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Spectral Analysis of One Class of Positive-Definite Operators and Their Application in Fractional Calculus

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Abstract: This paper is devoted to the spectral analysis of one class of integral operators, associated with the boundary-value problems for differential equations of fractional order. In particular, we show the positive definiteness of studying operators, which makes it possible to select areas in the complex plane where there are no eigenvalues for these operators.

Keywords: fractional derivative; eigenvalue; eigenfunction; Mittag–Leffler function; spectral analysis

MSC: 34A08; 34K37; 35R11

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1. Introduction

As known (this was noted in reference [1]) in fractional calculus and in the theory of mixed-type equations, an important role is played by the potential

$$\frac{1}{2}\int_0^1 \frac{u(t)dt}{|x-t|^{1/\rho}}$$

with density $u(t)$ and a power kernel $\frac{1}{|x-t|^{1/\rho}}$, which is positive-definite for $0 < \frac{1}{\rho} < 1$; this fact was established in [2] by Tricomi. There are papers where various generalizations of this result are given. First of all, we should note the paper of Gellerstedt [3], where an operator of the following form was investigated for positive definiteness

$$P_{01}^{\phi}u(x) = \int_0^1 \varphi(|x-t|)u(t)dt$$

where

$$\varphi(|x-t|) = |x-t|^{m/(m+2)}P_0(c|x-t|^{4/(m+2)})$$

which is a generalization of the operator

$$\frac{1}{2}\int_0^1 \frac{u(t)dt}{|x-t|^{1/\rho}}.$$

Another direction was started in [4,5], where, in particular, it was shown that the operator $\widetilde{A}_\rho : L_2 \to L_2$, for $0 < \frac{1}{\rho} < 1$, is sectorial and also the values of the form $(\widetilde{A}_\rho u, u)$, for $1 < \frac{1}{\rho} < \infty$, fill the whole complex plane [4]. This manuscript is devoted to studying the positive definiteness of operators in the form

$$A_{\gamma}^{[\alpha,\beta]}u(x) = c_\alpha \int_0^x (x-t)^{1/\alpha-1}u(t)dt + c_{\alpha,\beta}\int_0^1 x^{1/\rho-1}(1-t)^{1/\gamma-1}u(t)dt.$$

which are finite-dimensional perturbations (finite-dimensional perturbations of a special kind) of a fractional-integration operator of a special kind.

We suggest a principally new wide class of positive-definite operators, which play an important role in fractional calculus and their applications (the beginning of the spectral analysis is presented in [1]). Obtained results are used to study some very important properties of functions of the Mittag–Leffler type.

2. On the Positive Definiteness of Operators of the Kind $A_{\gamma}^{[\alpha,\beta]}$

Let us consider

$$A_{\gamma}^{[\alpha,\beta]}u(x) = c_\alpha \int_0^x (x-t)^{1/\alpha-1}u(t)dt + c_{\alpha,\beta}\int_0^1 x^{1/\rho-1}(1-t)^{1/\gamma-1}u(t)dt,$$

where α, β,γ, $c_{\alpha,\beta}$, c_γ are real numbers and α, β,γ are positive .

Consider the operator arising in the solution of boundary-value problems for fractional-differential equations [6].

Let us show that this operator (for specific α, γ, ρ) is positive-definite. To highlight the main ideas, et us consider the simplest cases. Let us consider in space $L_2(0,1)$ the operator $A_{\gamma}^{[\alpha,\beta]}$ for $\alpha = \beta = \gamma = \rho$, $0 < \rho < 2$, i.e., we consider the operator A_ρ (how significant the role of the operator A_ρ is in fractional calculus is described in detail in the monograph [7]). The case for $0 < \rho < 1$ is more important, as in this case the operator A_ρ corresponds to the differential equations of order more than 1. The case for $1 < \rho < \infty$ in fractional calculus is not so interesting, but to complete our investigation, we will consider some results for this case too.

First of all, we note that the first term of operator A_ρ is fractional integral $J^{1/\rho}$ of order $1/\rho$.

Let us designate

$$\widetilde{A}_\rho u = \int_0^x (x-t)^{1/\rho-1}u(t)dt.$$

It is obvious that the operator $\widetilde{A}_\rho$ is different from the operator $J^{1/\rho}$ by the positive constant. But in the future, to reduce the amount of text and thus make for easier reading, we will use the operator $\widetilde{A}_\rho$ and will not paid attention to this. As is known in fractional calculus and in the theory of mixed-type equations, an important role is played by the potential

$$\frac{1}{2}\int_0^1 \frac{u(t)dt}{|x-t|^{(1/\rho)}} \tag{1}$$

with density $u(t)$ and with a power kernel $\frac{1}{|x-t|^{(1/\rho)}}$ which is positive-definite, for $0 < 1/\rho < 1$; this fact was established by F. Tricomi [2].

F. Tricomi [2] showed that the real component of the operator A_ρ, i.e.,

$$\widetilde{A}_{\rho_R} u = \frac{1}{2}\int_0^1 \frac{u(t)dt}{|x-t|^{1/\rho}},$$

is fixed-sign, i.e.,

$$(\widetilde{A}_{\rho_R}u,u)=\frac{1}{2}\int\limits_0^1\int\limits_0^1\frac{u(t)\overline{u}(x)dtdx}{|x-t|^{1/\rho}}\geq 0$$

is positive-definite, i.e., $(\widetilde{A}_{\rho_R}u,u)\geq 0$. It should be noted that the operator $\widetilde{A}_\rho$ is strictly definite $(\widetilde{A}_\rho u,u)>0$ (the equality sign holds if and only if $u=0$). Aittleater, Matsaev and Palant [5] showed that the operator $\widetilde{A}_\rho$ is sectorial $(0<1/\rho<1)$; that is, the values of the form $(\widetilde{A}_\rho u,u)$ies in the angle

$$|arg\lambda|<\frac{\pi}{2\rho}.$$

Further, Gokhberg and Krein [4] showed that the values of the form $(\widetilde{A}_\rho u,u)$ for $(1<1/\rho<\infty)$ fill the whole complex plane. This paper provides further analysis of these operators. Let us formulate some theorems:

Theorem 1. *The operator A_ρ, for $1<\rho<2$ for $(A_\rho : L_2\to L_2)$, is positive-definite.*

Proof. We need to show that the numerical form $(A_\rho u,u)>0$. Let

$$v(x)=\int\limits_0^x(x-t)^{1/\rho-1}u(t)dt-\int\limits_0^1 x^{1/\rho-1}(1-t)^{1/\rho-1}u(t)dt. \quad (2)$$

As seen, the first term in expression (2) is a fractional-integration operator of order $1/\rho$ for $1/2<1/\rho<1$. We act on both sides of expression (8) by the fractional-differentiation operator $D^{(1/\rho)}u$ of order $1/\rho$, $1/2<1/\rho<1$.

Then, we obtain

$$D^{1/\rho}v(x)=D^{1/\rho}(\int\limits_0^x(x-t)^{1/\rho-1}u(t)dt)-D^{1/\rho}(\int\limits_0^1 x^{1/\rho-1}(1-t)^{1/\rho-1}u(t)dt)=$$

$$=u(t)-cD^{1/\rho}x^{1/\rho-1}$$

where $c=\int\limits_0^1(1-t)^{1/\rho-1}u(t)dt$ i.e., $D^{1/\rho}v(x)=u(t)$.

So

$$(A_\rho u,u)=(D^{1/\rho}v,v).$$

□

It is known [5] that for $1/2<1/\rho<1$, the numerical form $(D^{1/\rho}v,v)\geq 0$ $((D^{1/\rho}v,v)>0$ for $v\in A_0^\alpha[0,1]$ where $A_0^\alpha[0,1]$ is a set of all functions $v(x)$ having absolutely continuous fractional integral of order $1-\alpha$ on [0,1] and for $x=0$ equals 0. Therefore, the operator A_ρ is also positive-definite for $1<\rho<2$.

Remark 1. *This theorem shows that under a perturbation of the operator $\widetilde{A}_\rho$ (using a special finite-dimensional perturbation), the numerical form $(A_\rho u,u)>0$ preserves positive definiteness. Before formulating the next theorem, note that for $1/2<1/\rho<1$ the operator A_ρ accompanies the following boundary-value problem (boundary-value problem, for a "model" fractional differential equation):*

$$\frac{1}{\Gamma(1-(1/\rho))}\frac{d}{dx}\int\limits_0^x\frac{u'(t)dt}{(x-t)^{1/\rho-1}}=\lambda u, \quad (3)$$

$$u(0)=0,u(1)=0. \quad (4)$$

It was shown [8] that the kernel of the operator A_ρ is positive and persymmetric. Now we show that for $1/2 < 1/\rho < 1$, the operator $-A_\rho$ is positive-definite.

Theorem 2. *The operator $-A_\rho$, for $1/2 < 1/\rho < 1$, where $(A_\rho : L_2 \to L_2)$, is positive-definite.*

Proof. Let us carry out the proof of this theorem similarly to the proof of Theorem 1. Obviously, the first term in expression (2) is a fractional-integration operator of order $1/\rho$, where $1 < 1/\rho < 2$. Let us act on both sides of expression (2) with the fractional-differentiation operator $D^{(1/\rho)}$, where $1 < 1/\rho < 2$. We obtain

$$D^{(1/\rho)}v(x) = D^{(1/\rho)}(\int_0^x (x-t)^{(1/\rho-1)}u(t)dt) - D^{(1/\rho)}(\int_0^1 x^{(1/\rho-1)}(1-t)^{(1/\rho-1)}u(t)dt) =$$

$$= u(t) - cD^{(1/\rho)}x^{(1/\rho-1)},$$

where $c = \int_0^1 (1-t)^{(1/\rho-1)}u(t)dt$, i.e.,

$$D^{(1/\rho)}v(x) = u(t).$$

Thus, $(A_\rho u, u) = (D^{(1/\rho)}v, v)$, or $-(Au, u) = -(D^{(1/\rho)}v, v)$. Let us show that the form $-(D^{(1/\rho)}u, u) > 0$. We have

$$(D^{(1/\rho)}v, v) = \frac{1}{\Gamma(1/\rho)} \int_0^1 \frac{d}{dx} \int_0^x \frac{(v'(t)dt)}{(x-t)^{(1/\rho-1)}} v(t)dt =$$

$$= \int_0^1 (v(x)d(\int_0^x \frac{v'(t)dt}{(x-t)^{(1/\rho-1)}}) =$$

$$(v(x) \int_0^x \frac{v'(t)dt}{(x-t)^{(1/\rho-1)}} \Big|_0^1 - \int_0^1 (\int_0^x \frac{v'(t)dt}{(x-t)^{(1/\rho-1)})} dv(x)) =$$

$$- \int_0^1 (\int_0^x v'(t)dt/((x-t)^{(1/\rho-1)}))v'(x)dx =$$

$$- \int_0^1 (\int_0^x z(t)dt/((x-t)^{(1/\rho-1)}))z(t)dt \Leftarrow (J^{(2-1/\rho)}z, z) < 0,$$

(here $z = v'$). That is, by the theorem of V. I. Matsaev and Yu. A. Palant [5], the operator $-A_\rho$ is positive-definite for $1/2 < \rho < 1$. □

Remark 2. *Let us make a very interesting and important remark. A special finite-dimensional perturbation makes the indefinite form $(A_\rho u, u)$ definite.*

The trick proposed here can be used to prove that the operator $-A_\rho$ is positive-definite also in the cases $1/3 < \rho < 0$. In particular, we have the following theorem.

Theorem 3. *For $1/3 < \rho < 1/2$, the operator $-A_\rho$ where $(A_\rho : L_2 \to L_2)$ is positive-definite.*

Proof. Note that the operator A_ρ accompanies the following boundary-value problem

$$\frac{1}{(\Gamma(3-1/\rho))}\frac{d^3}{(dx^3)}\int\limits_0^x \frac{u(t)dt}{(x-t)^{(1/\rho-2)}} = \lambda u \tag{5}$$

$$u(0)=0, u'(0)=0, u(1)=0 \tag{6}$$

The first term in expression (2) is a fractional-integration operator of order $1/\rho$, where $2<1/\rho<3$. Let us act on both sides of expression (2) using the fractional-differentiation operator $D^{(1/\rho)}$ where $2<1/\rho<3$. We obtain

$$D^{(1/\rho)}v(x) = D^{(1/\rho)}(\int\limits_0^x (x-t)^{(1/\rho-1)}u(t)dt - D^{(1/\rho)}(\int\limits_0^1 x^{(1/\rho-1)}(1-t)^{(1/\rho-1)}u(t)dt) =$$

$$= u(t) - cD^{(1/\rho)}x^{(1/\rho-1)},$$

where $c=\int\limits_0^1 (1-t)^{(1/\rho-1)}u(t)dt$, i.e.,

$$D^{(1/\rho)}v(x) = u(t).$$

Thus,

$$(A_\rho u, u) = (D^{(1/\rho)}v, v).$$

Now we will show that with the form $-(D^{(1/\rho)}u,u)>0$ we have

$$(D^{(1/\rho)}v,v) = \int\limits_0^1 (\frac{d^3}{dx^3}\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1/\rho-2)}}v(x)dx) =$$

$$\int\limits_0^1 (\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1/\rho-2)}})'''v(x)dx = \int\limits_0^1 v(x)d(\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1/\rho-2)}})'' =$$

$$v(x)(\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1\rho-2)}}\Big|_0^1)'' - \int\limits_0^1(\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1/\rho-2)}})''v'(x)dx =$$

$$-\int\limits_0^x(\int\limits_0^1 \frac{v(t)dt}{(x-t)^{1/\rho-2}})'',$$

where

$$(\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1/\rho-2)}})'' = \frac{d^2}{dx^2}\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1/\rho-2)}} =$$

$$\frac{d}{dx}(\frac{d}{dx}\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1/\rho-2)}}).$$

According to the well-known formula

$$\frac{d}{dx}\int\limits_0^x \frac{v(t)dt}{(x-t)^{(1/\rho-2)}} = \frac{v(0)}{\Gamma(3-1/\rho)x^{-(1/\rho-2)}} +$$

$$+\frac{1}{\Gamma(3-1/\rho)}\int_0^x (x-t)^{-(1/\rho-2)}v'(t)dt =$$

$$=\int_0^x \frac{v'(t)dt}{(x-t)^{(1/\rho-2)}},$$

we obtain

$$-\int_0^1 \left(\frac{d}{dx}\int_0^x \frac{v'(t)dt}{(x-t)^{(1/\rho-2)}}\right)v'(x)dx =$$

$$-\int_0^1 \left(\frac{d}{dx}-\int_0^1 \frac{z(t)dt}{(x-t)^{(1/\rho-2)}}z(x)dx\right) = (J^{(3-1/\rho)}z,z) < 0.$$

□

Therefore, the number form, and hence the operator, is positive-definite for the following.

Remark 3. *Operators of the form*

$$A_\rho^{[\alpha^{-1},\rho]}u(x) = \frac{1}{\Gamma(\rho^{-1})}\int_0^x (x-t)^{1/\rho-1}u(t)dt -$$

$$-\frac{1}{\Gamma(\rho^{-1})}\int_0^1 x^{1/\rho-1}(1-t)^{\alpha-1}u(t)dt \tag{7}$$

can also be studied using the method given above. In the same way as in the case of the operator A_ρ, it can be shown that the operator $A_\rho^{[\alpha^{-1},\rho]}$ is positive-definite. Here, we note that it is known [6] that the number λ will be the eigenvalues of the operator $A_\rho^{[\alpha^{-1},\rho]}$ if and only if $E_\rho(\lambda,\alpha) = 0$ was used to study the distribution of the zeros of the function $E_\rho(\lambda,\alpha)$ [9].

3. Application of the Obtained Results to Study the Problem of Distribution of Zeros of the Mittag–Leffler Function

The distribution of the zeros of a function of the Mittag–Leffler type is the subject of works by many authors [10,11]. This paper also studies the distribution of zeros for functions of the Mittag–Leffler type. In fractional calculus, a special role is played by the function

$$e_{\frac{1}{\rho}}^{\lambda x} = x^{1/\rho-1}E_\rho(\lambda x^{1/\rho-1}, 1/\rho)$$

and zeros of the function

$$E_\rho(\lambda, 1/\rho).$$

Therefore, the main focus is on studying the zeros of the function $E_\rho(\lambda;1/\rho)$ as well as a system of eigenfunctions

$$\{x^{1/\rho-1}E_\rho(\lambda_j x^{1/\rho-1}, 1/\rho)\},\ j = 1,2,\ldots,\infty$$

of the operator A_ρ. Of course, the finite-dimensional perturbation with which the operator $A_\gamma^{([\alpha,\beta])}$ is obtained from the fractional-integration operator depends on two parameters, ρ and γ

$$A_\gamma^{([\alpha,\beta])}u(x) = c_\alpha\int_0^x (x-t)^{(1/\alpha-1)}u(t)dt + c_{(\beta,\gamma)}\int_0^1 x^{(1/\rho-1)}(1-t)^{(1/\gamma-1)}u(t)dt,$$

which allows us to study the distribution of zeros of a wide class of functions of the Mittag–Leffler type. In this section, the results obtained earlier in Sections 1 and 2 are applied to study the problem of the distribution of the zeros of a function of the Mittag–Leffler type. But we will note that it has been proved that the system of main functions of the operator A_ρ is complete in $L_2(0,1)$, or, which is the same, it has been proved that the system of functions

$$\{x^{1/\rho-1}E_\rho(\lambda_j x^{1/\rho-1}, 1/\rho)\},\ j = 1,2,\ldots,\infty$$

is complete in $L_2(0,1)$.

We shall note the papers of M.M. Malamud [12–15] and his students devoted to the study of the problem of completeness of systems of eigen and associated functions of boundary-value problems for fractional-differential equations. These studies are essentially based on the well-known analogue of M. A. Neimark's theorem [16]. The method presented here has not been previously cited by anyone.

4. Distribution of Eigenvalues and Zeros of the Function of Mittag–Leffler Type in Corner Regions

Next, we need the previously mentioned theorem of M.M. Dzhrbashjan.

Theorem 4 (M.M. Dzhrbashjan)**.** *Let $\rho > 1/2, \rho \neq 1, Im\mu = 0$; then, all sufficientlyarge in modulus zeros of the function $E_\rho(z,\mu)$ (where $\rho > 1/2, \rho \neq 1, Im\mu = 0$) are simple. The following asymptotic formulas are valid*

$$\gamma_k^{\pm} = e^{(\pm i\pi/(2\rho))}(2\pi k)^{(1/\rho)}(1 + 0(\log_k/k)), k \to \infty.$$

The question arises whether all zeros of the function $E_\rho(z,\mu)$ie in this domain, and for what ρ the operator (7) is trace class.

Remark 4. *All eigenvalues of the operator A_ρ, for $1 < \rho < \infty$,ie in the angle $|arg\lambda| < \pi/2$.*

Proof. Since the operator $-A_\rho$ is positive-definite, all characteristic numbers of the operator $A_\rho$ie in the same angle, which proves Theorem 3. □

Corollary 1. *All zeros of the function $E_{(1/\rho)}(\lambda, 1/\rho)$ for $1 < \rho < \infty$ie in the angle $|arg\lambda| < \pi/2$. The following theorem can be proved in the same way.*

Theorem 5. *All eigenvalues of the operator $-A_\rho$ for $1/2 < \rho < 1$ie in the angle $|arg\lambda| \leq \pi - \pi/(2\rho)$.*

Corollary 2. *All zeros of the function $E_{(1/\rho)}(-\lambda, 1/\rho)$ for $1/2 < \rho < 1$ie in the same angle. Finally, we give one more statement, which is a consequence of Theorem 2.*

Corollary 3. *Since for $1/3 < \rho < 1/2$ all zeros of the function are negative, all the eigenvalues of the operator A_ρ are negative, so there is no need to talk about the corner regions where the eigenvalues of this operatorie in this case. We have presented in detail the distribution of the zeros of the function $E_{(1/\rho)}\ (\lambda, 1/\rho)$ because, as noted earlier, in the same way we can consider the problems of the distribution and zeros of the function $E_\rho(\lambda, \alpha)$.*

Such questions were first studied in [2,5,8]. The first article known to the author [8] containing results on this topic appeared in 1993. Somewhatater, in 1997, a paper of Ostrovsky and his students appeared on the same topic. It should be noted that in paper [2] it was shown that all zeros of the function $E_{(1/\rho)}(\lambda, 1/\rho)$ for $1/2 < \rho < 1$,ie in the right half-plane, or all the eigenvalues of the operator $A_\rho$ie in the right half-plane. A much stronger result was published in [5], where it was shown that all the eigenvalues of the operator A_ρ for $1/2 < \rho < 1$ie in the angle $|arg\lambda| < \frac{\pi}{2\pi}$. As noted, these works were preceded by a 1993 paper [8] devoted to similar questions for the function $E_\rho(\lambda; 2)$.

Let us also note the paper of A.M.Sedletsky in 2004 [10], where it was shown that for $\mu > 1, \mu \in (1, 1 + 1/\rho)$ all roots of the function $E_\rho(z, \mu)$ie outside the angle $|argz| \leq \pi/2\rho$.

As noted in Remark 3, the author does not aim to describe the widest possible set of pairs of parameters ρ and μ such that all zeros $E_{(1/\rho)}(\lambda, 1/\rho)$ie in the angle $|argz| \leq \pi/2\rho$. But it is obvious that all the statements formulated and proved are valid for the operator

$$A_\rho^{([\alpha^{-1},\rho])}u(x) =$$

$$1/\Gamma(\rho^{-1}) \int_0^x (x-t)^{(1/\rho-1)}u(t)dt - 1/\Gamma(\rho^{-1}) \int_0^1 x^{(1/\rho-1)}(1-t)^{(\alpha-1)}u(t)dt.$$

Note that for $1 < \rho < \infty$ the operator A_ρ is completely non-self-adjoint, which implies that for $1 < \rho < \infty$ all eigenvalues of the operator A_ρ are complex, or all zeros of the function $E_{(1/\rho)}(\lambda, 1/\rho)$ are complex [17–20]. From the theorem formulated by M.M. Dzhrbashjan (directly from the asymptotics) follows Proposition 3.1. The operator A_ρ for $1/2 < \rho < 1$ is trace class. Nowet us study the completeness of the systems of eigenfunctions and associated functions of the considered operators. For this, we need the following theorem of M. S. Livshits [21,22].

Theorem 6 (Livshits)**.** *If it is a bounded kernel, the "real part" $\frac{(K+K^*)}{2}$ of which is a non-negative kernel, then the inequality*

$$\sum_{j=1}^{\infty} Re\frac{1}{\lambda_j} \leq \int_a^b ReK(t,t)dt,$$

holds, where λ_j are the characteristic numbers of the kernel K. The system of main eigenfunctions of the kernel K is complete in the range of values of the integral operator Kf if and only if the equality sign holds in the relation above.

We use the results obtained, along with the well-known theorem of M. S. Livshits, to prove that the system of eigenfunctions of this operator is complete in $L_2(0,1)$.

Theorem 7. *The system of eigenfunctions and associated functions of the operator A_ρ is complete in $L_2(0,1)$.*

Proof. Using the theorem of M. S. Livshits, we show that the system of eigenfunctions of the operator accompanying the boundary-value problem for the model fractional-differential fractional equation is complete in $L_2(0,1)$. The proof consists of these statements

(1) $(A_\rho u, u) > 0$. This inequality was proved in [16];

(2) and

$$\sum_{j=1}^{\infty} Re\frac{1}{\lambda_j} = \int_a^b ReK(t,t)dt,$$

where the eigenvalues are of the operator A_ρ.

□

We denote $\mu_j = 1/\lambda_j$—the eigenvalues of the operator A_ρ; then

$$\sum_{j=1}^{\infty} Re\mu_j = \int_a^b ReK(t,t)dt.$$

The sum of the eigenvalues μ_j is the trace of the operator A_ρ, i.e.,

$$spA = spA_0 + spA_1 = sp(\int_0^1 x^{1/\rho-1}(1-t)^{1/\rho-1}u(t)dt).$$

Let us find the sum of the eigenvalues (i.e., the operator's trace). Note the following important statement:

Let the number λ_j be an eigenvalue of boundary-value problems (3) and (4) if and only if it is the zero of the Mittag–Leffler function $E_\rho(z,\mu)$. This function can have both real and complex zeros $\lambda_j = \alpha_j + i\beta_j$, since function

$$E_\rho(z,\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k/\rho)}$$

is a whole function; then, the conjugate number $\overline{\lambda_j} = \alpha_j - i\beta_j$ will also be the root of this function. Since

$$sp(A) = \sum_{j=0}^{\infty} \lambda_j$$

(note that the trace includes all roots, both real and complex, taking into account their multiplicity), then

$$\sum_{j=0}^{\infty} \lambda_j = (\alpha + i\beta) + (\alpha - i\beta) + \ldots = 2\alpha + \gamma + \ldots = \sum_{j=0}^{\infty} Re\lambda_j$$

—is the sum of the real parts of the eigenvalues.

Nowet us show that

$$\sum_{j=1}^{\infty} Re\frac{1}{\lambda_j} = \int_a^b ReK(t,t)dt.$$

Beforeooking for the trace of the operator Au, it is necessary to show that the trace exists, i.e., a series of the form

$$\sum_{j=0}^{\infty} \lambda_j = \lambda_1 + \lambda_2 + \lambda_3 + \ldots$$

should converge. From the theorem of M. M. Dzhrbashjan [10], the following asymptotics of the zeros is known for the function $E_\rho(\lambda,\beta)$

$$\gamma_k^{\pm} = e^{(\pm i\pi/2\rho)}(2\pi k)^{(1/\rho)}(1 + 0(\ldots)), k \to \infty.$$

This asymptotics is also true for the eigenvalues of the following problem

$$D^\alpha u = \lambda u,\ (\sigma = 2-\alpha) \tag{8}$$

$$u(0) = 0, u(1) = 0. \tag{9}$$

Obviously, a series of the form $\sum \frac{1}{(2\pi k)^{1/\rho}}$ for $1/\rho > 1$ converges, and for $1/\rho < 1$ it diverges since

$$1/\rho - 1 = 1-\alpha, 1/\rho = 2-\alpha,$$

where $0 < \alpha < 1$ i.e., $(2-\alpha) > 1$. We have shown that the series converges, i.e., the operator A_ρ is nuclear. Nowet us calculate its trace. Obviously, the operators A_0 and A_1 are nuclear, so their sum is also a nuclear operator [23–29]. Therefore, it is enough for us to find traces of the operators A_0 and A_1, respectively. Since the operator A_0 is a Volterra operator, its trace is equal to 0, and the operator A_1 transforms the space L_2 in functions of the form $cx^{1/\rho-1}$ (the operator A_1 is one-dimensional). Thus, the problem was reduced to determining the only eigenvalue λ_1 of the operator A_1. It is obvious that

$$\lambda_1 = \int_0^1 (1-t)^{1/\rho-1} t^{1/\rho-1} dt = \frac{\Gamma(\frac{1}{\rho})}{\Gamma(\frac{2}{\rho})}.$$

Theorem 8. *The system of eigenfunctions (and not the system of eigenfunctions and associated functions) of the operator A_ρ for $0 < \rho < 1/2$ is complete in $L_2(0,1)$.*

Proof. As noted earlier, the zeros of the function $E_\rho(\lambda; \frac{1}{\rho})$ for $0 < \rho < 1/2$ are simple; therefore, the eigenvalues of the operator A_ρ are also simple. Therefore, in the cases when $0 < \rho < 1/2$, the operator A_ρ does not generate eigenfunctions. The positive definiteness was proved above, and it was also shown there that the relation

$$\sum_{j=1}^{\infty} Re\frac{1}{\lambda_j} = \int_a^b ReK(t,t)dt$$

holds for the eigenvalues of the operator A_ρ. Therefore, the proof of this theorem follows from the Livshits theorem. □

5. An Estimate for the Spectral Radius of the Operator $A_\rho^{([\alpha^{(-1)},\rho])}$ and Some Corollaries

First, et us study the spectral radius of the operator A_ρ. The following theorem holds.

Theorem 9. *Outside a circle centered at the origin and radius $(\rho(\rho+1))/\Gamma(\frac{1}{\rho})$, the operator A_ρ has no eigenvalues, or, which is the same, all zeros of the function $E_\rho(\lambda; \frac{1}{\rho})$ie outside the circle with this radius.*

Proof. Let us provide an upper estimation of the spectral radius for the operator A_ρ

$$||A_\rho||_{L_2} \leq \frac{1}{\Gamma(1/\rho)}(\int_0^1\int_0^1 |(x-t)^{1/\rho-1} - x^{1/\rho-1}(1-t)^{1/\rho-1}|dxdt) \leq$$
$$\leq \frac{\rho(\rho+1)}{\Gamma(1/\rho)}$$

from which follows the proof of this theorem. □

Note that the spectral radius of the operator $A_\rho^{[\alpha^{-1},\rho]}$ can be studied in a similar way, which allows us to calculate the radius of the circle, inside which the function $E_\rho(\lambda,\alpha) = 0$ has no zeros.

Note that these statements play an important role in the theory of inverse problems, and therefore this problem has attracted the attention of many authors.

For the function $E_\rho(\lambda,\alpha)$ with $\alpha = 2$ and $\frac{1}{2} < \rho < 1$, this problem was studied in 1983 [24,30,31] (it was proven that the function $E_\rho(\lambda,2)$ has no zeros in a circle of radius R). Further, for $1/2 < \rho < \infty$, the same result is obtained, as shown by A. M. Gachaev in 2005 [32].

We also note the paper of A. Yu. Popov, which also appeared in 2006 [33], where a similar result was obtained for the case $\frac{1}{2} < \rho < 1$ by other methods (it should be noted that this result is the basis for the proof of the main result of this (see [33]) and the paper of A. Yu. Popov and A. M. Sedletskiy, 2011 [34], where it was shown that for $\rho > 1, 1 \leq \mu \leq 1+1/\rho$ the function $E_\rho(z;\mu)$ has no roots in the circle $|z| \leq \pi^{\frac{1}{\rho}}$. If we take into account Remark 3, where it is said that the number λ will be the eigenvalues of the operator $A_\rho^{([\alpha^{(-1)},\rho])}$ if and only if $E_\rho(\lambda,\alpha) = 0$, it is clear that with the help of Theorem 7 similar results can be proved for a wide class of functions of the Mittag–Leffler type. Comparison of the above results shows how effective Theorem 7 is in solving problems of the distribution of zeros of a function of the Mittag–Leffler type. As noted in Remark 3, the author does not aim to describe the widest possible set of pairs of parameters ρ and μ such that all zeros $E_{(\frac{1}{\rho})}(\lambda, \frac{1}{\rho})$ie in the angle $|argz| \leq \frac{\pi}{2\rho}o$ or outside the circle centered at the origin. But it is obvious that all the statements formulated and proved are valid for operator (7). To confirm this fact, we present one theorem that generalizes all previously known results in this direction, and is proved according to the same scheme as Theorem 3.

Theorem 10. *All zeros of the function $E_\rho(\lambda;\mu)$ for $1 < \rho < 2$ and $\mu > \frac{1}{2}$ie in the angle $\frac{\pi}{2}$.*

Proof. It is necessary to show that the numerical form $(A_\rho^{([\alpha^{-1},\rho])}u,u) > 0$. Let

$$v(x) = \int_0^x (x-t)^{(\frac{1}{\rho}-1)}u(t)dt - \int_0^1 x^{(\frac{1}{\rho}-1)}(1-t)^{(\alpha-1))}u(t)dt.$$

As can be seen, the first term of theast expression is a fractional-integration operator of order $\frac{1}{\rho}$, where $\frac{1}{2} < 1/\rho < 1$. We act on both sides of theast expression with the fractional-differentiation operator $D^{\frac{1}{\rho}}u$ of order $\frac{1}{2} < 1/\rho < 1$. Then, we obtain

$$D^{\frac{1}{\rho}}v(x) = D^{\frac{1}{\rho}}(\int_0^x (x-t)^{(\frac{1}{\rho}-1)}u(t)dt) - D^{(1/\rho)}(\int_0^1 x^{(\frac{1}{\rho}-1)}(1-t)^{(\alpha-1)}u(t)dt) =$$

$$u(t) - cD^{\frac{1}{\rho}}x^{(1/\rho-1)}$$

where $c = \int_0^1 (1-t)^{(\alpha-1)}u(t)dt$; that is, $D^{\frac{1}{\rho}}v(x) = u(t)$.

Thus,

$$(A_\rho^{([\alpha^{-1},\rho])}u,u) = (D^{\frac{1}{\rho}}v,v).$$

It is known [5] that for $\frac{1}{2} < \frac{1}{\rho} < 1$ the numerical form $(D^{(\frac{1}{\rho})}v,v) \geq 0$ and, consequently, the operator $A_\rho^{([\alpha^{(-1)},\rho])}$ is positive-definite for $1 < \rho < 2$. □

6. Conclusions

In this work, we carry out a spectral analysis of one class of integral operators affected by boundary-value problems for fractional-differential equations. First of all, we note that the operators under study are non-self-adjoint and their spectral structure is very complex and little studied. The manuscript proves the positive definiteness of these operators, which made it possible to prove the completeness of the system of eigen and associated functions. The manuscript also devotes significant space to the problem of spectrum localization.

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