, then the Einstein metric g on a closed M is stable (see [6], p. 132). Moreover, according to the statements above, Δ_E is an example of the Lichnerowicz Laplacian, and the elements in $\ker(\Delta_{E|TT})$ are infinitesimal Einstein deformations (see also [6], pp. 54, 347). In accordance with the above, we will call Δ_E the Einstein Laplacian (see also [20]).

Next, via direct calculations from (14), we deduce the following formula:

$$1/2\Delta\,\|\varphi\|^2 = -g(\Delta_E\varphi, \varphi) + \|\nabla\varphi\|^2 + 2\,g(\overset{\circ}{R}(\varphi), \varphi). \tag{15}$$

Moreover, if $\varphi \in Ker\Delta_E$, then, from (15), we have the following inequality:

$$\|\varphi\|\,\Delta\|\varphi\| \geq 2g(\overset{\circ}{R}(\varphi), \varphi).$$

Recall that the sectional curvature of a Riemannian manifold (M, g) with the non-negative curvature operator of the second kind $\overset{\circ}{R}$ is also non-negative; see Remark 5. At the same time, a complete noncompact Riemannian manifold with a non-negative sectional curvature has infinite volume (see [17]). Therefore, we can conclude that the statement is true.

Theorem 6. *Let Δ_E be the Einstein Laplacian defined on a complete noncompact Riemannian manifold (M, g) with a non-negative curvature operator of the second kind. Then, there is no nonzero $\varphi \in L^q(Ker_E) \cap C^\infty(S^2_0 M)$ for any positive number $q > 1$.*

Remark 8. *If (M, g) is an n-dimensional Riemannian manifold with a non-negative curvature operator of the second kind, then $sec \geq 0$ and its scalar curvature $s(x) = \sum_{i \neq j} sec(e_i, e_j) \geq 0$ for an arbitrary orthonormal frame $\{e_1, \ldots, e_n\}$ of $T_x M$ at any point $x \in M$. Therefore, if (M, g) is a manifold with zero scalar curvature, then it is locally flat. In this case, if (M, g) is also simply connected, noncompact, and complete, then it is isometric to a Euclidean space $\mathbb{R}^n$.*

Let Δ_E: $C^\infty(S^2 M) \to C^\infty(S^2 M)$ be the Einstein Laplacian acting on C^∞-sections of the bundle $S^2 M$ of covariant symmetric two-tensor fields on a closed manifold (M, g). This is a self-adjoint elliptic operator, and by the compactness of (M, g) it has a discrete spectrum $Spec^{(2)}\Delta_L = \{\mu_1^{(2)}, \mu_2^{(2)}, \ldots\}$. Let $\mu_a^{(2)} \in Spec^{(2)}\Delta_L$ for some $a = 1, 2, \ldots$ that correspond to the nonzero eigentensor such that $\varphi \neq \mu \cdot g$, then from (15) we obtain the integral inequality

$$\mu_a^{(2)} \int_M \|\varphi\|^2 \, dv_g \geq 2 \int_M g(\overset{\circ}{R}(\varphi), \varphi) \, dv_g. \tag{16}$$

Further, let the curvature operator of the second kind $\overset{\circ}{R}$ be a positive definite, and let there exist a number $\rho_{\min} > 0$ such that $g(\overset{\circ}{R}(\varphi), \varphi) \geq \rho_{\min} \cdot \|\varphi\|^2$, i.e., $\rho_{\min}$ is the smallest eigenvalue of $\overset{\circ}{R}$. Then, from (16), we deduce $\mu_a^{(2)} \geq 2\rho_{\min}$.

Theorem 7. *Let Δ_E: $C^\infty(S^2 M) \to C^\infty(S^2 M)$ be the Einstein Laplacian defined on a closed Riemannian manifold (M, g) with a positive curvature operator of the second kind, i.e.,*

$$g(\overset{\circ}{R}(\varphi), \varphi) \geq \rho_{\min} \cdot \|\varphi\|^2$$

for some $\rho_{\min} > 0$ and any nonzero $\varphi \in C^\infty(S^2 M)$. Then, the first nonzero eigenvalue $\mu_1^{(2)}$ of Δ_E corresponds to the nonzero eigentensor such that $\varphi \neq \mu \circ g$ satisfies the inequality $\mu_1^{(2)} \geq 2\rho_{\min}$.

4. Conclusions

In this article, we have considered the generalized Bochner technique, which is a natural development of the classical Bochner technique. That is, we have proven a number of vanishing theorems for the symmetric two-tensors that form the kernel of the Lichnerowicz Laplacian using the L_p-Liouville theorems for subharmonic functions. Moreover, we have estimated the lowest eigenvalues of the Lichnerowicz Laplacian on the compact Riemannian manifolds of sign-definite curvature operator. As an application of the obtained results, we have considered the theory of infinitesimal Einstein deformations and the analysis of the theory of stability of the Einstein metrics. All our results are new. They supplement and generalize the results obtained earlier by various authors.

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