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Approximation of Brownian Motion on Simple Graphs

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Abstract: This article is based on chapters 9 and 19 of the new neural network approximation monograph written by the first author. We use the approximation properties coming from the parametrized and deformed neural networks based on the parametrized error and q-deformed and β-parametrized half-hyperbolic tangent activation functions. So, we implement a univariate theory on a compact interval that is ordinary and fractional. The result is the quantitative approximation of Brownian motion on simple graphs: in particular over a system S of semiaxes emanating from a common origin radially arranged and a particle moving randomly on S. We produce a large variety of Jackson-type inequalities, calculating the degree of approximation of the engaged neural network operators to a general expectation function of this kind of Brownian motion. We finish with a detailed list of approximation applications related to the expectation of important functions of this Brownian motion. The differentiability of our functions is taken into account, producing higher speeds of approximation.

Keywords: neural network operators; Brownian motion on simple graphs; expectation; quantitative approximation

MSC: 26A33; 41A17; 41A25; 60G15; 60G22

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1. Introduction

The first author in [1,2], see Sections 2–5, was the first researcher to derive quantitative neural network approximation to continuous functions by precisely defined neural network operators of Cardaliaguet–Euvrard and 'Squasing' types, by using the modulus of continuity of the engaged function or its high-order derivative and obtaining almost attained Jackson-type inequalities. He took care of both the univariate and multivariate cases. Defining these operators as 'bell-shaped' and 'squashing' functions is supposed to provide compact support.

Furthermore, the first author (motivated by [3]) continued his studies on neural networks approximation by introducing and using the appropriate quasi-interpolation operators of sigmoidal and hyperbolic tangent type, which resulted in [4], by treating both the univariate and multivariate cases. He dealt also with the corresponding fractional cases [5–7]. The authors also are motivated by the seminal works [8–13].

In [14,15], the first author extended his studies for Banach space-valued functions for activation functions induced by the parametrized error and q-deformed and β-parametrized half-hyperbolic tangent sigmoid functions. The authors, motivated by [16], created neural network quantitative approximations to Brownian motion over a simple graph of a system of semiaxes.

They obtained a collection of Jackson-type inequalities, calculating the error of approximation to a general expectation function of this Brownian motion and its derivative. They present ordinary and fractional calculus results. They finish with a plethora of interesting applications.

2. Basics

2.1. About the Parametrized (Gauss) Error Special Activation Function

Here, we follow [17].

We consider here the parametrized (Gauss) error special activation function

$$erf\lambda z = \frac{2}{\sqrt{\pi}} \int_0^{\lambda z} e^{-t^2} dt, \ \ \lambda > 0, z \in \mathbb{R}, \tag{1}$$

which is a sigmoidal-type function and a strictly increasing function.

Of special interest in neural network theory is when $0 < \lambda < 1$; see Section 1—Introduction.

It has the basic properties

$$\begin{gathered} erf(0) = 0, \ \ erf(-\lambda x) = -erf(\lambda x), \text{for every } 0 < \lambda < 1 \\ erf(+\infty) = 1, \ erf(-\infty) = -1, \end{gathered} \tag{2}$$

and

$$(erf(\lambda x))' = \frac{2\lambda}{\sqrt{\pi}} e^{-(\lambda x)^2}, \ x \in \mathbb{R}. \tag{3}$$

We consider the function

$$\chi(x) = \frac{1}{4}(erf(\lambda(x+1)) - erf(\lambda(x-1))), \ x \in \mathbb{R}, \tag{4}$$

and we notice that

$$\chi(-x) = \chi(x). \tag{5}$$

Thus, χ is an even function.

Since $x+1 > x-1$, then $erf(\lambda(x+1)) > erf(\lambda(x-1))$, and $\chi(x) > 0$, all $x \in \mathbb{R}$.

We see

$$\chi(0) = \frac{erf\lambda}{2}. \tag{6}$$

Let $x > 0$; then, we have that

$$\chi'(x) = \frac{\lambda}{2\sqrt{\pi}} \left(\frac{e^{\lambda^2(x-1)^2} - e^{\lambda^2(x+1)^2}}{e^{\lambda^2(x+1)^2} e^{\lambda^2(x-1)^2}} \right) < 0, \tag{7}$$

proving $\chi'(x) < 0$, for $x > 0$. That is, χ is strictly decreasing on $[0, \infty)$, and it is strictly increasing on $(-\infty, 0]$, and $\chi'(0) = 0$.

Clearly, the x-axis is the horizontal asymptote of χ.

Concluding, χ is a bell symmetric function with maximum

$$\chi(0) = \frac{erf\lambda}{2}.$$

Theorem 1. *It holds*

$$\sum_{i=-\infty}^{\infty} \chi(x-i) = 1, \ \forall\, x \in \mathbb{R}. \tag{8}$$

We have

Theorem 2. *We have that*

$$\int_{-\infty}^{\infty} \chi(x) dx = 1. \tag{9}$$

Hence, $\chi(x)$ is a density function on $\mathbb{R}$. We need

Theorem 3. *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$, $\lambda > 0$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(nx - k) < \frac{\left(1 - erf\left(\lambda\left(n^{1-\alpha} - 2\right)\right)\right)}{2}, \tag{10}$$

with

$$\lim_{n \to +\infty} \frac{\left(1 - erf\left(\lambda\left(n^{1-\alpha} - 2\right)\right)\right)}{2} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part and by $\lceil \cdot \rceil$ the ceiling of a number. Furthermore, we need

Theorem 4. *Let $x \in [a, b] \subset \mathbb{R}$, $\lambda > 0$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then,*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k)} < \frac{1}{\chi(1)} = \frac{4}{erf(2\lambda)}. \tag{11}$$

Remark 1. *As in [18], we have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k) \neq 1. \tag{12}$$

Note 1. *For large enough n, we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also, $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. As in [18], we obtain that*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k) \leq 1. \tag{13}$$

Definition 1. *Let $f \in C([a, b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the X-valued linear neural network operators*

$$A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \chi(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k)}, \quad x \in [a, b]. \tag{14}$$

Clearly here, $A_n(f, x) \in C([a, b])$. We study here the pointwise and uniform convergence of $A_n(f, x)$ to $f(x)$ with rates.

For convenience, also, we call

$$A_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \chi(nx - k), \tag{15}$$

that is

$$A_n(f, x) = \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k)}. \tag{16}$$

So that

$$\begin{aligned} A_n(f, x) - f(x) &= \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k)} - f(x) \\ &= \frac{A_n^*(f, x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k)}. \end{aligned} \tag{17}$$

Consequently, we derive

$$|A_n(f,x) - f(x)| \leq \frac{4}{erf(2\lambda)} \left| A_n^*(f,x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) \right) \right|. \tag{18}$$

That is

$$|A_n(f,x) - f(x)| \leq \frac{4}{erf(2\lambda)} \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \chi(nx-k) \right|. \tag{19}$$

We will estimate the right-hand side of (19).

For that, we need, for $f \in C([a,b])$, the first modulus of continuity

$$\omega_1(f,\delta)_{[a,b]} := \omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|, \ \ \delta > 0. \tag{20}$$

The fact $f \in C([a,b])$ is equivalent to $\lim_{\delta \to 0} \omega_1(f,\delta) = 0$; see [19].

We present a series of real-valued neural network approximations to a function given with rates.

We first give

Theorem 5. *Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $\lambda > 0$, $x \in [a,b]$. Then,*

(i)

$$|A_n(f,x) - f(x)| \leq \frac{4}{erf(2\lambda)} \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - erf\left(\lambda\left(n^{1-\alpha} - 2\right)\right)\right) \|f\|_\infty \right] =: \rho, \tag{21}$$

and

(ii)

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{22}$$

We notice $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly.

The speed of convergence is $\max\left\{ \frac{1}{n^\alpha}, 1 - erf\left(\lambda\left(n^{1-\alpha} - 2\right)\right) \right\}$.

We need

Definition 2 ([20])**.** *Let $[a,b] \subset \mathbb{R}$, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to \mathbb{R}$. We assume that $f^{(m)} \in L_1([a,b])$. We call the left Caputo fractional derivative of order α:*

$$(D_{*a}^\alpha f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \ \ \forall \, x \in [a,b]. \tag{23}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^\alpha f := f^{(m)}$ the ordinary real-valued derivative (defined similar to numerical one, see [21], p. 83), and set $D_{*a}^0 f := f$. See also [22–25].

By [20], $(D_{*a}^\alpha f)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^\alpha f \in L_1([a,b])$.

If $\left| f^{(m)} \right|_{L_\infty([a,b])} < \infty$, then by [20], $D_{*a}^\alpha f \in C([a,b])$, hence $|D_{*a}^\alpha f| \in C([a,b])$.

We mention the following.

Lemma 1 ([19])**.** *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$ and $f^{(m)} \in L_\infty([a,b])$. Then, $D_{*a}^\alpha f(a) = 0$.*

We also mention the following.

Definition 3 ([26])**.** *Let $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b])$, where $f : [a,b] \to \mathbb{R}$. We call the right Caputo fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z) dz, \quad \forall\, x \in [a,b]. \tag{24}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [26], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b])$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b])} < \infty$, and $\alpha \notin \mathbb{N}$, by [26], $D_{b-}^{\alpha} f \in C([a,b])$, hence $\left| D_{b-}^{\alpha} f \right| \in C([a,b])$.

See also [27].

We need

Lemma 2 ([19])**.** *Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then, $D_{b-}^{\alpha} f(b) = 0$.*

We present the following real-valued fractional approximation result by $erf\lambda$-based neural networks.

Theorem 6. *Let $0 < \alpha, \beta^* < 1$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $\lambda > 0$. Then,*

(i)

$$|A_n(f,x) - f(x)| \leq$$

$$\frac{4}{erf(2\lambda)\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta^*}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta^*}}\right)_{[x,b]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \left(\frac{1 - erf\left(\lambda\left(n^{1-\beta^*} - 2\right)\right)}{2} \right) \left(\|D_{x-}^{\alpha} f\|_{\infty,[a,x]} (x-a)^{\alpha} + \|D_{*x}^{\alpha} f\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \tag{25}$$

and

(ii)

$$\|A_n f - f\|_\infty \leq \frac{4}{\Gamma(\alpha+1) erf(2\lambda)}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta^*}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta^*}}\right)_{[x,b]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \left(\frac{1 - erf\left(\lambda\left(n^{1-\beta^*} - 2\right)\right)}{2} \right) (b-a)^{\alpha} \left(\sup_{x\in[a,b]} \|D_{x-}^{\alpha} f\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \|D_{*x}^{\alpha} f\|_{\infty,[x,b]} \right) \right\}. \tag{26}$$

When $\alpha = \frac{1}{2}$, we derive

Corollary 1. *Let $0 < \beta^* < 1$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $\lambda > 0$. Then,*

(i)

$$|A_n(f,x) - f(x)| \leq$$

$$\frac{8}{erf(2\lambda)\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\left(\frac{1-erf\left(\lambda\left(n^{1-\beta^*}-2\right)\right)}{2}\right)\left(\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}\sqrt{(x-a)}+\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\sqrt{(b-x)}\right)\right\}, \tag{27}$$

and

(ii)

$$\|A_n f - f\|_\infty \leq \frac{8}{erf(2\lambda)\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\left(\frac{1-erf\left(\lambda\left(n^{1-\beta^*}-2\right)\right)}{2}\right)\sqrt{(b-a)}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\right\}<\infty. \tag{28}$$

2.2. About q-Deformed and β-Parametrized Half-Hyperbolic Tangent Function φ_q

All the next background comes from [28].

Here, we describe the properties of the activation function

$$\varphi_q(t) := \frac{1-qe^{-\beta t}}{1+qe^{-\beta t}}, \quad \forall\, t \in \mathbb{R}, \tag{29}$$

where $q, \beta > 0$.

We have that

$$\varphi_q(0) = \frac{1-q}{1+q}$$

and

$$\varphi_q(-t) = -\varphi_{\frac{1}{q}}(t), \ \forall\, t \in \mathbb{R}, \tag{30}$$

hence

$$\varphi'_{\frac{1}{q}}(t) = \varphi'_q(-t). \tag{31}$$

It is

$$\lim_{t\to+\infty}\varphi_q(t) = \varphi_q(+\infty) = 1, \tag{32}$$

and

$$\lim_{t\to-\infty}\varphi_q(t) = \varphi_q(-\infty) = -1. \tag{33}$$

Furthermore

$$\varphi'_q(t) = \frac{2\beta q e^{\beta t}}{\left(e^{\beta t}+q\right)^2} > 0, \quad \forall\, t \in \mathbb{R}, \tag{34}$$

therefore, φ_q is striclty increasing. Moreover, in case of $t < \frac{\ln q}{\beta}$, we have that φ_q is strictly concave up, with $\varphi''_q\left(\frac{\ln q}{\beta}\right) = 0$.

And in case of $t > \frac{\ln q}{\beta}$, we have that φ_q is strictly concave down.

Clearly, φ_q is a shifted sigmoid function with $\varphi_q(0) = \frac{1-q}{1+q}$, and $\varphi_q(-x) = -\varphi_{q^{-1}}(x)$, $\forall\, x \in \mathbb{R}$, (a semi-odd function); see also [28].

We consider the function

$$\phi_q(x) := \frac{1}{4}\big(\varphi_q(x+1) - \varphi_q(x-1)\big) > 0, \tag{35}$$

$\forall x \in \mathbb{R}$; $\beta, q > 0$. Notice that $\phi_q(\pm\infty) = 0$, so the x-axis is horizontal asymptote. We have that

$$\phi_q(-x) = \phi_{\frac{1}{q}}(x),\ \forall\, x \in \mathbb{R}, \tag{36}$$

which is a deformed symmetry.

Next, we have that

φ_q is striclty increasing over $\left(-\infty, \frac{\ln q}{\beta} - 1\right)$ and it is strictly decreasing over $\left(\frac{\ln q}{\beta} + 1, +\infty\right)$.

Moreover, ϕ_q is concave down over $\left[\frac{\ln q}{\beta} - 1, \frac{\ln q}{\beta} + 1\right]$, and it is strictly concave down over $\left(\frac{\ln q}{\beta} - 1, \frac{\ln q}{\beta} + 1\right)$.

Consequently, ϕ_q has a bell-type shape over $\mathbb{R}$.

Of course, it holds $\phi_q''\left(\frac{\ln q}{\beta}\right) < 0$. Thus, at $x = \frac{\ln q}{\beta}$, we have the maximum value of ϕ_q, which is

$$\phi_q\left(\frac{\ln q}{\beta}\right) = \frac{(1-e^{-\beta})}{2(1+e^{-\beta})} = \frac{\varphi_1(1)}{2}. \tag{37}$$

We mention

Theorem 7 ([29])**.** *We have that*

$$\sum_{i=-\infty}^{\infty} \phi_q(x-i) = 1,\ \forall\, x \in \mathbb{R}, \forall\, q, \beta > 0. \tag{38}$$

It follows

Theorem 8 ([29])**.** *It holds*

$$\int_{-\infty}^{\infty} \phi_q(x)dx = 1,\ \ q, \beta > 0. \tag{39}$$

So that ϕ_q is a density function on $\mathbb{R}$; $q, \beta > 0$.

We need the following result,

Theorem 9 ([29])**.** *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \beta > 0$. Then,*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \phi_q(nx-k) < \max\left\{q, \frac{1}{q}\right\} e^{2\beta} e^{-\beta n^{(1-\alpha)}} = K e^{-\beta n^{(1-\alpha)}}, \tag{40}$$

where $K := \max\left\{q, \frac{1}{q}\right\} e^{2\beta}$.

Let $\lceil \cdot \rceil$ be the ceiling of the number, and let $\lfloor \cdot \rfloor$ be the integral part of the number.

We mention the following result:

Theorem 10 ([29]). *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, we consider the number $\lambda_q > z_0 > 0$ with $\phi_q(z_0) = \phi_q(0)$ and $\beta, \lambda_q > 1$. Then,*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_q(nx-k)} < max\left\{ \frac{1}{\phi_q(\lambda_q)}, \frac{1}{\phi_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)} \right\} =: \theta(q). \tag{41}$$

We also mention

Remark 2 ([29]). *(i) We have that*

$$\lim_{n \to +\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_q(nx-k) \neq 1, \ \textit{for at least some } x \in [a,b], \tag{42}$$

where $\beta, q > 0$.

(ii) Let $[a,b] \subset \mathbb{R}$. For large n, we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also, $a \leq \frac{k}{n} \leq b$, if $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \phi_q(nx-k) \leq 1. \tag{43}$$

We need

Definition 4. *Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the real-valued linear neural network operators*

$$H_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi_q(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)}, \ x \in [a,b]; q, \beta > 0. \tag{44}$$

Clearly, $H_n(f) \in C([a,b])$.

We study here the pointwise and uniform convergence of $H_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$H_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi_q(nx-k),. \tag{45}$$

That is

$$H_n(f,x) := \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)}. \tag{46}$$

So that

$$H_n(f,x) - f(x) = \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)} - f(x) = \tag{47}$$

$$\frac{H_n^*(f,x) - f(x)\left(\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)\right)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)}.$$

Consequently, we derive that

$$|H_n(f,x) - f(x)| \leq \theta(q)\left|H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi_q(nx-k)\right)\right| =$$

$$\theta(q)\left|\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right)\Phi_q(nx-k)\right|, \tag{48}$$

where $\theta(q)$ as in (41). We will estimate the right-hand side of the last quantity.

We present a set of real-valued neural network approximations to a function given with rates.

Theorem 11. *Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $q, \beta > 0$, $x \in [a,b]$. Then,*

(i)

$$|H_n(f,x) - f(x)| \leq \theta(q)\left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty K e^{-\beta n^{(1-\alpha)}}\right] =: \tau, \tag{49}$$

where K as in (40),

and

(ii)

$$\|H_n(f) - f\|_\infty \leq \tau. \tag{50}$$

We observe that $\lim\limits_{n\to\infty} H_n(f) = f$, pointwise and uniformly.

Next, we present the following.

Theorem 12. *Let $0 < \alpha, \beta^* < 1$, $q, \beta > 0$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then,*

(i)

$$|H_n(f,x) - f(x)| \leq$$

$$\frac{\theta(q)}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^{\beta^*}}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. K e^{-\beta n^{(1-\beta^*)}}\left(\|D_{x-}^\alpha f\|_{\infty,[a,x]}(x-a)^\alpha + \|D_{*x}^\alpha f\|_{\infty,[x,b]}(b-x)^\alpha\right)\right\}, \tag{51}$$

and

(ii)

$$\|H_n f - f\|_\infty \leq \frac{\theta(q)}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup\limits_{x\in[a,b]} \omega_1\left(D_{x-}^\alpha f, \frac{1}{n^{\beta^*}}\right)_{[a,x]} + \sup\limits_{x\in[a,b]} \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.(b-a)^\alpha K e^{-\beta n^{(1-\beta^*)}}\left(\sup_{x\in[a,b]}\|D_{x-}^\alpha f\|_{\infty,[a,x]} + \sup_{x\in[a,b]}\|D_{*x}^\alpha f\|_{\infty,[x,b]}\right)\right\}. \tag{52}$$

When $\alpha = \frac{1}{2}$, we derive

Corollary 2. *Let $0 < \beta^* < 1$, $q, \beta > 0$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then,*
(i)

$$|H_n(f,x) - f(x)| \leq$$

$$\frac{2\theta(q)}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta^*}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta^*}} \right)_{[x,b]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. K e^{-\beta n^{(1-\beta^*)}} \left(\left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \sqrt{(b-x)} \right) \right\}, \tag{53}$$

and
(ii)

$$\|H_n f - f\|_\infty \leq \frac{2\theta(q)}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta^*}} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta^*}} \right)_{[x,b]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \sqrt{(b-a)} K e^{-\beta n^{(1-\beta^*)}} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} < \infty. \tag{54}$$

3. Combine 2.1 and 2.2

Let $a, b \in \mathbb{R}$ with $a < b$, $f \in C([a,b])$. Let also $q, \lambda, \beta > 0$, $\gamma = \max\left\{q, \frac{1}{q}\right\}$.
For the next theorems, we call

$$_1L_n(f,x) := A_n(f,x), x \in [a,b]$$

$$_2L_n(f,x) := H_n(f,x), x \in [a,b].$$

Also, we set

$$K_1 = K_1(\lambda) = \frac{4}{erf(2\lambda)}$$

$$K_2 = K_2(q) = \theta(q).$$

Furthermore, we set

$$\hat{\beta}_{1,n} = \hat{\beta}_{1,n}(\lambda, \beta^*) = 1 - erf\left(\lambda\left(n^{1-\beta^*} - 2\right)\right), n \in \mathbb{N}, \lambda > 0, 0 < \beta^* < 1.$$

$$\hat{\beta}_{2,n} = \hat{\beta}_{2,n}(q, \beta, \beta^*) = 2\gamma e^{2\beta - \beta n^{1-\beta^*}}, n \in \mathbb{N}, q, \beta > 0, 0 < \beta^* < 1$$

We present the following.

Theorem 13. *Let $f \in C([a,b])$, $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $x \in [a,b]$. Then, for $i = 1, 2$,*
(i)

$$|_iL_n(f,x) - f(x)| \leq K_i \cdot \left[\omega_1\left(f, \frac{1}{n^{\beta^*}}\right) + \hat{\beta}_{i,n} \|f\|_\infty \right] =: \rho_i, \tag{55}$$

and

(ii)

$$\| {}_iL_n(f) - f \|_\infty \leq \rho_i. \tag{56}$$

We observe that $\lim_{n\to\infty} {}_iL_n(f) = f$, *pointwise and uniformly.*

Proof. From Theorems 5 and 11. □

Next, we present

Theorem 14. *Let* $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then, for* $i = 1,2$

(i)

$$\left| {}_iL_n(f,x) - f(x) \right| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D^\alpha_{x-} f, \frac{1}{n^{\beta^*}} \right)_{[a,x]} + \omega_1 \left(D^\alpha_{*x} f, \frac{1}{n^{\beta^*}} \right)_{[x,b]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\| D^\alpha_{x-} f \|_{\infty,[a,x]} (x-a)^\alpha + \| D^\alpha_{*x} f \|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \tag{57}$$

and

(ii)

$$\| {}_iL_n f - f \|_\infty \leq \frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D^\alpha_{x-} f, \frac{1}{n^{\beta^*}} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D^\alpha_{*x} f, \frac{1}{n^{\beta^*}} \right)_{[x,b]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{(b-a)^\alpha \hat{\beta}_{i,n}}{2} \left(\sup_{x\in[a,b]} \| D^\alpha_{x-} f \|_{\infty,[a,x]} + \sup_{x\in[a,b]} \| D^\alpha_{*x} f \|_{\infty,[x,b]} \right) \right\}. \tag{58}$$

Proof. From Theorems 6 and 12. □

When $\alpha = \frac{1}{2}$, we derive

Corollary 3. *Let* $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $f \in C^1([a,b])$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then, for* $i = 1,2$

(i)

$$\left| {}_iL_n(f,x) - f(x) \right| \leq$$

$$\frac{2K_i}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D^{\frac{1}{2}}_{x-} f, \frac{1}{n^{\beta^*}} \right)_{[a,x]} + \omega_1 \left(D^{\frac{1}{2}}_{*x} f, \frac{1}{n^{\beta^*}} \right)_{[x,b]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\left\| D^{\frac{1}{2}}_{x-} f \right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\| D^{\frac{1}{2}}_{*x} f \right\|_{\infty,[x,b]} \sqrt{(b-x)} \right) \right\}, \tag{59}$$

and

(ii)

$$\| {}_iL_n f - f \|_\infty \leq \frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta^*}}\right)_{[x,b]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\sqrt{(b-a)}\hat{\beta}_{i,n}}{2}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\right\}<\infty. \tag{60}$$

4. About Random Motion on Simple Graphs

Here, we follow [16].

Suppose we have a system of S semiaxes with a common origin radially arranged and a particle moving randomly on S. Possible applications include the spread of toxic particles in a system of channels or vessels or the propagation of information in networks.

The mathematical model is the following: Let S be the set consisting of n semiaxes $S_1,\ldots,S_n, n\geq 2$, with a common origin 0 and let X_t be the Brownian motion process on S: namely, the diffusion process on S whose infinitesimal generator L is

$$Lu=\frac{1}{2}u'', \tag{61}$$

where

$$u=(u_1,\ldots,u_n),$$

together with the continuity conditions (a total of $n-1$ equations),

$$u_1(0)=\ldots=u_n(0) \tag{62}$$

and the so-called "Kirchoff condition"

$$u_1'(0)+\ldots+u_n'(0)=0. \tag{63}$$

This is a Walsh-type Brownian motion (see [30]).

The process X_t has a standard Brownian motion on each of the semiaxes and, when it hits 0, it continues its motion on the j-th semiaxis, $1\leq j\leq n$, with probability $\frac{1}{n}$.

For each semiaxis $S_j, 1\leq j\leq n$, it is convenient to use the coordinate $x_j, 0\leq x_j\leq\infty$. Notice that if $u=(u_1,\ldots,u_n)$ is a function on S, then its j-th component, u_j, is a function on S_j; thus, $u_j=u_j(x_j)$.

The transition density of X_t is

$$p(t,x_k,y_j)=\frac{2}{n\sqrt{2\pi t}}e^{-\frac{(x_k+y_j)^2}{2t}},\text{ if } k\neq j,$$

and

$$p(t,x_k,y_k)=\frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(x_k-y_k)^2}{2t}}-\frac{n-2}{n}e^{-\frac{(x_k+y_k)^2}{2t}}\right). \tag{64}$$

We need the following result.

Theorem 15. *Let $t\in[t_1,t_2]$, where $t_1,t_2\in(0,\infty)$ with $t_1<t_2$ fixed. We consider the function $g:\mathbb{R}\to\mathbb{R}$, which is bounded on $(0,\infty)$, i.e., there exists $M>0$ such that $|g(x)|\leq M$, for every $x\in(0,\infty)$, and it is Lebesgue measurable on $\mathbb{R}$. Let also X_t be the standard Brownian motion on each of the semiaxes $j=1,\ldots,n$ as described above. Here, x_k is fixed on S_k semiaxes, $k\in\{1,\ldots,n\}$. We consider the related expected value function*

$$r(t):=E_k(|g(X_t)|)=\int_0^\infty|g(y_k)|p(t,x_k,y_k)dy_k+\sum_{j=1,j\neq k}^n\int_0^\infty\left|g(y_j)\right|p(t,x_k,y_j)dy_j,\ t\in[t_1,t_2].$$

The function $r(t)$ is continuous in t and differentiable.

Proof. First, we observe that for $t \in [t_1, t_2]$ and $k, j \in \{1, \ldots, n\}$ with $k \neq j$

$$0 < p\big(t, x_k, y_j\big) < \frac{2}{\sqrt{2\pi t_1}}.$$

Also, for $t \in [t_1, t_2]$, and $k \in \{1, \ldots, n\}$, it is

$$0 < p(t, x_k, y_k) < \frac{2}{\sqrt{2\pi t_1}}.$$

It is enough to prove that

$$I(t) := \int_0^\infty |g(y_k)| p(t, x_k, y_k) dy_k$$

is continuous in $t \in [t_1, t_2]$.
We have that

$$|g(y_k)| \leq M.$$

Thus,

$$|g(y_k)| p(t, x_k, y_k) \leq M \frac{2}{\sqrt{2\pi t_1}}.$$

Furthermore, as $0 < t_N \to t$, with $N \longrightarrow \infty$, we obtain

$$|g(y_k)| p(t_N, x_k, y_k) \longrightarrow |g(y_k)| p(t, x_k, y_k), \text{ for every } y_k \geq 0.$$

By the dominating convergence theorem $I(t_n) \longrightarrow I(t)$ and thus, $I(t)$ is continuous in t; consequently, the function

$$r(t) := E_k(|g(X_t)|)$$

is continuous in t. □

We also need the next theorem.

Theorem 16. *Let $t \in [t_1, t_2]$, where $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$ are fixed. We consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $(0, \infty)$ and Lebesgue measurable on $\mathbb{R}$. Let also X_t be the standard Brownian motion on each of the semiaxes $j = 1, \ldots, n$ as described above. Here, x_k is fixed on S_k semiaxes, $k \in \{1, \ldots, n\}$. Then, the related expected value function*

$$r(t) := E_k(|g(X_t)|) = \int_0^\infty |g(y_k)| p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty \Big|g(y_j)\Big| p(t, x_k, y_j) dy_j, \ t \in [t_1, t_2],$$

is differentiable in t, and

$$\frac{\partial r(t)}{\partial t} = \int_0^\infty |g(y_k)| \frac{\partial p(t, x_k, y_k)}{\partial t} dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |g(y_j)| \frac{\partial p(t, x_k, y_j)}{\partial t} dy_j, \ t \in [t_1, t_2], \quad (65)$$

which is continuous in t.

Proof. First, we observe that for $t \in [t_1, t_2]$ and $k, j \in \{1, \ldots, n\}$ with $k \neq j$,

$$\frac{\partial p\big(t, x_k, y_j\big)}{\partial t} = \frac{1}{nt\sqrt{2\pi t}} e^{-\frac{\left(x_k + y_j\right)^2}{2t}} \left(\frac{\left(x_k + y_j\right)^2}{t} - 1 \right).$$

Also, for $t \in [t_1, t_2]$, and $k \in \{1, \ldots, n\}$, it is

$$\frac{\partial p(t,x_k,y_k)}{\partial t}=\frac{1}{2t\sqrt{2\pi t}}\left[e^{-\frac{(x_k-y_k)^2}{2t}}\left(\frac{(x_k-y_k)^2}{t}-1\right)-\frac{(n-2)}{n}e^{-\frac{(x_k+y_k)^2}{2t}}\left(\frac{(x_k+y_k)^2}{t}-1\right)\right].$$

Furthermore, for $k\neq j$,

$$\left|\frac{\partial p(t,x_k,y_j)}{\partial t}\right|\leq\frac{1}{nt_1\sqrt{2\pi t_1}}\left(\frac{(x_k+y_j)^2}{t_1}+1\right)$$

for every $y_j\in(0,\infty)$,
and

$$\left|\frac{\partial p(t,x_k,y_k)}{\partial t}\right|\leq\frac{1}{2t_1\sqrt{2\pi t_1}}\left[\left(\frac{(x_k-y_k)^2}{t_1}+1\right)+\frac{(n-2)}{n}\left(\frac{(x_k+y_k)^2}{t_1}+1\right)\right]$$

for every $y_k\in(0,\infty)$.

So, $\frac{\partial p(t,x_k,y_j)}{\partial t}$ and $\frac{\partial p(t,x_k,y_k)}{\partial t}$ are bounded with respect to t. The bounds are integrable with respect to y_j and y_k, respectively.

We have

$$r(t):=E_k(|g(X_t)|)=\int_0^\infty|g(y_k)|p(t,x_k,y_k)dy_k+\sum_{j=1,j\neq k}^{n}\int_0^\infty\left|g(y_j)\right|p(t,x_k,y_j)dy_j,\ t\in[t_1,t_2].$$

We apply differentiation under the integral sign.
We notice that

$$|g(y_k)|p(t,x_k,y_k)\leq M\frac{1}{2t_1\sqrt{2\pi t_1}}\left[\left(\frac{(x_k-y_k)^2}{t_1}+1\right)+\frac{(n-2)}{n}\left(\frac{(x_k+y_k)^2}{t_1}+1\right)\right].$$

and

$$|g(y_k)|p(t,x_k,y_j)\leq M\frac{1}{nt_1\sqrt{2\pi t_1}}\left(\frac{(x_k+y_j)^2}{t_1}+1\right).$$

Therefore, there exists

$$\frac{\partial r(t)}{\partial t}=\int_0^\infty|g(y_k)|\frac{\partial p(t,x_k,y_k)}{\partial t}dy_k+\sum_{j=1,j\neq k}^{n}\int_0^\infty|g(y_j)|\frac{\partial p(t,x_k,y_j)}{\partial t}dy_j,\ t\in[t_1,t_2],$$

which is continuous in t (same proof as in Theorem 15). □

5. Main Results

We present the following general approximation results of Brownian motion on simple graphs.

Theorem 17. *We consider function $g:\mathbb{R}\to\mathbb{R}$, which is bounded on $(0,\infty)$ and Lebesgue measurable on $\mathbb{R}$. Let also $r(t):=E_k[g(X_t)]$ be the related expected value function.*

If $0<\beta^*<1$, $n\in\mathbb{N}: n^{1-\beta^*}>2$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, where $t_1,t_2\in(0,\infty)$ with $t_1<t_2$, then for $i=1,2$

(i)

$$|{}_iL_n(r(t))-r(t)|\leq K_i\cdot\left[\omega_1\left(r,\frac{1}{n^{\beta^*}}\right)+\hat{\beta}_{i,n}\|r\|_{\infty,[t_1,t_2]}\right]=:\rho_i, \tag{66}$$

and

(ii)

$$\|{}_iL_n(r(t))-r(t)\|_{\infty,[t_1,t_2]}\leq\rho_i. \tag{67}$$

We observe that $\lim_{n\to\infty}{}_iL_n(r)=r$, pointwise and uniformly.

Proof. From Theorem 13. □

Next, we present

Theorem 18. *We consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $(0, \infty)$ and Lebesgue measurable on $\mathbb{R}$. Let also $r(t) := E_k[g(X_t)]$ be the related expected value function.*

If $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, where $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$, and $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then, for $i = 1, 2$

(i)

$$|{}_iL_n(r(t)) - r(t)| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D^{\alpha}_{t-} r, \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \omega_1\left(D^{\alpha}_{*t} r, \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\|D^{\alpha}_{t-} r\|_{\infty,[t_1,t]} (t - t_1)^{\alpha} + \|D^{\alpha}_{*t} r\|_{\infty,[t,t_2]} (t_2 - t)^{\alpha} \right) \right\}, \tag{68}$$

and

(ii)

$$\|{}_iL_n r - r\|_{\infty,[t_1,t_2]} \leq \frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1\left(D^{\alpha}_{t-} r, \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1\left(D^{\alpha}_{*t} r, \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\alpha\beta^*}} + \right.$$

$$\left. \frac{(t_2 - t_1)^{\alpha} \hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \|D^{\alpha}_{t-} r\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \|D^{\alpha}_{*t} r\|_{\infty,[t,t_2]} \right) \right\}. \tag{69}$$

Proof. From Theorem 14. □

When $\alpha = \frac{1}{2}$, we derive

Corollary 4. *We consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $(0, \infty)$ and Lebesgue measurable on $\mathbb{R}$. Let also $r(t) := E_k[g(X_t)]$ be the related expected value function.*

If $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$ and $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then, for $i = 1, 2$,

(i)

$$|{}_iL_n(r(t)) - r(t)| \leq$$

$$\frac{2K_i}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1\left(D^{\frac{1}{2}}_{t-} r, \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \omega_1\left(D^{\frac{1}{2}}_{*t} r, \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\left\| D^{\frac{1}{2}}_{t-} r \right\|_{\infty,[t_1,t]} \sqrt{(t - t_1)} + \left\| D^{\frac{1}{2}}_{*t} r \right\|_{\infty,[t,t_2]} \sqrt{(t_2 - t)} \right) \right\}, \tag{70}$$

and

(ii)

$$\|{}_iL_n r - r\|_{\infty,[t_1,t_2]} \leq \frac{2K_i}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1\left(D_{t-}^{\frac{1}{2}}r, \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1\left(D_{*t}^{\frac{1}{2}}r, \frac{1}{n^{\beta^*}}\right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\frac{1}{2}}r \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\frac{1}{2}}r \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \tag{71}$$

Proof. From Corollary 3. □

We continue with

Theorem 19. *We consider function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $(0,\infty)$ and Lebesgue measurable on $\mathbb{R}$. Let also $r(t) := E_k[g(X_t)]$ be the related expected value function.*

If $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, where $t_1, t_2 \in (0,\infty)$ with $t_1 < t_2$, then for $i = 1, 2$,

(i)

$$\left| {}_iL_n\left(\frac{\partial r(t)}{\partial t}\right) - \frac{\partial r(t)}{\partial t} \right| \le K_i \cdot \left[\omega_1\left(\frac{\partial r}{\partial t}, \frac{1}{n^{\beta^*}}\right) + \hat{\beta}_{i,n} \left\| \frac{\partial r}{\partial t} \right\|_{\infty,[t_1,t_2]} \right] =: \rho_i, \tag{72}$$

and

(ii)

$$\left\| {}_iL_n\left(\frac{\partial r}{\partial t}\right) - \frac{\partial r}{\partial t} \right\|_{\infty,[t_1,t_2]} \le \rho_i. \tag{73}$$

We observe that $\lim_{n\to\infty} {}_iL_n\left(\frac{\partial r}{\partial t}\right) = \frac{\partial r}{\partial t}$, pointwise and uniformly.

Proof. From Theorem 13. □

6. Applications

Let a function $g : \mathbb{R} \to \mathbb{R}$, which is bounded on $[t_1, t_2]$, where $t_1, t_2 > 0$ with $t_1 < t_2$ and is Lebesgue measurable on $\mathbb{R}$. For the Brownian Motion on simple graphs X_t, we will use the following notations

$$r(t) := E_k(|g(X_t)|) =$$

$$\int_0^\infty |g(y_k)| p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |g(y_j)| p(t, x_k, y_j) dy_j := E_k(|g(X_t)|)^{(0)}. \tag{74}$$

and

$$\frac{\partial r(t)}{\partial t} = \int_0^\infty |g(y_k)| \frac{\partial p(t, x_k, y_k)}{\partial t} dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |g(y_j)| \frac{\partial p(t, x_k, y_j)}{\partial t} dy_j := E_k(|g(X_t)|)^{(1)}. \tag{75}$$

We can apply our main results to the function $g(W) = W$. Consider the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = x$ for every $x \in \mathbb{R}$. Let also $W = X_t$ be the Brownian motion on simple graphs. Then, the expectation

$$E_k(|W|)(t) = \int_0^\infty |y_k| p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |y_j| p(t, x_k, y_j) dy_j$$

is continuous in t.

Moreover,

Corollary 5. *Let* $0<\beta^*<1$, $n\in\mathbb{N}: n^{1-\beta^*}>2$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in(0,\infty)$ *with* $t_1<t_2$*; then, for* $i=1,2$ *and* $j=0,1$

(i)

$$\left|{}_iL_n\left(E_k(|W|)^{(j)}\right)(t)-E_k(|W|)^{(j)}(t)\right|\le K_i\cdot\left[\omega_1\left(E_k(|W|)^{(j)},\frac{1}{n^{\beta^*}}\right)+\hat{\beta}_{i,n}\left\|E_k(|W|)^{(j)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_i, \tag{76}$$

and

(ii)

$$\left\|{}_iL_n\left(E_k(|W|)^{(j)}\right)(t)-E_k(|W|)^{(j)}(t)\right\|_{\infty,[t_1,t_2]}\le\rho_i. \tag{77}$$

We observe that $\lim_{n\to\infty}{}_iL_nE_k(|W|)^{(j)}(t)=E_k(|W|)^{(j)}$*, pointwise and uniformly.*

Proof. From Theorems 17 and 19. □

Next, we present

Corollary 6. *Let* $0<\alpha,\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in(0,\infty)$ *with* $t_1<t_2$, *and* $n\in\mathbb{N}: n^{1-\beta^*}>2$. *Then, for* $i=1,2$,

(i)

$$|{}_iL_n(E_k(|W|)(t))-E_k(|W|)(t)|\le$$
$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{t-}E_k(|W|),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D^{\alpha}_{*t}E_k(|W|),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$
$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\|D^{\alpha}_{t-}E_k(|W|)\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\|D^{\alpha}_{*t}E_k(|W|)\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \tag{78}$$

and

(ii)

$$\|{}_iL_nE_k(|W|)-E_k(|W|)\|_{\infty,[t_1,t_2]}\le\frac{K_i}{\Gamma(\alpha+1)}$$
$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D^{\alpha}_{t-}E_k(|W|),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D^{\alpha}_{*t}E_k(|W|)\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$
$$\left.\frac{(t_2-t_1)^{\alpha}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\|D^{\alpha}_{t-}E_k(|W|)\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\|D^{\alpha}_{*t}E_k(|W|)\|_{\infty,[t,t_2]}\right)\right\}. \tag{79}$$

Proof. From Theorem 18. □

When $\alpha=\frac{1}{2}$, we derive

Corollary 7. *Let* $0<\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$ *and* $n\in\mathbb{N}: n^{1-\beta^*}>2$. *Then, for* $i=1,2$,

(i)

$$|{}_iL_n(E_k(|W|)(t))-E_k(|W|)(t)|\le$$
$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D^{\frac{1}{2}}_{t-}E_k(|W|),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D^{\frac{1}{2}}_{*t}E_k(|W|),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\frac{1}{2}}E_k(|W|)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)}+\left\|D_{*t}^{\frac{1}{2}}E_k(|W|)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\Bigg\}, \tag{80}$$

and

(ii)

$$\|{}_iL_nE_k(|W|)-E_k(|W|)\|_{\infty,[t_1,t_2]}\leq\frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\frac{1}{2}}E_k(|W|),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\frac{1}{2}}E_k(|W|),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\frac{1}{2}}E_k(|W|)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\frac{1}{2}}E_k(|W|)\right\|_{\infty,[t,t_2]}\right)\Bigg\}<\infty. \tag{81}$$

Proof. From Corollary 4. □

For the next application, we consider the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=\cos x$ for every $x\in\mathbb{R}$. Let also $W=X_t$ be the Brownian motion on simple graphs. Then, the expectation

$$E_k(|\cos W|)(t)=\int_0^\infty|\cos(y_k)|p(t,x_k,y_k)dy_k+\sum_{j=1,j\neq k}^{n}\int_0^\infty|\cos(y_j)|p(t,x_k,y_j)dy_j$$

is continuous in t.

Moreover,

Corollary 8. *Let* $0<\beta^*<1$, $n\in\mathbb{N}:n^{1-\beta^*}>2$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in(0,\infty)$ *with* $t_1<t_2$; *then, for* $i=1,2$ *and* $j=0,1$

(i)

$$\left|{}_iL_n\left(E_k(|\cos W|)^{(j)}\right)(t)-E_k(|\cos W|)^{(j)}(t)\right|\leq$$

$$K_i\cdot\left[\omega_1\left(E_k(|\cos W|)^{(j)},\frac{1}{n^{\beta^*}}\right)+\hat{\beta}_{i,n}\left\|E_k(|\cos W|)^{(j)}\right\|_{\infty,[t_1,t_2]}\right]=:\rho_i, \tag{82}$$

and

(ii)

$$\left\|{}_iL_n\left(E_k(|\cos W|)^{(j)}\right)(t)-E_k(|\cos W|)^{(j)}(t)\right\|_{\infty,[t_1,t_2]}\leq\rho_i. \tag{83}$$

We observe that $\lim_{n\to\infty}{}_iL_nE_k(|\cos W|)^{(j)}(t)=E_k(|\cos W|)^{(j)}$, *pointwise and uniformly.*

Proof. From Theorems 17 and 19. □

Next, we present

Corollary 9. *Let* $0<\alpha,\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in(0,\infty)$ *with* $t_1<t_2$, *and* $n\in\mathbb{N}:n^{1-\beta^*}>2$. *Then, for* $i=1,2$,

(i)

$$|{}_iL_n(E_k(|\cos W|)(t))-E_k(|\cos W|)(t)|\leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{t-}E_k(|\cos W|),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D^{\alpha}_{*t}E_k(|\cos W|),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\|D^{\alpha}_{t-}E_k(|\cos W|)\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\|D^{\alpha}_{*t}E_k(|\cos W|)\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \tag{84}$$

and
(ii)

$$\|{}_iL_nE_k(|\cos W|)-E_k(|\cos W|)\|_{\infty,[t_1,t_2]}\leq\frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D^{\alpha}_{t-}E_k(|\cos W|),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D^{\alpha}_{*t}E_k(|\cos W|)\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{(t_2-t_1)^{\alpha}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\|D^{\alpha}_{t-}E_k(|\cos W|)\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\|D^{\alpha}_{*t}E_k(|\cos W|)\|_{\infty,[t,t_2]}\right)\right\}. \tag{85}$$

Proof. From Theorem 18. □

When $\alpha=\frac{1}{2}$, we derive

Corollary 10. *Let $0<\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$ and $n\in\mathbb{N}:n^{1-\beta^*}>2$. Then, for $i=1,2$,*
(i)

$$|{}_iL_n(E_k(|\cos W|)(t))-E_k(|\cos W|)(t)|\leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D^{\frac{1}{2}}_{t-}E_k(|\cos W|),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D^{\frac{1}{2}}_{*t}E_k(|\cos W|),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D^{\frac{1}{2}}_{t-}E_k(|\cos W|)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)}+\left\|D^{\frac{1}{2}}_{*t}E_k(|\cos W|)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\right\}, \tag{86}$$

and
(ii)

$$\|{}_iL_nE_k(|\cos W|)-E_k(|\cos W|)\|_{\infty,[t_1,t_2]}\leq\frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D^{\frac{1}{2}}_{t-}E_k(|\cos W|),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D^{\frac{1}{2}}_{*t}E_k(|\cos W|),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D^{\frac{1}{2}}_{t-}E_k(|\cos W|)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D^{\frac{1}{2}}_{*t}E_k(|\cos W|)\right\|_{\infty,[t,t_2]}\right)\right\}<\infty. \tag{87}$$

Proof. From Corollary 4. □

Let us consider now the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = \tanh x$ for every $x \in \mathbb{R}$. Let also $W = X_t$ be the Brownian motion on simple graphs. Then, the expectation

$$E_k(|\tanh W|)(t) = \int_0^\infty |\tanh(y_k)|p(t, x_k, y_k)dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty |\tanh(y_j)|p(t, x_k, y_j)dy_j$$

is continuous in t.

Moreover,

Corollary 11. *Let* $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$; *then, for* $i = 1, 2$ *and* $j = 0, 1$,

(i)

$$\left| {}_iL_n\left(E_k(|\tanh W|)^{(j)}\right)(t) - E_k(|\tanh W|)^{(j)}(t)\right| \leq$$

$$K_i \cdot \left[\omega_1\left(E_k(|\tanh W|)^{(j)}, \frac{1}{n^{\beta^*}}\right) + \hat{\beta}_{i,n}\left\|E_k(|\tanh W|)^{(j)}\right\|_{\infty,[t_1,t_2]}\right] =: \rho_i, \tag{88}$$

and

(ii)

$$\left\|{}_iL_n\left(E_k(|\tanh W|)^{(j)}\right)(t) - E_k(|\tanh W|)^{(j)}(t)\right\|_{\infty,[t_1,t_2]} \leq \rho_i. \tag{89}$$

We observe that $\lim_{n\to\infty} {}_iL_nE_k(|\tanh W|)^{(j)}(t) = E_k(|\tanh W|)^{(j)}$, *pointwise and uniformly.*

Proof. From Theorems 17 and 19. □

Next, we present

Corollary 12. *Let* $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$, *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then, for* $i = 1, 2$,

(i)

$$|{}_iL_n(E_k(|\tanh W|)(t)) - E_k(|\tanh W|)(t)| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^\alpha_{t-}E_k(|\tanh W|), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \omega_1\left(D^\alpha_{*t}E_k(|\tanh W|), \frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\|D^\alpha_{t-}E_k(|\tanh W|)\|_{\infty,[t_1,t]}(t-t_1)^\alpha + \|D^\alpha_{*t}E_k(|\tanh W|)\|_{\infty,[t,t_2]}(t_2-t)^\alpha\right)\right\}, \tag{90}$$

and

(ii)

$$\|{}_iL_nE_k(|\tanh W|) - E_k(|\tanh W|)\|_{\infty,[t_1,t_2]} \leq \frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D^\alpha_{t-}E_k(|\tanh W|), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]}\omega_1\left(D^\alpha_{*t}E_k(|\tanh W|)\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.\frac{(t_2-t_1)^\alpha\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\|D^\alpha_{t-}E_k(|\tanh W|)\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]}\|D^\alpha_{*t}E_k(|\tanh W|)\|_{\infty,[t,t_2]}\right)\right\}. \tag{91}$$

Proof. From Theorem 18. □

When $\alpha = \frac{1}{2}$, we derive

Corollary 13. *Let* $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$ *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then, for* $i = 1, 2$,

(i)

$$|{}_iL_n(E_k(|\tanh W|)(t)) - E_k(|\tanh W|)(t)| \leq$$

$$\frac{2K_i}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1\left(D_{t-}^{\frac{1}{2}} E_k(|\tanh W|), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \omega_1\left(D_{*t}^{\frac{1}{2}} E_k(|\tanh W|), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\hat{\beta}_{i,n}}{2} \left(\left\| D_{t-}^{\frac{1}{2}} E_k(|\tanh W|) \right\|_{\infty,[t_1,t]} \sqrt{(t - t_1)} + \left\| D_{*t}^{\frac{1}{2}} E_k(|\tanh W|) \right\|_{\infty,[t,t_2]} \sqrt{(t_2 - t)} \right) \right\}, \tag{92}$$

and

(ii)

$$\|{}_iL_n E_k(|\tanh W|) - E_k(|\tanh W|)\|_{\infty,[t_1,t_2]} \leq \frac{2K_i}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{t\in[t_1,t_2]} \omega_1\left(D_{t-}^{\frac{1}{2}} E_k(|\tanh W|), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1\left(D_{*t}^{\frac{1}{2}} E_k(|\tanh W|), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\sqrt{(t_2 - t_1)}\hat{\beta}_{i,n}}{2} \left(\sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\frac{1}{2}} E_k(|\tanh W|) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\frac{1}{2}} E_k(|\tanh W|) \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \tag{93}$$

Proof. From Corollary 4. □

In the following, we consider the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = e^{-\ell x}, \ell > 0$ for every $x \in \mathbb{R}$. Let also $W = X_t$ be the Brownian motion on simple graphs. Then, the expectation

$$E_k\left(e^{-\ell W}\right)(t) = \int_0^\infty e^{-\ell y_k} p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty e^{-\ell y_j} p(t, x_k, y_j) dy_j$$

is continuous in t.

Moreover,

Corollary 14. *Let* $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, *where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$; *then, for* $i = 1, 2$ *and* $j = 0, 1$,

(i)

$$\left| {}_iL_n\left(E_k\left(e^{-\ell W}\right)^{(j)} \right)(t) - E_k\left(e^{-\ell W}\right)^{(j)}(t) \right| \leq$$

$$K_i \cdot \left[\omega_1\left(E_k\left(e^{-\ell W}\right)^{(j)}, \frac{1}{n^{\beta^*}} \right) + \hat{\beta}_{i,n} \left\| E_k\left(e^{-\ell W}\right)^{(j)} \right\|_{\infty,[t_1,t_2]} \right] =: \rho_i, \tag{94}$$

and

(ii)

$$\left\| {}_iL_n\left(E_k\left(e^{-\ell W}\right)^{(j)} \right)(t) - E_k\left(e^{-\ell W}\right)^{(j)}(t) \right\|_{\infty,[t_1,t_2]} \leq \rho_i. \tag{95}$$

We observe that $\lim_{n\to\infty} {}_iL_nE_k\left(e^{-\ell W}\right)^{(j)}(t) = E_k\left(e^{-\ell W}\right)^{(j)}$*, pointwise and uniformly.*

Proof. From Theorems 17 and 19. □

Next, we present

Corollary 15. *Let* $0 < \alpha, \beta^* < 1$*,* $q, \lambda, \beta > 0$*,* $t \in [t_1, t_2]$*, where* $t_1, t_2 \in (0, \infty)$ *with* $t_1 < t_2$*, and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$*. Then, for* $i = 1, 2$*,*

(i)

$$\left|{}_iL_n\left(E_k\left(e^{-\ell W}\right)(t)\right) - E_k\left(e^{-\ell W}\right)(t)\right| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{t-}E_k\left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \omega_1\left(D^{\alpha}_{*t}E_k\left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D^{\alpha}_{t-}E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha} + \left\|D^{\alpha}_{*t}E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \tag{96}$$

and

(ii)

$$\left\|{}_iL_nE_k\left(e^{-\ell W}\right) - E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t_1,t_2]} \leq \frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D^{\alpha}_{t-}E_k\left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]}\omega_1\left(D^{\alpha}_{*t}E_k\left(e^{-\ell W}\right)\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.\frac{(t_2-t_1)^{\alpha}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D^{\alpha}_{t-}E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]}\left\|D^{\alpha}_{*t}E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{97}$$

Proof. From Theorem 18. □

When $\alpha = \frac{1}{2}$, we derive

Corollary 16. *Let* $0 < \beta^* < 1$*,* $q, \lambda, \beta > 0$*,* $t \in [t_1, t_2]$ *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$*. Then, for* $i = 1, 2$*,*

(i)

$$\left|{}_iL_n\left(E_k\left(e^{-\ell W}\right)(t)\right) - E_k\left(e^{-\ell W}\right)(t)\right| \leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D^{\frac{1}{2}}_{t-}E_k\left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \omega_1\left(D^{\frac{1}{2}}_{*t}E_k\left(e^{-\ell W}\right), \frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D^{\frac{1}{2}}_{t-}E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)} + \left\|D^{\frac{1}{2}}_{*t}E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\right\}, \tag{98}$$

and

(ii)

$$\left\|{}_iL_nE_k\left(e^{-\ell W}\right) - E_k\left(e^{-\ell W}\right)\right\|_{\infty,[t_1,t_2]} \leq \frac{2K_i}{\sqrt{\pi}}$$

$$\left\{ \frac{\left(\sup_{t \in [t_1,t_2]} \omega_1 \left(D_{t-}^{\frac{1}{2}} E_k \left(e^{-\ell W} \right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \sup_{t \in [t_1,t_2]} \omega_1 \left(D_{*t}^{\frac{1}{2}} E_k \left(e^{-\ell W} \right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left. \frac{\sqrt{(t_2 - t_1)} \hat{\beta}_{i,n}}{2} \left(\sup_{t \in [t_1,t_2]} \left\| D_{t-}^{\frac{1}{2}} E_k \left(e^{-\ell W} \right) \right\|_{\infty,[t_1,t]} + \sup_{t \in [t_1,t_2]} \left\| D_{*t}^{\frac{1}{2}} E_k \left(e^{-\ell W} \right) \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \quad (99)$$

Proof. From Corollary 4. □

Let the generalized logistic sigmoid function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = (1 + e^{-x})^{\delta}$, $\delta > 0$ for every $x \in \mathbb{R}$. Let also $W = X_t$ be the Brownian motion on simple graphs. Then, the expectation

$$E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)(t) = \int_0^{\infty} \left(1 + e^{-y_k} \right)^{\delta} p(t, x_k, y_k) dy_k + \sum_{j=1, j \neq k}^{n} \int_0^{\infty} \left(1 + e^{-y_j} \right)^{\delta} p(t, x_k, y_j) dy_j$$

is continuous in t.

Moreover,

Corollary 17. *Let $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, where $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$; then, for $i = 1, 2$ and $j = 0, 1$,*

(i)

$$\left| {}_iL_n \left(E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)^{(j)} \right)(t) - E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)^{(j)}(t) \right| \leq$$

$$K_i \cdot \left[\omega_1 \left(E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)^{(j)}, \frac{1}{n^{\beta^*}} \right) + \hat{\beta}_{i,n} \left\| E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)^{(j)} \right\|_{\infty,[t_1,t_2]} \right] =: \rho_i, \quad (100)$$

and

(ii)

$$\left\| {}_iL_n \left(E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)^{(j)} \right)(t) - E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)^{(j)}(t) \right\|_{\infty,[t_1,t_2]} \leq \rho_i. \quad (101)$$

We observe that $\lim_{n \to \infty} {}_iL_n E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)^{(j)}(t) = E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)^{(j)}$*, pointwise and uniformly.*

Proof. From Theorems 17 and 19. □

Next, we present

Corollary 18. *Let $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, where $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$, and $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then, for $i = 1, 2$,*

(i)

$$\left| {}_iL_n \left(E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)(t) \right) - E_k \left(\left(1 + e^{-W} \right)^{\delta} \right)(t) \right| \leq$$

$$\frac{K_i}{\Gamma(\alpha + 1)} \left\{ \frac{\left(\omega_1 \left(D_{t-}^{\alpha} E_k \left(\left(1 + e^{-W} \right)^{\delta} \right), \frac{1}{n^{\beta^*}} \right)_{[t_1,t]} + \omega_1 \left(D_{*t}^{\alpha} E_k \left(\left(1 + e^{-W} \right)^{\delta} \right), \frac{1}{n^{\beta^*}} \right)_{[t,t_2]} \right)}{n^{\alpha \beta^*}} + \right.$$

$$\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\alpha}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\Bigg\}, \tag{102}$$

and

(ii)

$$\left\|{}_iL_nE_k\left(\left(1+e^{-W}\right)^{\delta}\right)-E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t_1,t_2]}\leq\frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}E_k\left(\left(1+e^{-W}\right)^{\delta}\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}}+\right.$$

$$\left.\frac{(t_2-t_1)^{\alpha}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{103}$$

Proof. From Theorem 18. □

When $\alpha=\frac{1}{2}$, we derive

Corollary 19. *Let $0<\beta^*<1$, $q,\lambda,\beta>0$, $t\in[t_1,t_2]$ and $n\in\mathbb{N}:n^{1-\beta^*}>2$. Then, for $i=1,2$,*

(i)

$$\left|{}_iL_n\left(E_k\left(\left(1+e^{-W}\right)^{\delta}\right)(t)\right)-E_k\left(\left(1+e^{-W}\right)^{\delta}\right)(t)\right|\leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(\left(1+e^{-W}\right)^{\delta}\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(\left(1+e^{-W}\right)^{\delta}\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D_{t-}^{\frac{1}{2}}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)}+\left\|D_{*t}^{\frac{1}{2}}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\right\}, \tag{104}$$

and

(ii)

$$\left\|{}_iL_nE_k\left(\left(1+e^{-W}\right)^{\delta}\right)-E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t_1,t_2]}\leq\frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(\left(1+e^{-W}\right)^{\delta}\right),\frac{1}{n^{\beta^*}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(\left(1+e^{-W}\right)^{\delta}\right),\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}}+\right.$$

$$\left.\frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\frac{1}{2}}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\frac{1}{2}}E_k\left(\left(1+e^{-W}\right)^{\delta}\right)\right\|_{\infty,[t,t_2]}\right)\right\}<\infty. \tag{105}$$

Proof. From Corollary 4. □

When $\delta=1$, we have the usual logistic sigmoid function.

For the last application, we consider the Gompertz function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=e^{\mu e^{-x}}\mu<0$ for every $x\in\mathbb{R}$. The Gompertz function is also a sigmoid function which describes growth as being slowest at the start and end of a given time period. Let also $W = X_t$ be the Brownian motion on simple graphs. Then, the expectation

$$E_k\left(e^{\mu e^{-W}}\right)(t) = \int_0^\infty e^{\mu e^{-y_k}} p(t, x_k, y_k) dy_k + \sum_{j=1, j\neq k}^{n} \int_0^\infty e^{\mu e^{-y_j}} p(t, x_k, y_j) dy_j$$

is continuous in t.

Moreover,

Corollary 20. *Let $0 < \beta^* < 1$, $n \in \mathbb{N} : n^{1-\beta^*} > 2$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, where $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$, then for $i = 1, 2$ and $j = 0, 1$,*

(i)

$$\left| {}_iL_n\left(E_k\left(e^{\mu e^{-W}}\right)^{(j)}\right)(t) - E_k\left(e^{\mu e^{-W}}\right)^{(j)}(t)\right| \leq$$

$$K_i \cdot \left[\omega_1\left(E_k\left(e^{\mu e^{-W}}\right)^{(j)}, \frac{1}{n^{\beta^*}}\right) + \hat{\beta}_{i,n}\left\|E_k\left(e^{\mu e^{-W}}\right)^{(j)}\right\|_{\infty,[t_1,t_2]}\right] =: \rho_i, \tag{106}$$

and

(ii)

$$\left\| {}_iL_n\left(E_k\left(e^{\mu e^{-W}}\right)^{(j)}\right)(t) - E_k\left(e^{\mu e^{-W}}\right)^{(j)}(t)\right\|_{\infty,[t_1,t_2]} \leq \rho_i. \tag{107}$$

We observe that $\lim_{n\to\infty} {}_iL_nE_k\left(e^{\mu e^{-W}}\right)^{(j)}(t) = E_k\left(e^{\mu e^{-W}}\right)^{(j)}$, pointwise and uniformly.

Proof. From Theorems 17 and 19. □

Next, we present

Corollary 21. *Let $0 < \alpha, \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$, where $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$, and $n \in \mathbb{N} : n^{1-\beta^*} > 2$. Then, for $i = 1, 2$,*

(i)

$$\left| {}_iL_n\left(E_k\left(e^{\mu e^{-W}}\right)(t)\right) - E_k\left(e^{\mu e^{-W}}\right)(t)\right| \leq$$

$$\frac{K_i}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^\alpha_{t-}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \omega_1\left(D^\alpha_{*t}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\|D^\alpha_{t-}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]}(t-t_1)^\alpha + \left\|D^\alpha_{*t}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}(t_2-t)^\alpha\right)\right\}, \tag{108}$$

and

(ii)

$$\left\| {}_iL_nE_k\left(e^{\mu e^{-W}}\right) - E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t_2]} \leq \frac{K_i}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D^\alpha_{t-}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]}\omega_1\left(D^\alpha_{*t}E_k\left(e^{\mu e^{-W}}\right)\frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta^*}} + \right.$$

$$\left.\frac{(t_2-t_1)^\alpha\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\|D^\alpha_{t-}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]}\left\|D^\alpha_{*t}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{109}$$

Proof. From Theorem 18. □

When $\alpha = \frac{1}{2}$, we derive

Corollary 22. *Let* $0 < \beta^* < 1$, $q, \lambda, \beta > 0$, $t \in [t_1, t_2]$ *and* $n \in \mathbb{N} : n^{1-\beta^*} > 2$. *Then, for* $i = 1, 2$,

(i)

$$\left| {}_{i}L_n\left(E_k\left(e^{\mu e^{-W}}\right)(t)\right) - E_k\left(e^{\mu e^{-W}}\right)(t)\right| \leq$$

$$\frac{2K_i}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left.\frac{\hat{\beta}_{i,n}}{2}\left(\left\| D_{t-}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)} + \left\| D_{*t}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\right\}, \tag{110}$$

and

(ii)

$$\left\| {}_{i}L_n E_k\left(e^{\mu e^{-W}}\right) - E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t_2]} \leq \frac{2K_i}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right), \frac{1}{n^{\beta^*}}\right)_{[t,t_2]}\right)}{n^{\frac{\beta^*}{2}}} + \right.$$

$$\left.\frac{\sqrt{(t_2-t_1)}\hat{\beta}_{i,n}}{2}\left(\sup_{t\in[t_1,t_2]}\left\| D_{t-}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]}\left\| D_{*t}^{\frac{1}{2}}E_k\left(e^{\mu e^{-W}}\right)\right\|_{\infty,[t,t_2]}\right)\right\} < \infty. \tag{111}$$

Proof. From Corollary 4. □

7. Conclusions

Here, we employ two important parametrized and deformed activation function neural network approximators with their establish approximation properties. The parametrized activation functions kill far fewer neurons than the original ones. The asymmetry of the brain is best described by deformed activation functions. We derive quantitative stochastic approximations to Brownian motion over a set of semiaxes emanating from a fixed point. We finish with a very wide variety of interesting applications. This article is intended for interested mathematicians, probabilists and engineers.

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