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A Family of Truncated Positive Distributions

Héctor J. Gómez [1], Karol I. Santoro [2], Inmaculada Barranco-Chamorro [3], Osvaldo Venegas [1,*], Diego I. Gallardo [4] and Héctor W. Gómez [2]

1 Departamento de Ciencias Matemáticas y Físicas, Facultad de Ingeniería, Universidad Católica de Temuco, Temuco 4780000, Chile; hgomez@uct.cl
2 Departamento de Estadística y Ciencias de Datos, Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta 1240000, Chile; karol.santoro@uantof.cl (K.I.S.); hector.gomez@uantof.cl (H.W.G.)
3 Departamento de Estadística e Investigación Operativa, Facultad de Matemáticas, Universidad de Sevilla, 41012 Sevilla, Spain; chamorro@us.es
4 Departamento de Estadística, Facultad de Ciencias, Universidad del Bío-Bío, Concepción 4081112, Chile; dgallardo@ubiobio.cl
* Correspondence: ovenegas@uct.cl

Abstract: In this paper, a new family of continuous distributions with positive support is introduced. This family is generated by a truncation of the family of univariate symmetrical distributions. In this new family of distributions, general properties, such as moments, asymmetry and kurtosis coefficients, are derived. Particular cases of interest based on the normal, logistic, Laplace and Cauchy models are discussed in depth. The estimation of parameters is carried out by applying moments and maximum likelihood methods. Also, a simulation study was conducted to illustrate the good performance of estimators. An application to the Survival Times (in days) of Guinea Pigs dataset is included, where the special cases of distributions in this family are fitted. The option which provides the best fit is ultimately chosen. An R package, called "tpn", has been implemented, which includes the relevant cases of interest in this family.

Keywords: positive truncation; likelihood; symmetric distributions; tpn package

MSC: 62E15; 62E20

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1. Introduction

In modeling continuous real data, we frequently encounter real data that assume values only in $(0, \infty)$; models with positive support must therefore be used with these data. In this context, the traditional distributions used in the literature are the gamma, Weibull, log-normal and half-normal models, among others. Models of these types are readily applicable in areas associated with medicine, climatology, the environment, and marine sciences, to name just a few. In the more general context of distributions, we can cite the family of spherical distributions, which is a particular case of the family of elliptic distributions. The univariate case falls into a class of symmetric distributions. If the probability density function (pdf) of a random variable (r.v) X satisfies the condition of being symmetric about zero, then X itself is also symmetric about zero. In other words, its pdf—say, $g_0(\cdot)$—complies with

$$g_0(-x) = g_0(x), \quad \forall x \in \mathbb{R}. \tag{1}$$

Let $G_0(\cdot)$ be the cumulative distribution function (cdf) associated with the r.v. X, which satisfies $G_0(0) = \frac{1}{2}$, $G_0(-x) = 1 - G_0(x)$ and $G_0'(x) = g_0(x)$. For the multivariate case (considering also location and scale), the elliptic family of distributions is obtained; see Kelker [1], Cambanis et al. [2], Fang et al. [3] and Gupta and Varga [4], among others. A family of models with positive support that is generated based on the family of symmetric

distributions is the family of half-symmetric (HS) distributions. If the r.v. X is symmetric about zero, then the r.v. $Y = |X|$ has an HS distribution. The pdf of the r.v. Y is given by

$$h_X(x) = 2g_0(x), \quad \forall x \in \mathbb{R}^+. \tag{2}$$

Incorporating a scale parameter $\sigma > 0$, we obtain

$$h_X(x;\sigma) = \frac{2}{\sigma} g_0\left(\frac{x}{\sigma}\right), \quad \forall x \in \mathbb{R}^+. \tag{3}$$

We will denote this as $Y \sim HS(\sigma)$. The HS family of distributions is a positive truncation of symmetrical distributions. The half-normal (HN) model is a member of this family, which was studied in detail by Pewsey [5,6]. Some extensions of the HN distribution have been introduced by Chou and Liu [7], Barranco-Chamorro et al. [8], Cooray and Ananda [9] and Olmos et al. [10], among others. Other authors who have developed extensions of distributions contained in the HS family are Balakrishnan and Puthenpura [11] for the half-logistic (HL) distribution, Wiper et al. [12] for the HN and half-t (Ht) distributions, Polson and Scott [13] for the half-Cauchy (HC) distribution and Gui [14] for the half-power exponential (HPE) distribution, among others. The aim of the present paper is to use univariate symmetric distributions to generate a new family of distributions with positive support that contains the HS family as a particular case and offers an alternative to distributions with two parameters for modeling data on reliability, survival or any area with continuous positive observations. Also, an R package is available, which can be used by the readers for modeling positive continuous data. The truncation of distributions or families of distributions has grown in recent years; see Almarashi et al. [15], Bantan et al. [16], Morán et al. [17] and Alotaibi et al. [18].

In this context, a key tool is presented, which significantly improves the analysis of truncated distributions: the R package "tpn". The "tpn" package allows the application of four particular cases of truncated distributions: Truncated Positive Laplace (TPLa), Truncated Positive Logistic (TPL), Truncated Positive Cauchy (TPC) and, finally, Truncated Positive Normal (TPN); this latter was previously studied by Gómez et al. [19]. What sets "tpn" apart is its ability to assist in the selection of the optimal model among these four particular cases by using fitting criteria, providing a valuable tool for researchers and practitioners working in data analysis and statistical modeling.

The paper is organized as follows. In Section 2, we present the new family with positive support and its properties. The methods for estimation and computational implementation of the model in an R package [20] are discussed in Section 3. Section 4 presents a simulation study that evaluates the performance of the maximum likelihood estimators in finite samples. In Section 5, we present an application to a set of real data, showing the fits of four sub-models of the family of distributions. Finally, Section 6 provides the main conclusions drawn from this research.

2. Truncated Positive Symmetrical Distributions

In this section, we present a novel family of distributions with positive support. Its principal associated functions and properties are discussed in detail.

2.1. pdf, cdf and Hazard Function

The following definition presents a new family of distributions with positive support based on a positive truncation of the symmetric distributions, which we call truncated positive symmetrical (TPS).

Definition 1. *Let X be an r.v. with pdf symmetric around zero $g_0(x)$ and respective cdf $G_0(x)$. By using the location-scale transformation, a new r.v. $Y = \sigma X + \mu$ is defined, with $\sigma > 0$ and $\mu \in \mathbb{R}$. If $Z = Y \mid Y > 0$ (in other words, Z is defined as the r.v. Y, but only in its positive part), then $Z \sim TPS(\sigma, \lambda)$, where $\lambda = \mu/\sigma \in \mathbb{R}$ is a shape parameter.*

The pdf related to the TPS(σ, λ) distribution is presented in the following proposition.

Proposition 1. *For $Z \sim TPS(\sigma, \lambda)$, its pdf is given by*

$$f_Z(z; \sigma, \lambda) = \frac{1}{\sigma G_0(\lambda)} g_0\Big(\frac{z}{\sigma} - \lambda\Big), \quad z > 0, \tag{4}$$

where $\sigma > 0$ and $\lambda \in \mathbb{R}$ are scale and shape parameters, respectively.

Proof. From the definition itself, it is clear that the pdf of Z has the form

$$f_Z(z; \sigma, \lambda) = \frac{K}{\sigma} g_0\left(\frac{z - \mu}{\sigma}\right), \quad z > 0,$$

where K is the normalization constant. Taking $K = 1/G_0(\lambda)$ and making $\lambda = \mu/\sigma$, it can directly be seen that

$$\frac{1}{\sigma G_0(\lambda)} \int_0^\infty g_0\Big(\frac{z}{\sigma} - \lambda\Big) dz = \frac{1}{\sigma G_0(\lambda)} \int_{-\lambda}^\infty g_0(u) \sigma du = \frac{1}{G_0(\lambda)} (1 - G_0(-\lambda)) = 1.$$

□

Proposition 2. *Let $Z \sim TPS(\sigma, \lambda)$. Then, the cdf of Z is given by*

$$F_Z(z; \sigma, \lambda) = \frac{1}{G_0(\lambda)} \Big[G_0\Big(\frac{z}{\sigma} - \lambda\Big) + G_0(\lambda) - 1\Big], \quad z > 0. \tag{5}$$

Proposition 3. *Let $Z \sim TPS(\sigma, \lambda)$. Then, the hazard function of Z is given by*

$$h(z) = \frac{g_0\Big(\frac{z}{\sigma} - \lambda\Big)}{\sigma[2G_0(\lambda) - G_0(\frac{z}{\sigma} - \lambda) - 1]}, \quad z > 0. \tag{6}$$

The proofs of Propositions 2 and 3 are directly obtained by using the corresponding definitions and algebraic manipulations.

2.2. Particular Cases

The TPS distribution encompasses various models, which are detailed as follows:

1. TPS$(\sigma, \lambda = 0) \equiv$ HS(σ);
2. TPS$_{g_0 = normal} \equiv$ TPN(σ, λ);
3. TPS$_{g_0 = laplace} \equiv$ TPLa(σ, λ);
4. TPS$_{g_0 = cauchy} \equiv$ TPC(σ, λ);
5. TPS$_{g_0 = logistic} \equiv$ TPL(σ, λ).

The relationships among the TPS and its specific cases are summarized in Figure 1.

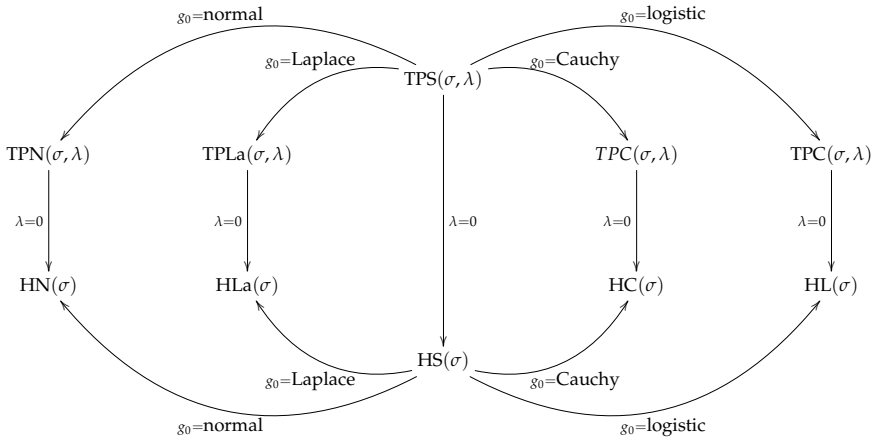


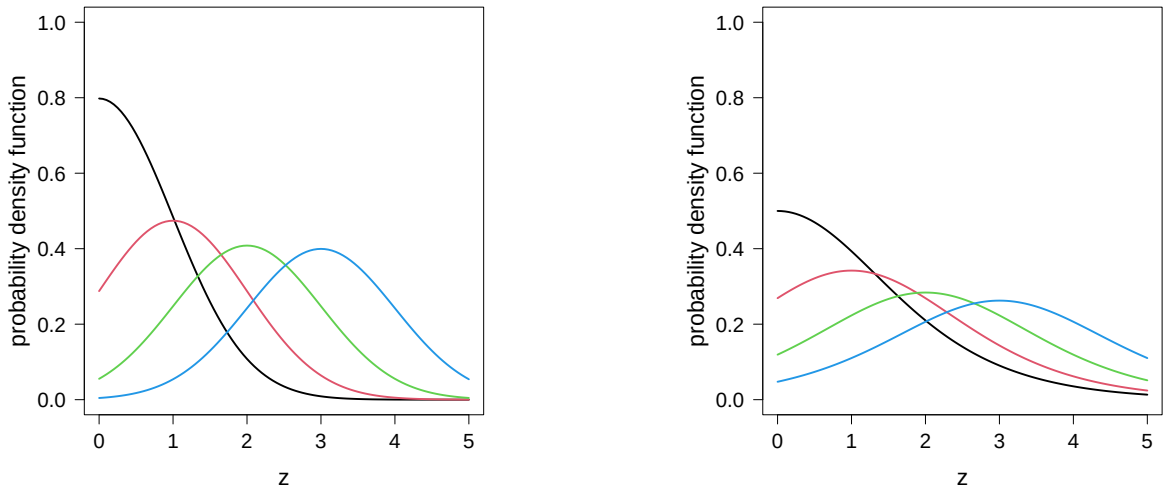
Figure 1. Some particular cases for the TPS distribution.

The pdf, cdf and hazard function for some particular cases of the TPS model are presented in Table 1. Illustrative plots for the pdf and hazard function are given in Figure 2 and Figure 3, respectively.

Table 1. pdf, cdf and hazard function for some particular cases of the TPS model.

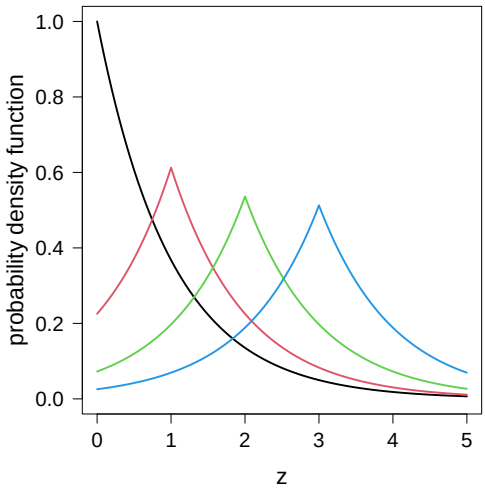
g_0	pdf	cdf	Hazard Function
Normal	$\frac{1}{\sigma\Phi(\lambda)}\phi\left(\frac{z}{\sigma}-\lambda\right)$	$\frac{\Phi\left(\frac{z}{\sigma}-\lambda\right)+\Phi(\lambda)-1}{\Phi(\lambda)}$	$\frac{\phi\left(\frac{z}{\sigma}-\lambda\right)}{\sigma\left[1-\Phi\left(\frac{z}{\sigma}-\lambda\right)\right]}$
Logistic	$\frac{(1+e^{-\lambda})e^{-(\frac{z}{\sigma}-\lambda)}}{\sigma(1+e^{-(\frac{z}{\sigma}-\lambda)})^2}$	$1-(1+e^{\lambda})\left[\frac{e^{\frac{z}{\sigma}-\lambda}}{1+e^{\frac{z}{\sigma}-\lambda}}\right]$	$\frac{1}{\sigma\left[1+e^{-\left(\frac{z}{\sigma}-\lambda\right)}\right]}$
Laplace	$\frac{e^{-\lvert\frac{z}{\sigma}-\lambda\rvert}}{\sigma(1+\mathrm{sgn}(\lambda)[1+e^{-\lvert\lambda\rvert}])}$	$\frac{\mathrm{sgn}\left(\frac{z}{\sigma}-\lambda\right)\left(1-e^{-\lvert\frac{z}{\sigma}-\lambda\rvert}\right)+\mathrm{sgn}(\lambda)(1-e^{-\lvert\lambda\rvert})}{1+\mathrm{sgn}(\lambda)(1-e^{-\lvert\lambda\rvert})}$	$\frac{e^{\lvert\frac{z}{\sigma}-\lambda\rvert}}{2\sigma\left[1-\left(\frac{1}{2}+\frac{1}{2}\mathrm{sgn}\left(\frac{z}{\sigma}-\lambda\right)\left(1-e^{\lvert\frac{z}{\sigma}-\lambda\rvert}\right)\right)\right]}$
Cauchy	$\frac{(1+(\frac{z}{\sigma}-\lambda)^2)^{-1}}{\sigma(\arctan(\lambda)+\frac{\pi}{2})}$	$\frac{\frac{1}{\pi}\left(\arctan\left(\frac{z}{\sigma}-\lambda\right)+\arctan(\lambda)\right)}{\frac{1}{2}+\frac{1}{\pi}\arctan(\lambda)}$	$\frac{\left[\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{z}{\sigma}-\lambda\right)\right]^{-1}}{\sigma\pi\left(1+\left(\frac{z}{\sigma}-\lambda\right)^2\right)}$

$\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cdf of the standard normal distribution, respectively, and $\mathrm{sgn}(\cdot)$ denotes the sign function.

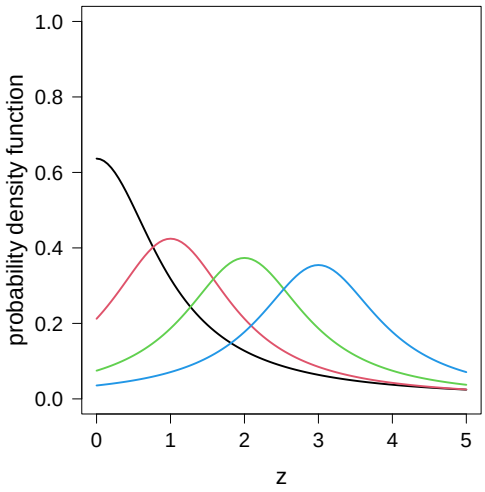


(**a**) TPN$(\sigma=1,\lambda)$ (**b**) TPL$(\sigma=1,\lambda)$

Figure 2. *Cont.*

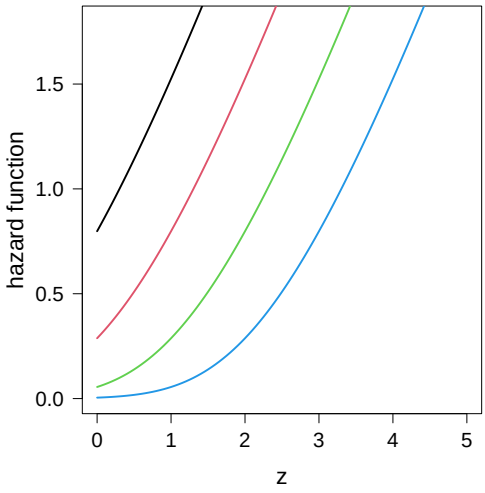


(**c**) TPLa$(\sigma = 1, \lambda)$

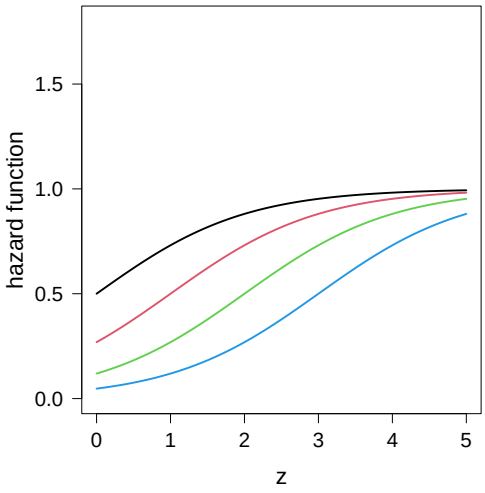


(**d**) TPC$(\sigma = 1, \lambda)$

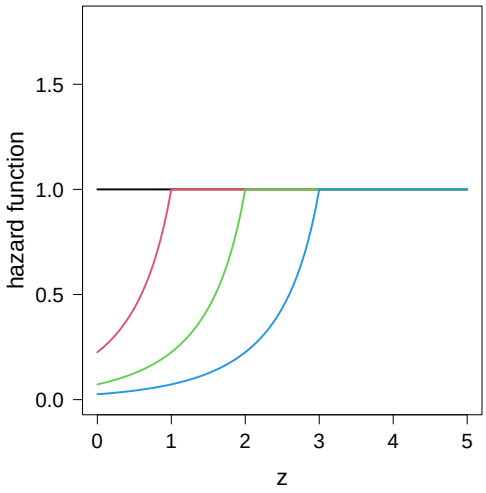
Figure 2. pdf for some particular cases of $TPS_{g_0}(\sigma = 1, \lambda)$, with $\lambda = 0$ (black line), $\lambda = 1$ (red line), $\lambda = 2$ (green line) and $\lambda = 3$ (blue line).



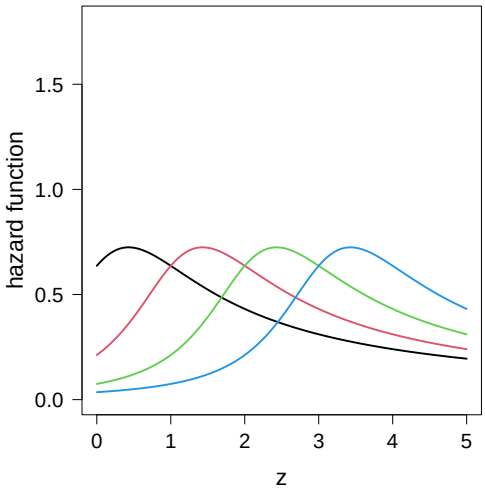
(**a**) TPN$(\sigma = 1, \lambda)$



(**b**) TPL$(\sigma = 1, \lambda)$



(**c**) TPLa$(\sigma = 1, \lambda)$



(**d**) TPC$(\sigma = 1, \lambda)$

Figure 3. Hazard function for some particular cases of $TPS_{g_0}(\sigma = 1, \lambda)$, with $\lambda = 0$ (black line), $\lambda = 1$ (red line), $\lambda = 2$ (green line) and $\lambda = 3$ (blue line).

2.3. Moments

In this subsection, we discuss the moment-generating function (mgf) and the non-central moments for the TPS model.

Proposition 4. *Let $Z \sim TPS(\sigma, \lambda)$. Then, the mgf of Z is given by*

$$M_Z(t) = \frac{e^{t\sigma\lambda}}{G_0(\lambda)} \int_{-\lambda}^{\infty} e^{t\sigma u} g_0(u)\, du, \tag{7}$$

provided that the previous integral exists for t in some neighborhood of zero.

Proof. By definition,

$$M_Z(t) = \int_0^{\infty} e^{tz} \frac{1}{\sigma G_0(\lambda)} g_0\Big(\frac{z}{\sigma} - \lambda\Big) dz.$$

Using the change $u = z/\sigma - \lambda$, we obtain the result. □

Proposition 5. *Let $Z \sim TPS(\sigma, \lambda)$ and assume that the moments associated with the pdf $g_0(\cdot)$ exist. Then, for $r = 1, 2, \ldots$, the r-th non-central moment of Z is given by*

$$\mu_r = E(Z^r) = \frac{\sigma^r}{G_0(\lambda)} \rho_r(\lambda),$$

where $\rho_r(\lambda) := \sum_{k=0}^{r} \binom{r}{k} \lambda^{r-k} \int_{-\lambda}^{\infty} u^k g_0(u) du.$

Proof. Using the definition, the non-central moment of the TPS model is

$$\mu_r = \int_0^{\infty} \frac{z^r}{\sigma G_0(\lambda)} g_0\Big(\frac{z}{\sigma} - \lambda\Big) dz.$$

Using the substitution $u = \frac{z}{\sigma} - \lambda$, we obtain

$$\mu_r = \frac{\sigma^r}{G_0(\lambda)} \int_{-\lambda}^{\infty} (u + \lambda)^r g_0(u) du.$$

Since $r \in \mathbb{N}$, the binomial theorem can be applied for the term $(u + \lambda)^r$, obtaining

$$\mu_r = \frac{\sigma^r}{G_0(\lambda)} \int_{-\lambda}^{\infty} \sum_{k=0}^{r} \binom{r}{k} u^k \lambda^{r-k} g_0(u) du.$$

As the sum converges uniformly in any closer interval $[a, b] \subset (-\lambda, \infty)$, we can then exchange the order of the integral and the sum

$$\mu_r = \frac{\sigma^r}{G_0(\lambda)} \sum_{k=0}^{r} \binom{r}{k} \lambda^{r-k} \int_{-\lambda}^{\infty} u^k g_0(u) du.$$

□

Remark 1. *Note that the two first cases for $\rho_r(\lambda)$ are given by*

$$\rho_1(\lambda) = \lambda G_0(\lambda) + \int_{-\lambda}^{\infty} u g_0(u) du, \quad \textit{and}$$
$$\rho_2(\lambda) = \lambda^2 G_0(\lambda) + 2\lambda \int_{-\lambda}^{\infty} u g_0(u) du + \int_{-\lambda}^{\infty} u^2 g_0(u) du.$$

Corollary 1. *Let $Z \sim TPS(\sigma, \lambda)$. Then, the skewness ($\sqrt{\beta_1}$) and kurtosis ($\beta_2$) coefficients of Z are given by*

$$\sqrt{\beta_1} = \frac{G_0(\lambda)\rho_3(\lambda) - 3G_0(\lambda)\rho_1(\lambda)\rho_2(\lambda) + 2\rho_1^3(\lambda)}{[G_0(\lambda)\rho_2(\lambda) - \rho_1^2(\lambda)]^{3/2}}, \quad \textit{and}$$

$$\beta_2 = \frac{G_0^3(\lambda)\rho_4(\lambda) - 4G_0^2(\lambda)\rho_1(\lambda)\rho_3(\lambda) + 6G_0(\lambda)\rho_1^2(\lambda)\rho_2(\lambda) - 3\rho_1^4(\lambda)}{[G_0(\lambda)\rho_2(\lambda) - \rho_1^2(\lambda)]^2}.$$

Remark 2. *Corollary 1 shows that the coefficients of asymmetry and kurtosis depend only on the parameter λ. Table 2 presents the values of the asymmetry and kurtosis coefficients of the TP-normal, TP-logistic and TP-Laplace models for some values of λ. Plots are given in Figure 4.*

Table 2. Asymmetry and kurtosis coefficients for different values of λ.

	TPN		**TPL**		**TPLa**	
λ	$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2
−5	1.83	7.76	1.99	8.97	2.00	9.00
−2	1.54	6.02	1.91	8.46	2.00	9.00
−1	1.32	4.99	1.77	7.74	2.00	9.00
0	0.99	3.87	1.54	6.58	2.00	9.00
1	0.59	3.00	1.21	5.28	1.69	7.76
2	0.22	2.76	0.88	4.32	1.15	6.13
5	0.00	2.99	0.23	3.73	0.26	5.29

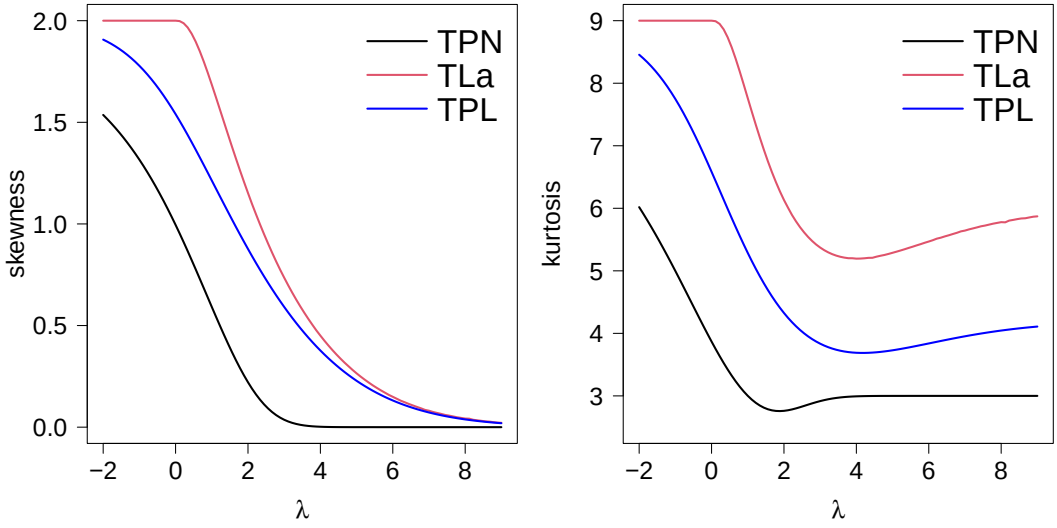


Figure 4. Skewness and kurtosis coefficients for some particular cases of the $TPS_{g_0}(\sigma = 1, \lambda)$ model.

2.4. Shannon Entropy

The Shannon entropy [21] quantifies the level of uncertainty associated with a r.v. Z. It is defined as

$$S(Z) = -E(\log f(Z)).$$

Proposition 6. *For the TPS model, the Shannon entropy is given by*

$$S(Z) = \log(\sigma G_0(\lambda)) - \frac{1}{G_0(\lambda)} \int_{-\lambda}^{\infty} \log(g_0(u)) g_0(u)\, du.$$

Proof. Using the definition,

$$\begin{aligned} S(Z) &= -E\left(\log\left\{\frac{1}{\sigma G_0(\lambda)} g_0\left(\frac{Z}{\sigma}-\lambda\right)\right\}\right) \\ &= E(\log(\sigma G_0(\lambda))) - E\left(\log g_0\left(\frac{Z}{\sigma}-\lambda\right)\right) \\ &= \log(\sigma G_0(\lambda)) - \frac{1}{\sigma G_0(\lambda)} \int_0^\infty \log g_0\Big(\frac{z}{\sigma}-\lambda\Big) g_0\Big(\frac{z}{\sigma}-\lambda\Big) dz. \end{aligned}$$

With the change $u = z/\sigma - \lambda$, we obtain

$$S(Z) = \log(\sigma G_0(\lambda)) - \frac{1}{G_0(\lambda)} \int_{-\lambda}^\infty \log g_0(u) g_0(u) dz.$$

□

2.5. Quantiles

In this subsection, we present some quantiles of the TPS distribution.

Proposition 7. *Let $Z \sim TPS(\sigma, \lambda)$. Then, the quantile function for Z is given by*

$$Q(p) = \sigma[G_0^{-1}(1 + (p-1)G_0(\lambda)) + \lambda], \quad 0 < p < 1,$$

where $G_0^{-1}(\cdot)$ denotes the inverse function of $G_0(\cdot)$.

Corollary 2. *The quartiles of the TPS distribution are given by the following:*

1. *First quartile:* $\sigma\left[G_0^{-1}\left(1 - \frac{3}{4}G_0(\lambda)\right) + \lambda\right]$;
2. *Median:* $\sigma\left[G_0^{-1}\left(1 - \frac{1}{2}G_0(\lambda)\right) + \lambda\right]$;
3. *Third quartile:* $\sigma\left[G_0^{-1}\left(1 - \frac{1}{4}G_0(\lambda)\right) + \lambda\right]$.

2.6. Mode

We now present the mode of the TPS distribution.

Proposition 8. *The mode of the $TPS(\sigma, \lambda)$ model is as follows:*

1. $z = \sigma\lambda$ *when* $\lambda \geq 0$;
2. $z = 0$ *when* $\lambda < 0$,

Proof. Since the mode of g_0 is attained at 0, the corresponding mode of the TPS satisfies $z/\sigma - \lambda = 0$. If $\lambda \geq 0$, then the mode will be attained at $z = \sigma\lambda$. If $\lambda < 0$, the pdf of the TPS model will be strictly decreasing in $(0, \infty)$; then, the mode of the model is 0. □

Remark 3. *Since $\lambda = \frac{\mu}{\sigma}$, Proposition 8 can be rewritten in terms of μ. That is the mode of the TPS model is reached at μ if $\mu \geq 0$ and at 0 if $\mu < 0$.*

2.7. Order Statistics

One of the models that naturally arises when the observations are placed in increasing order is the order statistics (see Balakrishnan and Clifford [22]). When the r.v.s are sampled from a continuous distribution with common pdf and cdf $f(\cdot)$ and $F(\cdot)$, respectively, then the pdf of the k-th order statistic is given by

$$f_{k:n}(z) = \frac{n!}{(k-1)!(n-k)!} f(z)[G(z)]^{k-1}[1-G(z)]^{n-k}, \quad z \in \mathcal{Z}, \quad k = 1, \ldots, n, \qquad (8)$$

where $\mathcal{Z}$ is the support related to the distribution. The following proposition particularizes this result for the TPS distribution.

Proposition 9. *Suppose the r.v.s $Z_1, Z_2, \ldots, \ldots, Z_n$ are independent and identically distributed TPS(σ, λ). Then, the pdf of the k-th order statistic is given by*

$$f_{k:n}(z) = \frac{n!}{\sigma(k-1)!(n-k)!G_0^n(\lambda)} g_0\Big(\frac{z}{\sigma} - \lambda\Big)\Big[G_0\Big(\frac{z}{\sigma} - \lambda\Big) + G_0(\lambda) - 1\Big]^{k-1}\Big[1 - G_0\Big(\frac{z}{\sigma} - \lambda\Big)\Big]^{n-k}.$$

Proof. This is a direct consequence of Equation (8) taking $f(z)$ as the pdf of the TPS model. □

Corollary 3. *For the minimum and maximum, the pdf is reduced to the following cases:*

1. *Minimum ($k = 1$).*

$$f_{1:n}(z) = \frac{n}{\sigma G_0^n(\lambda)} g_0\Big(\frac{z}{\sigma} - \lambda\Big)\Big[1 - G_0\Big(\frac{z}{\sigma} - \lambda\Big)\Big]^{n-1};$$

2. *Maximum ($k = n$).*

$$f_{n:n}(z) = \frac{n}{\sigma G_0^n(\lambda)} g_0\Big(\frac{z}{\sigma} - \lambda\Big)\Big[G_0\Big(\frac{z}{\sigma} - \lambda\Big) + G_0(\lambda) - 1\Big]^{n-1}. \tag{9}$$

3. Inference

In this section, we explore a traditional method for making inferences about the TPS distribution. Specifically, we delve into the estimation of moments and maximum likelihood (ML) techniques.

3.1. Moment Estimators

The moment estimators arise from solving the equations $E(Z^r) = \overline{Z^r}$ for $r = 1, 2$, where $\overline{Z^r} = n^{-1}\sum_{i=1}^n z_i^r$ refers to the r-th sample moment. Solving $E(Z) = \overline{Z}$, one obtains

$$\sigma = \frac{\overline{Z} G_0(\lambda)}{\rho_1(\lambda)}. \tag{10}$$

Replacing the first and second sample moments in the expressions given in Proposition 5, it follows that

$$\overline{Z}^2 G_0(\lambda)\rho_2(\lambda) - \overline{Z^2}\rho_1^2(\lambda) = 0. \tag{11}$$

Various software options are available to solve Equation (11). For instance, in R, the uniroot function can be utilized to calculate the moment estimator $\widehat{\lambda}_M$. The moment estimator $\widehat{\sigma}_M$ is obtained by replacing $\widehat{\lambda}_M$ in Equation (10).

Remark 4. *The application of the method of moments assumes the existence of at least the first two moments of the distribution. For instance, for $g_0(\cdot)$, the pdf of the standard Cauchy is not fulfilled.*

3.2. Maximum Likelihood Estimation

Given $z_1, \ldots, z_n$, i.e., a random sample from the TPS(σ, λ) distribution based on the logarithm of the pdf given in Equation (4), the log-likelihood function for $\boldsymbol{\theta} = (\sigma, \lambda)$ is given by

$$\ell(\boldsymbol{\theta}) = -n\log(\sigma) - n\log(G_0(\lambda)) - \sum_{i=1}^n \log\Big(g_0\Big(\frac{z_i}{\sigma} - \lambda\Big)\Big). \tag{12}$$

To compute the maximum likelihood estimator (MLE) for $\boldsymbol{\theta}$, (12) must be maximized. An equivalent approach is to solve the following system of equations: $\frac{\partial \ell(\boldsymbol{\theta})}{\partial \sigma} = 0$ and $\frac{\partial \ell(\boldsymbol{\theta})}{\partial \lambda} = 0$, namely,

$$\sum_{i=1}^{n} \frac{g_0^{(\sigma)}(v_i)}{g_0(v_i)} = -\frac{n}{\sigma}, \tag{13}$$

$$\sum_{i=1}^{n} \frac{g_0^{(\lambda)}(v_i)}{g_0(v_i)} = n\frac{g_0(\lambda)}{G_0(\lambda)}, \tag{14}$$

where $v_i = \frac{z_i}{\sigma} - \lambda$, $g_0^{(\sigma)}(v_i) = \frac{\partial g_0(v_i)}{\partial \sigma}$ and $g_0^{(\lambda)}(v_i) = \frac{\partial g_0(v_i)}{\partial \lambda}$. Equations (13) and (14) can be numerically solved by using the R-4.3.1 software [20].

3.3. Observed Fisher Information Matrix

The asymptotic variance of MLEs, $\widehat{\boldsymbol{\theta}} = (\widehat{\sigma}, \widehat{\lambda})$, can be estimated from the Fisher information matrix, given by $\mathcal{I}(\boldsymbol{\theta}) = -E\left[\partial^2 \ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top\right]$, with $\ell(\boldsymbol{\theta})$ given in (12). Recall that, under regularity conditions,

$$\mathcal{I}(\boldsymbol{\theta})^{-1/2}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \xrightarrow{\mathcal{D}} N_2(\mathbf{0}_2, \boldsymbol{I}_2), \quad \text{as } n \to +\infty, \tag{15}$$

where $\mathcal{D}$ stands for convergence in distribution and $N_2(\mathbf{0}_2, \boldsymbol{I}_2)$ denotes the standard bivariate normal distribution. Moreover, $\mathcal{I}(\boldsymbol{\theta})$ can be obtained from the matrix $-\partial^2 \ell(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top$, whose elements are given by $I_{\sigma\sigma} = -\partial^2 \ell(\boldsymbol{\theta})/\partial\sigma^2$, $I_{\sigma\lambda} = -\partial^2 \ell(\boldsymbol{\theta})/\partial\sigma\partial\lambda$ and so on. Explicitly, we have

$$\begin{aligned}
I_{\sigma\sigma} &= \frac{n}{\sigma^2} - \sum_{i=1}^{n} \frac{g_0^{(\sigma\sigma)}(v_i)g_0(v_i) - (g_0^{(\sigma)}(v_i))^2}{g_0^2(v_i)}, \\
I_{\sigma\lambda} &= \sum_{i=1}^{n} \frac{g_0^{(\sigma)}(v_i)g_0^{(\lambda)}(v_i) - g_0^{(\sigma\lambda)}(v_i)g_0(v_i)}{g_0^2(v_i)}, \\
I_{\lambda\lambda} &= \frac{n(g_0^2(\lambda) - g_0^{(\lambda)}(v_i)G(\lambda))}{G(\lambda)} + \sum_{i=1}^{n} \frac{g_0^{(\lambda\lambda)}(v_i)g_0(v_i) - (g_0^{(\lambda)}(v_i))^2}{g_0^2(v_i)},
\end{aligned}$$

where $g_0^{(\sigma\sigma)}(v_i) = \frac{\partial^2 g_0(v_i)}{\partial\sigma^2}$, $g_0^{(\sigma\lambda)}(v_i) = \frac{\partial^2 g_0(v_i)}{\partial\sigma\partial\lambda}$ and $g_0^{(\lambda\lambda)}(v_i) = \frac{\partial^2 g_0(v_i)}{\partial\lambda^2}$.

3.4. Residuals

The quantile residuals (QRs) introduced by Dunn and Smyth [23] are an ad hoc method to assess the validity of a model. For the TPS distribution, such residuals are given by

$$\widehat{r}_i = \Phi^{-1}\left(1 - \frac{1 - G_0\left(\frac{z_i}{\widehat{\sigma}} - \widehat{\lambda}\right)}{G_0(\widehat{\lambda})}\right), \quad i = 1, \ldots, n,$$

where $\widehat{\sigma}$ and $\widehat{\lambda}$ denote the ML estimators of σ and λ, respectively. If the model is correctly specified, then $\widehat{r}_1, \widehat{r}_2, \ldots, \widehat{r}_n$ will be a random sample from the standard normal distribution, which can be checked using traditional normality tests such as Anderson–Darling (AD), Cramér–von Mises (CVM) and Shapiro–Wilks (SW).

3.5. Software

The ML estimation method for the TPS model was implemented in the `tpn` package [24] from R [20], version 1.7. To achieve these outcomes, the subsequent function can be utilized:

```
est.fts(y, dist="norm")
```

where `y` is the response variable and dist should be any of the following options: `norm` (default), `logis`, `cauchy` and `laplace` for the normal, logistic, Cauchy and Laplace distributions, respectively. The TPN package provides additional functions such as `dfts`, `pfts`, `qfts` and `rfts`. These functions allow for the calculation of the pdf, cdf, quantile function and generated data for the TPS model using any of the available distributions.

4. Simulation

In this section, we assess the performance of the ML estimators for the TPS distribution in finite samples. Since the normal distribution was studied in Gómez et al. [19], we focus here on the logistic and Cauchy models. We examined two values for σ, namely, 1 and 10, as well as four values for λ, which were 0.5, 1, 2 and 3. Additionally, we considered three different sample sizes, namely, 50, 100 and 200. In total, we had 48 combinations of σ, λ and n as models. For each combination, we performed 1000 replicates and utilized the `est.tps` function from Section 3.5 to obtain the maximum likelihood estimators for the corresponding model. Table 3 provides a summary of the average bias, standard errors, root mean squared errors and coverage probabilities of the ML estimators at a 95% confidence level based on the asymptotic distribution. It is worth noting that increasing the sample size leads to a reduction in both bias and RMSE, indicating that the estimators are consistent. Moreover, as the sample size increases, the standard errors and RMSE become more similar, suggesting that the standard errors are accurately estimated. Finally, the coverage probabilities approach the nominal values used to construct the confidence intervals, indicating that the normal model-based approximation for the distribution of the ML estimators performs well, even for finite samples.

Table 3. Empirical bias, SE, RMSE and 95% CP for the ML estimators of σ and λ in the TPS distribution using the logistic and Cauchy as basal models.

	True Values			$n = 50$				$n = 100$				$n = 200$			
Model	σ	λ	**Estimator**	**Bias**	**SE**	**RMSE**	**CP**	**Bias**	**SE**	**RMSE**	**CP**	**Bias**	**SE**	**RMSE**	**CP**
logistic	1	0.5	$\widehat{\sigma}$	−0.010	0.210	0.213	0.897	−0.003	0.150	0.151	0.922	0.000	0.107	0.109	0.940
			$\widehat{\lambda}$	−0.059	1.338	1.129	0.938	0.004	0.622	0.623	0.945	−0.005	0.395	0.396	0.961
		1	$\widehat{\sigma}$	−0.005	0.197	0.196	0.919	−0.004	0.138	0.138	0.934	−0.002	0.098	0.097	0.943
			$\widehat{\lambda}$	0.017	0.760	0.751	0.958	0.022	0.475	0.480	0.959	0.011	0.330	0.329	0.955
		2	$\widehat{\sigma}$	−0.006	0.167	0.168	0.925	0.001	0.118	0.118	0.941	−0.002	0.083	0.084	0.943
			$\widehat{\lambda}$	0.068	0.577	0.594	0.956	0.021	0.403	0.403	0.958	0.018	0.283	0.288	0.951
		3	$\widehat{\sigma}$	−0.009	0.145	0.149	0.925	−0.005	0.102	0.104	0.933	−0.002	0.073	0.074	0.944
			$\widehat{\lambda}$	0.091	0.577	0.595	0.950	0.050	0.404	0.409	0.949	0.024	0.284	0.288	0.949
	10	0.5	$\widehat{\sigma}$	−0.075	2.110	2.105	0.905	−0.024	1.504	1.505	0.928	−0.008	1.064	1.078	0.934
			$\widehat{\lambda}$	−0.049	1.186	1.058	0.944	0.002	0.608	0.611	0.952	−0.001	0.394	0.405	0.953
		1	$\widehat{\sigma}$	−0.015	1.978	2.060	0.912	0.003	1.392	1.416	0.934	−0.029	0.977	0.993	0.936
			$\widehat{\lambda}$	0.029	0.746	0.764	0.950	0.013	0.476	0.482	0.959	0.007	0.330	0.333	0.955
		2	$\widehat{\sigma}$	−0.040	1.670	1.695	0.923	−0.027	1.175	1.171	0.942	−0.010	0.830	0.842	0.944
			$\widehat{\lambda}$	0.060	0.577	0.585	0.961	0.030	0.403	0.404	0.956	0.017	0.283	0.286	0.954
		3	$\widehat{\sigma}$	−0.053	1.457	1.466	0.929	−0.024	1.029	1.011	0.946	−0.010	0.727	0.720	0.944
			$\widehat{\lambda}$	0.084	0.577	0.601	0.956	0.039	0.404	0.400	0.953	0.018	0.284	0.281	0.954
Cauchy	1	0.5	$\widehat{\sigma}$	−0.032	0.219	0.213	0.924	−0.014	0.150	0.149	0.940	−0.014	0.105	0.108	0.933
			$\widehat{\lambda}$	0.028	0.506	0.523	0.977	0.007	0.322	0.551	0.974	0.029	0.176	0.176	0.956
		1	$\widehat{\sigma}$	−0.010	0.220	0.225	0.918	−0.004	0.156	0.155	0.936	−0.006	0.110	0.111	0.941
			$\widehat{\lambda}$	0.072	0.390	0.404	0.969	0.033	0.264	0.267	0.960	0.024	0.184	0.185	0.956
		2	$\widehat{\sigma}$	0.000	0.219	0.227	0.929	−0.002	0.154	0.155	0.938	−0.002	0.109	0.109	0.942
			$\widehat{\lambda}$	0.104	0.527	0.552	0.954	0.052	0.364	0.375	0.952	0.029	0.254	0.257	0.952
		3	$\widehat{\sigma}$	−0.002	0.214	0.221	0.929	0.000	0.151	0.153	0.940	−0.003	0.106	0.105	0.945
			$\widehat{\lambda}$	0.153	0.716	0.750	0.950	0.070	0.492	0.503	0.950	0.043	0.344	0.340	0.957

Table 3. *Cont.*

Model	True Values σ	True Values λ	Estimator	$n = 50$ Bias	SE	RMSE	CP	$n = 100$ Bias	SE	RMSE	CP	$n = 200$ Bias	SE	RMSE	CP
	10	0.5	$\widehat{\sigma}$	−0.345	2.203	2.167	0.914	−0.088	1.502	1.498	0.940	−0.149	1.052	1.047	0.943
			$\widehat{\lambda}$	0.032	0.499	0.518	0.975	0.002	0.366	0.549	0.978	0.031	0.175	0.175	0.965
		1	$\widehat{\sigma}$	−0.134	2.184	2.207	0.923	−0.071	1.560	1.576	0.930	−0.083	1.100	1.095	0.944
			$\widehat{\lambda}$	0.071	0.384	0.392	0.975	0.030	0.326	0.532	0.959	0.027	0.184	0.184	0.959
		2	$\widehat{\sigma}$	−0.048	2.176	2.172	0.925	−0.003	1.544	1.556	0.940	−0.015	1.087	1.079	0.944
			$\widehat{\lambda}$	0.112	0.528	0.548	0.959	0.050	0.364	0.371	0.953	0.027	0.254	0.254	0.958
		3	$\widehat{\sigma}$	−0.009	2.149	2.218	0.928	−0.004	1.507	1.540	0.939	−0.018	1.062	1.079	0.941
			$\widehat{\lambda}$	0.149	0.717	0.751	0.953	0.074	0.492	0.502	0.952	0.041	0.344	0.353	0.944

5. Application to Real Data

In this section, we illustrate our family of models using a database associated with survival times of Guinea pigs injected with different doses of tubercle bacilli, specifically, regimen 66; these data were studied by Bjerkedal [25]. The study of the survival time of Guinea pigs under different diets is crucial for identifying patterns and ratios, quantifying uncertainty in predictions and comparing the impact of the different diets on the animals' longevity. This approach contributes to progress in investigation into animal nutrition and welfare. Table 4 shows a descriptive analysis of the data.

Table 4. Descriptive statistics of the Survival Times (in days) of Guinea Pigs dataset.

Dataset	n	$\overline{X}$	S^2	$\sqrt{b_1}$	b_2
Survival Times of Guinea Pigs in Days	72	99.82	6580.12	1.80	5.61

Table 5 shows the estimations for the TPN, TPLa, TPC and TPL models in the Guinea pigs data and the respective AIC and BIC criteria. Note that the TPC model provides the lowest AIC; this is corroborated by Figure 5, which shows that the TPC model provides better performance for this dataset than competing models and a better fit in the accumulated distribution (green line cdf TPC) than the empirical fit (black line). The Kolmogorov–Smirnov (KS) distance and the corresponding p-values are also included. These summaries further support our conclusions.

Table 5. Estimated parameters and their standard errors (in parentheses) for the TPN, TPLa, TPC and TPL models for the Survival Times (in days) of Guinea Pigs dataset. The AIC and BIC criteria are also presented.

Parameters	TPN	TPLa	TPC	TPL
σ	154.944 (47.298)	67.772 (0.035)	29.926 (5.895)	57.513 (10.091)
λ	−0.488 (0.762)	0.885 (0.004)	2.133 (0.451)	0.749 (0.556)
log-likelihood	−401.46	−392.27	−391.16	−399.38
AIC	806.92	788.54	786.33	802.75
BIC	811.47	793.09	790.88	807.31
KS (p-value)	0.152 (0.073)	0.105 (0.400)	0.085 (0.675)	0.126 (0.200)

In order to assess the suitability of each model, we conducted an analysis using the QRs introduced in Section 3.4. Based on the findings presented in Figure 6 (TPN, TPLa, TPC and TPL), it can be concluded that, among all the fitted models, only the TPC model is suitable for this particular dataset.

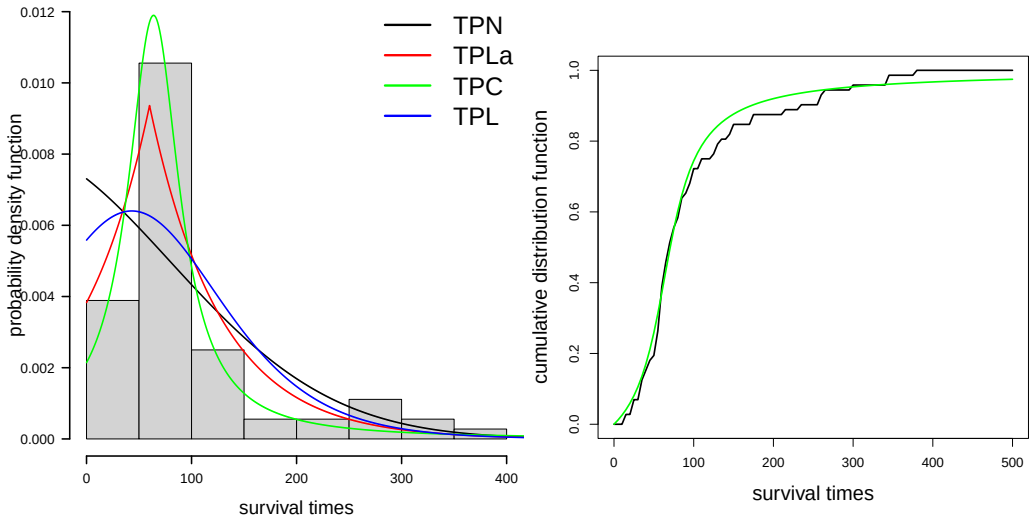
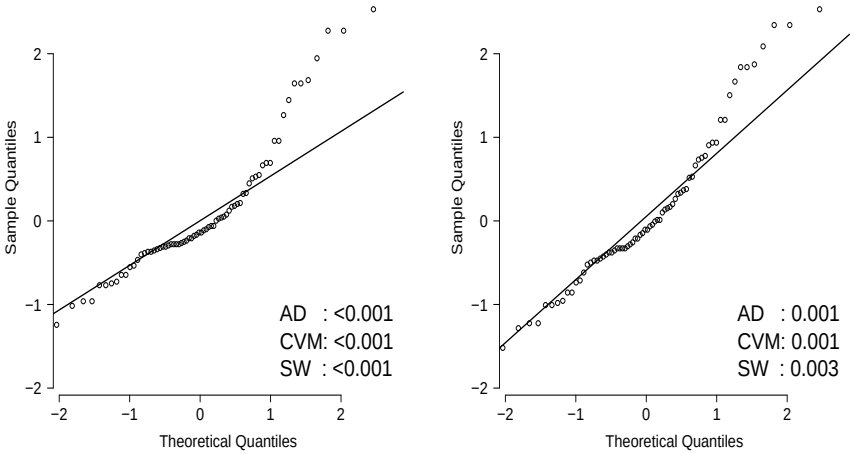
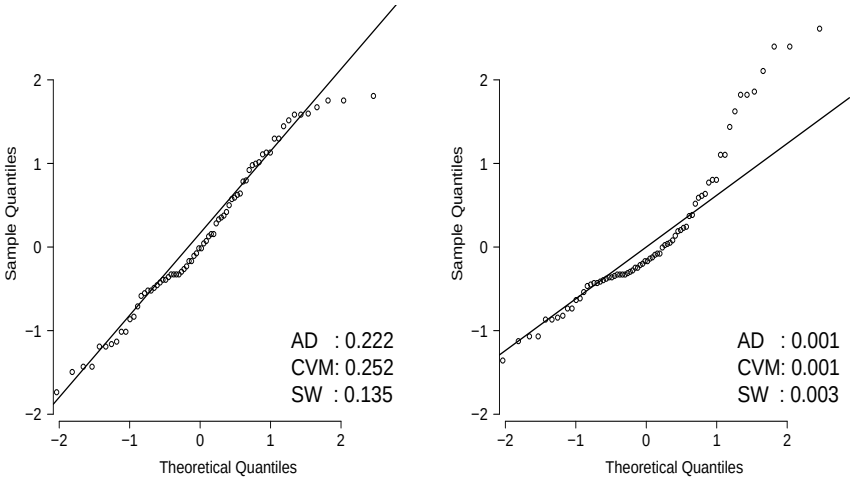


Figure 5. Family models adjusted by the maximum likelihood method for the Survival Times (in days) of Guinea Pigs dataset and the fit of the cumulative distribution of the TPC model with the empirical one.



(**a**) TPN (**b**) TPLa



(**c**) TPC (**d**) TPL

Figure 6. QRs for the fitted models of the Survival Times (in days) of Guinea Pigs dataset. The *p*-values for the AD, CVM and SW normality tests are also presented.

6. Conclusions

This paper presents the TPS family of distributions based on univariate symmetric distributions, which contains, as a special case, the HS family. The TPS family has two parameters: scale and shape. This fact makes it an attractive competitor to several two-parameter models used in different areas of knowledge. The TPS family is a viable alternative to the HS family of distributions for fitting continuous positive data. Relevant characteristics of the TPS family studied in this paper are as follows:

- The TPS family of distributions has turned out to be more flexible than the HS family, since it has a shape parameter;
- The TPS family of distributions is based on the univariate symmetric family of distributions. It has been proven that its properties are manageable and depend to a large extent on the properties of the pdf and cdf of the basal symmetric distribution;
- pdf, cdf and risk functions are represented by very well-known functions;
- A simulation study was carried out for some members of the TPS family. The ML estimators showed good performance for moderate sample sizes;
- A real application is given, in which several members of the TPS family were considered and fitted. The AIC and BIC model selection criteria showed which member had the best fit to this dataset. The AD, CVM and SW tests were carried out to validate the normality of the QR;
- As for the applicability of our results, we highlight that our proposal allowed us to incorporate a variety of shapes for modeling positive data, as has been established in Section 2. Therefore, these models can be considered as competitors to those commonly used for modeling lifetime data and other disciplines with positive data. On the other hand, an important advantage is that the new models inherit theoretical properties from the base models on which they are built, which increases their interest from an applied point of view;
- Practitioners may use the R package tpn for modeling positive continuous data.

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