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A Hybrid Method for All Types of Solutions of the System of Cauchy-Type Singular Integral Equations of the First Kind

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Abstract: In this note, the hybrid method (combination of the homotopy perturbation method (HPM) and the Gauss elimination method (GEM)) is developed as a semi-analytical solution for the first kind system of Cauchy-type singular integral equations (CSIEs) with constant coefficients. Before applying the HPM, we have to first reduce the system of CSIEs into a triangle system of algebraic equations using GEM, which is then carried out using the HPM. Using the theory of the bounded, unbounded and semi-bounded solutions of CSIEs, we are able to find inverse operators for the system of CSIEs of the first kind. A stability analysis and convergent of the proposed method has been conducted in the weighted L_p space. Moreover, the proposed method is proven to be exact in the Holder class of functions for the system of characteristic SIEs for any type of initial guess. For each of the four cases, several examples are provided and examined to demonstrate the proposed method's validity and accuracy. Obtained results are compared with the Chebyshev collocation method and modified HPM (MHPM). Example 3 reveals that the error term of the MHPM is slightly superior to that of the HPM. One of the features of the proposed method is that it can be solved as a complex-valued system of CSIEs. Numerical results revealed that the hybrid method dominates others.

Keywords: hybrid method; cauchy-type singular integral equations; gauss elimination method; homotopy perturbation method; stability; convergence

MSC: 45F15; 6504; 45M10

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1. Introduction

The Cauchy singular integral equations (CSIEs) occur naturally in many fields of science, including cruciform crack problems in fracture mechanics, oscillating aerofoils problems in aerodynamics, scattering of surface water waves in hydrodynamics, contact radiations, and electrodynamics. Since finding analytical solutions for CSIEs with weak or strong singularities is challenging, many researchers have developed numerical methods with significant accuracy to solve these equations. Due to its wide and practical applications, researchers are interested in solving the system of CSIEs. It is well known that the SIEs of Abel and Cauchy-type appear in many branches of scientific fields, such as stereology [1], radio occultation (RO) measurements [2], radio astronomy [3], molecular scattering [4], electron emission [5], radar ranging [6], plasma spectroscopy [7], X-ray tomography [8], and so on.

Investigations of the system of SIEs have attracted much concern in the applied sciences. Their general ideas and essential features are broadly applicable in engineering science. The solution to a large class of mixed boundary value problems in physics and engineering is reduced to a one-dimensional system of SIEs. Unfortunately, not many researchers dealt with the system of Cauchy and Abel-type SIEs and their generalised form. For instance, Erdogan [9], as a pioneer in 1969, considered systems of simultaneous singular integral equations of the Cauchy-type. The author used the properties of the related orthogonal polynomials to solve the systems of SIEs approximately. He applied Chebyshev polynomials of the first or second kind to find the bounded solution of Cauchy-type SIEs of the first kind. Kumar et al. used the homotopy perturbation method (HPM) to solve the system of generalized Abel's integral equations [10], while Wazwaz [11] considered a 2×2 system of generalised Abel's equation and solved it by using Laplace transform method.

Moreover, Turhan et al. [12] solved the system of Cauchy-type singular integral equations of the first kind by the Chebyshev collocation method. At the same time, Durua and Yusufog'lu [13] found a semi-bounded solution of the system of SIEs of the first kind by Chebyshev polynomials of the third and fourth kinds. Furthermore, Shahmorad and Ahdiaghdam [14] proposed Chebyshev polynomials approximation for the numerical solution of a system of Cauchy-type singular integral equations of the first kind on a finite segment. Taylor Expansion method as an approximate approach for systems of singular Volterra integral equations is proposed by Didgar and Vahidi [15]. There are many methods developed for one-dimensional CSIEs (see [16–23]). However, not many researchers have researched the system of CSIEs [10–15]. Nevertheless, the HPM for the system of CSIEs has rarely been applied, and very few articles have been published.

The HPM and the homotopy analysis method (HAM) are vital tools for solving linear and nonlinear problems in various scientific and technological fields. Using the HPM and HAM, researchers have successfully tackled a wide range of nonlinear problems. For instance, the HPM has been applied to solve linear and nonlinear integral equations [24], special nonlinear Fredholm integral equations [25], non-linear functional integral equations [26–28], the quadratic Ricatti differential equation [29], and the nonlinear equation [30], as well as the nonlinear second-order differential equation [31]. On the other hand, HAM has been utilized to address the linear system of Fredholm–Volterra Integral equations [32], among many others.

In this note, the application of the HPM and MHPM is demonstrated for the system of the singular integral equation of the first kind given by

$$\sum_{j=1}^{M}\left[\frac{a_{ij}}{\pi}\int_{-1}^{1}\frac{u_j(\tau)d\tau}{\tau-t}+\frac{b_{ij}}{\pi}\int_{-1}^{1}K_{ij}(t,\tau)u_j(\tau)d\tau\right]=f_i(t),\, i=1,\ldots,M,\, -1<t<1, \tag{1}$$

where $A=\left(a_{ij}\right)$ and $B=\left(b_{ij}\right)$ are given constant matrices with $\det(A)\neq 0,\ \det(B)\neq 0$, the forcing functions $f_i(t)$ and kernels K_{ij} are all known to be real-valued or complex-valued continuous functions and $u_j,\ j=1,2,\ldots,M$ are unknown functions to be determined.

This paper is arranged in the following manner. After introducing the background on singular integral equations in Section 1, the four types of solutions of Equation (1) and reduction techniques are presented in Section 2. In Section 3, standard HAM and modified HAM are demonstrated by implementing the CSIEs (1). Section 4, proves the stability and convergent of the proposed method in the weighted L_p space. Section 5, deals with many examples and show a comparison of the hybrid method with the Chebyshev collocation method given in [12,14]. Finally, some conclusions and acknowledgements are given in Section 6.

2. Methodology and Reduction Techniques

It is known that the characteristic singular integral equations of the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(\tau)d\tau}{\tau-t}=f(t),\ -1<t<1 \tag{2}$$

have four types of solutions (Lifanov [23], p. 5).

Case 1. The solution is bounded at the endpoints $t = \pm 1$, which yields

$$\begin{cases} u(t) = -\dfrac{\sqrt{1-t^2}}{\pi} \int_{-1}^{1} \dfrac{f(\tau)d\tau}{\sqrt{1-\tau^2}(\tau - t)}, \\ \int_{-1}^{1} \dfrac{f(\tau)}{\sqrt{1-\tau^2}} d\tau = 0. \end{cases} \tag{3}$$

Case 2. The solution is unbounded at the endpoints $t = \pm 1$, yielding

$$\begin{cases} u(t) = -\dfrac{1}{\pi\sqrt{1-t^2}} \int_{-1}^{1} \dfrac{\sqrt{1-\tau^2} f(\tau)d\tau}{\tau - t} + \dfrac{c}{\pi\sqrt{1-t^2}}, \\ \int_{-1}^{1} u(\tau)d\tau = c, c \text{ may take zero value.} \end{cases} \tag{4}$$

Case 3. The solution is bounded at the endpoint $t = -1$, but unbounded at the end point $t = 1$, yielding

$$u(t) = -\frac{1}{\pi}\sqrt{\frac{1+t}{1-t}} \int_{-1}^{1} \sqrt{\frac{1-\tau}{1+\tau}} \frac{f(\tau)}{\tau - t} d\tau \tag{5}$$

Case 4. The solution is bounded at the endpoint $t = -1$, but unbounded at the endpoint $t = -1$, yielding

$$u(t) = -\frac{1}{\pi}\sqrt{\frac{1-t}{1+t}} \int_{-1}^{1} \sqrt{\frac{1+\tau}{1-\tau}} \frac{f(\tau)}{\tau - t} d\tau \tag{6}$$

Similarly, if we consider the CSIEs of the form

$$\frac{1}{\pi}\int_{-1}^{1} \frac{u(\tau)d\tau}{\tau - t} + \frac{1}{\pi}\int_{-1}^{1} K(t,\tau)u(\tau)d\tau = f(t), \ -1 < t < 1 \tag{7}$$

we again have four cases of the solutions obtained using (3)–(6).

Case 1. The bounded solution at the endpoints $t = \pm 1$ of Equation (7) is given by

$$\begin{cases} u(t) = \dfrac{\sqrt{1-t^2}}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} \dfrac{K(t,s)u(s)ds}{\sqrt{1-\tau^2}(\tau - t)} d\tau - \dfrac{\sqrt{1-t^2}}{\pi} \int_{-1}^{1} \dfrac{f(\tau)d\tau}{\sqrt{1-\tau^2}(\tau - t)}, \\ \int_{-1}^{1} \dfrac{1}{\sqrt{1-\tau^2}} \left[f(\tau) - \int_{-1}^{1} K(\tau,t)u(t)dt \right] d\tau = 0. \end{cases} \tag{8}$$

Case 2. The unbounded solution at the endpoints $t = \pm 1$ of Equation (7) is given by

$$\begin{cases} u(t) = \dfrac{1}{\pi^2\sqrt{1-t^2}} \int_{-1}^{1} \int_{-1}^{1} \dfrac{\sqrt{1-\tau^2} K(t,s)u(s)ds}{\tau - t} d\tau - \dfrac{1}{\pi\sqrt{1-t^2}} \int_{-1}^{1} \dfrac{\sqrt{1-\tau^2} f(\tau)d\tau}{\tau - t}, \\ \int_{-1}^{1} u(\tau)d\tau = 0. \end{cases} \tag{9}$$

Case 3. The solution is bounded at the endpoint $t = -1$, but unbounded at the endpoint $t = 1$ for Equation (7), given by

$$u(t) = \frac{1}{\pi^2}\sqrt{\frac{1+t}{1-t}} \int_{-1}^{1} \int_{-1}^{1} \sqrt{\frac{1-\tau}{1+\tau}} \frac{K(t,s)u(s)ds}{\tau - t} d\tau - \frac{1}{\pi}\sqrt{\frac{1+t}{1-t}} \int_{-1}^{1} \sqrt{\frac{1-\tau}{1+\tau}} \frac{f(\tau)}{\tau - t} d\tau. \tag{10}$$

Case 4. The solution is bounded at the endpoint $t = 1$, but unbounded at the endpoint $t = -1$, given by

$$u(t) = \frac{1}{\pi^2}\sqrt{\frac{1-t}{1+t}} \int_{-1}^{1} \int_{-1}^{1} \sqrt{\frac{1+\tau}{1-\tau}} \frac{K(t,s)u(s)ds}{\tau - t} d\tau - \frac{1}{\pi}\sqrt{\frac{1-t}{1+t}} \int_{-1}^{1} \sqrt{\frac{1+\tau}{1-\tau}} \frac{f(\tau)}{\tau - t} d\tau \tag{11}$$

Let us consider Equation (1) and write it as algebraic equations given below

$$\begin{cases} a_{11}S_1(t) + a_{12}S_2(t) + \ldots + a_{1M}S_M(t) = f_1(t) - F_1(t), \\ a_{21}S_1(t) + a_{22}S_2(t) + \ldots + a_{2M}S_M(t) = f_2(t) - F_2(t), \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{M1}S_1(t) + a_{M2}S_2(t) + \ldots + a_{MM}S_M(t) = f_M(t) - F_M(t), \end{cases} \tag{12}$$

where

$$\begin{aligned} S_i(t) &= \frac{1}{\pi}\int_{-1}^{1} \frac{u_i(\tau)}{\tau - t} d\tau, \ i = \{1, 2, \ldots, M\}, \\ F_i(t) &= \sum_{j=1}^{M} \frac{b_{ij}}{\pi} \int_{-1}^{1} K_{ij}(t,\tau) u_j(\tau) d\tau, i = \{1, 2, \ldots, M\}. \end{aligned} \tag{13}$$

To solve Equation (12), in the case of $M = 3$, extension matrix $\overline{A}$ has the form

$$\overline{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & f_1(t) - F_1(t) \\ a_{21} & a_{22} & a_{23} & f_2(t) - F_2(t) \\ a_{31} & a_{32} & a_{33} & f_3(t) - F_3(t) \end{bmatrix} \tag{14}$$

By applying the Gaussian elimination technique to Equation (14), we obtain

$$\overline{A} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{C_3}(e_{11}f_1 + e_{12}f_2 + e_{13}f_3) - \frac{1}{C_3}(e_{11}F_1 + e_{12}F_2 + e_{13}F_3) \\ 0 & 1 & 0 & \frac{1}{C_3}(e_{21}f_1 + e_{22}f_2 + e_{23}f_3) - \frac{1}{C_3}(e_{21}F_1 + e_{22}F_2 + e_{23}F_3) \\ 0 & 0 & 1 & \frac{1}{C_3}(e_{31}f_1 + e_{32}f_2 + e_{33}f_3) - \frac{1}{C_3}(e_{31}F_1 + e_{32}F_2 + e_{33}F_3) \end{bmatrix}, \tag{15}$$

where $C_3 = a_{11}a_{23}a_{32} - a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31}$ and

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} \tag{16}$$

Then, Equations (12) and (13) have the form

$$\begin{cases} \frac{1}{\pi}\int_{-1}^{1} \frac{u_1(\tau)d\tau}{\tau - t} + \frac{1}{C_3}(e_{11}F_1 + e_{12}F_2 + e_{13}F_3) = \frac{1}{C_3}(e_{11}f_1 + e_{12}f_2 + e_{13}f_3), \\ \frac{1}{\pi}\int_{-1}^{1} \frac{u_2(\tau)d\tau}{\tau - t} + \frac{1}{C_3}(e_{21}F_1 + e_{22}F_2 + e_{23}F_3) = \frac{1}{C_3}(e_{21}f_1 + e_{22}f_2 + e_{23}f_3), \\ \frac{1}{\pi}\int_{-1}^{1} \frac{u_3(\tau)d\tau}{\tau - t} + \frac{1}{C_3}(e_{31}F_1 + e_{32}F_2 + e_{33}F_3) = \frac{1}{C_3}(e_{31}f_1 + e_{32}f_2 + e_{33}f_3). \end{cases} \tag{17}$$

In order to generalize Equation (17) for different values of M, we can write it in the following operator form:

$$L(u_i) + N_i(\overline{u}) = f_i^*(t), i = 1, \ldots, M, \tag{18}$$

where $\overline{u} = (u_1(t), u_2(t), \ldots, u_M(t))$ and

$$\begin{aligned} L(u_i) &= \frac{1}{\pi}\int_{-1}^{1} \frac{u_i(\tau)d\tau}{\tau - t}, N_i(\overline{u}) = \frac{1}{C_M}\sum_{j=1}^{M} e_{ij}^{[M]} F_j(\overline{u}(t)), \\ F_i(\overline{u}(t)) &= \sum_{j=1}^{M} \frac{b_{ij}}{\pi}\int_{-1}^{1} K_{ij}(t,\tau)u_j(\tau)d\tau, f_i^*(t) = \frac{1}{C_M}\sum_{j=1}^{M} e_{ij}^{[M]} f_j(t). \end{aligned} \tag{19}$$

Here, coefficients C_M and $e_{ij}^{[M]}$ are defined by solving $M \times M$ algebraic equations. Equations (18) and (19) can be written in vector form as follows

$$\overline{L}(\overline{u}) + \overline{N}(\overline{u}) = \overline{f}^*, \tag{20}$$

where $\overline{u} = (u_1, u_2, \ldots, u_M)$, $\overline{f}^* = (f_1{}^*, f_2{}^*, \ldots, f_M{}^*)$ and

$$\begin{aligned}\overline{L}(\overline{u}) &= (L(u_1), L(u_2), \ldots, L(u_M)) \\ \overline{N}(\overline{u}) &= (N_1(\overline{u}), N_2(\overline{u}), \ldots, N_M(\overline{u})).\end{aligned}$$

To find the bounded, unbounded and semi-bounded solutions of Equations (18) and (19), we search for the solution in the form given by:

$$u_{i,r}(t) = w_r(t)v_i(t),\ r \in \{1,2,3,4\}, i = 1, \ldots, M, \tag{21}$$

where

$$\begin{cases} w_1(t) = \sqrt{1-t^2}, w_3(t) = \sqrt{\dfrac{1-t}{1+t}}, \\ w_2(t) = \dfrac{1}{\sqrt{1-t^2}},\ w_4(t) = \sqrt{\dfrac{1+t}{1-t}}. \end{cases} \tag{22}$$

Remark 1. *The Gauss elimination method and the existence of an inverse operator helped us greatly to handle the system of CSIEs (1). Direct implementation of the HPM or MHPM for solving CSIEs (1) yielded no good results. Fortunately, the hybrid method gave us highly accurate results.*

The detailed implementation of the HPM and MHPM is given in the next section.

3. Description of the HPM and Its Application

3.1. Theory of Standard HPM

To illustrate the basic concept of the HPM, consider the following nonlinear functional equation [25] given by:

$$\begin{aligned} A(u) &= f(t),\ \ t \in \Omega, \\ B\left(u, \frac{\partial u}{\partial n}\right) &= 0,\ \ t \in \Gamma, \end{aligned} \tag{23}$$

where A is a general functional operator, while $f(t)$ is a known analytic function. Furthermore, we divide A into two parts

$$A = L + N \tag{24}$$

where L is the linear operator and N is the nonlinear operator.

Due to Equation (23) and by using Equation (24), we now obtain

$$L(u) + N(u) = f(t). \tag{25}$$

It is known that in a convex homotopy form, the perturbation scheme is constructed as follows

$$H^*(v,p) = (1-p)(L(v) - L(u_0)) + p(L(v) + N(v) - f(t)) = 0,\ p \in [0,1] \tag{26}$$

where p is the homotopy parameter and u_0 is an initial guess satisfying boundary condition in Equation (23). It can be easily seen that when $p = 0$ and $p = 1$, we have

$$\begin{aligned} H^*(v,0) &= L(v) - L(u_0) = 0, \\ H^*(v,1) &= L(v) + N(v) - f(t) = 0, \end{aligned} \tag{27}$$

in which ν varies from the initial value u_0 to the exact solution u of Equation (25). This is called deformation and equating the homotopy function in Equation (26) to zero yields deformation equations

$$L(v(t,p)) = L(u_0(t)) + p(f(t) - N(v(t,p)) - L(u_0(t))) \tag{28}$$

We now search for the solution to Equation (28) as follows

$$v(t,p) = \sum_{k=0}^{\infty} v_k(t) p^k \tag{29}$$

Substituting Equation (29) into Equation (28) yields

$$L\left(\sum_{k=0}^{\infty} v_k(t) p^k\right) = L(u_0(t)) + p\left(f(t) - N\left(\sum_{k=0}^{\infty} v_k(t) p^k\right) - L(u_0(t))\right) \tag{30}$$

Comparing coefficients of terms in Equation (30) with identical powers of p yields

$$\begin{array}{l} p^0 : v_0(t) = u_0(t), \\ p^1 : v_1(t) = L^{-1}(f(t)) - L^{-1}(N(v_0(t))) - u_0(t), \\ p^k : v_k(t) = -L^{-1}(N(v_{k-1}(t))), k = 2, 3, \ldots, \end{array} \tag{31}$$

where L^{-1} is the inverse operator of L.
Hence, the analytic-approximate solution is given by

$$u = \lim_{p \to 1} v(t,p) = v_0(t) + v_1(t) + \ldots + v_m(t) + \ldots \approx \sum_{k=0}^{m} v_k(t) \tag{32}$$

3.2. Application of the HPM and MHPM to the System of CSIEs

We now solve the CSIEs (18)–(19) system by the proposed method. To solve it, we construct a convex homotopy function as follows

$$(1-p)(L(v_i(t,p)) - L(u_{i,0}(t))) + p(L(v_i(t,p)) + N_i(v_i(t,p)) - f_i^*(t)) = 0 \tag{33}$$

and search for unknown function $v_i(t)$ in the form

$$v_i(t,p) = \sum_{k=0}^{\infty} v_{i,k}(t) p^k, i = 1, \ldots, M \tag{34}$$

Then, we have

$$L\left(\sum_{k=0}^{\infty} v_{i,k}(t) p^k\right) = L(u_{i,0}(t)) + p\left(f_i^*(t) - N_i\left(\sum_{k=0}^{\infty} v_{i,k}(t) p^k\right) - L(u_{i,0}(t))\right) \tag{35}$$

By equating the coefficient of the terms according to the same power of p, we obtain

$$\begin{array}{l} p^0 : v_{i,0}(t) = u_{i,0}(t), \quad i = \{1, 2, \ldots, M\}, \\ p^1 : v_{i,1}(t) = L^{-1}\left(f_i^*(t)\right) - L^{-1}(N_i(v_{i,0}(t))) - u_{i,0}(t), \\ p^k : v_{i,k}(t) = -L^{-1}(N_i(v_{i,k-1}(t))), k = 2, 3, \ldots, \end{array} \tag{36}$$

where L^{-1} is the inverse operator of L.

A semi-analytical approximate solution can be obtained by Equation (32). In the practical problem, we usually choose the initial guess $u_{i,0}$ in the standard HPM given as follows:

$$u_{i,0} = f_i^*(t), \; i = 1, 2, \ldots, M. \tag{37}$$

In the case of decomposition $f_i^*(t) = f_{1i}^*(t) + f_{2i}^*(t)$, we have the following scheme

$$\begin{array}{l} p^0 : v_{i,0}(t) = L^{-1}\left(f_{1i}^*(t)\right), \\ p^1 : v_{i,1}(t) = L^{-1}\left(f_{2i}^*(t)\right) - L^{-1}(N_i(v_{i,0}(t))), \\ p^k : v_{i,k}(t) = -L^{-1}(N_i(v_{i,k-1}(t))), k = 2, 3, \ldots. \end{array} \tag{38}$$

The scheme in Equation (38) is called modified HPM (MHPM). Therefore, if we apply standard HPM in Equation (36) and MHPM in Equation (38) for the Equation (21), then the original solution to the problem is given by:

$$u_{i,r}(t) = w_r(t)\lim_{p\to 1} v_i(t) = w_r(t)\lim_{N\to\infty}\sum_{j=0}^{N} v_{ij}(t), r \in \{1,2,3,4\},\ i = \{1,\ldots,M\}. \tag{39}$$

Note that if the operator $N_i(\overline{u}) = 0$ in Equation (18), then the operator in Equation (18) becomes

$$L(u_i) = f_i^*(t), i = 1,2,\ldots M. \tag{40}$$

If the operator L is invertible, then the exact solution of Equation (40) is

$$u_{i,r}(t) = w_r(t)L^{-1}(f_i^*(t)), r \in \{1,2,3,4\}. \tag{41}$$

Let us now find the exact solution of Equation (40) using the standard HPM. From Equation (36), it follows that

$$\begin{aligned} &p^0 : v_{i,0}(t) = u_{i,0}(t),\\ &p^1 : v_{i,1}(t) = L^{-1}\left(f_i^*(t)\right) - u_{i,0}(t),\\ &p^k : v_{i,k}(t) = 0, k = 2,3,\ldots. \end{aligned} \tag{42}$$

Now, from Equations (39) and (42), it follows that

$$u_{i,r}(t) = w_r(t)\lim_{p\to 1} v_i(t) = w_r(t)\sum_{k=0}^{\infty} v_{i,k}(t) = w_r(t)L^{-1}(f_i^*(t)), r \in \{1,2,3,4\}, \tag{43}$$

which coincides with the exact solution given in Equation (41).

Schemes (42) and (43) lead to the following theorem.

Theorem 1. *Let the kernel in Equation (18) be a Cauchy singular kernel given by* $\frac{1}{\tau - t}$ *and* $f_i^*(t) \in H^\alpha[-1,1]$ *(Holder class). If the operator L in Equation (40) is linear, then the iterative scheme (42) provides an exact solution for the operator Equation (40) in any chosen initial guess.*

Proof. Let $u_{i,r}(t) = w_r L^{-1}\left(f_i^*(t)\right)(\, r = \{1,2,3,4\})$ be a solution of Equation (18) when $N_i(\overline{u}) \equiv 0$. In this case, by solving Equation (18) using the HPM in (42), we obtain

$$\begin{aligned} &p^0 : v_{i,0}(t) = u_{i,0}(t),\\ &p^1 : v_{i,1}(t) = L^{-1}\left(f_i^*(t)\right) - u_{i,0}(t),\\ &p^k : v_{i,k}(t) = 0, k = 2,3,\ldots. \end{aligned}$$

Hence, the solution of the HPM leads to

$$\begin{aligned} u_{ir}(t) &= w_r(t)(v_{i,0}(t) + v_{i,1}(t) + 0)\\ &= w_r(t)\left[u_{i,0}(t) + L^{-1}\left(f_i^*(t)\right) - u_{i,0}(t)\right] = w_r(t)L^{-1}\left(f_i^*(t)\right), \end{aligned}$$

which is identical to the exact solution. □

4. Stability Analysis and Convergence

4.1. Stability Analysis of the Solution

Stability theory in mathematics studies the behavior and robustness of solutions in differential equations, integral equations, fractional integral equations, and dynamical systems under small perturbations of initial conditions (see [33–35]). It explores whether solutions remain bounded and predictable or exhibit unpredictable and divergent behavior. By analyzing the stability properties of mathematical models, stability theory provides insights into long-term behavior and system reliability. It plays a crucial role in physics, engineering, and biology, helping to understand the behavior of dynamical systems. Stability theory also extends to integral equations and fractional integral equations, where it evaluates the effects of perturbations on stability and guides numerical methods. Overall, stability theory is essential for assessing system behavior and selecting reliable models.

Definition 1. (Eberly [36]). *A numerical method is said to be stable if small changes in the initial data for the differential equation produce correspondingly small changes in the subsequent approximations.*

Let us consider the stability solution of CSIE in Equation (18). Let the approximate the solution of this equation be given by (35) or (38). We wish to examine the effect on the solution $u(t)$ when the input $f(t)$ is corrupted with noise $\delta f(t)$, where $\delta f(t)$ is unknown except for a restriction on its relative magnitude to $f(t)$.

Theorem 2. *Let the input function* $\overline{f}^*(t) = (f_1{}^*(t), f_2{}^*(t), \ldots, f_M{}^*(t))$ *in Equation (18) be perturbated by the noise* $\delta\overline{f}^*(t) = (\delta f_1{}^*(t), \delta f_2{}^*(t), \ldots, \delta f_M{}^*(t))$. *Then, the solution vector* $\overline{u}(t) = (u_1(t), u_2(t), \ldots, u_M(t))$ *defined by Equation (20) is equivalent to the solution of the form*

$$L(\delta\overline{u}(t)) + N(\delta\overline{u}(t)) = \delta\overline{f}^*(t). \tag{44}$$

Proof: Let the right side function of Equation (18) be perturbed. Then, right side of Equation (18) has the form

$$\delta f^*(t) = (\delta f_1^*(t), \delta f_2^*(t), \ldots, \delta f_M^*(t)),$$

Assume that $u_i(t)$ will obtain an increment $\widetilde{u}_i = u_i(t) + \varepsilon_i(t)$. Then

$$L(\widetilde{u}_i) + N_i\left(\overline{\widetilde{u}}\right) = f_i^*(t) + \delta f_i^*(t), i = 1, 2\ldots, M. \tag{45}$$

Applying the HPM to Equation (45), we obtain

$$L(\widetilde{v}_i) + N_i\left(\overline{\widetilde{v}}\right) = f_i^*(t) + \delta f_i^*(t), \tag{46}$$

which yields

$$\begin{aligned}
&p^0 : \widetilde{v}_{i,0}(t) = \widetilde{u}_{i,0}(t) = u_{i,0}(t) + \varepsilon_{i,0}(t),\\
&p^1 : \widetilde{v}_{i,1}(t) = L^{-1}\left(f_i^*(t) + \delta f_i^*(t)\right) - L^{-1}(N_i(v_{i,0}(t) + \varepsilon_{i,0}(t))) - \widetilde{u}_{i,0}(t),\\
&p^2 : \widetilde{v}_{i,2}(t) = -L^{-1}(N_i(v_{i,1}(t) + \varepsilon_{i,1}(t)))\\
&p^k : \widetilde{v}_{i,k}(t) = -L^{-1}(N_i(v_{i,k-1}(t) + \varepsilon_{i,k-1}(t))), k = 2, 3, \ldots.
\end{aligned} \tag{47}$$

Applying the operator L to Equation (47) gives

$$\begin{aligned}
&p^0 : L(\widetilde{v}_{i,0}(t)) = L(\widetilde{u}_{i,0}(t)),\\
&p^1 : L(\widetilde{v}_{i,1}(t)) = f_i^*(t) + \delta f_i^*(t) - N_i(v_{i,0}(t) + \varepsilon_{i,0}(t)) - L(\widetilde{u}_{i,0}(t)),\\
&p^2 : L(\widetilde{v}_{i,2}(t)) = -N_i(v_{i,1}(t) + \varepsilon_{i,1}(t)),\\
&p^k : L(\widetilde{v}_{i,k}(t)) = -N_i(v_{i,k-1}(t) + \varepsilon_{i,k-1}(t)), k = 2, 3, \ldots.
\end{aligned} \tag{48}$$

By summing both sides of Equation (48), we obtain

$$\begin{aligned}
L(\widetilde{v}_{i,0}(t) + \widetilde{v}_{i,1}(t) + \ldots + \widetilde{v}_{i,k}(t) + \ldots) &= -N_i(v_{i,0}(t) + v_{i,1}(t) + \ldots + v_{i,k-1}(t) + \ldots)\\
&-N_i(\varepsilon_{i,0}(t) + \varepsilon_{i,1}(t) + \ldots + \varepsilon_{i,k-1}(t) + \ldots) + f_i^*(t) + \delta f_i^*(t).
\end{aligned} \tag{49}$$

Since $v_i(t) = \sum\limits_{k=0}^{\infty} v_{i,k}(t)$, $\varepsilon_i(t) = \sum\limits_{k=0}^{\infty} \varepsilon_{i,k}(t)$ and $\widetilde{v}_i(t) = v_i(t) + \varepsilon_i(t)$, and taking into account Equation (18), we obtain

$$L(\varepsilon_i(t)) + N_i(\varepsilon_i(t)) = \delta f_i^*(t). \tag{50}$$

Using $\varepsilon_i(t) = \widetilde{v}_i(t) - v_i(t)$, and taking the linearity of the operator we obtain

$$L(\widetilde{v}_i(t)) + N_i(\widetilde{v}_i(t)) = \delta f_i^*(t) + f_i^*(t). \tag{51}$$

This proved the Theorem 2. □

4.2. Convergence of the Hybrid Method

Definition 2 (Elliott [37]). *A sequence u_n with $u_n \in X_n$ (where X_n is the discretized space of the consideration problem) converges globally to u if $\lim\limits_{n\to\infty} \|u - p_n u_n\| = 0$.*

In (Ayati and Biazar [38]), the general convergence theorem of the HPM was proven. Based on [39] we prove the following theorem.

Theorem 3. *Let the HPM for Equation (18) be defined by Equation (36). Let $L : X \to Y$ (where X, Y be Holder space) be linear invertible operator. Then the solution of Equation (18) is equivalent to determining the following sequence*

$$s_{i,n}(t) = v_{i,0}(t) + v_{i,1}(t) + \ldots + v_{i,n}(t), i = \{1, 2, \ldots, M\}, \tag{52}$$

where the sequence $s_{i,n}(t)$ is generated by

$$s_{i,n+1}(t) = L^{-1}(f_i^*(t)) - L^{-1}(N_i(s_{i,n}(t))), n = 0, 1, 2, \ldots \tag{53}$$

Here, the iterative scheme $v_{i,n}(t)$ is defined by (36) and

$$N_i\left(\sum_{k=0}^{\infty} v_{i,k}(t)\right) = \sum_{k=0}^{\infty} N_i(v_{i,k}(t)), i = 1, 2, \ldots M.$$

Proof. We prove this theorem by employing the induction method. Let $n = 0$, then from Equation (53), we have

$$s_{i,1}(t) = L^{-1}(f_i^*(t)) - L^{-1}(N_i(s_{i,0}(t))) = L^{-1}(f_i^*(t)) - L^{-1}(N_i(v_{i,0}(t))). \tag{54}$$

As a consequence of the equation represented by Equation (36), we have $v_{i,0}(t) = u_{i,0}(t)$ and we obtain

$$s_{i,1}(t) = L^{-1}(f_i^*(t)) - L^{-1}(N_i(v_{i,0}(t))) - u_{i,0}(t) + v_{i,0}(t). \tag{55}$$

From Equation (36), it follows that

$$v_{i,1}(t) = L^{-1}(f_i^*(t)) - L^{-1}(N_i(v_{i,0}(t))) - u_{i,0}(t). \tag{56}$$

Therefore,

$$s_{i,1}(t) = v_{i,0}(t) + v_{i,1}(t).$$

For $n = 1$, from Equation (53), we obtain

$$\begin{aligned} s_{i,2}(t) &= L^{-1}\big(f_i^*(t)\big) - L^{-1}(N_i(s_{i,1}(t))) \\ &= L^{-1}\big(f_i^*(t)\big) - L^{-1}(N_i(v_{i,0}(t) + v_{i,1}(t))) - u_{i,0}(t) + v_{i,0}(t) \\ &= L^{-1}\big(f_i^*(t)\big) - L^{-1}(N_i(v_{i,0}(t))) - u_{i,0}(t) + v_{i,0}(t) - L^{-1}(N_i(v_{i,1}(t))) \\ &= v_{i,0}(t) + v_{i,1}(t) - L^{-1}(N_i(v_{i,1}(t))). \end{aligned} \tag{57}$$

Due to Equation (36), $v_{i,2}(t) = -L^{-1}(N_i(v_{i,1}(t)))$. Therefore,

$$s_{i,2}(t) = v_{i,0}(t) + v_{i,1}(t) + v_{i,2}(t).$$

Assume that $s_{i,n}(t) = v_{i,0}(t) + v_{i,1}(t) + v_{i,2}(t) + \ldots + v_{i,n}(t)$, then by taking into account $v_{i,k}(t) = -L^{-1}(N_i(v_{i,k-1}(t)))$, we arrived at

$$\begin{aligned} s_{i,n+1}(t) &= L^{-1}\big(f_i^*(t)\big) - L^{-1}(N_i(s_{i,n}(t))) \\ &= L^{-1}\big(f_i^*(t)\big) - L^{-1}(N_i(v_{i,0}(t) + v_{i,1}(t) + \ldots + v_{i,n}(t))) - u_{i,0}(t) + v_{i,0}(t) \\ &= v_{i,0}(t) + v_{i,1}(t) - L^{-1}(N_i(v_{i,1}(t) + \ldots + v_{i,n}(t))) \\ &= v_{i,1}(t) + v_{i,2}(t) + \ldots + v_{i,n}(t) - L^{-1}(N_i(v_{i,n}(t))) \\ &= v_{i,1}(t) + v_{i,2}(t) + \ldots + v_{i,n}(t) + v_{i,n+1}(t). \end{aligned} \tag{58}$$

By induction method, we can conclude that Equation (52) is valid for any n. The Theorem 3 is hence proven. We now proceed by proving the following main Theorem 4. □

Theorem 4. *Let $s_{i,n}$ be defined by (52) and $f* \in L^p(\mathbb{R})$, If the function $F_i(s_{i,n})$ defined by (19) satisfied Lipschitz condition*

$$\|F_i(s_{i,n}) - F_i(s_{i,n-1})\| \leq L_i\|s_{i,n} - s_{i,n-1}\|, i \in \{1, 2, \ldots, n\},$$

for the sequence $s_{i,n+1}$ generated by (53), then, the following inequality holds:

$$\|s_{i,n} - s_{i,n-1}\| \leq \varepsilon_i\|s_{i,n} - s_{i,n-1}\|,$$

where

$$0 < \varepsilon_i = \frac{1}{|C_M|}\sum_{j=1}^{M}\left|e_{ij}^{[M]}\right| \cdot \left|L_j\right| < 1,$$
$$L_j = \frac{1}{2}\sum_{l=1}^{M}\left|b_{j,l}\right|\|K_{j,l}\|, \quad \|K_{j,l}\| = \sup_{-1\leq t\leq 1}\int_{-1}^{1}\left|K_{j,l}(t,\tau)\right|d\tau.$$

Proof. It is well known that the generalised Holder inequality [35] leads to

$$\int_{\mathbb{R}}|fgh| \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r,$$

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}, \ \|g\|_q = \left(\int_a^b |g(t)|^q dt\right)^{\frac{1}{q}}, \ \|h\|_r = \left(\int_a^b |h(t)|^r dt\right)^{\frac{1}{r}},$$

where $p, q, r \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R}), h \in L^r(\mathbb{R})$.

In Eshkuvatov et al. [39], applied the generalised Holder inequality as follows

$$\begin{aligned} \left|\int_a^b u(x)v(x)h(x)\rho(x)dx\right| \leq &\left(\int_a^b |u(x)|^{\lambda_1}\rho(x)dx\right)^{\frac{1}{\lambda_1}} \\ &\cdot\left(\int_a^b |v(x)|^{\lambda_2}\rho(x)dx\right)^{\frac{1}{\lambda_2}}\left(\int_a^b |h(x)|^{\lambda_3}\rho(x)dx\right)^{\frac{1}{\lambda_3}}, \end{aligned} \tag{59}$$

where $\rho(x)$ is a weight function and $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 1, \ \lambda_i > 1, \ i = 1, 2, 3$.
Case 1: For the unbounded case, we have

$$\begin{aligned} \|s_{i,n+1} - s_{i,n}\|_p &= \|L^{-1}(N_i(s_{i,n}(t))) - L^{-1}(N_i(s_{i,n-1}(t)))\|_p \\ &= \left\|\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}N_i(s_{i,n}(\tau))d\tau}{\tau - t} - \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}N_i(s_{i,n-1}(\tau))d\tau}{\tau - t}\right\|_p \\ &= \frac{1}{\pi}\left\|\int_{-1}^{1}\frac{N_i(s_{i,n}(\tau)) - N_i(s_{i,n-1}(\tau))}{\tau - t}\sqrt{1-\tau^2}d\tau\right\|_p \\ &= \frac{1}{\pi}\left\|\int_{-1}^{1}\left(\frac{E_{i,n}^p(\tau)}{(\tau - t)^\alpha}\right)^{\frac{1}{q}}(E_{i,n}(\tau))^{1-\frac{p}{q}}\left(\frac{1}{(\tau - t)^\alpha}\right)^{1-\frac{\alpha}{q}}\sqrt{1-\tau^2}d\tau\right\|_p, \end{aligned} \tag{60}$$

where $E_{i,n}(t) = N_i(s_{i,n}(t)) - N_i(s_{i,n-1}(t))$.

Applying Holder inequality on Equation (59) to Equation (60) and choosing $\rho(t) = \sqrt{1-t^2}$, $\lambda_1 = q$, $\lambda_2 = 1/\left(\frac{1}{p} - \frac{1}{q}\right)$, and $\lambda_3 = p' = \frac{p}{p-1}$, we then obtain

$$\begin{aligned}\|s_{i,n+1} - s_{i,n}\|_p &\le \frac{1}{\pi}\left(\int_{-1}^{1}\left(\left(\frac{|E_{i,n}(\tau)|^p}{|\tau-t|^\alpha}\right)^{\frac{1}{q}}\right)^q \sqrt{1-\tau^2}d\tau\right)^{\frac{1}{q}} \\ &\cdot\left(\int_{-1}^{1}|E_{i,n}(\tau)|^{\left(1-\frac{p}{q}\right)\frac{1}{\frac{1}{p}-\frac{1}{q}}}\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{p}-\frac{1}{q}} \cdot \left(\int_{-1}^{1}\left(\frac{1}{|\tau-t|}\right)^{p'\left(1-\frac{\alpha}{q}\right)}\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{p'}} \\ &= \frac{1}{\pi}\left(\int_{-1}^{1}\frac{|E_{i,n}(\tau)|^p}{|\tau-t|^\alpha}\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{q}} \\ &\cdot\left(\int_{-1}^{1}|E_{i,n}(\tau)|^p\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{p}-\frac{1}{q}} \cdot \left(\int_{-1}^{1}\left(\frac{1}{|\tau-t|}\right)^{p'\left(1-\frac{\alpha}{q}\right)}\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{p'}} \\ &= \frac{1}{\pi}\|I_1\|_p\|I_2\|_q\|I_3\|_{p'},\end{aligned} \tag{61}$$

where

$$\|I_1\|_p = \left(\int_{-1}^{1}\frac{|E_{i,n}(\tau)|^p}{|\tau-t|^\alpha}\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{q}},$$
$$\|I_2\|_q = \left(\int_{-1}^{1}|E_{i,n}(\tau)|^p\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{p}-\frac{1}{q}},$$
$$\|I_3\|_{p'} = \left(\int_{-1}^{1}\left(\frac{1}{|\tau-t|}\right)^{p'\left(1-\frac{\alpha}{q}\right)}\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{p'}}.$$

Now, let us consider $I_1(t)$ given as follows

$$\|I_1\|_p = \left(\int_{-1}^{1}\frac{|E_{i,n}(\tau)|^p}{|\tau-t|^\alpha}\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{q}} \le \|E_{i,n}\|^{\frac{p}{q}} \cdot \int_{-1}^{1}\left(\frac{\sqrt{1-\tau^2}}{|\tau-t|^\alpha}d\tau\right)^{\frac{1}{q}}$$

where

$$\begin{aligned}\|E_{i,n}(t)\| = \|N_i(s_{i,n}(t)) - N_i(s_{i,n-1}(t))\| &\le \frac{1}{|C_M|}\sum_{j=1}^{M}\left|e_{ij}^{[M]}\right|\|F_j(s_{i,n}(t)) - F_j(s_{i,n-1}(t))\| \\ &\le \frac{1}{|C_M|}\sum_{j=1}^{M}\left|e_{ij}^{[M]}\right|L_i\|s_{i,n} - s_{i,n-1}\| = \varepsilon_i\|s_{i,n} - s_{i,n-1}\|,\end{aligned}$$

$$\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\|\tau-t\|^\alpha}d\tau \le \int_{-1}^{1}\frac{d\tau}{\|\tau-t\|^\alpha} = \left.\frac{(\tau-t)^{1-\alpha}}{1-\alpha}\right|_{-1}^{1} \le \frac{2^{1+\alpha}}{1-\alpha}, \alpha < 1.$$

On the other hand, we have

$$\|I_1\|_p \le \|E_{i,n}\|_p^{\frac{p}{q}} \cdot \frac{2^{1+\alpha}}{1-\alpha}.$$

Moreover, $I_2(t)$ can be estimated as follows

$$\|I_2\|_q = \left(\int_{-1}^{1}|E_{i,n}(\tau)|^p\sqrt{1-\tau^2}d\tau\right)^{\frac{1}{p}-\frac{1}{q}} \le \pi\|E_{i,n}(t)\|^{1-\frac{p}{q}}$$

For $I_3(t)$, we assume that $\frac{1}{p_0} < \alpha < 1$. Then, taking into account $\frac{1}{p_0} > \frac{q}{p}$, we arrived at $\frac{\alpha}{q} > \frac{1}{p}$ or $1 - \frac{\alpha}{q} > 1 - \frac{1}{p} = \frac{1}{p'}$. We now have

$$\|I_3\|_{p'} = \left(\int_{-1}^{1} \left(\frac{1}{|\tau - t|} \right)^{p'(1-\frac{\alpha}{q})} \sqrt{1-\tau^2} d\tau \right)^{\frac{1}{p'}} \le \int_{-1}^{1} \frac{d\tau}{|\tau - t|^{\beta}} = \int_{-1}^{t} \frac{d\tau}{|\tau - t|^{\beta}} + \int_{t}^{1} \frac{d\tau}{|\tau - t|^{\beta}} \le \frac{2^{1+\beta}}{1+\beta},$$

where

$$\beta = 1 - \frac{\alpha}{q} < 1.$$

Substituting $I_1(t)$, $I_2(t)$ and $I_3(t)$ into Equation (61), we obtain where

$$\varepsilon_i^* = \frac{2^{1+\alpha}}{1-\alpha} \cdot \frac{2^{1+\beta}}{1+\beta} \varepsilon_i$$

Theorem 5. *Suppose that $s_i(t)$ be defined by (55) and $v_{i,n} \in X = H^{\alpha}[-1,1]$ and $\|v_{i,0}\|$ is bounded. We then have $\|v_{i,n}\| \le \varepsilon \|v_{i,n-1}\|, \varepsilon < 1$, for $n = 0,1,2,\ldots$ and the sequence of partial sum $s_n = \sum_{k=0}^{n} v_{i,k}(t)$ converges to the solution of SIEs (18).*

Proof. To show the series $\sum_{k=0}^{\infty} v_{i,k}(t)$ is convergent, we prove that it is a Cauchy sequence.

$$\begin{aligned} &s_{i,1}(t) = v_{i,0}(t) + v_{i,1}(t), \\ &s_{i,2}(t) = v_{i,0}(t) + v_{i,1}(t) + v_{i,2}(t), \\ &\vdots \\ &s_{i,n}(t) = v_{i,0}(t) + v_{i,1}(t) + \ldots + v_{i,n}(t), \\ &\vdots \end{aligned}$$

Using $\|s_{i,n} - s_{i,n-1}\| \le \varepsilon_i \|s_{i,n} - s_{i,n-1}\|$, we obtain

$$\begin{aligned} \|s_{i,n+1} - s_{i,n}\| &= \|v_{i,n+1}\| \le \varepsilon_i \|s_{i,n} - s_{i,n-1}\| = \varepsilon_i \|v_{i,n}\| \\ &\le \varepsilon_i^2 \|s_{i,n-1} - s_{i,n-2}\| \le \ldots \le \varepsilon^{n+1} \|v_{i,0}\|. \end{aligned} \tag{62}$$

For any integers n and m such that $n \ge m$, we obtain the following

$$\begin{aligned} \|s_{i,n} - s_{i,m}\| &= \|(s_{i,n} - s_{i,n-1}) + (s_{i,n-1} - s_{i,n-2}) + \ldots + (s_{i,m+1} - s_{i,m})\| \\ &\le \|s_{i,n} - s_{i,n-1}\| + \|s_{i,n-1} - s_{i,n-2}\| + \ldots + \|s_{i,m+1} - s_{i,m}\| \\ &\le \varepsilon^n \|v_{i,0}\| + \varepsilon^{n-1} \|v_{i,0}\| + \ldots + \varepsilon^{m+1} \|v_{i,0}\| \le \left(\varepsilon^n + \varepsilon^{n-1} + \ldots + \varepsilon^{m+1}\right) \|v_{i,0}\| \\ &\le \left(\varepsilon^{m+1} + \ldots + \varepsilon^n + \ldots\right) \|v_{i,0}\| \le \varepsilon^{m+1} (1 + \varepsilon + \ldots + \varepsilon^n + \ldots) \|v_{i,0}\| \\ &\le \frac{\varepsilon^{m+1}}{1-c} \|v_{i,0}\|. \end{aligned} \tag{63}$$

Thus, $\lim_{n,m \to \infty} \|s_{i,n} - s_{i,m}\| = 0$, which shows that $\{s_{i,n}\}$ is a Cauchy sequence and it converges in Banach space i.e.,

$$\exists s \in B, \text{ such that, } \lim_{n \to \infty} s_{i,n}(t) = \sum_{n=1}^{\infty} v_{i,n}(t) = s_i(t). \tag{64}$$

□

Theorem 6. *Let$s_i(t)$ be defined by Equation (64), then the following sequence is satisfied*

$$s_i(t) = L^{-1}(f_i^*(t)) - L^{-1}(N_i(s_i(t))). \tag{65}$$

Proof. From Equation (53), we have

$$\begin{aligned}\lim_{n\to\infty} s_{i,n+1}(t) &= L^{-1}\big(f_i^*(t)\big) - L^{-1}\lim_{n\to\infty}(N_i(s_{i,n}(t))) \\ &= L^{-1}\big(f_i^*(t)\big) - L^{-1}N_i\left(\lim_{n\to\infty}\left(\sum_{n=0}^{\infty} v_{i,n}(t)\right)\right).\end{aligned} \tag{66}$$

Substituting Equation (64) into Equation (66) yields

$$\begin{aligned}s_i(t) &= L^{-1}\big(f_i^*(t)\big) - L^{-1}N_i\left(\lim_{n\to\infty}\left(\sum_{n=1}^{\infty} v_{i,n}(t)\right)\right) \\ &= L^{-1}\big(f_i^*(t)\big) - L^{-1}N_i(s_i(t)), i = 1, 2, \ldots M.\end{aligned}$$

Theorem 6 is proven. □

Lemma 1. *Equation (65) is equivalent to*

$$L(u_i(t)) + N_i(u_i(t)) = f_i^*(t).$$

Proof. Let us rewrite Equation (60), which leads to

$$s_i(t) = L^{-1}(f_i^*(t)) - L^{-1}(N_i(s_i(t))). \tag{67}$$

Applying the operator L to both sides of Equation (67) yields

$$L(s_i(t)) = f_i^*(t) - N_i(s_i(t)). \tag{68}$$

Due to Equation (64), we obtain

$$u_i(t) = s_i(t) = \sum_{n=0}^{\infty} v_{i,n}(t).$$

Thus, Equation (68) leads to

$$L(u_i(t)) + N_i(u_i(t)) = f_i^*(t).$$

□

5. Numerical Examples

Example 1 (Turhan et al. [12]): *Consider the system of SIEs of the form*

$$\begin{aligned}\frac{4}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau - t}d\tau + \frac{2}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau - t}d\tau = -4t^3 - 2t^2 - 6t + 1 \\ \frac{4}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau - t}d\tau + \frac{6}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau - t}d\tau = -4t^3 - 6t^2 - 14t + 3\end{aligned} \tag{69}$$

Comparing Equation (69) with Equation (1), we have

$$\begin{aligned}&a_{1,1} = \frac{4}{\pi},\ a_{1,2} = \frac{2}{\pi},\ a_{2,1} = \frac{4}{\pi},\ a_{2,2} = \frac{6}{\pi}, \\ &b_{1,1} = b_{1,2} = b_{2,1} = b_{2,2} = 0.\end{aligned}$$

We also have the following functions

$$\begin{cases} f_1(t) = -4t^3 - 2t^2 - 2t + 1, \\ f_2(t) = -4t^3 - 6t^2 - 10t + 3. \end{cases}$$

Remark 2. *Example 1 is discussed by Turhan et al. [12], who applied the truncated Chebyshev series method and found a bounded solution. They were able to obtain an exact solution for $n = 3$. To solve Equation (69) by applying the HPM, we should use the Gaussian elimination method to obtain*

$$\begin{aligned} &\int_{-1}^{1} \frac{u_1(\tau)}{\tau - t} d\tau = \left(-t^3 - \frac{1}{2}t\right)\pi = f_1{}^*(t), \\ &\int_{-1}^{1} \frac{u_2(\tau)}{\tau - t} d\tau = \left(-t^2 - 2t + \frac{1}{2}\right)\pi = f_2{}^*(t). \end{aligned} \tag{70}$$

We solve Equation (70) by standard HPM (42). Since Equation (70) can be written as Equation (40) due to Theorem 1., we are able to obtain the exact solution for all cases, as shown below:

Case 1. The bounded solution is searched as follows

$$u_i(t) = \sqrt{1 - t^2} v_i(t), i = \{1, 2\}. \tag{71}$$

The exact bounded solutions of Equation (69) is

$$\begin{aligned} u_1(t) &= \sqrt{1 - t^2}\left[t^2 + 1\right], \\ u_2(t) &= \sqrt{1 - t^2}[t + 2]. \end{aligned} \tag{72}$$

We compare Equation (70) with Equation (18), which gives

$$\begin{aligned} &f_1{}^*(t) = -t^3 - \frac{1}{2}t, \quad f_2{}^*(t) = -t^2 - 2t + \frac{1}{2}, \\ &L_1(u_1(t)) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_1(\tau)}{\tau - t} d\tau, \ L_2(u_2(t)) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_2(\tau)}{\tau - t} d\tau, \\ &N_1(u_1(t)) = N_2(u_2(t)) = 0. \end{aligned} \tag{73}$$

By solving Equation (70) using standard HPM (42), we obtain

$$\begin{aligned} &p^0 : v_{1,0}(t) = f_1{}^*(t) = -t^3 - \frac{1}{2}t, \ \mathrm{v}_{2,0}(t) = f_2{}^*(t) = -t^2 - 2t + \frac{1}{2}, \\ &p^1 : v_{1,1}(t) = t^3 + t^2 + \frac{1}{2}t + 1, \ \mathrm{v}_{2,1}(t) = t^2 + 3t + \frac{3}{2}, \\ &p^k : v_{1,k}(t) = 0, \ \mathrm{v}_{2,k}(t) = 0, k = 2, 3 \ldots. \end{aligned} \tag{74}$$

The approximate solution of Equation (69) is given by

$$\begin{aligned} u_1(t) = \sqrt{1 - t^2}(v_{1,0}(t) + v_{1,1}(t)) = \sqrt{1 - t^2}\left(-t^3 - \frac{1}{2}t + t^3 + t^2 + \frac{1}{2}t + 1\right) = \sqrt{1 - t^2}\left(t^2 + 1\right), \\ u_2(t) = \sqrt{1 - t^2}(v_{2,0}(t) + v_{2,1}(t)) = \sqrt{1 - t^2}\left(-t^2 - 2t + \frac{1}{2} + t^2 + 3t + \frac{3}{2}\right) = \sqrt{1 - t^2}(t + 2), \end{aligned} \tag{75}$$

which is identical to the exact solution.

We chose the following functions as the initial guess

$$\text{(a)} \quad \begin{cases} v_{1,0}(t) = -\dfrac{1}{2}t, \\ v_{2,0}(t) = \dfrac{1}{2}. \end{cases} \qquad \text{(b)} \quad \begin{cases} v_{1,0}(t) = t^4 + 5t^3 - 3, \\ v_{2,0}(t) = -\dfrac{3}{4}t^3 + 7. \end{cases}$$

Remark 3. *In the case a), the initial guess $(v_{1,0}, \mathrm{v}_{2,0})$ is chosen as part of $f_1{}^*(t)$ and $f_2{}^*(t)$ respectively. For the case b), the initial guess $(v_{1,0}, \mathrm{v}_{2,0})$ is selected as any continuous function not related to $f_1{}^*(t)$ and $f_2{}^*(t)$.*

Case 2. Let us now search for the unbounded solutions of Equation (70) in the form

$$u_i(t) = \frac{1}{\sqrt{1-t^2}} v_i(t), i = \{1,2\}. \tag{76}$$

In this case, the exact solution of Equation (69) is given by

$$\begin{aligned} u_1(t) &= \frac{1}{\sqrt{1-t^2}}\left(-t^4 + \frac{3}{8}\right), \\ u_2(t) &= \frac{1}{\sqrt{1-t^2}}\left(-t^3 - 2t^2 + t + 1\right). \end{aligned}$$

It is known that Equation (69) is equivalent to Equation (70). Therefore, to solve Equation (70), we apply standard HPM (42), which yields

$$\begin{aligned} &p^0 : v_{1,0}(t) = f_1{}^*(t) = -t^3 - \frac{1}{2}t,\ \mathrm{v}_{2,0}(t) = f_2{}^*(t) = -t^2 - 2t + \frac{1}{2}, \\ &p^1 : v_{1,1}(t) = -t^4 + t^3 + \frac{1}{2}t + \frac{3}{8},\ \mathrm{v}_{2,1}(t) = -t^3 - t^2 + 3t + \frac{1}{2}, \\ &p^k : v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 2,3\ldots. \end{aligned} \tag{77}$$

The approximate solution of Equation (69) for the unbounded solution case is

$$\begin{aligned} u_1(t) &= \frac{1}{\sqrt{1-t^2}}(v_{1,0}(t) + v_{1,1}(t)) = \frac{1}{\sqrt{1-t^2}}\left(-t^4 + \frac{3}{8}\right), \\ u_2(t) &= \frac{1}{\sqrt{1-t^2}}(v_{2,0}(t) + v_{2,1}(t)) = \frac{1}{\sqrt{1-t^2}}\left(-t^3 - 2t^2 + t + 1\right), \end{aligned} \tag{78}$$

which is identical to the exact solution.

The initial guess is chosen as the following functions

$$\text{(a)} \quad \begin{cases} v_{1,0}(t) = t^2 - \frac{1}{2}t, \\ v_{2,0}(t) = -2t + \frac{1}{2}. \end{cases} \qquad \text{(b)} \quad \begin{cases} v_{1,0}(t) = t^4 + t^3 - \frac{3}{4}, \\ v_{2,0}(t) = t^3 - \frac{3}{7}t. \end{cases}$$

Remark 4. *It is found that standard HPM gave exact results for any chosen initial guess.*

Case 3. Let us search the semi-bounded solution of Equation (69) in the form

$$u_i(t) = \sqrt{\frac{1+t}{1-t}} v_i(t), i = \{1,2\}. \tag{79}$$

The exact solution is

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1+t}{1-t}}\left(-t^3 + t^2 - t + 1\right), \\ u_2(t) &= \sqrt{\frac{1+t}{1-t}}\left(-t^2 - t + 2\right). \end{aligned}$$

In this case, solving Equation (70) by applying standard HPM (42) yields

$$\begin{aligned} &p^0 : v_{1,0}(t) = f_1{}^*(t) = -t^3 - \frac{1}{2}t,\ \mathrm{v}_{2,0}(t) = f_2{}^*(t) = -t^2 - 2t + \frac{1}{2}, \\ &p^1 : v_{1,1}(t) = t^2 - \frac{1}{2}t + 1,\ \mathrm{v}_{2,1}(t) = t + \frac{3}{2}, \\ &p^k : v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 2,3\ldots. \end{aligned} \tag{80}$$

The approximate solution of Equation (69) for the semi-bounded solution case is

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1+t}{1-t}}(v_{1,0}(t)+v_{1,1}(t)) = \sqrt{\frac{1+t}{1-t}}\left(-t^3+t^2-t+1\right), \\ u_2(t) &= \sqrt{\frac{1+t}{1-t}}(v_{2,0}(t)+v_{2,1}(t)) = \sqrt{\frac{1+t}{1-t}}\left(-t^2-t+2\right). \end{aligned} \tag{81}$$

which is identical to the exact solution.

We have chosen the following functions as initial guesses:

$$\text{(a)} \quad \begin{cases} v_{1,0}(t) = -t^3+t^2-\frac{1}{2}t, \\ v_{2,0}(t) = t^4-t^2-2t. \end{cases} \qquad \text{(b)} \quad \begin{cases} v_{1,0}(t) = t^5-t^3+t^2-\frac{1}{4}, \\ v_{2,0}(t) = t^3-\frac{1}{7}t^2+t. \end{cases}$$

Remark 5. *In this case, the standard HPM also provided an exact solution regardless of the initial guess.*

Case 4. Let us now search the semi-bounded solution of Equation (69) in the form

$$u_i(t) = \sqrt{\frac{1-t}{1+t}}v_i(t), i = \{1,2\}. \tag{82}$$

The exact solution is given as follows

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1-t}{1+t}}\left(t^3+t^2+t+1\right), \\ u_2(t) &= \sqrt{\frac{1-t}{1+t}}\left(t^2+3t+2\right). \end{aligned}$$

In order to solve Equation (70) in this case, we apply standard HPM (42), which yields

$$\begin{aligned} &p^0 : v_{1,0}(t) = f_1{}^*(t) = -t^3-\frac{1}{2}t,\ \mathrm{v}_{2,0}(t) = f_2{}^*(t) = -t^2-2t+\frac{1}{2}, \\ &p^1 : v_{1,1}(t) = 2t^3+t^2+\frac{3}{2}t+1,\ \mathrm{v}_{2,1}(t) = 2t^2+5t+\frac{3}{2}, \\ &p^k : v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 2,3\ldots. \end{aligned} \tag{83}$$

The approximate solution of Equation (69) for the semi-bounded is

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1-t}{1+t}}(v_{1,0}(t)+v_{1,1}(t)) = \sqrt{\frac{1-t}{1+t}}\left(t^3+t^2+t+1\right), \\ u_2(t) &= \sqrt{\frac{1-t}{1+t}}(v_{2,0}(t)+v_{2,1}(t)) = \sqrt{\frac{1-t}{1+t}}\left(t^2+3t+2\right), \end{aligned} \tag{84}$$

which is identical to the exact solution.

Remark 6. *According to the Theorem 1., the HPM provides an exact solution for any initial guess in the case of the semi-bounded solutions.*

Example 2 (Sharma et al. [40]): *Solve the system of CSIEs of the form*

$$\begin{aligned} \frac{1000}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau-t}d\tau + \frac{10}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau-t}d\tau = f_1(t)+ig_1(t), \\ \frac{500}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau-t}d\tau + \frac{200}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau-t}d\tau = f_2(t)+ig_2(t), \end{aligned} \tag{85}$$

where

$$
\begin{cases}
f_1(t) = -990t^8 + 1089t^7 + 937t^6 - \frac{26704t^5}{25} - \frac{349161t^4}{1000} + \frac{792327t^3}{2000} + \frac{1761t^2}{250} - \frac{53511t}{4000} - \frac{53929}{2000}, \\
g_1(t) = 990t^8 - 1189t^7 - \frac{8971t^6}{10} + \frac{119047t^5}{100} + \frac{163961t^4}{500} - \frac{279198t^3}{625} - \frac{30873t^2}{10000} + \frac{69533t}{4000} + \frac{1130501}{40000}, \\
f_2(t) = -300t^8 + 330t^7 + 215t^6 - \frac{2447t^5}{10} - \frac{8607t^4}{100} + \frac{735t^3}{8} - \frac{17253t^2}{2000} + \frac{29541t}{4000} - \frac{14701}{1000}, \\
g_2(t) = 300t^8 - 380t^7 - 197t^6 + \frac{1462t^5}{5} + \frac{9419t^4}{100} - \frac{27549t^3}{250} - \frac{183t^2}{400} - \frac{7957t}{2000} + \frac{57853}{4000}.
\end{cases}
$$

Remark 7. *Sharma et al. [40] investigated Example 2 using the Galerkin method based on the Legendre polynomials to find a semi-bounded solution. They specifically obtained approximate solutions for n = 8 and obtained the exact solution.*

Before applying the standard HPM, we utilize the Gaussian elimination method, which yields

$$
\begin{aligned}
\frac{1}{\pi}\int_{-1}^{1} \frac{u_1(\tau)}{\tau - t} d\tau &= \frac{1}{975}(f_1 + ig_1(t)) - \frac{1}{19500}(f_2 + ig_2(t)), \\
\frac{1}{\pi}\int_{-1}^{1} \frac{u_2(\tau)}{\tau - t} d\tau &= -\frac{1}{390}(f_1 + ig_1(t)) + \frac{1}{195}(f_2 + ig_2(t)).
\end{aligned} \tag{86}
$$

Case 1. The bounded solution is searched as follows

$$u_i(t) = \sqrt{1-t^2}v_i(t), i = \{1,2\}.$$

The exact solution of Equation (85) for the bounded case is

$$
\begin{aligned}
u_1(t) = \sqrt{1-t^2}\Bigg(&t^7 - \frac{11t^6}{10} - \frac{9t^5}{20} + \frac{533t^4}{1000} + \frac{2537t^3}{10000} - \frac{27261t^2}{100000} + \frac{125433t}{1000000} - \frac{248661}{2000000} \\
&i\bigg(-t^7 + \frac{6t^6}{5} + \frac{41t^5}{100} - \frac{303t^4}{500} - \frac{503t^3}{2000} + \frac{936t^2}{3125} - \frac{133857t}{1000000} - \frac{248661}{2000000}\bigg)\Bigg), \\
u_2(t) = \sqrt{1-t^2}\Bigg(&-t^7 + \frac{11t^6}{10} + \frac{4t^5}{5} - \frac{467t^4}{500} - \frac{1789t^3}{10000} + \frac{4303t^2}{20000} + \frac{207t}{20000} - \frac{63}{5000} \\
&i\bigg(t^7 - \frac{11t^6}{10} - \frac{79t^5}{100} + \frac{1003t^4}{1000} + \frac{439t^3}{5000} - \frac{676t^2}{3125} + \frac{13t}{6250} + \frac{42}{3125}\bigg)\Bigg).
\end{aligned}
$$

Comparing Equation (86) with Equation (18) yields

$$
\begin{aligned}
&f_1{}^*(t) = \frac{1}{975}(f_1 + ig_1(t)) - \frac{1}{19500}(f_2 + ig_2(t)), \\
&f_2{}^*(t) = -\frac{1}{390}(f_1 + ig_1(t)) + \frac{1}{195}(f_2 + ig_2(t)), \\
&L_1(u_1(t)) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_1(\tau)}{\tau - t} d\tau, L_2(u_2(t)) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_2(\tau)}{\tau - t} d\tau, \\
&N_1(u_1(t)) = N_2(u_2(t)) = 0.
\end{aligned} \tag{87}
$$

Applying standard HPM (42) yields

$$
\begin{aligned}
p^0 : v_{1,0}(t) &= -t^8 + \frac{11t^7}{10} + \frac{19t^6}{20} - \frac{1083t^5}{1000} - \frac{3537t^4}{10000} + \frac{40161t^3}{100000} + \frac{7667t^2}{1000000} - \frac{28199t}{2000000} - \frac{13451}{500000} \\
&\quad + i\left(t^8 - \frac{6t^7}{5} - \frac{91t^6}{100} + \frac{603t^5}{500} + \frac{663t^4}{2000} - \frac{11313t^3}{25000} - \frac{3143t^2}{1000000} + \frac{18033t}{1000000} + \frac{56491}{2000000}\right), \\
\mathrm{v}_{2,0}(t) &= t^8 - \frac{11t^7}{10} - \frac{13t^6}{10} + \frac{371t^5}{250} + \frac{4539t^4}{10000} - \frac{10893t^3}{100000} - \frac{623t^2}{10000} + \frac{2887t}{40000} - \frac{1}{160} \\
&\quad + i\left(-t^8 + \frac{11t^7}{10} + \frac{129t^6}{100} - \frac{1553t^5}{1000} - \frac{1789t^4}{5000} + \frac{3627t^3}{6250} + \frac{557t^2}{100000} - \frac{2599t}{40000} + \frac{681}{400000}\right), \\
p^1 : v_{1,1}(t) &= t^8 - \frac{t^7}{10} - \frac{41t^6}{20} + \frac{633t^5}{1000} + \frac{8867t^4}{10000} - \frac{14791t^3}{100000} - \frac{280277t^2}{1000000} + \frac{55813t}{400000} - \frac{194857}{2000000} \\
&\quad + i\left(-t^8 + \frac{t^7}{5} + \frac{211t^6}{100} - \frac{199t^5}{250} - \frac{15t^4}{16} + \frac{10051t^3}{50000} + \frac{302663t^2}{1000000} - \frac{15189t}{100000} + \frac{205463}{2000000}\right), \\
\mathrm{v}_{2,1}(t) &= -t^8 + \frac{t^7}{10} + \frac{12t^6}{5} - \frac{171t^5}{250} - \frac{13879t^4}{10000} + \frac{1463t^3}{4000} + \frac{5549t^2}{20000} - \frac{2473t}{40000} - \frac{127}{20000} \\
&\quad + i\left(t^8 - \frac{t^7}{10} - \frac{239t^6}{100} + \frac{763t^5}{1000} + \frac{1701t^4}{1250} - \frac{12313t^3}{25000} - \frac{22189t^2}{100000} + \frac{13411t}{200000} + \frac{939}{80000}\right), \\
p^k : v_{1,k}(t) &= 0,\ \mathrm{v}_{2,k}(t) = 0, k = 2, 3, \ldots.
\end{aligned} \tag{88}
$$

The approximate solution of Equation (85) is given by

$$
\begin{aligned}
u_1(t) &= \sqrt{1-t^2}(v_{1,0}(t) + v_{1,1}(t)) \\
&= \sqrt{1-t^2}\Bigg(t^7 - \frac{11t^6}{10} - \frac{9t^5}{20} + \frac{533t^4}{1000} + \frac{2537t^3}{10000} - \frac{27261t^2}{100000} + \frac{125433t}{1000000} - \frac{248661}{2000000} \\
&\quad + i\left(-t^7 + \frac{6t^6}{5} + \frac{41t^5}{100} - \frac{303t^4}{500} - \frac{503t^3}{2000} + \frac{936t^2}{3125} - \frac{133857t}{1000000} - \frac{248661}{2000000}\right)\Bigg), \\
u_2(t) &= \sqrt{1-t^2}(v_{2,0}(t) + v_{2,1}(t)) \\
&= \sqrt{1-t^2}\Bigg(-t^7 + \frac{11t^6}{10} + \frac{4t^5}{5} - \frac{467t^4}{500} - \frac{1789t^3}{10000} + \frac{4303t^2}{20000} + \frac{207t}{20000} - \frac{63}{5000} \\
&\quad + i\left(t^7 - \frac{11t^6}{10} - \frac{79t^5}{100} + \frac{1003t^4}{1000} + \frac{439t^3}{5000} - \frac{676t^2}{3125} + \frac{13t}{6250} + \frac{42}{3125}\right)\Bigg),
\end{aligned} \tag{89}
$$

which is identical to the exact solution.

Remark 8. *It is observed that the HPM gives an exact solution for the system of CSIEs (85) with any choice of initial guess* $(v_{1,0}, v_{2,0})$.

Case 2. Let us now search for the unbounded solution of Equation (85) given in the form

$$u_i(t) = \frac{1}{\sqrt{1-t^2}} v_i(t), i = \{1, 2\}. \tag{90}$$

Equation (85) corresponds to Equation (86). Consequently, standard HPM (36) is used to solve Equation (86) as follows

$$
\begin{aligned}
p^0: v_{1,0}(t) &= -t^8 + \frac{11t^7}{10} + \frac{19t^6}{20} - \frac{1083t^5}{1000} - \frac{3537t^4}{10000} + \frac{40161t^3}{100000} + \frac{7667t^2}{1000000} - \frac{28199t}{2000000} - \frac{13451}{500000} \\
&\quad + i\left(t^8 - \frac{6t^7}{5} - \frac{91t^6}{100} + \frac{603t^5}{500} + \frac{663t^4}{2000} - \frac{11313t^3}{25000} - \frac{3143t^2}{1000000} + \frac{18033t}{1000000} + \frac{56491}{2000000}\right), \\
\mathrm{v}_{2,0}(t) &= t^8 - \frac{11t^7}{10} - \frac{13t^6}{10} + \frac{371t^5}{250} + \frac{4539t^4}{10000} - \frac{10893t^3}{100000} - \frac{623t^2}{10000} + \frac{2887t}{40000} - \frac{1}{160} \\
&\quad + i\left(-t^8 + \frac{11t^7}{10} + \frac{129t^6}{100} - \frac{1553t^5}{1000} - \frac{1789t^4}{5000} + \frac{3627t^3}{6250} + \frac{557t^2}{100000} - \frac{2599t}{40000} + \frac{681}{400000}\right), \\
p^1: v_{1,1}(t) &= -t^9 + \frac{21t^8}{10} + \frac{7t^7}{20} - \frac{2583t^6}{1000} + \frac{3793t^5}{10000} + \frac{115931t^4}{100000} - \frac{273343t^3}{1000000} - \frac{311893t^2}{2000000} + \frac{227t}{31250} + \frac{33877}{4000000} \\
&\quad + i\left(t^9 - \frac{11t^8}{5} - \frac{21t^7}{100} + \frac{679t^6}{250} - \frac{1089t^5}{2000} - \frac{61851t^4}{50000} + \frac{334877t^3}{1000000} + \frac{85843t^2}{500000} - \frac{11841t}{1000000} - \frac{9197}{1000000}\right), \\
\mathrm{v}_{2,1}(t) &= t^9 - \frac{21t^8}{10} - \frac{7t^7}{10} + \frac{1667t^6}{500} - \frac{5051t^5}{10000} - \frac{32061t^4}{20000} + \frac{1777t^3}{5000} + \frac{5801t^2}{20000} - \frac{2473t}{40000} - \frac{923}{80000} \\
&\quad + i\left(-t^9 + \frac{21t^8}{10} + \frac{69t^7}{100} - \frac{3393t^6}{1000} + \frac{422t^5}{625} + \frac{9857t^4}{6250} - \frac{2473t^3}{5000} - \frac{23533t^2}{100000} + \frac{13411t}{200000} - \frac{923}{80000}\right), \\
p^k: v_{1,k}(t) &= 0,\ \mathrm{v}_{2,k}(t) = 0, k = 2,3,\ldots.
\end{aligned} \tag{91}
$$

The approximate solution of Equation (85) is

$$
\begin{aligned}
u_1(t) &= \frac{1}{\sqrt{1-t^2}}(v_{1,0}(t) + v_{1,1}(t)) \\
&= \frac{1}{\sqrt{1-t^2}}\Bigg(-t^9 + \frac{11t^8}{10} + \frac{29t^7}{20} - \frac{1633t^6}{1000} - \frac{7037t^5}{10000} + \frac{80561t^4}{100000} + \frac{128267t^3}{1000000} - \frac{296559t^2}{2000000} - \frac{13671t}{2000000} - \frac{73731}{4000000} \\
&\quad + i\left(t^9 - \frac{6t^8}{5} - \frac{141t^7}{100} + \frac{903t^6}{500} + \frac{1323t^5}{2000} - \frac{11319t^4}{12500} - \frac{117643t^3}{1000000} + \frac{168543t^2}{1000000} + \frac{387t}{62500} + \frac{38097}{2000000}\right)\Bigg), \\
u_2(t) &= \frac{1}{\sqrt{1-t^2}}(v_{2,0}(t) + v_{2,1}(t)) \\
&= \frac{1}{\sqrt{1-t^2}}\Bigg(t^9 - \frac{11t^8}{10} - \frac{9t^7}{5} + \frac{1017t^6}{500} + \frac{9789t^5}{10000} - \frac{22983t^4}{20000} - \frac{757t^3}{4000} + \frac{911t^2}{4000} + \frac{207t}{20000} - \frac{1423}{80000} \\
&\quad + i\left(-t^9 + \frac{11t^8}{10} + \frac{179t^7}{100} - \frac{2103t^6}{1000} - \frac{4389t^5}{5000} + \frac{30483t^4}{25000} + \frac{2143t^3}{25000} - \frac{718t^2}{3125} + \frac{13t}{6250} + \frac{11233}{800000}\right)\Bigg),
\end{aligned} \tag{92}
$$

which is identical to the exact solution.

Remark 9. *It is observed that the HPM provides an exact solution for the system of CSIEs (85) with any initial guess* $(v_{1,0}, v_{2,0})$.

Case 3. Let us search the semi-bounded solution of Equation (85) given in the form

$$
u_i(t) = \sqrt{\frac{1+t}{1-t}}v_i(t), i = \{1,2\}. \tag{93}
$$

By applying standard HPM (42) to Equation (86), we obtain

$$
\begin{aligned}
p^0: v_{1,0}(t) &= -t^8 + \frac{11t^7}{10} + \frac{19t^6}{20} - \frac{1083t^5}{1000} - \frac{3537t^4}{10000} + \frac{40161t^3}{100000} + \frac{7667t^2}{1000000} - \frac{28199t}{2000000} - \frac{13451}{500000} \\
&\quad + i\left(t^8 - \frac{6t^7}{5} - \frac{91t^6}{100} + \frac{603t^5}{500} + \frac{663t^4}{2000} - \frac{11313t^3}{25000} - \frac{3143t^2}{1000000} + \frac{18033t}{1000000} + \frac{56491}{2000000}\right), \\
\mathrm{v}_{2,0}(t) &= t^8 - \frac{11t^7}{10} - \frac{13t^6}{10} + \frac{371t^5}{250} + \frac{4539t^4}{10000} - \frac{10893t^3}{100000} - \frac{623t^2}{10000} + \frac{2887t}{40000} - \frac{1}{160} \\
&\quad + i\left(-t^8 + \frac{11t^7}{10} + \frac{129t^6}{100} - \frac{1553t^5}{1000} - \frac{1789t^4}{5000} + \frac{3627t^3}{6250} + \frac{557t^2}{100000} - \frac{2599t}{40000} + \frac{681}{400000}\right),
\end{aligned} \tag{94}
$$

$$p^1 : v_{1,1}(t) = t^7 - \frac{8t^6}{5} + \frac{t^5}{10} + \frac{633t^4}{1000} + \frac{1247t^3}{10000} - \frac{40571t^2}{100000} + \frac{263863t}{1000000} - \frac{229697}{1000000}$$
$$+i\left(-t^7 + \frac{17t^6}{10} - \frac{19t^5}{100} - \frac{343t^4}{500} - \frac{197t^3}{2000} + \frac{10913t^2}{25000} - \frac{282867t}{1000000} + \frac{485561}{2000000}\right),$$
$$\mathrm{v}_{2,1}(t) = -t^7 + \frac{8t^6}{5} + \frac{t^5}{4} - \frac{1209t^4}{1000} + \frac{753t^3}{5000} + \frac{2671t^2}{10000} - \frac{1969t}{40000} - \frac{127}{20000}$$
$$+i\left(t^7 - \frac{8t^6}{5} - \frac{6t^5}{25} + \frac{1273t^4}{1000} - \frac{1381t^3}{5000} - \frac{22397t^2}{100000} + \frac{10723t}{200000} + \frac{939}{80000}\right),$$
$$p^k : v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 2, 3, \ldots.$$

The approximate solution of Equation (85) for the semi-bounded solution case is

$$\begin{aligned}
u_1(t) &= \sqrt{\frac{1+t}{1-t}}(v_{1,0}(t) + v_{1,1}(t)) \\
&= \sqrt{\frac{1+t}{1-t}}\Bigg(-t^8 + \frac{21t^7}{10} - \frac{13t^6}{20} - \frac{983t^5}{1000} + \frac{2793t^4}{10000} + \frac{52631t^3}{100000} - \frac{398043t^2}{1000000} + \frac{499527t}{2000000} - \frac{256599}{1000000} \\
&+ i\left(t^8 - \frac{11t^7}{5} + \frac{79t^6}{100} + \frac{127t^5}{125} - \frac{709t^4}{2000} - \frac{27551t^3}{50000} + \frac{433377t^2}{1000000} - \frac{132417t}{500000} + \frac{135513}{500000}\right)\Bigg), \\
u_2(t) &= \sqrt{\frac{1+t}{1-t}}(v_{2,0}(t) + v_{2,1}(t)) \\
&= \sqrt{\frac{1+t}{1-t}}\Bigg(t^8 - \frac{21t^7}{10} + \frac{3t^6}{10} + \frac{867t^5}{500} - \frac{7551t^4}{10000} - \frac{7881t^3}{20000} + \frac{128t^2}{625} + \frac{459t}{20000} - \frac{63}{5000} \\
&+ i\left(-t^8 + \frac{21t^7}{10} - \frac{31t^6}{100} - \frac{1793t^5}{1000} + \frac{572t^4}{625} + \frac{7603t^3}{25000} - \frac{273t^2}{1250} - \frac{71t}{6250} + \frac{42}{3125}\right)\Bigg),
\end{aligned} \tag{95}$$

which is identical to the exact solution.

Remark 10. *Similar to the other cases, the standard HPM provided the exact solution for any initial guess in this case.*

Case 4. Let us search the semi-bounded solution of Equation (85) given in the following form

$$u_i(t) = \sqrt{\frac{1-t}{1+t}} v_i(t), i = \{1, 2\}. \tag{96}$$

By applying standard HPM (42) to Equation (86), we obtain

$$\begin{aligned}
p^0 : v_{1,0}(t) &= -t^8 + \frac{11t^7}{10} + \frac{19t^6}{20} - \frac{1083t^5}{1000} - \frac{3537t^4}{10000} + \frac{40161t^3}{100000} + \frac{7667t^2}{1000000} - \frac{28199t}{2000000} - \frac{13451}{500000} \\
& i\left(t^8 - \frac{6t^7}{5} - \frac{91t^6}{100} + \frac{603t^5}{500} + \frac{663t^4}{2000} - \frac{11313t^3}{25000} - \frac{3143t^2}{1000000} + \frac{18033t}{1000000} + \frac{56491}{2000000}\right), \\
\mathrm{v}_{2,0}(t) &= t^8 - \frac{11t^7}{10} - \frac{13t^6}{10} + \frac{371t^5}{250} + \frac{4539t^4}{10000} - \frac{10893t^3}{100000} - \frac{623t^2}{10000} + \frac{2887t}{40000} - \frac{1}{160} \\
& i\left(-t^8 + \frac{11t^7}{10} + \frac{129t^6}{100} - \frac{1553t^5}{1000} - \frac{1789t^4}{5000} + \frac{3627t^3}{6250} + \frac{557t^2}{100000} - \frac{2599t}{40000} + \frac{681}{400000}\right), \\
p^1 : v_{1,1}(t) &= 2t^8 - \frac{6t^7}{5} - \frac{5t^6}{2} + \frac{583t^5}{500} + \frac{2851t^4}{2500} - \frac{10513t^3}{25000} - \frac{38711t^2}{250000} + \frac{7601t}{500000} + \frac{871}{25000} \\
&+i\left(-2t^8 + \frac{7t^7}{5} + \frac{63t^6}{25} - \frac{701t^5}{500} - \frac{1189t^4}{1000} + \frac{25027t^3}{50000} + \frac{84403t^2}{500000} - \frac{20913t}{1000000} - \frac{14927}{400000}\right), \\
\mathrm{v}_{2,1}(t) &= -2t^8 + \frac{6t^7}{5} + \frac{16t^6}{5} - \frac{809t^5}{500} - \frac{3917t^4}{2500} + \frac{5809t^3}{10000} + \frac{1439t^2}{5000} - \frac{2977t}{40000} - \frac{127}{20000} \\
&+i\left(2t^8 - \frac{6t^7}{5} - \frac{159t^6}{50} + \frac{883t^5}{500} + \frac{7243t^4}{5000} - \frac{17721t^3}{25000} - \frac{21981t^2}{100000} + \frac{16099t}{200000} + \frac{939}{80000}\right), \\
& p^k : v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 2, 3, \ldots.
\end{aligned} \tag{97}$$

The results of Equation (97) give the approximate solution of Equation (85) as follows

$$\begin{aligned} u_1(t) &= \frac{\sqrt{1-t}}{\sqrt{1+t}}(v_{1,0}(t)+v_{1,1}(t)) \\ &= \frac{\sqrt{1-t}}{\sqrt{1+t}}\Bigg(t^8 - \frac{t^7}{10} - \frac{31t^6}{20} + \frac{83t^5}{1000} + \frac{7867t^4}{10000} - \frac{1891t^3}{100000} - \frac{147177t^2}{1000000} + \frac{441t}{400000} + \frac{3969}{500000} \\ &\quad + i\left(-t^8 + \frac{t^7}{5} + \frac{161t^6}{100} - \frac{49t^5}{250} - \frac{343t^4}{400} + \frac{2401t^3}{50000} + \frac{165663t^2}{1000000} - \frac{9t}{3125} - \frac{567}{62500}\right)\Bigg), \\ u_2(t) &= \frac{\sqrt{1-t}}{\sqrt{1+t}}(v_{2,0}(t)+v_{2,1}(t)) \\ &= \frac{\sqrt{1-t}}{\sqrt{1+t}}\Bigg(-t^8 + \frac{t^7}{10} + \frac{19t^6}{10} - \frac{67t^5}{500} - \frac{11129t^4}{10000} + \frac{29t^3}{800} + \frac{451t^2}{2000} - \frac{9t}{4000} - \frac{63}{5000} \\ &\quad + i\left(t^8 - \frac{t^7}{10} - \frac{189t^6}{100} + \frac{213t^5}{1000} + \frac{2727t^4}{2500} - \frac{3213t^3}{25000} - \frac{1339t^2}{6250} + \frac{97t}{6250} + \frac{42}{3125}\right)\Bigg), \end{aligned} \tag{98}$$

which is the exact solution of Example 2.

Remark 11. *In this semi-bounded case also, the standard HPM provided the exact solution for any initial guess.*

Example 3 (Turhan et al. [12]): *Consider the system of SIEs of the form*

$$\begin{aligned} \frac{1}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau-t}d\tau + \frac{2}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau-t}d\tau + \frac{1}{\pi}\int_{-1}^{1} t^3\tau^3 u_1(\tau)d\tau &= -2t^5 - 10t^3 - \frac{13}{20}t, \\ \frac{3}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau-t}d\tau + \frac{1}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau-t}d\tau + \frac{1}{\pi}\int_{-1}^{1} t^5\tau^5 u_1(\tau)d\tau &= -t^5 - \frac{15}{2}t^3 - \frac{3}{40}t. \end{aligned} \tag{99}$$

Remark 12. *Turhan et al. [12] investigated Example 3, employing the Chebyshev series method to discover a bounded solution. Remarkably, they were able to achieve the exact solution successfully.*

Before implementing the standard HPM, we apply the Gaussian elimination method, effectively reducing Equation (99) to the desired form

$$\begin{aligned} \frac{1}{\pi}\int_{-1}^{1}\frac{u_1(\tau)}{\tau-t}d\tau + \frac{2}{5\pi}\int_{-1}^{1} t^5\tau^5 u_1(\tau)d\tau - \frac{1}{5\pi}\int_{-1}^{1} t^3\tau^3 u_1(\tau)d\tau &= -t^3 + \frac{1}{10}t \\ \frac{1}{\pi}\int_{-1}^{1}\frac{u_2(\tau)}{\tau-t}d\tau - \frac{1}{5\pi}\int_{-1}^{1} t^5\tau^5 u_1(\tau)d\tau + \frac{3}{5\pi}\int_{-1}^{1} t^3\tau^3 u_1(\tau)d\tau &= -t^5 - \frac{9}{2}t^3 - \frac{3}{8}t \end{aligned} \tag{100}$$

Case 1. We know that the bounded solution is searched in the form

$$u_i(t) = \sqrt{1-t^2}v_i(t), i = \{1,2\}. \tag{101}$$

In this case, the exact solution of Equation (99) is

$$\begin{aligned} u_1(t) &= \sqrt{1-t^2}\left(t^2 + \frac{2}{5}\right), \\ u_2(t) &= \sqrt{1-t^2}\left(t^4 + 5t^2 + 3\right). \end{aligned}$$

By comparing Equation (100) to Equation (18), we obtain the following

$$\begin{aligned}
&f_1{}^*(t) = -t^3 + \frac{1}{10}t, f_2{}^*(t) = -t^5 - \frac{9}{2}t^3 - \frac{3}{8}t,\\
&L_1(u_1(t)) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_1(\tau)}{\tau - t}d\tau, L_2(u_2(t)) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_2(\tau)}{\tau - t}d\tau,\\
&N_1(u_1(t)) = \frac{2}{5\pi}\int_{-1}^{1} t^5\tau^5 u_1(\tau)d\tau - \frac{1}{5\pi}\int_{-1}^{1} t^3\tau^3 u_1(\tau)d\tau,\\
&N_2(u_2(t)) = -\frac{1}{5\pi}\int_{-1}^{1} t^5\tau^5 u_1(\tau)d\tau + \frac{3}{5\pi}\int_{-1}^{1} t^3\tau^3 u_1(\tau)d\tau.
\end{aligned} \tag{102}$$

Using the standard HPM (36), we solve Equation (100) as follows

$$\begin{aligned}
&p^0 : v_{1,0}(t) = -t^3 + \frac{1}{10}t,\ \mathrm{v}_{2,0}(t) = -t^5 - \frac{9}{2}t^3 - \frac{3}{8}t,\\
&p^1 : v_{1,1}(t) = -\frac{3}{320}t^4 - \frac{1603}{1600}t^2 - \frac{5117}{12800},\\
&\quad \mathrm{v}_{2,1}(t) = \frac{643}{640}t^4 + \frac{31889}{6400}t^2 + \frac{76593}{25600},\\
&p^2 : v_{1,2}(t) = \frac{3}{320}t^4 - \frac{3}{1600}t^2 + \frac{3}{12800},\\
&\quad \mathrm{v}_{2,2}(t) = -\frac{3}{640}t^4 + \frac{111}{6400}t^2 + \frac{207}{25600},\\
&p^k : v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 3, 4 \ldots.
\end{aligned} \tag{103}$$

The approximate solution of Equation (99) is given by

$$\begin{aligned}
u_1(t) &= \sqrt{1-t^2}(v_{1,0}(t) + v_{1,1}(t) + v_{1,2}(t)) = \sqrt{1-t^2}\big(t^2 + \tfrac{2}{5}\big),\\
u_2(t) &= \sqrt{1-t^2}(v_{2,0}(t) + v_{2,1}(t) + v_{2,2}(t)) = \sqrt{1-t^2}\big(t^4 + 5t^2 + 3\big),
\end{aligned} \tag{104}$$

which is identical to the exact solution.

Remark 13. *In the bounded case, the standard HPM (36), gave an exact solution for any initial guess.*

Case 2. Let us search the unbounded solution of Equation (99) by using standard HPM (36) of the form

$$u_i(t) = \frac{1}{\sqrt{1-t^2}}v_i(t), i = \{1, 2\}. \tag{105}$$

The exact unbounded solution of Equation (99) is

$$\begin{aligned}
u_1(t) &= \frac{1}{\sqrt{1-t^2}}\left(-t^4 + \frac{3}{5}t^2 + \frac{3}{40}\right),\\
u_2(t) &= \frac{1}{\sqrt{1-t^2}}\left(-t^6 - 4t^4 + 2t^2 + \frac{13}{16}\right).
\end{aligned}$$

It is known that Equation (99) is equivalent to Equation (100). Therefore, we can apply standard HPM (36) to Equation (100), which yields

$$
\begin{aligned}
&p^0: v_{1,0}(t) = \frac{t}{10} + 2,\ \mathrm{v}_{2,0}(t) = -\frac{3t}{8} + 2,\\
&p^1: v_{1,1}(t) = -\frac{1}{8}t^6 - \frac{789}{800}t^4 + \frac{1913}{3200}t^2 - \frac{1}{10}t - \frac{12321}{6400},\\
&\quad \mathrm{v}_{2,1}(t) = -\frac{159}{160}t^6 - \frac{6441}{1600}t^4 + \frac{12867}{6400}t^2 + \frac{3}{8}t - \frac{15169}{12800},\\
&p^2: v_{1,1}(t) = \frac{1}{80}t^6 - \frac{11}{800}t^4 + \frac{7}{3200}t^2 + \frac{1}{6400},\\
&\quad \mathrm{v}_{2,1}(t) = -\frac{1}{160}t^6 + \frac{41}{1600}t^4 - \frac{67}{6400}t^2 - \frac{31}{12800},\\
&p^k: v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 3, 4 \ldots.
\end{aligned}
\tag{106}
$$

The approximate solution of Equation (99) for the unbounded case is

$$
\begin{aligned}
u_1(t) &= \frac{1}{\sqrt{1-t^2}}(v_{1,0}(t) + v_{1,1}(t) + v_{1,2}(t)) = \frac{1}{\sqrt{1-t^2}}\left(-t^4 + \frac{3}{5}t^2 + \frac{3}{40}\right),\\
u_2(t) &= \frac{1}{\sqrt{1-t^2}}(v_{2,0}(t) + v_{2,1}(t) + v_{2,2}(t)) = \frac{1}{\sqrt{1-t^2}}\left(-t^6 - 4t^4 + 2t^2 + \frac{13}{16}\right),
\end{aligned}
\tag{107}
$$

which is identical to the exact solution.

Remark 14. *In the unbounded case, the standard HPM (36) also gave an exact solution for any initial guess.*

Case 3. Let us now search for the semi-bounded solution of Equation (99) given in the form

$$
u_i(t) = \sqrt{\frac{1+t}{1-t}}v_i(t), i = \{1, 2\}. \tag{108}
$$

The exact semi-bounded solution of Equation (99) is

$$
\begin{aligned}
u_1(t) &= \sqrt{\frac{1+t}{1-t}}\left(-t^3 + t^2 - \frac{2}{5}t + \frac{2}{5}\right),\\
u_2(t) &= \sqrt{\frac{1+t}{1-t}}\left(-t^5 + t^4 - 5t^3 + 5t^2 - 3t + 3\right).
\end{aligned}
$$

In this case, we apply standard HPM (36) to solve Equation (100), yielding

$$
\begin{aligned}
&p^0: v_{1,0}(t) = \frac{t}{10},\ \mathrm{v}_{2,0}(t) = \left(-\frac{9}{2}t^3 - \frac{3}{8}t\right),\\
&p^1: v_{1,1}(t) = -\frac{40t^5 - 40t^4 + 3196t^3 - 3196t^2 + 1603t - 1283}{3200},\\
&\quad \mathrm{v}_{2,1}(t) = -\frac{6360t^5 - 6360t^4 + 3324t^3 - 32124t^2 + 16857t - 19257}{6400},\\
&p^2: v_{1,2}(t) = \frac{40t^5 - 40t^4 - 4t^3 + 4t^2 + 3t - 3}{3200},\\
&\quad \mathrm{v}_{2,2}(t) = -\frac{40t^5 - 40t^4 - 124t^3 + 124t^2 - 57t + 57}{6400}\\
&p^k: v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 2, 3 \ldots.
\end{aligned}
\tag{109}
$$

The approximate solution of Equation (99) for the semi-bounded is

$$
\begin{aligned}
u_1(t) &= \sqrt{\frac{1+t}{1-t}}(v_{1,0}(t) + v_{1,1}(t) + v_{1,2}(t)) = \sqrt{\frac{1+t}{1-t}}\left(-t^3 + t^2 - \frac{2}{5}t + \frac{2}{5}\right),\\
u_2(t) &= \sqrt{\frac{1+t}{1-t}}(v_{2,0}(t) + v_{2,1}(t) + v_{2,2}(t)) = \sqrt{\frac{1+t}{1-t}}\left(-t^5 + t^4 - 5t^3 + 5t^2 - 3t + 3\right),
\end{aligned}
\tag{110}
$$

which is identical to the exact solution.

Remark 15. *In the semi-bounded case, the standard HPM (36) gave an exact solution for any initial guess.*

Case 4. Let us search the semi-bounded solution of Equation (99) given by

$$u_i(t) = \sqrt{\frac{1-t}{1+t}}v_i(t), i = \{1,2\}. \tag{111}$$

The exact semi-bounded solution of Equation (99) is

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1-t}{1+t}}\left(t^3 + t^2 + \frac{2}{5}t + \frac{2}{5}\right), \\ u_2(t) &= \sqrt{\frac{1-t}{1+t}}\left(t^5 + t^4 + 5t^3 + 5t^2 + 3t + 3\right). \end{aligned}$$

By applying standard HPM (36), we obtain

$$\begin{aligned} p^0 &: v_{1,0}(t) = \frac{t}{10}, \ \mathrm{v}_{2,0}(t) = -\frac{9}{2}t^3 - \frac{3}{8}t, \\ p^1 &: v_{1,1}(t) = \frac{40t^5 + 40t^4 + 3196t^3 + 3196t^2 + 963t + 1283}{3200}, \\ &\mathrm{v}_{2,1}(t) = \frac{6360t^5 + 6360t^4 + 60924t^3 + 32124t^2 + 21657t + 19257}{6400}, \\ p^2 &: v_{1,2}(t) = \frac{-40t^5 - 40t^4 + 4t^3 + 4t^2 - 3t - 3}{3200}, \\ &\mathrm{v}_{2,2}(t) = \frac{40t^5 + 40t^4 - 124t^3 - 124t^2 - 57t - 57}{6400} \\ p^k &: v_{1,k}(t) = 0, \ \mathrm{v}_{2,k}(t) = 0, k = 2, 3 \ldots. \end{aligned} \tag{112}$$

The approximate solution of Equation (99) for the semi-bounded is

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1-t}{1+t}}(v_{1,0}(t) + v_{1,1}(t) + v_{1,2}(t)) = \sqrt{\frac{1-t}{1+t}}\left(t^3 + t^2 + \frac{2}{5}t + \frac{2}{5}\right), \\ u_2(t) &= \sqrt{\frac{1-t}{1+t}}(v_{2,0}(t) + v_{2,1}(t) + v_{2,2}(t)) = \sqrt{\frac{1-t}{1+t}}\left(t^5 + t^4 + 5t^3 + 5t^2 + 3t + 3\right), \end{aligned} \tag{113}$$

which is identical to the exact solution.

Remark 16. *The standard HPM (36) gave an exact solution for any initial guess.*

Example 4 (Ahdiaghdam and Shahmorad [14]): *Consider the system of SIEs of the form*

$$\begin{aligned} &\int_{-1}^{1} \frac{u_1(\tau)}{\tau - t}d\tau + \int_{-1}^{1} (\tau - t)u_1(\tau)d\tau + \int_{-1}^{1} tu_2(\tau)d\tau = \pi, \\ &\int_{-1}^{1} \frac{u_2(\tau)}{\tau - t}d\tau + \int_{-1}^{1} \tau u_1(\tau)d\tau + \int_{-1}^{1} (\tau + t)u_2(\tau)d\tau = 2\pi t, -1 < t < 1. \end{aligned} \tag{114}$$

The exact unbounded solution is given by

$$\begin{aligned} u_1(t) &= \frac{2t}{3\sqrt{1-t^2}}, \\ u_2(t) &= \frac{18t^2 - 2t - 9}{9\sqrt{1-t^2}}. \end{aligned} \tag{115}$$

Remark 17. *Example 4 is considered by Ahdiaghdam and Shahmorad [14], who found the unbounded solution as follows*

$$u_1(t) = \frac{2t+6}{3\sqrt{1-t^2}},$$
$$u_2(t) = -\frac{2t-18}{9\sqrt{1-t^2}}. \tag{116}$$

Solution (116) does not satisfy the condition in Equation (9). Fortunately, solution (115) satisfies the condition in Equation (9). Therefore, the real exact solution of Equation (114) should be given as Equation (115). Authors in [14] found exact solutions for the unbounded and semi-bounded cases.

Case 2. Following is the search for the unbounded solution.

$$u_i(t) = \frac{1}{\sqrt{1-t^2}} v_i(t), i = \{1,2\}. \tag{117}$$

Comparing Equation (114) with Equation (18), we obtain

$$\begin{aligned}
&f_1(t) = 1, f_2(t) = 2t,\\
&L_1(u_1(t)) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_1(\tau)}{\tau - t} d\tau, L_2(u_2(t)) = \frac{1}{\pi}\int_{-1}^{1} \frac{u_2(\tau)}{\tau - t} d\tau,\\
&N_1(u_1(t)) = \frac{1}{\pi}\int_{-1}^{1} (\tau - t) u_1(\tau) d\tau + \frac{1}{\pi}\int_{-1}^{1} t u_2(\tau) d\tau\\
&N_2(u_2(t)) = \frac{1}{\pi}\int_{-1}^{1} \tau u_1(\tau) d\tau + \frac{1}{\pi}\int_{-1}^{1} (\tau + t) u_2(\tau) d\tau.
\end{aligned} \tag{118}$$

The initial guess is chosen as the following function

$$u_{1,0}(t) = \frac{2t+6}{3},$$
$$u_{2,0}(t) = -\frac{2t-18}{9}. \tag{119}$$

Standard HPM (36) is then applied, where the results are

$$\begin{aligned}
&p^0 : v_{1,0}(t) = \frac{2t+6}{3},\ \mathrm{v}_{2,0}(t) = -\frac{2t-18}{9},\\
&p^1 : v_{1,1}(t) = -2,\ \mathrm{v}_{2,1}(t) = -2,\\
&p^2 : v_{1,2}(t) = 0,\ \mathrm{v}_{2,2}(t) = 2t^2 - 1,\\
&p^k : v_{1,k}(t) = 0,\ \mathrm{v}_{2,k}(t) = 0, k = 3,4\ldots.
\end{aligned} \tag{120}$$

The approximate solution of Equation (114) is given by

$$\begin{aligned}
u_1(t) &= \frac{1}{\sqrt{1-t^2}}(v_{1,0}(t) + v_{1,1}(t) + v_{1,2}(t)) = \frac{2t}{3\sqrt{1-t^2}},\\
u_2(t) &= \frac{1}{\sqrt{1-t^2}}(v_{2,0}(t) + v_{2,1}(t) + v_{2,2}(t)) = \frac{18t^2 - 2t - 9}{9\sqrt{1-t^2}},
\end{aligned} \tag{121}$$

which is identical to the exact solution.

Let us choose the following initial guess for Equation (114) that is far from the exact solution

$$u_{1,0}(t) = 1,$$
$$u_{2,0}(t) = 1. \tag{122}$$

By the chosen initial guess, we solve Equation (114) using the standard HPM. Then, we calculate the error functions by using Maple. Here, the error function is $\mathrm{E}_i(t) = |u_i(t) - v_i(t)|, i = \{1,2\}$.

In Table 1, the results demonstrate the robustness and convergence of the method employed in our study.

Table 1. Errors term (unbounded solution) of Equation (114) for standard HPM at M = 50.

t	$U_1(t)$	$U_2(t)$	$E_1(t)$	$E_2(t)$
−0.95	−2.02828994819777	3.25415992566116	$1.8014833608 \times 10^{-15}$	$8.947367359 \times 10^{-14}$
−0.70	−0.65346403921307	0.18981574472379	$5.8039265767 \times 10^{-16}$	$2.882616866 \times 10^{-14}$
−0.30	−0.20965696734438	−0.78970791033051	$1.8621279393 \times 10^{-16}$	$9.248568765 \times 10^{-15}$
0.00	0.00000000000000	−1.00000000000000	0.0000000000000	0.000000000000
0.50	0.38490017945975	−0.70565032900954	$3.418600331 \times 10^{-16}$	$1.697904831 \times 10^{-14}$
0.70	0.65346403921307	−0.24582694808491	$5.803926576 \times 10^{-16}$	$2.882616866 \times 10^{-14}$
0.95	2.02828994819777	1.90196662686264	$1.801483360 \times 10^{-15}$	$8.947367359 \times 10^{-14}$

Regardless of the initial guess chosen, whether close to or far from the exact solution, we observe that the obtained results consistently approach the exact solution. This finding is significant because it indicates the stability and effectiveness of the method in finding the solution, regardless of the starting point. Furthermore, it suggests that the algorithm used in our study has a strong capability to converge towards the desired solution, even if the initial guess is not particularly accurate.

Remark 18. *In Example 4, we have chosen two types of initial guesses (119) and (122). Since the initial guess (119) satisfies Equation (114), we obtain the exact solution. On the other hand, the initial guess (122) does not satisfy Equation (114). Hence, we obtain high accurate approximation solution.*

Case 3. Let us search the semi-bounded solution of Equation (114) given in the form

$$u_i(t) = \sqrt{\frac{1+t}{1-t}} v_i(t), i = \{1,2\}. \tag{123}$$

In this case, we define the exact solution for Equation (114) as

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1+t}{1-t}} \left(\frac{56}{27} t - \frac{38}{27} \right), \\ u_2(t) &= \sqrt{\frac{1+t}{1-t}} \left(\frac{40}{9} t - \frac{14}{3} \right). \end{aligned} \tag{124}$$

Let us choose the following initial guess given by

$$\begin{aligned} u_{1.0}(t) &= f_1(t) = 1, \\ u_{2.0}(t) &= f_2(t) = 2t. \end{aligned}$$

We then solve Equation (114) via the standard HPM.

In Table 2, displays the error term of Equation (114) for the semi-bounded case. It indicates a significant reduction in the error term as the number of iteration increases. The proposed method approaches the exact solution with fifty iterations.

Table 2. Errors term (semi-bounded solution) of Equation (114) for standard HPM at M = 50.

t	$U_1(t)$	$U_2(t)$	$E_1(t)$	$E_2(t)$
−0.95	−0.5408773195194	−1.4233613671563	$1.3650345104957 \times 10^{-13}$	$1.2578778554944 \times 10^{-15}$
−0.70	−1.2011291387440	−3.2673201960653	$3.1217939169280 \times 10^{-13}$	$1.9733350361090 \times 10^{-15}$
−0.30	−1.4893409754315	−4.4027963142320	$4.1695251528505 \times 10^{-13}$	$2.6069791150908 \times 10^{-16}$
0.00	−1.4074074074074	−4.6666666666666	$4.3701667798948 \times 10^{-13}$	$4.1448326252672 \times 10^{-15}$
0.50	−0.6415002990995	−4.2339019740572	$3.7821114999152 \times 10^{-13}$	$1.8118581756136 \times 10^{-14}$
0.70	0.1057989396821	−3.7029628888740	$3.1159899903512 \times 10^{-13}$	$3.0915582232374 \times 10^{-14}$
0.95	3.5157025768761	−2.7755546659548	$1.3470196768869 \times 10^{-13}$	$1.0082617926126 \times 10^{-13}$

Using Equation (37) for choosing the initial guess by decomposition function, we obtain the following

$$\begin{aligned} f_1(t) &= f_{1,1}(t) + f_{1,2}(t) = 1, \ \ f_{1,1}(t) = \frac{1}{2}, f_{1,2}(t) = \frac{1}{2}, \\ f_2(t) &= f_{2,1}(t) + f_{2,2}(t) = 2x, \ f_{2,1}(t) = x, f_{2,2}(t) = x. \end{aligned}$$

We solve Equation (114) using MHPM (38) with the chosen initial guess, then compare the results obtained from the HPM and MHPM.

In Table 3, we can observe the error term of Equation (114) for both the MHPM and the HPM. The comparison highlights that the error of MHPM is slightly better than the error of the HPM. It shows that the decomposition function allows the MHPM approaches to obtain the exact solution faster than standard HPM.

Table 3. The error term of MHPM $(\varepsilon_i(t))$ and HPM $(E_i(t))$ for M = 50.

t	$\varepsilon_1(t)$	$\varepsilon_2(t)$	$E_1(t)$	$E_2(t)$
−0.95	$3.6481618306462 \times 10^{-14}$	$2.2550146982292 \times 10^{-15}$	$1.3650345104957 \times 10^{-13}$	$1.2578778554944 \times 10^{-15}$
−0.70	$8.3428680766814 \times 10^{-14}$	$4.8172590587367 \times 10^{-15}$	$3.1217939169280 \times 10^{-13}$	$1.9733350361090 \times 10^{-15}$
−0.30	$1.1141697686596 \times 10^{-13}$	$5.3443071859360 \times 10^{-15}$	$4.1695251528505 \times 10^{-13}$	$2.6069791150908 \times 10^{-16}$
0.00	$1.1676256669354 \times 10^{-13}$	$4.1448326252672 \times 10^{-15}$	$4.3701667798948 \times 10^{-13}$	$4.1448326252672 \times 10^{-15}$
0.50	$1.0099115145525 \times 10^{-13}$	$1.8802301822417 \times 10^{-15}$	$3.7821114999152 \times 10^{-13}$	$1.8118581756136 \times 10^{-14}$
0.70	$8.3138484437974 \times 10^{-14}$	$7.5644509717512 \times 10^{-15}$	$3.1159899903512 \times 10^{-13}$	$3.0915582232374 \times 10^{-14}$
0.95	$3.5580876626020 \times 10^{-14}$	$3.6176630333961 \times 10^{-14}$	$1.3470196768869 \times 10^{-13}$	$1.0082617926126 \times 10^{-13}$

Case 4. We can search the semi-bounded solution of Equation (114) given in the form

$$u_i(t) = \sqrt{\frac{1-t}{1+t}}v_i(t), i = \{1, 2\}. \tag{125}$$

In this case, the exact solution for Equation (114) is defined as

$$\begin{aligned} u_1(t) &= \sqrt{\frac{1-t}{1+t}}\left(-\frac{16}{27}t - \frac{34}{27}\right), \\ u_2(t) &= \sqrt{\frac{1-t}{1+t}}\left(-\frac{32}{9}t - \frac{10}{3}\right). \end{aligned} \tag{126}$$

Let us choose the following initial guess given by

$$\begin{aligned} u_{1,0}(t) &= -(t+1), \\ u_{2,0}(t) &= -2t. \end{aligned}$$

By the chosen initial guess, we solve Equation (114) using the standard HPM.

In Table 4, presents the error term of Equation (114) for the semi-bounded case. It becomes evident that as the number of iterations increases, the error term experiences a significant reduction.

Table 4. Errors term of standard HPM for Equation (114) with M = 50.

t	$U_1(t)$	$U_2(t)$	$E_1(t)$	$E_2(t)$
−0.95	−4.34836897666258	0.2775554665954	$2.3368952774 \times 10^{-13}$	$1.8593678657 \times 10^{-13}$
−0.70	−2.01017985396020	−2.01017985396ther02	$5.4151602282 \times 10^{-13}$	$5.4078776822 \times 10^{-14}$
−0.30	−1.47381082970237	−3.08894598554058	$7.24423287407 \times 10^{-13}$	$1.5573596666 \times 10^{-14}$
0.00	−1.25925925925925	−3.33333333333333	$7.59655712819 \times 10^{-13}$	$2.9605947323 \times 10^{-15}$
0.50	−0.89810041873941	−2.95090137585808	$6.580520754485 \times 10^{-13}$	$6.4383639573 \times 10^{-15}$
0.70	−0.70325177553406	−2.44582254676892	$5.426768081409 \times 10^{-13}$	$7.0559164526 \times 10^{-15}$
0.95	−0.29178908026704	−1.07463783220302	$2.372924944696 \times 10^{-13}$	$3.8194607747 \times 10^{-15}$

6. Conclusions

In this paper, we have introduced and developed the homotopy perturbation method (HPM) as a semi-analytical solution technique for the system of singular integral equations of the first kind. This method offers a novel approach to tackling this challenging class of equations and has shown promising results in our study. Furthermore, we have applied the HPM to several illustrative examples, as demonstrated earlier. These examples provided valuable insights into the performance and effectiveness of the HPM compared to the Chebyshev series method, which is commonly used for solving singular integral equations. The results obtained from the numerical examples clearly indicated that the HPM outperformed the Chebyshev series method in terms of accuracy and convergence. The solutions obtained through the HPM were in excellent agreement with the exact solutions, validating the reliability and robustness of the method.

By showcasing the numerical examples, we have not only demonstrated the capability of the HPM to produce accurate results but also emphasized its superiority over existing methods. The fact that the HPM coincided with the exact solutions in the examples further strengthens our confidence in its applicability and effectiveness for solving the system of singular integral equations of the first kind. Overall, our study highlights the successful development and application of the HPM for solving a challenging class of equations. The numerical examples presented in this paper provide compelling evidence of the HPM's superiority over alternative methods, as it consistently yields accurate solutions that coincide with the exact solutions. This suggests that the HPM has the potential to be a valuable tool in various scientific and engineering applications involving singular integral equations.

This characteristic of the method is crucial in practical applications, as it provides reassurance that the algorithm will consistently yield reliable results, irrespective of the initial conditions. It demonstrates the method's ability to overcome potential errors or uncertainties associated with the initial guess and ensures the accuracy of the final solution. The convergence of the results, regardless of the initial guess, provides confidence in the reliability and robustness of the method. Furthermore, it implies that the method is not sensitive to the choice of initial conditions and can efficiently find the solution under various circumstances. Overall, the findings in Tables 1–4 reinforce the effectiveness and suitability of the method used in our study, as it consistently approaches the exact solution regardless of the initial guess, validating its stability and robustness in practical applications.

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