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Statistical Analysis and Theoretical Framework for a Partially Accelerated Life Test Model with Progressive First Failure Censoring Utilizing a Power Hazard Distribution

Amel Abd-El-Monem [1], Mohamed S. Eliwa [2,3,*], Mahmoud El-Morshedy [4,5], Afrah Al-Bossly [4] and Rashad M. EL-Sagheer [6,7]

1 Department of Mathematics, Faculty of Education, Ain-Shams University, Cairo 11566, Egypt
2 Department of Statistics and Operation Research, College of Science, Qassim University, Buraydah 51482, Saudi Arabia
3 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
4 Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
5 Department of Statistics and Computer Science, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
6 Mathematics Department, Faculty of Science, Al-Azhar University, Naser City, Cairo 11884, Egypt
7 High Institute of Computer and Management Information System, First Statement, New Cairo, Cairo 11865, Egypt
* Correspondence: m.eliwa@qu.edu.sa

Abstract: Monitoring life-testing trials for a product or substance often demands significant time and effort. To expedite this process, sometimes units are subjected to more severe conditions in what is known as accelerated life tests. This paper is dedicated to addressing the challenge of estimating the power hazard distribution, both in terms of point and interval estimations, during constant- stress partially accelerated life tests using progressive first failure censored samples. Three techniques are employed for this purpose: maximum likelihood, two parametric bootstraps, and Bayesian methods. These techniques yield point estimates for unknown parameters and the acceleration factor. Additionally, we construct approximate confidence intervals and highest posterior density credible intervals for both the parameters and acceleration factor. The former relies on the asymptotic distribution of maximum likelihood estimators, while the latter employs the Markov chain Monte Carlo technique and focuses on the squared error loss function. To assess the effectiveness of these estimation methods and compare the performance of their respective confidence intervals, a simulation study is conducted. Finally, we validate these inference techniques using real-life engineering data.

Keywords: statistical model; power hazard distribution; constant stress partially accelerated life tests; Bayes theorem; progressive first failure censored; parametric bootstrap; computer simulation; statistics and numerical data

MSC: 62N05; 62F10

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1. Introduction

Most manufacturers are currently dedicated to optimizing their product performance to increase demand and establish trust with their customers. However, during the product development process, producers encounter several challenges, including difficulties in managing product failures within the allocated test duration for reliability estimation. In industrial operations, typical operating conditions often result in long periods required to observe unit failures, leading to extended average product failure times. This misalignment with modern industrial practices and technology standards prompted the adoption of accelerated life testing (ALT) by experimenters to expedite responses in such scenarios. ALT involves subjecting the test units to stress levels higher than standard values to accelerate the failure process. Typically, experimenters use data from accelerated tests to estimate the failure distribution of these units. Consequently, there are two categories of ALTs: fully accelerated life tests, where the relationship between life and stress is known, and partially accelerated life tests, where this relationship is either unknown or cannot be assumed. To estimate the lifespan distribution under typical usage conditions, a statistically relevant model is employed to extrapolate data obtained from these accelerated settings.

As outlined by [1], ALT encompasses various stress loading methods, namely constant stress, step stress, and progressive stress. In constant-stress ALT, sample units endure a sustained stress level until they either fail or undergo censoring, whichever occurs first. However, constant-stress testing may become impractical in certain scenarios due to the broad spectrum of failure times. In such cases, there is a need for a method that ensures faster failure occurrence. Step-stress testing, which proves to be more efficient and practical compared to continuous stress, appears to address this issue effectively. In step-stress testing, the test unit is exposed to a specific stress level for a predefined duration until it fails. Should it not fail within this period, the stress level is incrementally increased until the unit eventually fails or reaches the censored condition. In progressive-stress ALT, test units experience continuously increasing stress levels over time. Various researchers have investigated these three stress-loading methods using a variety of distributions (refer to [2–7]).

Given the known or assumed relationship between product life and stress, the fundamental presumption in ALT is that data acquired under accelerated conditions can be extrapolated to reflect performance under normal usage conditions. Nevertheless, it has been observed that, in certain situations, particularly when dealing with new test units, it becomes challenging to ascertain or make a reliable assumption about this relationship. Consequently, in such cases, partial accelerated life tests (PALT) are frequently employed. PALT finds its utility in test environments where it is challenging to collect lifetimes for highly reliable items with extended lifespans using conventional test conditions. PALT typically falls into two distinct categories: step-stress PALT (SS-PALT) and constant-stress PALT (CS-PALT). In CS-PALT, all groups of test units are individually exposed to accelerated conditions and usage profiles. Conversely, in SS-PALT, the usage conditions for the remaining components of the experiment transition from normal use to higher stress levels at predetermined times or after a predefined number of failures. Recent research in the field of PALT has yielded numerous studies, some of which are exemplified by references [8–14].

While the primary objective of PALT is to shorten the duration of the testing experiment, experimenters often face substantial downtime as they wait for all test units to fail. To mitigate this challenge, working with censored data becomes essential, aiming to reduce both the cost and duration of the test. Two commonly employed types of censorship are type-I and type-II censoring. In the former, units are simultaneously tested for a predetermined duration, and during this period, some units experience failure; subsequently, the remaining units are withdrawn from the test at its conclusion (refer to [15,16]). Conversely, in the latter, units are tested concurrently until a predefined number of failures occur, at which point the remaining units are removed (as described in [17]). However, these earlier approaches lack flexibility in terms of removing test units mid-test. To address this limitation, a progressive type-II censoring (PTIIC) scheme is proposed as a more versatile censoring method to overcome this challenge. In PTIIC, predetermined units are removed from the test at the moment of a single unit's failure, and the test proceeds at this pace until a fixed number of units experience failure. Upon reaching this point (the last failed unit), the remaining surviving units are then removed (refer to [18–20]).

At times, the duration of the control experiment can become excessively long due to product aging issues. A life test method introduced by [21] offers experimenters the flexibility to segregate the test units into distinct groups and simultaneously run each group until the first failure occurs within each group. This form of censorship is referred to as 'first-failure censoring'. However, under this censoring approach, the researcher cannot remove experimental groups from the test until the first failure is observed. To address this limitation, ref. [22] devised a life testing approach that combines first-failure censoring with progressive type-II censoring, resulting in what is known as a 'progressive first-failure censoring' (PFFC) scheme, which will be discussed in the upcoming section. Let us now briefly delve into the progressive first-failure filtering system. Let us put n-independent groups each with k units to the test in real life. Start removing R_1 number of groups as soon as $X_{1:m:n:k}$ encounters its first failure. Repeat after the second failure time $X_{2:m:n:k}$, deleting R_2 groups at random from the experiment as well as the group where the second failure was noticed. The experimenter keeps going in the same way until all live R_m groups that are still active and the group where the m^{th} failure has taken place have been eliminated. The observed failures $X_{1:m:n:k} < X_{2:m:n:k} < \cdots < X_{m:m:n:k}$ are referred to as progressive first failure censored order statistics, whereas the progressive censoring scheme is known as $R = (R_1, \ldots, R_m)$. Recent years have seen an increase in the amount of literature on PFFC, including in [23–30].

In the realm of lifetime data analysis and the modeling of failure processes, parametric models play a crucial role and are widely employed due to their demonstrated utility across diverse scenarios. Among the various univariate models, a select few distributions hold a prominent position for their proven effectiveness in a wide array of situations. Notably, the exponential, Weibull, gamma, and log-normal distributions stand out in this regard. Another versatile model for lifetime distribution, capable of fitting well with certain sets of failure data, is the power hazard function distribution (PHFD). Reference [31] delved into the application of the PHFD and illustrated its suitability for assessing the reliability of electrical components. Through analyses of reliability and hazard functions, they demonstrated that the PHFD outperforms the exponential, log-normal, and Weibull distributions in this context. As an alternative to the Weibull, Rayleigh, and exponential distributions, Reference [32] explored the two-parameter version of PHFD, denoted as PHFD(δ, ρ), and investigated its various characteristics. If X is a continuous random variable that obeys a PHFD with shape and scale parameters δ and ρ, respectively, the probability density function (PDF) and its related cumulative distribution function (CDF) can be written as

$$f(x;\delta,\rho) = \rho x^{\delta} \exp\left\{-\frac{\rho}{\delta+1}x^{\delta+1}\right\},\ x > 0,$$

and

$$F(x;\delta,\rho) = 1 - \exp\left\{-\frac{\rho}{\delta+1}x^{\delta+1}\right\},\ x > 0,$$

respectively. Additionally, the failure rate (FR) and survival functions (SF) can be represented as

$$h(x;\delta,\rho) = \rho x^{\delta},\quad x > 0,$$

and

$$S(x;\delta,\rho) = \exp\left\{-\frac{\rho}{\delta+1}x^{\delta+1}\right\},\ x > 0,$$

where $\rho > 0$, and $\delta > -1$. When $\delta > 0$, this distribution's FR function increases, and when $-1 < \delta < 0$, it decreases. This distribution is a very adaptable model, and when its parameters are altered, it approaches various models. It includes the following specific models: PHFD relates to Rayleigh(α) when $\rho = 1/\alpha^2$ and $\delta = 1$, PHFD lowers to Weibull(ρ, 1) when $\delta = \rho - 1$, and PHFD is an exponential distribution with mean $1/\rho$ when $\delta = 0$. Because of these characteristics, this model was utilized by multiple writers to model data, particularly censored observations. Due to its practical significance in the wide range of alternative fields, as noted in numerous references, this distribution's verification throughout this work serves as our inspiration. These references include in [33–37].

The paper is structured and organized as follows: Section 2 introduces the model description and lays out the fundamental assumptions. In Section 3, we delve into the most common estimations, including maximum likelihood estimates (MLEs) and the construction of approximate confidence intervals for unknown parameters. Section 4 is dedicated to the discussion of percentile bootstrap and bootstrap-t algorithms. Section 5 outlines the process of generating Bayes point estimates using the squared error loss function and provides insights into the associated credible intervals. In Section 6, we conduct a simulation study using Monte Carlo methods. Section 7 illustrates the application of our methodology with a real engineering example. Finally, Section 8 presents some concluding remarks.

2. Model Descriptions and Assumptions

PALT is often employed in testing scenarios where it is challenging to collect data on the lifetimes of exceptionally reliable units with long lifespans under typical testing conditions. In PALT, certain test units are subjected to elevated stress levels, while others are placed in a standard testing environment. This study specifically focuses on the CS-PALT criterion, where some test units operate under normal stress conditions, while others are subjected to a continuous elevated stress level.

2.1. Model Description

As previously mentioned, in CS-PALT, N_1 test units are randomly chosen from the total number of units available N, and they are run under normal conditions, while the remaining $N_2 = N - N_1$ test units are operated under accelerated settings. Assuming the test unit's lifespan complies with the PHFD, the PDF, CDF, and FR function under typical circumstances are provided, respectively, by

$$f_1(x_1;\delta,\rho) = \rho x_1^{\delta} \exp\left\{-\frac{\rho}{\delta+1} x_1^{\delta+1}\right\}, \; x_1 > 0, \tag{1}$$

$$F_1(x_1;\delta,\rho) = 1 - \exp\left\{-\frac{\rho}{\delta+1} x_1^{\delta+1}\right\}, \; x_1 > 0, \tag{2}$$

and

$$h_1(x_1;\delta,\rho) = \rho x_1^{\delta}, \quad x_1 > 0, \tag{3}$$

where $\rho > 0$, and $\delta > -1$. When a unit is tested under accelerated conditions, the FR function is provided by the formula

$$h_2(x_2;\beta,\delta,\rho) = \beta h_1(x_1) = \beta\rho x_2^{\delta}, \quad x_2 > 0, \tag{4}$$

where β is an acceleration factor that meets the criterion $\beta > 1$. As a result, the SF, CDF, and PDF may each be expressed as follows

$$S_2(x_2;\beta,\delta,\rho) = \exp\left\{-\frac{\beta\rho}{\delta+1} x_2^{\delta+1}\right\}, \; x_2 > 0, \tag{5}$$

$$F_2(x_2;\beta,\delta,\rho) = 1 - \exp\left\{-\frac{\beta\rho}{\delta+1} x_2^{\delta+1}\right\}, \; x_2 > 0, \tag{6}$$

and

$$f_2(x_2;\beta,\delta,\rho) = \beta\rho x_2^{\delta} \exp\left\{-\frac{\beta\rho}{\delta+1} x_2^{\delta+1}\right\}, \; x_1 > 0. \tag{7}$$

Figure 1 plots the PDFs under normal and accelerated conditions. In this instance, CS-PALT and PFFC are coupled. As a result, Group 1 and Group 2 will be created from the entire N test units. The first group's components $(N_1 = n_1 k_1)$ are categorized as belonging to normal conditions, whereas the second group's components $(N_2 = n_2 k_2)$ are categorized as belonging to stress conditions. Each group is split into a number of groups with k_j,

$j = 1, 2$ test units under either normal or accelerated settings. The progressive censoring plans for the normal and accelerated tests in this approach are R_{1i} and R_{2i}, respectively. This technique continues to operate until m_j, where $j = 1, 2$ failures are noticed in each test condition. The likelihood function of the observed sample of PFFC scheme under CS-PALT can be expressed as follows:

$$L(\beta, \delta, \rho | \underline{x}) \propto \prod_{i=1}^{m_1} f_1(x_{1i}; \delta, \rho)(1 - F_1(x_{1i}; \delta, \rho))^{k_1(R_{1i}+1)-1} \times \prod_{i=1}^{m_2} f_2(x_{2i}; \beta, \delta, \rho)(1 - F_2(x_{2i}; \beta, \delta, \rho))^{k_2(R_{2i}+1)-1}. \tag{8}$$

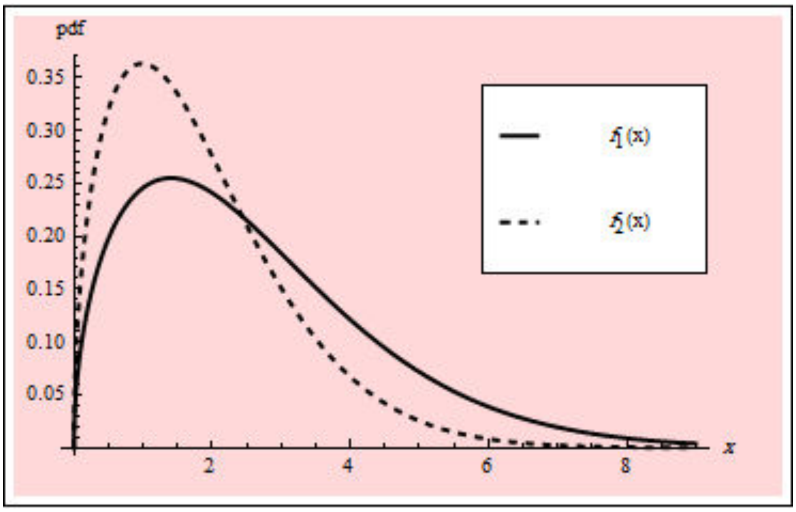

Figure 1. PDFs under normal and accelerated conditions.

2.2. Assumptions

These presumptions are made in relation to the proposed PALT methodology:

- The lifetime of all units tested under various normal or accelerated conditions follows the PHFD.
- The lifetimes of test units are independent identically distributed random variables.
- The total number of units under test is $N = N_1 + N_2 = n_1 k_1 + n_2 k_2$.
- Any unit has a lifetime of $X_2 = \beta^{-1} X_1$ under accelerated conditions.
- The lifetimes $X_{1i}, i = 1, 2, \ldots, m_1$ of units assigned to the normal condition, while the lifetimes $X_{2i}, i = 1, 2, \ldots, m_2$ of units assigned to the accelerated condition are independent of one another.

3. Maximum Likelihood Estimation

One of the most significant and popular statistical techniques is the maximum likelihood estimate (MLE). The maximum likelihood (ML) technique produces estimates of parameters with favorable statistical properties, such as consistency, asymptotic unbiasedness, asymptotic efficiency, and asymptotic normality. To obtain the parameter estimates with the maximum likelihood, one must calculate the estimates of the parameter that maximizes the probability of the sample data. The MLEs are consistent and asymptotically normal for large samples, which are other desired characteristics. Let $X^{R_j}_{j1:m_j:n_j:k_j} < X^{R_j}_{j2:m_j:n_j:k_j} < \cdots < X^{R_j}_{jm_j:m_j:n_j:k_j}$ for $j = 1, 2$ represent the two PFFC samples from the two populations whose PDFs and CDFs are as indicated in (1), (2), and (6), (7) with censoring scheme $R_j = (R_{j1}, R_{j2}, \ldots, R_{jm})$. Without a normalized constant, the logarithm likelihood function can be written as

$$\ell(\beta,\delta,\rho|\underline{x}) \propto (m_1+m_2)\log\rho + m_2\log\beta + \delta\left(\sum_{i=1}^{m_1}\log x_{1i} + \sum_{i=1}^{m_2}\log x_{2i}\right) - \rho(\Phi_1+\beta\Phi_2), \quad (9)$$

where

$$\Phi_s = \frac{1}{\delta+1}\sum_{i=1}^{m_s} k_s(R_{si}+1)x_{si}^{(\delta+1)}, s=1,2. \quad (10)$$

By computing the first derivatives of (9) with respect to β, δ, and ρ and then setting them equal to zero, the resulting simultaneous equations are represented as follows

$$\frac{\partial\ell(\beta,\delta,\rho|\underline{x})}{\partial\beta} = \frac{m_2}{\beta} - \rho\Phi_2 = 0, \quad (11)$$

$$\frac{\partial\ell(\beta,\delta,\rho|\underline{x})}{\partial\delta} = \sum_{i=1}^{m_1}\log x_{1i} + \sum_{i=1}^{m_2}\log x_{2i} - \rho\left[\left(\Phi_3 - \frac{\Phi_1}{\delta+1}\right) + \beta\left(\Phi_4 - \frac{\Phi_2}{\delta+1}\right)\right] = 0, \quad (12)$$

where

$$\Phi_t = \frac{1}{\delta+1}\sum_{i=1}^{m_t} k_t(R_{ti}+1)x_{ti}^{(\delta+1)}\log x_{ti}, t=3,4, \quad (13)$$

and

$$\frac{\partial\ell(\beta,\delta,\rho|\underline{x})}{\partial\rho} = \frac{m_1+m_2}{\rho} - (\Phi_1+\beta\Phi_2) = 0. \quad (14)$$

A system of three non-linear equations in three unknowns β, δ, and ρ are formally represented by the equations that come before them. The previous non-linear equations are challenging to provide closed-form solutions to theoretically. In order to obtain the MLEs $(\hat{\beta}_{ML}, \hat{\delta}_{ML}, \hat{\rho}_{ML})$ of (β,δ,ρ), the numerical Newton–Raphson approach will be used to solve these simultaneous equations to obtain the estimates. The algorithm is described as follows:

(1) Use the method of moments or any other methods to estimate the parameters β, δ and ρ as starting point of iteration, denote the estimates as $(\beta_0,\delta_0,\rho_0)$, and set $l=0$.
(2) Calculate $\left(\frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial\rho}\right)_{(\beta_l,\delta_l,\rho_l)}$ and the observed Fisher information matrix $I^{-1}(\beta,\delta,\rho)$, given in Section 3.
(3) Update (β,δ,ρ) as
$$(\beta_{l+1},\delta_{l+1},\rho_{l+1}) = (\beta_l,\delta_l,\rho_l) + \left(\frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial\rho}\right)_{(\beta_l,\delta_l,\rho_l)} \times I^{-1}(\beta,\delta,\rho).$$
(4) Set $l=l+1$, and then go back to Step (1).
(5) Continue the iterative steps until $|(\beta_{l+1},\delta_{l+1},\rho_{l+1}) - (\beta_l,\delta_l,\rho_l)|$ is smaller than a threshold value. The final estimates of β, δ, and ρ are the MLE of the parameters, denoted as $\hat{\beta}$, $\hat{\delta}$ and $\hat{\rho}$.

To delve deeper into the topic, refer to [20] for additional information. For distributions that are expressed using more than one parameter, the second derivatives are crucial for a number of reasons. They will confirm that maxima have been found for one of those reasons. The second partial derivatives of the likelihood function in our situation can be written as

$$\frac{\partial^2\ell(\beta,\delta,\rho|\underline{x})}{\partial\beta^2} = \frac{-m_2}{\beta^2}, \quad (15)$$

$$\frac{\partial^2\ell(\beta,\delta,\rho|\underline{x})}{\partial\beta\partial\delta} = \frac{\partial^2\ell(\beta,\delta,\rho|\underline{x})}{\partial\delta\partial\beta} = -\rho\left(\Phi_4 - \frac{\Phi_2}{\delta+1}\right), \quad (16)$$

$$\frac{\partial^2\ell(\beta,\delta,\rho|\underline{x})}{\partial\beta\partial\rho} = \frac{\partial^2\ell(\beta,\delta,\rho|\underline{x})}{\partial\rho\partial\beta} = -\Phi_2, \quad (17)$$

$$\frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \rho^2} = \frac{-(m_1+m_2)}{\rho^2}, \tag{18}$$

$$\frac{\partial \ell(\beta,\delta,\rho|\underline{x})}{\partial \rho \partial \delta} = \frac{\partial \ell(\beta,\delta,\rho|\underline{x})}{\partial \delta \partial \rho} = -\left(\Phi_3 - \frac{\Phi_1}{\delta+1}\right) - \beta\left(\Phi_4 - \frac{\Phi_2}{\delta+1}\right), \tag{19}$$

and

$$\frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \delta^2} = \frac{2\rho}{\delta+1}[\Phi_3 - \Phi_1 + \beta(\Phi_4 - \Phi_2)] - \rho(\Phi_5 - \beta\Phi_6), \tag{20}$$

where

$$\Phi_v = \frac{1}{\delta+1}\sum_{i=1}^{m_v} k_v(R_{vi}+1)x_{vi}^{(\delta+1)}(\log x_{vi})^2, v = 5, 6. \tag{21}$$

The Fisher information matrix (FIM) is obtained by arranging the second partial derivatives (15)–(20) in a matrix structure. The FIM being negative semi-definite is a necessary requirement in an optimization context for a stationary point to be a maximum. The asymptotic variances–covariances of the maximum likelihood estimators $\hat{\beta}_{ML}$, $\hat{\delta}_{ML}$, and $\hat{\rho}_{ML}$ of the parameters β, δ and ρ are obtained by the elements of the inverse of the FIM. The observed asymptotic variance-covariance matrix for the ML estimators is obtained as

$$\begin{aligned} I^{-1}(\beta,\delta,\rho) &= \begin{bmatrix} \frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \beta^2} & \frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \beta \partial \delta} & \frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \beta \partial \rho} \\ \frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \delta \partial \beta} & \frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \delta^2} & \frac{\partial \ell(\beta,\delta,\rho|\underline{x})}{\partial \delta \partial \rho} \\ \frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \rho \partial \beta} & \frac{\partial \ell(\beta,\delta,\rho|\underline{x})}{\partial \rho \partial \delta} & \frac{\partial^2 \ell(\beta,\delta,\rho|\underline{x})}{\partial \rho^2} \end{bmatrix}^{-1}_{(\hat{\beta}_{ML},\hat{\delta}_{ML},\hat{\rho}_{ML})} \\ &= \begin{bmatrix} Var(\hat{\beta}_{ML}) & Cov(\hat{\beta}_{ML},\hat{\delta}_{ML}) & Cov(\hat{\beta}_{ML},\hat{\rho}_{ML}) \\ Cov(\hat{\delta}_{ML},\hat{\beta}_{ML}) & Var(\hat{\delta}_{ML}) & Cov(\hat{\delta}_{ML},\hat{\rho}_{ML}) \\ Cov(\hat{\rho}_{ML},\hat{\beta}_{ML}) & Cov(\hat{\rho}_{ML},\hat{\delta}_{ML}) & Var(\hat{\rho}_{ML}) \end{bmatrix}. \end{aligned} \tag{22}$$

Therefore, using the asymptotic normality of the ML findings of intervals determined, the approximate $(1-\alpha)100\%$ confidence intervals (ACIs) for β, δ, and ρ are obtained according to

$$\hat{\beta}_{ML} \mp z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\beta}_{ML})}, \;\; \hat{\delta}_{ML} \mp z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\delta}_{ML})}, \;\; \hat{\rho}_{ML} \mp z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\rho}_{ML})}. \tag{23}$$

Here, $z_{\frac{\alpha}{2}}$ is the percentile of the conventional normal model with a right-tail probability of $\frac{\alpha}{2}$. The problem with applying a normal approximation of the MLE is that when the sample size is small, the normal approximation may be poor. However, a different transformation of the MLE can be used to correct the inadequate performance of the normal approximation. Reference [38] presented a log-transformation as a way to enhance the performance of the normal approximation. Therefore, for the parameters being considered, ACIs of $(1-\alpha)100\%$ are provided as

$$\hat{\beta}_{ML}\exp\left(\mp\frac{z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\beta}_{ML})}}{\hat{\beta}_{ML}}\right), \; \hat{\delta}_{ML}\exp\left(\mp\frac{z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\delta}_{ML})}}{\hat{\delta}_{ML}}\right), \; \hat{\rho}_{ML}\exp\left(\mp\frac{z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\rho}_{ML})}}{\hat{\rho}_{ML}}\right). \tag{24}$$

3.1. Consistent and Asymptotically Normal Estimators

3.1.1. Consistency characteristic

Consider $\boldsymbol{\theta} = (\beta,\delta,\rho)$ as the true parameter value of a statistical model, and let $\widehat{\boldsymbol{\theta}}$ represent the MLE of $\boldsymbol{\theta}$. The MLE is considered consistent when $\widehat{\boldsymbol{\theta}}$ converges to $\boldsymbol{\theta}$ in probability as the sample size n grows. To establish the MLE's consistency, we can employ the following theorem.

Theorem 1. *Assuming that the log-likelihood function $\ell(\boldsymbol{\theta}|\underline{x})$ exhibits continuity with respect to $\boldsymbol{\theta}$ and meets the subsequent criteria:*

1. *$\ell(\boldsymbol{\theta}|\underline{x})$ is differentiable in $\boldsymbol{\theta}$ for all x in the sample space.*
2. *The expected value of the score function $\Omega(\underline{x}, \boldsymbol{\theta}) = \dfrac{\partial \ell(\boldsymbol{\theta}|\underline{x})}{\partial \boldsymbol{\theta}}$ is zero at the true parameter value, i.e., $E[\Omega(\underline{x}, \boldsymbol{\theta})] = 0$ for $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.*
3. *The FIM $I^{-1}(\boldsymbol{\theta}) = E\left[\Omega(\underline{x}, \boldsymbol{\theta})\Omega(\underline{x}, \boldsymbol{\theta})^T\right]$ is positive definite at the true parameter value, i.e., $I^{-1}(\boldsymbol{\theta}_0) > 0$.*

Then, the $\widehat{\boldsymbol{\theta}}$ is a consistent estimator of $\boldsymbol{\theta}$.

Proof. Consider $\varepsilon > 0$ as any chosen positive value. Applying the Chebyshev inequality, we obtain

$$P\left(\left|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right| > \epsilon\right) \preceq \frac{Var\left(\widehat{\boldsymbol{\theta}}\right)}{\varepsilon^2}.$$

Utilizing the central limit theorem, it's established that the distribution of $\widehat{\boldsymbol{\theta}}$ tends toward a normal distribution with a mean of $\boldsymbol{\theta}$ and a variance of $I^{-1}(\boldsymbol{\theta})$ as the sample size n grows. Consequently, we can express this as

$$Var\left(\widehat{\boldsymbol{\theta}}\right) = I^{-1}(\boldsymbol{\theta}) + o(1),$$

where $o(1)$ is a term that goes to zero as n increases. Substituting this into the Chebyshev inequality, we obtain

$$P\left(\left|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right| > \epsilon\right) \preceq \frac{I^{-1}(\boldsymbol{\theta}) + o(1)}{\varepsilon^2}.$$

As the sample size n increases, the term $o(1)$ diminishes to zero, and the denominator in the inequality grows towards infinity. Consequently, the probability of $\left|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right| > \varepsilon$ approaches zero, thereby confirming the consistency of the $\widehat{\boldsymbol{\theta}}$ with respect to $\boldsymbol{\theta}$. □

3.1.2. Asymptotic Normality Characteristic

We describe the $\widehat{\boldsymbol{\theta}}$ as exhibiting asymptotic normality when its distribution approximates a normal distribution with a mean of $\boldsymbol{\theta}$ and a variance of $I^{-1}(\boldsymbol{\theta})$ as the sample size n grows. To establish the MLE's asymptotic normality, we can utilize the following theorem.

Theorem 2. *If the log-likelihood function $\ell(\boldsymbol{\theta}|\underline{x})$ fulfills the conditions outlined in the consistency theorem mentioned earlier, then the $\widehat{\boldsymbol{\theta}}$ demonstrates asymptotic normality.*

Proof. According to the central limit theorem, it is established that the distribution of the score function $\Omega(\underline{x}, \boldsymbol{\theta})$ tends towards a normal distribution with a mean of zero and a variance of $I^{-1}(\boldsymbol{\theta})$ as the sample size n grows. As a result, we can express it as follows:

$$\Omega(\underline{x}, \boldsymbol{\theta}) = N\left(0, I^{-1}(\boldsymbol{\theta})\right) + o(1),$$

here, $o(1)$ represents a term that diminishes to zero with the increasing value of n. Utilizing the Taylor series expansion, we can express this as

$$\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta} = I^{-1}(\boldsymbol{\theta})\Omega(\underline{x}, \boldsymbol{\theta}) + o(1).$$

Replacing the preceding equation into the score function's asymptotic normality, we obtain

$$\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta} = I^{-1}(\boldsymbol{\theta})N\left(0, I^{-1}(\boldsymbol{\theta})\right) + o(1).$$

This demonstrates that as the sample size n increases, the distribution of $\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ tends toward a normal distribution with a mean of zero and a variance of $I^{-1}(\boldsymbol{\theta})$, thereby confirming the asymptotic normality of the $\widehat{\boldsymbol{\theta}}$. It is essential to emphasize that the consistency and asymptotic normality properties of MLEs are valid under specific regularity conditions. While these conditions are generally met in numerous statistical models, it is crucial to verify their satisfaction before employing the MLE approach. $\square$

4. Parametric Bootstrap

As mentioned earlier, normal approximations are effective when dealing with large sample sizes. However, when working with small sample sizes, the assumption of normality may not hold. In such cases, the use of resampling techniques like bootstrapping can provide more precise approximations for confidence intervals. Bootstrapping has gained popularity in recent times due to its capability to offer a robust and accurate means of assessing the reliability of a specific model. Bootstrapping entails repeatedly resampling data from a population to obtain more accurate estimates of the population's true mean and variance. To achieve this, we recommend employing confidence intervals based on two parametric bootstrap methods: the percentile bootstrap technique (Boot-p), which relies on the theory of [39], and the bootstrap-t method (Boot-t), which is grounded in the theory of [40]. To generate bootstrap samples for both approaches, the following procedures are employed:

1. Using the original PFFC sample as a foundation, $x^{R_j}_{j1:m_j:n_j:k_j}, x^{R_j}_{j2:m_j:n_j:k_j}, \cdots, x^{R_j}_{jm_j:m_j:n_j:k_j}$ for $j = 1, 2$, obtain $\hat{\beta}_{ML}$, $\hat{\delta}_{ML}$ and $\hat{\rho}_{ML}$.
2. Employ the censoring plan (n_j, m_j, k_j, R_{ji}) and $(\hat{\beta}_{ML}, \hat{\delta}_{ML}, \hat{\rho}_{ML})$ to generate a PFFC bootstrap sample $x^{*R_j}_{j1:m_j:n_j:k_j}, x^{*R_j}_{j2:m_j:n_j:k_j}, \cdots, x^{*R_j}_{jm_j:m_j:n_j:k_j}$ for $j = 1, 2$.
3. From $x^{*R_j}_{j1:m_j:n_j:k_j}, x^{*R_j}_{j2:m_j:n_j:k_j}, \cdots, x^{*R_j}_{jm_j:m_j:n_j:k_j}$ calculate the bootstrap estimates, which are indicated by the symbol $\hat{\eta}^*$, where $\hat{\eta}^* = \hat{\beta}^*_{ML}, \hat{\delta}^*_{ML}$and $\hat{\rho}^*_{ML}$.
4. Steps 2 and 3 should be repeated NB times to produce $\hat{\eta}^*_1, \hat{\eta}^*_2, \ldots, \hat{\eta}^*_{NB}$.
5. Sort $\hat{\eta}^*_j$, $j = 1, 2, \ldots, NB$, ascendingly as $\hat{\eta}^*_{(j)}$, $j = 1, 2, \ldots, NB$.

4.1. Parametric Boot-p

Let $\varphi_1(z) = P(\hat{\eta}^* \le z)$ be the CDF of $\hat{\eta}^*$. Define $\hat{\eta}^*_{boot-p}(z) = \varphi^{-1}(z)$ for given z. Then, the approximate $100(1-\alpha)\%$ Boot-p CI of $\hat{\eta}^*$is given by

$$\left[\hat{\eta}^*_{boot-p}\left(\frac{\alpha}{2}\right),\ \hat{\eta}^*_{boot-p}\left(1-\frac{\alpha}{2}\right)\right]. \tag{25}$$

4.2. Parametric Boot-t

We discover the ordering statistics $\hat{\eta}^*$ as

$$\hat{\vartheta}^{*\eta} = \frac{\hat{\eta}^* - \hat{\eta}}{\sqrt{Var(\hat{\eta}^*)}},$$

where $Var(\hat{\eta}^*)$ obtained using FIM for $\hat{\eta}^* = \hat{\beta}^*_{ML}, \hat{\delta}^*_{ML}$and $\hat{\rho}^*_{ML}$. Let $\varphi_2(z) = P\left(\hat{\vartheta}^{*\eta} \le z\right)$ be the CDF of $\hat{\vartheta}^{*\eta}$. For a given z , define

$$\hat{\eta}_{boot-t}(z) = \hat{\eta} + \sqrt{Var(\hat{\eta})}\varphi_2^{-1}(z).$$

Thus, the approximate $100(1-\alpha)\%$ Boot-t CI of $\hat{\eta}$ is given by

$$\left[\hat{\eta}_{boot-t}\left(\frac{\alpha}{2}\right),\ \hat{\eta}_{boot-t}\left(1-\frac{\alpha}{2}\right)\right]. \tag{26}$$

5. Bayesian Estimation

Bayesian estimation is a powerful technique for determining unknown parameters from measurable data. Its foundation is the Bayes theorem, a concept in probability theory that allows the probability of a hypothesis to be updated as new information is gathered.

This approach provides a number of benefits over traditional MLE strategies because it may account for prior knowledge while estimating. It also has the ability to assess the degree of uncertainty surrounding each parameter. For Bayesian deduction to work, the priors for the parameters must be chosen correctly. The authors of Reference [41] argue that it is evident that from a properly Bayesian standpoint, one cannot assert that one prior is superior to all others. One must undoubtedly accept their own subjective past with all of its flaws. However, if we have enough information about the parameter(s), employing informative priors that are unquestionably preferred over all other options is preferable. If not, using ambiguous or non-descriptive priors may be appropriate; for more details, see [42]. The family of gamma distributions is known to be simple and flexible enough to suit a variety of the experimenter's preexisting ideas, according to [43]. Consider the case in which the unknown parameters, δ and ρ, are stochastically independent and have conjugate gamma priors. Specifically, gamma(a_1, b_1) and gamma(a_2, b_2). Additionally, a vague prior is selected for the acceleration factor β with the following PDF

$$\pi_\beta = \frac{1}{\beta},\ \beta > 0. \tag{27}$$

As a result, the joint prior of the parameters δ, ρ, and β together can be stated as follows

$$\pi(\beta, \delta, \rho) \propto \delta^{a_1-1}\rho^{a_2-1}\beta^{-1}\exp\{-b_1\delta - b_2\rho\}. \tag{28}$$

In order to present the joint posterior distribution of δ, ρ, and β, one must combine the joint prior distribution $\pi(\beta, \delta, \rho)$ in (38) with the likelihood function $L(\beta, \delta, \rho|\underline{x})$ supplied in (8) as

$$\begin{aligned}\pi^*(\beta, \delta, \rho|\underline{x}) =& \frac{L(\beta, \delta, \rho|\underline{x}) \times \pi(\beta, \delta, \rho)}{\int_1^\infty \int_0^\infty \int_0^\infty L(\beta, \delta, \rho|\underline{x}) \times \pi(\beta, \delta, \rho) d\beta d\delta d\rho} \\ &\propto \beta^{m_2-1}\delta^{a_1-1}\rho^{m_1+m_2+a_2-1}\exp\{-\delta b_1 - \rho(\Phi_1 + \beta\Phi_2 + b_2)\}\left(\prod_{i=1}^{m_1} x_{1i}^\delta\right)\left(\prod_{i=1}^{m_2} x_{2i}^\delta\right),\end{aligned} \tag{29}$$

where Φ_1 and Φ_2 are given in (10). In the Bayes technique, one should select a loss function that corresponds to each of the potential estimators in order to arrive at the best estimator. Here, the squared error loss function estimations, which we can express as $\varphi(\hat{\eta}, \eta) = (\hat{\eta} - \eta)^2$, and Bayes estimate $E_\eta(\eta|\underline{x})$ are calculated. The inability to derive the joint posterior in a closed form, which would allow us to compute Bayes estimates of the unknown parameters δ, ρ, and β, may be seen in relation (29). The MCMC technique, which enables us to acquire simulated samples from the posterior distributions of the parameters, will therefore be used in order to obtain these estimations. Calculations for the point and interval estimate of unidentified parameters will be made using these generated samples. As for how this approach operates, it is based on the calculation of conditional posterior functions, where the conditional distribution of β given δ and ρ can be represented as

$$\begin{aligned}\pi_1^*(\beta|\delta, \rho, \underline{x}) &\propto \beta^{m_2-1}\exp\{-\beta(\rho\Phi_2)\} \\ &\sim Gamma[m_2, \rho\Phi_2].\end{aligned} \tag{30}$$

Similarly, the conditional distribution of δ given β and ρ can be reported as

$$\pi_2^*(\delta|\beta, \rho, \underline{x}) \propto \left(\prod_{i=1}^{m_1} x_{1i}^\delta\right)\left(\prod_{i=1}^{m_2} x_{2i}^\delta\right)\delta^{a_1-1}\exp\{-\delta b_1 - \rho(\Phi_1 + \beta\Phi_2)\}. \tag{31}$$

Additionally, the conditional distribution of ρ given β and δ can be stated as

$$\begin{aligned}\pi_3^*(\rho|\beta, \delta, \underline{x}) &\propto \rho^{m_1+m_2+a_2-1}\exp\{-\rho(\Phi_1 + \beta\Phi_2 + b_2)\} \\ &\sim Gamma[m_1 + m_2 + a_2, \Phi_1 + \beta\Phi_2 + b_2].\end{aligned} \tag{32}$$

Gamma densities $\pi_1^*(\beta|\delta,\rho,\underline{x})$ and $\pi_3^*(\rho|\beta,\delta,\underline{x})$ are evident. As a result, samples of β and ρ can be produced using a gamma generator. Additionally, $\pi_2^*(\delta|\beta,\rho,\underline{x})$ cannot be reduced for directly drawing samples using conventional techniques. The gamma distribution was chosen as the prior distribution of the parameters because it is the most appropriate one that matches the maximum likelihood function. Moreover, they are from the same family. The evidence for this is that two of the full conditional posterior distributions of the parameters $\pi_1^*(\beta|\delta,\rho,\underline{x})$ and $\pi_3^*(\rho|\beta,\delta,\underline{x})$ resulted in a gamma distribution, which proves the validity of the choice. In addition, choosing another prior distribution or dependent prior will increase the complexity and difficulty of mathematical equations. Gamma distribution is one of the rich distributions, as when changing its parameters (hyper-parameters), we obtain new data with new information, so it is the focus of attention of most statisticians. A special case is that when all hyper-parameters of gamma distribution are zero, we obtain the Jaffrey prior in the form$\frac{1}{\beta}$, $\frac{1}{\delta}$, and $\frac{1}{\rho}$.

In this scenario, we can utilize the Metropolis–Hastings (M-H) algorithm model, which is suggested by [44], to derive Bayes' estimate for using one of the well-known MCMC methods. To reduce the rejection rate as much as feasible in this algorithm, we can select either a symmetric or non-symmetric proposal distribution. The normal distribution is included as a symmetric proposal distribution since the marginal distribution of δ is not well known. The M-H steps are additionally incorporated into the Gibbs sampler to update δ, while β as well as ρ is updated straight from its full conditional; see [45] as follows:

1. Start with an $(\beta,\delta,\rho)=(\hat{\beta}_{ML},\hat{\delta}_{ML},\hat{\rho}_{ML})$, and set $J=1$.
2. Generate $\beta^{(J)}$ from $Gamma\left[m_2,\rho^{(J-1)}\Phi_2\right]$.
3. Generate $\delta^{(J)}$ according to the following :
   (a) Generate δ^* from normal distribution $N\left[\delta^{(J-1)},Var(\hat{\delta}_{ML})\right]$ where $Var(\hat{\delta}_{ML})$ the variance of δ given in (22).
   (b) Compute $r=\min\left[1,\frac{\pi_2^*(\delta^*|\beta^{(J)},\rho^{(J-1)},\underline{x})}{\pi_2^*(\delta^{(J-1)}|\beta^{(J)},\rho^{(J-1)},\underline{x})}\right]$.
   (c) Generate a sample μ from the $U[0,1]$ distribution.
   (d) If$\mu\preceq r$ set $\delta^{(J)}=\delta^*$; otherwise, $\delta^{(J)}=\delta^{(J-1)}$.
4. Generate $\rho^{(J)}$ from $Gamma\left[m_1+m_2+a_2,\Phi_1+\beta^{(J)}\Phi_2+b_2\right]$.
5. Set $J=J+1$.
6. To collect the required number of samples, repeat Steps 2–5 M times.

The original M_0 sample count from the burn-in process is discarded, and we use the $M-M_0$ samples that are still there to derive estimations. As a result, the Bayes estimate of $\zeta=(\beta,\delta\ or\ \rho)$ under the squared error loss function can be viewed as the average of the samples that were obtained from the posterior densities as follows:

$$\hat{\zeta}_{BS}=\frac{1}{M-M_0}\sum_{J=M_0+1}^{M}\zeta^{(J)}. \tag{33}$$

In order to create the highest posterior density (HPD) credible intervals (CRIs) of $\zeta=(\beta,\delta\ or\ \rho)$ using generated MCMC sampling procedure, we first refer to the ordered random sample produced by the previous algorithm in the form $\zeta^{(1)}<\zeta^{(2)}<\cdots<\zeta^{(M)}$. Then, the $100(1-\alpha)\%$ two-sided CRIs of ζ can be constructed as

$$\hat{\zeta}_{\left((M-M_0)\frac{\alpha}{2}\right)},\hat{\zeta}_{\left((M-M_0)\left(1-\frac{\alpha}{2}\right)\right)}. \tag{34}$$

6. Simulation Study

In this section, some computations in line with Monte Carlo simulation experiments are carried out using *Mathematica ver. 13* in an effort to assess the performance of the offered approaches. In light of the proposed algorithm proposed in [18] with the distribution function $1-(1-F(x))^k$, 1000 PFFC samples were generated under both normal and acceleration conditions from the PHF(δ,ρ) and PHF(β,δ,ρ) distributions, respectively, with the parameters $(\beta,\delta,\rho)=(2,1.5,2.5)$. The effectiveness of the obtained estimates of β, δ, and ρ from the various proposed approaches (MLE, two parametric bootstrap, and MCMC technique) is compared in terms of point and interval estimates. In order to achieve this, mean squared errors (MSEs) are taken into account for point estimates, whereas the average widths (AWs) of 95% confidence/HPD credible intervals and 95% coverage probabilities (CPs) of the parameters based on the simulation are taken into account for interval estimates. For the purpose of conducting our investigation, multiple combinations of $k_1=k_2=k$ (group size), $n_1=n_2=n$ (number of groups), and m_j, $j=1,2$ (observed data) are taken into consideration with various censoring schemes (CSs) R_j, $j=1,2$. For ease, three categories of CSs are taken into consideration, namely $R_j=(R_{j1},R_{j2},\ldots,R_{jm})$

$$\text{CS I:} R_{j1}=n_j-m_j,\ R_{j2}=R_{j3}=\cdots=R_{jm_j}=0,\ j=1,2.$$

$$\text{CS II:}\begin{cases} R_{j\left(\frac{m_j+1}{2}\right)}=n_j-m_j, R_{ji}=0, \text{ for } i\neq\frac{m_j+1}{2} \text{ if } m_j \text{ is odd} \\ R_{j\left(\frac{m_j}{2}\right)}=n_j-m_j, R_{ji}=0, \text{ for } i\neq\frac{m_j}{2} \text{ if } m_j \text{ is even} \end{cases}.$$

$$\text{CS III:} R_{j1}=R_{j2}=\cdots=R_{j(m_j-1)}=0, R_{jm_j}=n_j-m_j,\ j=1,2.$$

To resolve the non-linear Equations (11)–(14) and obtain the MLEs of the parameter values, we used the NMaximize command of the Mathematica 13 package. Additionally, the $\hat{\beta}_{ML}$, $\hat{\delta}_{ML}$, and $\hat{\rho}_{ML}$ are produced utilizing the MLE's invariance feature. A total of 1000 replicates were used in the investigation. Each replication makes use of 1000 bootstrap (Boot-p and Boot-t) samples. The first 2000 values are deleted as "burn-in" while computing Bayes estimates (BEs) and highest posterior density CRIs in a Bayesian framework utilizing 12,000 MCMC samples. Furthermore, we take into account informative gamma priors with the following hyper-parameter values: $a_1=2, b_1=1, a_2=3$, and $b_2=2$. The parameter values for the informative priors are chosen such that their mean is equal to the parameter values themselves. Tables 1–5 show the outcomes of the Monte Carlo simulation study. These tables allow us to draw the following conclusions:

1. In every instance, as would be expected, the MSEs and AWs of all estimates decrease as sample sizes increase. It verifies the consistency features of each estimation method.
2. With n and m keeping invariant, k increases both MSEs and AWs increase.
3. In terms of decreased MSEs and AWs, the first scheme (I) performs the best when sample sizes are fixed and failures are observed.
4. The MSE and AW both increase when removals are delayed.
5. In terms of MSEs and AWs, Bayes estimation using MCMC performs better than the other approaches (ML, Boot-p, Boot-t).
6. Due to having the smallest MSE and narrowest width, MCMC CRIs are, overall, the most satisfactory.
5. Bootstrap methods outperform the ML approach in terms of MSEs and AWs. Furthermore, Boot-t performs better than Boot-p in terms of MSEs and AWs.
8. The estimates produced by the ML, bootstrap, and Bayesian approaches are highly similar and have high CPs (around 0.95).
9. In spite of the fact that the Bayes estimators perform better than all other estimators, the simulation results show that all point and interval estimator approaches are efficient. The Bayes technique may be chosen if one has sufficient prior knowledge. If past knowledge of the topic being studied cannot be accessed, bootstrap approaches that primarily rely on MLEs are preferred.

Table 1. MSE of estimates for the parameters β and δ.

		β				δ			
(k,n,m_1,m_2)	CS	ML	Boot-p	Boot-t	Bayes	ML	Boot-p	Boot-t	Bayes
$(2,40,15,15)$	I	0.22546	0.23547	0.20417	0.18652	0.33457	0.32485	0.30635	0.28965
	II	0.24687	0.25473	0.22563	0.19968	0.35647	0.34783	0.32968	0.30124
	III	0.27365	0.26475	0.23984	0.22364	0.37856	0.36451	0.34789	0.32658
$(2,40,20,25)$	I	0.17635	0.17335	0.15364	0.13478	0.29365	0.28647	0.26455	0.24365
	II	0.19635	0.19124	0.17365	0.15879	0.31254	0.30654	0.28574	0.26458
	III	0.23345	0.22654	0.20658	0.18657	0.35647	0.34998	0.32456	0.29571
$(2,60,30,30)$	I	0.14532	0.14110	0.12365	0.10859	0.25648	0.24782	0.23011	0.21109
	II	0.15998	0.15366	0.13948	0.11932	0.27457	0.26475	0.24986	0.22145
	III	0.16997	0.16984	0.15621	0.13265	0.30564	0.29658	0.27694	0.24362
$(2,60,35,40)$	I	0.11997	0.10999	0.09587	0.91124	0.21456	0.20548	0.18325	0.15964
	II	0.12658	0.11965	0.10689	0.09463	0.23547	0.22457	0.19875	0.17145
	III	0.13475	0.12968	0.11654	0.10556	0.25639	0.24573	0.22143	0.20325
$(2,90,45,45)$	I	0.09967	0.09745	0.08869	0.08234	0.18635	0.18002	0.15968	0.13124
	II	0.10568	0.11002	0.09378	0.08968	0.21245	0.20321	0.17663	0.15347
	III	0.12065	0.11890	0.10554	0.99976	0.23475	0.22657	0.19661	0.16999
$(2,90,60,75)$	I	0.09345	0.09164	0.08345	0.07965	0.15635	0.14658	0.13001	0.11475
	II	0.97367	0.09535	0.09128	0.08467	0.17658	0.16584	0.15348	0.13554
	III	0.10369	0.10024	0.09786	0.09164	0.20214	0.19648	0.17653	0.14587
$(4,40,15,15)$	I	0.26548	0.25545	0.22416	0.20653	0.35456	0.34487	0.32636	0.29999
	II	0.27655	0.26547	0.24635	0.21455	0.37365	0.36784	0.34621	0.32632
	III	0.29365	0.28456	0.26459	0.23587	0.39652	0.38843	0.36427	0.33994
$(4,40,20,25)$	I	0.19639	0.19339	0.17368	0.15479	0.33364	0.32646	0.29454	0.27361
	II	0.21356	0.21012	0.19875	0.17348	0.35652	0.34721	0.31265	0.29478
	III	0.23854	0.23187	0.21365	0.19597	0.37452	0.36543	0.34652	0.31247
$(4,60,30,30)$	I	0.17534	0.17113	0.15364	0.13852	0.28646	0.27783	0.25024	0.23128
	II	0.19658	0.18635	0.16996	0.15023	0.30245	0.29654	0.27683	0.25362
	III	0.21345	0.20689	0.18965	0.17647	0.32654	0.31475	0.29012	0.26948
$(4,60,35,40)$	I	0.13998	0.12996	0.11581	0.10125	0.24453	0.23547	0.20326	0.17965
	II	0.15234	0.14687	0.12897	0.11012	0.26481	0.25644	0.22345	0.19634
	III	0.17124	0.16589	0.14658	0.13011	0.28635	0.27461	0.24867	0.21543
$(4,90,45,45)$	I	0.11968	0.10744	0.09866	0.09233	0.21636	0.20003	0.17969	0.14125
	II	0.13546	0.12896	0.10554	0.09989	0.23154	0.22547	0.19568	0.16532
	III	0.14896	0.13997	0.11856	0.10743	0.25473	0.24516	0.22341	0.18678
$(4,90,60,75)$	I	0.10346	0.10005	0.09344	0.08966	0.18634	0.17657	0.15002	0.12671
	II	0.11597	0.11063	0.10557	0.09764	0.21548	0.20362	0.17695	0.14635
	III	0.13124	0.12897	0.11869	0.10323	0.23684	0.22457	0.19632	0.17021

Table 2. MSE of estimates for the parameter ρ.

(k,n,m_1,m_2)	CS	ML	Boot-p	Boot-t	Bayes
$(2,40,15,15)$	I	0.52634	0.51635	0.47654	0.42658
	II	0.53642	0.52369	0.49652	0.45234
	III	0.56471	0.55632	0.52362	0.47685
$(2,40,20,25)$	I	0.49632	0.48657	0.42364	0.39874
	II	0.51243	0.50247	0.45632	0.41867
	III	0.53624	0.52463	0.47562	0.43869
$(2,60,30,30)$	I	0.45783	0.44568	0.38745	0.35476
	II	0.47695	0.46357	0.40693	0.37985
	III	0.49863	0.48655	0.43675	0.40127
$(2,60,35,40)$	I	0.41236	0.40238	0.35968	0.32154
	II	0.43658	0.42563	0.37454	0.34578
	III	0.46112	0.45027	0.40321	0.36942

Table 2. *Cont.*

(k, n, m_1, m_2)	CS	ML	Boot-p	Boot-t	Bayes
$(2, 90, 45, 45)$	I	0.37695	0.36546	0.31258	0.28994
	II	0.39542	0.38456	0.34127	0.31253
	III	0.41258	0.40357	0.37124	0.34624
$(2, 90, 60, 75)$	I	0.33642	0.32145	0.28635	0.24751
	II	0.35628	0.34658	0.30547	0.27136
	III	0.37564	0.36472	0.33453	0.29954
$(4, 40, 15, 15)$	I	0.54635	0.53636	0.49655	0.44657
	II	0.56243	0.55364	0.51247	0.47635
	III	0.58672	0.57463	0.53241	0.49356
$(4, 40, 20, 25)$	I	0.51634	0.50655	0.44366	0.42875
	II	0.53624	0.52471	0.46572	0.44632
	III	0.56328	0.55473	0.48652	0.46211
$(4, 60, 30, 30)$	I	0.47782	0.46569	0.40746	0.37475
	II	0.49363	0.48657	0.42869	0.39674
	III	0.51364	0.50472	0.45362	0.42578
$(4, 60, 35, 40)$	I	0.43237	0.42239	0.37967	0.34155
	II	0.45362	0.44572	0.39452	0.36973
	III	0.48965	0.47658	0.42583	0.39112
$(4, 90, 45, 45)$	I	0.39696	0.38545	0.33257	0.30995
	II	0.41283	0.40324	0.36254	0.32164
	III	0.44658	0.43657	0.39655	0.36442
$(4, 90, 60, 75)$	I	0.35642	0.34145	0.30635	0.26751
	II	0.37625	0.37001	0.32487	0.28974
	III	0.41235	0.40586	0.35646	0.31587

Table 3. AWs and CPs of estimates for the parameter β.

(k, n, m_1, m_2)	CS	MLE		Boot-p		Boot-t		Bayes	
		ACIs		ACIs		ACIs		CRIs	
		AWs	CPs	AWs	CPs	AWs	CPs	AWs	CPs
$(2, 40, 15, 15)$	I	3.2536	0.941	3.1562	0.951	2.9974	0.951	2.9265	0.959
	II	3.2785	0.939	3.1847	0.941	3.1047	0.954	2.9847	0.954
	III	3.3246	0.938	3.2354	0.945	3.1648	0.949	3.1025	0.951
$(2, 40, 20, 25)$	I	3.1346	0.941	3.0994	0.943	2.8575	0.947	2.7996	0.961
	II	3.1954	0.942	3.1168	0.950	2.9347	0.948	2.8465	0.974
	III	3.2246	0.929	3.1648	0.939	2.9877	0.951	2.9364	0.963
$(2, 60, 30, 30)$	I	3.0258	0.938	2.9578	0.954	2.7754	0.950	2.6789	0.955
	II	3.1045	0.937	3.0987	0.941	2.8346	0.955	2.7245	0.958
	III	3.1567	0.941	3.1011	0.939	2.9124	0.954	2.8654	0.963
$(2, 60, 35, 40)$	I	2.9567	0.938	2.8836	0.938	2.6648	0.949	2.5763	0.964
	II	2.9997	0.937	2.9475	0.941	2.7135	0.942	2.6345	0.954
	III	3.1245	0.941	3.0899	0.942	2.8366	0.947	2.7541	0.971
$(2, 90, 45, 45)$	I	2.8746	0.943	2.7986	0.947	2.5564	0.951	2.4975	0.972
	II	2.9257	0.939	2.8345	0.951	2.5987	0.946	2.5563	0.966
	III	2.9765	0.937	2.8841	0.946	2.6634	0.955	2.5946	0.967
$(2, 90, 60, 75)$	I	2.7568	0.951	2.6899	0.942	2.4757	0.953	2.3999	0.958
	II	2.8364	0.948	2.7246	0.943	2.5376	0.955	2.4462	0.955
	III	2.8822	0.943	2.7864	0.941	2.5947	0.951	2.5146	0.957
$(4, 40, 15, 15)$	I	3.3536	0.952	3.2562	0.951	3.1974	0.949	3.0926	0.962
	II	3.4125	0.954	3.3154	0.952	3.2246	0.948	3.1124	0.958
	III	3.4855	0.947	3.3698	0.944	3.2997	0.951	3.1994	0.962
$(4, 40, 20, 25)$	I	3.2347	0.946	3.1993	0.937	2.9576	0.953	2.8395	0.958
	II	3.3145	0.937	3.2564	0.951	3.1045	0.952	2.9457	0.961
	III	3.3765	0.938	3.3145	0.947	3.2247	0.956	3.1046	0.962

Table 3. *Cont.*

(k, n, m_1, m_2)	CS	MLE		Boot-p		Boot-t		Bayes	
		ACIs		ACIs		ACIs		CRIs	
		AWs	CPs	AWs	CPs	AWs	CPs	AWs	CPs
$(4, 60, 30, 30)$	I	3.1254	0.941	3.0579	0.939	2.8757	0.957	2.7786	0.958
	II	3.2547	0.944	3.1147	0.938	2.9456	0.955	2.8359	0.956
	III	3.3254	0.940	3.2169	0.941	3.1456	0.954	2.9248	0.955
$(4, 60, 35, 40)$	I	3.0766	0.951	2.9837	0.936	2.7649	0.953	2.6462	0.961
	II	3.1365	0.949	3.0689	0.941	2.8365	0.949	2.7154	0.960
	III	3.2355	0.950	3.1347	0.945	2.9446	0.955	2.8122	0.958
$(4, 90, 45, 45)$	I	2.9745	0.929	2.8985	0.943	2.6565	0.948	2.5976	0.957
	II	3.0997	0.955	2.9648	0.933	2.7253	0.959	2.6541	0.964
	III	3.1605	0.941	3.0765	0.941	2.8223	0.958	2.7446	0.955
$(4, 90, 60, 75)$	I	2.8567	0.949	2.7898	0.951	2.5758	0.954	2.4995	0.974
	II	2.9124	0.934	2.8247	0.952	2.6647	0.953	2.5731	0.966
	III	2.9999	0.941	2.8976	0.949	2.7764	0.955	2.6474	0.962

Table 4. AWs and CPs of estimates for the parameter δ.

(k, n, m_1, m_2)	CS	MLE		Boot-p		Boot-t		Bayes	
		ACIs		ACIs		ACIs		CRIs	
		AWs	CPs	AWs	CPs	AWs	CPs	AWs	CPs
$(2, 40, 15, 15)$	I	2.0456	0.939	1.8947	0.941	1.7458	0.941	1.6978	0.958
	II	2.1354	0.937	1.9365	0.945	1.8657	0.945	1.7245	0.956
	III	2.3654	0.951	2.0584	0.943	1.9658	0.948	1.8654	0.955
$(2, 40, 20, 25)$	I	1.8654	0.948	1.7548	0.950	1.6645	0.946	1.5471	0.961
	II	1.9345	0.943	1.8673	0.939	1.7654	0.944	1.6745	0.960
	III	2.1346	0.952	1.9694	0.954	1.8568	0.952	1.7589	0.958
$(2, 60, 30, 30)$	I	1.6547	0.954	1.5694	0.941	1.4573	0.951	1.3365	0.957
	II	1.7548	0.939	1.6947	0.939	1.5576	0.952	1.4289	0.964
	III	1.8345	0.937	1.7784	0.941	1.6532	0.954	1.5364	0.955
$(2, 60, 35, 40)$	I	1.4698	0.951	1.3687	0.945	1.3001	0.951	1.2874	0.974
	II	1.5643	0.948	1.4568	0.943	1.4999	0.954	1.3568	0.958
	III	1.6637	0.943	1.5587	0.950	1.5003	0.949	1.4346	0.956
$(2, 90, 45, 45)$	I	1.2654	0.952	1.1547	0.939	1.1136	0.947	1.0996	0.955
	II	1.3654	0.954	1.2756	0.954	1.2136	0.948	1.1564	0.961
	III	1.4587	0.939	1.3654	0.941	1.3122	0.951	1.2456	0.960
$(2, 90, 60, 75)$	I	1.1769	0.937	1.0987	0.939	1.0778	0.950	1.0658	0.958
	II	1.2365	0.951	1.1647	0.941	1.1034	0.955	1.1001	0.957
	III	1.3124	0.948	1.2546	0.945	1.1865	0.954	1.1236	0.964
$(4, 40, 15, 15)$	I	2.1454	0.943	1.9948	0.943	1.8457	0.951	1.7979	0.955
	II	2.2365	0.952	2.1457	0.950	1.9658	0.954	1.8547	0.974
	III	2.3691	0.954	2.2574	0.939	2.1365	0.949	1.9432	0.958
$(4, 40, 20, 25)$	I	1.9655	0.939	1.8549	0.954	1.7644	0.947	1.6472	0.956
	II	2.0547	0.937	1.9358	0.941	1.8576	0.948	1.7569	0.955
	III	2.1369	0.951	2.0965	0.939	1.9658	0.951	1.8694	0.961
$(4, 60, 30, 30)$	I	1.7548	0.948	1.6695	0.941	1.5574	0.950	1.4366	0.960
	II	1.8476	0.943	1.7764	0.945	1.6547	0.955	1.5467	0.958
	III	1.9568	0.952	1.8649	0.943	1.7466	0.954	1.6573	0.957
$(4, 60, 35, 40)$	I	1.5697	0.954	1.4686	0.950	1.4002	0.951	1.3875	0.964
	II	1.6694	0.941	1.5473	0.939	1.5012	0.954	1.4768	0.955
	III	1.7589	0.939	1.6377	0.954	1.5999	0.949	1.5152	0.974
$(4, 90, 45, 45)$	I	1.3652	0.937	1.2548	0.941	1.2137	0.947	1.1593	0.961
	II	1.4586	0.951	1.3475	0.939	1.3104	0.948	1.2468	0.960
	III	1.5624	0.948	1.4586	0.929	1.4007	0.951	1.3567	0.959
$(4, 90, 60, 75)$	I	1.2767	0.943	1.1988	0.941	1.1576	0.950	1.1051	0.953
	II	1.3654	0.952	1.2689	0.939	1.2145	0.955	1.1698	0.961
	III	1.4652	0.954	1.3584	0.941	1.2997	0.954	1.2563	0.959

Table 5. AWs and CPs of estimates for the parameter ρ.

(k, n, m_1, m_2)	CS	MLE ACIs AWs	MLE ACIs CPs	Boot-p ACIs AWs	Boot-p ACIs CPs	Boot-t ACIs AWs	Boot-t ACIs CPs	Bayes CRIs AWs	Bayes CRIs CPs
$(2, 40, 15, 15)$	I	4.0564	0.929	3.8654	0.939	3.6548	0.947	2.9584	0.951
	II	4.2658	0.955	4.0569	0.954	3.8614	0.948	3.1653	0.958
	III	4.4635	0.941	4.2355	0.941	4.0563	0.951	3.4658	0.957
$(2, 40, 20, 25)$	I	3.7365	0.949	3.5468	0.939	3.3659	0.950	3.0125	0.964
	II	3.9652	0.934	3.7659	0.941	3.5476	0.955	3.2486	0.955
	III	4.1365	0.929	3.9254	0.939	3.7463	0.954	3.3192	0.974
$(2, 60, 30, 30)$	I	3.5564	0.955	3.3684	0.954	3.1468	0.947	2.8567	0.958
	II	3.7685	0.941	3.5477	0.941	3.3698	0.948	3.0568	0.957
	III	3.9476	0.949	3.7666	0.939	3.5574	0.951	3.2695	0.964
$(2, 60, 35, 40)$	I	3.3659	0.934	3.1457	0.941	2.8745	0.950	2.6547	0.955
	II	3.5687	0.929	3.3394	0.939	3.1692	0.955	2.7954	0.974
	III	3.6985	0.955	3.5147	0.954	3.2998	0.954	3.0045	0.958
$(2, 90, 45, 45)$	I	3.1254	0.941	2.7994	0.941	2.5998	0.947	2.4577	0.957
	II	3.3258	0.949	2.9957	0.939	2.7984	0.948	2.5969	0.964
	III	3.5462	0.934	3.2598	0.941	3.0243	0.951	2.8635	0.955
$(2, 90, 60, 75)$	I	2.9965	0.929	2.6954	0.939	2.3874	0.950	2.2466	0.974
	II	3.1564	0.955	2.8547	0.954	2.5146	0.955	2.3721	0.958
	III	3.3467	0.941	3.0119	0.941	2.7568	0.954	2.5599	0.957
$(4, 40, 15, 15)$	I	4.2564	0.949	4.0654	0.939	3.8548	0.947	3.1584	0.964
	II	4.4689	0.934	4.2658	0.941	4.0654	0.948	3.2581	0.955
	III	4.5956	0.929	4.4365	0.939	4.1997	0.951	3.5876	0.974
$(4, 40, 20, 25)$	I	3.9365	0.955	3.7468	0.954	3.5659	0.950	3.2125	0.958
	II	4.1358	0.941	3.9524	0.941	3.7456	0.955	3.4658	0.957
	III	4.3692	0.949	4.1365	0.939	3.9647	0.954	3.6524	0.964
$(4, 60, 30, 30)$	I	3.7563	0.934	3.5685	0.941	3.2467	0.947	3.0568	0.955
	II	3.9658	0.929	3.7458	0.939	3.4578	0.948	3.1995	0.974
	III	4.1568	0.955	3.9651	0.954	3.6654	0.951	3.4554	0.958
$(4, 60, 35, 40)$	I	3.5657	0.941	3.3454	0.941	3.0746	0.950	2.8548	0.957
	II	3.7441	0.949	3.5662	0.939	3.2313	0.955	3.0533	0.964
	III	3.8954	0.934	3.7441	0.941	3.4225	0.954	3.1899	0.955
$(4, 90, 45, 45)$	I	3.3256	0.929	2.9895	0.939	2.7999	0.947	2.6578	0.974
	II	3.4996	0.955	3.1645	0.954	2.8974	0.948	2.8557	0.958
	III	3.7154	0.941	3.4571	0.941	3.1824	0.951	3.0079	0.957
$(4, 90, 60, 75)$	I	3.1063	0.949	2.8956	0.939	2.5875	0.950	2.4467	0.964
	II	3.3651	0.938	3.1587	0.943	2.7458	0.955	2.6552	0.955
	III	3.4985	0.397	3.3334	0.942	2.8965	0.954	2.8324	0.974

7. Practical Analysis of Engineering Data

In this section, we want to see how the estimate algorithms suggested for the accelerated data set perform as described in the aforementioned sections. The effectiveness of the suggested inferential approaches is displayed and demonstrated using a genuine data set that represents the observed failure rates in a life test of the light-emitting diode (LED). References [46,47] recently conducted an analysis of this data that was initially conducted by [48]. The observed failure samples were created in both normal and accelerated conditions, and they include the following:

Normal use condition: 0.18, 0.19, 0.19, 0.34, 0.36, 0.40, 0.44, 0.44, 0.45, 0.46, 0.47,0.53, 0.57, 0.57, 0.63, 0.65, 0.70, 0.71, 0.71, 0.75, 0.76, 0.76, 0.79,0.80, 0.85, 0.98, 1.01, 1.07, 1.12, 1.14, 1.15, 1.17, 1.20, 1.23, 1.24,1.25, 1.26, 1.32, 1.33, 1.33, 1.39, 1.42, 1.50, 1.55, 1.58, 1.59, 1.62, 1.68, 1.70, 1.79, 2.00, 2.01, 2.04, 2.54, 3.61, 3.76, 4.65, 8.97.

Accelerated stress condition: 0.13, 0.16, 0.20, 0.20, 0.21, 0.25, 0.26, 0.28, 0.28, 0.30, 0.31, 0.33, 0.35, 0.35, 0.35, 0.39, 0.50, 0.52, 0.58, 0.60, 0.60, 0.62, 0.63, 0.67, 0.71, 0.73, 0.75, 0.75, 0.78, 0.80, 0.80, 0.86, 0.90, 0.91, 0.93, 0.93, 0.94, 0.98, 0.99, 1.01, 1.03, 1.06, 1.06, 1.10, 1.22, 1.22, 1.24, 1.28, 1.39, 1.39, 1.46, 1.48, 1.52, 1.74, 1.95, 2.46, 3.02, 5.16.

Before moving on, we first determine whether the PHFD can be employed as a suitable model to match the data set using the goodness-of-fit statistic, known as the Kolmogorov–

Smirnov (K-S) statistic. The calculated K-S distances and p-values for the data set under the normal and accelerated stress conditions are 0.136924 (0.226930) and 0.092780 (0.700232), respectively. The PHFD was found to be a suitable model for this set of data. Further, the empirical PDF, P-P, and SF plots which are shown in Figures 2 and 3 provide additional proof that the PHFD provides a strong fit to the data. Non-parametric approaches, such as histograms, kernel densities, box, violin, TTT, and standard Q-Q plots, are used in Figures 4 and 5 to depict the initial shape. The asymmetry of the data and the validity of some outlier observations should be highlighted.

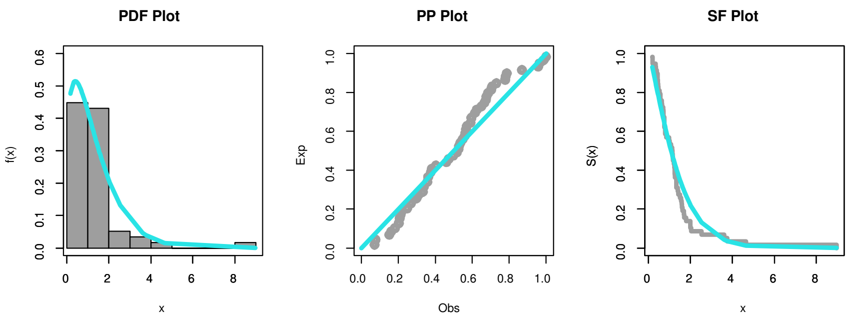

Figure 2. Empirical PDF, P-P, and SF plots for normal condition.

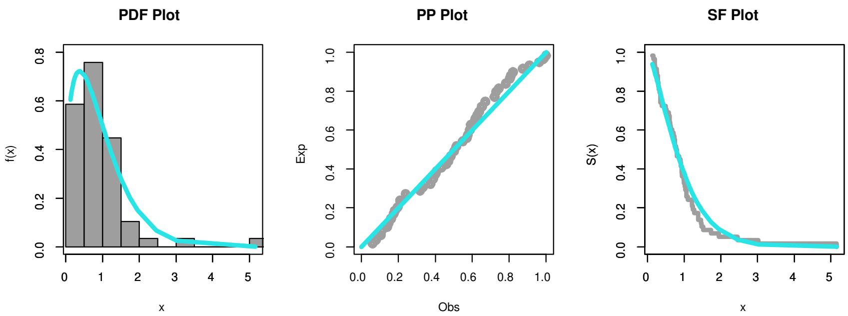

Figure 3. Empirical PDF, P-P, and SF plots for accelerated condition.

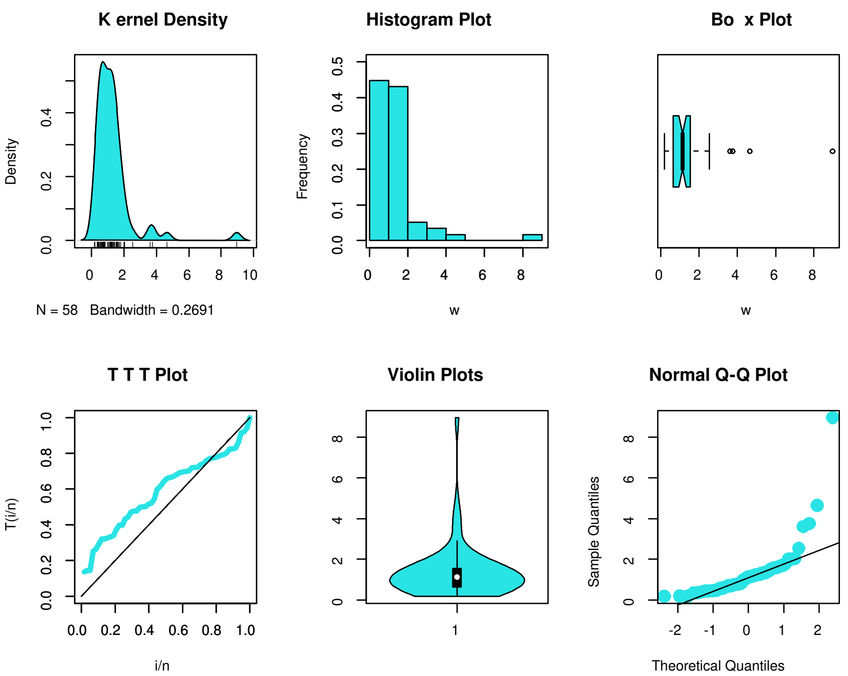

Figure 4. Non-parametric plots for normal condition.

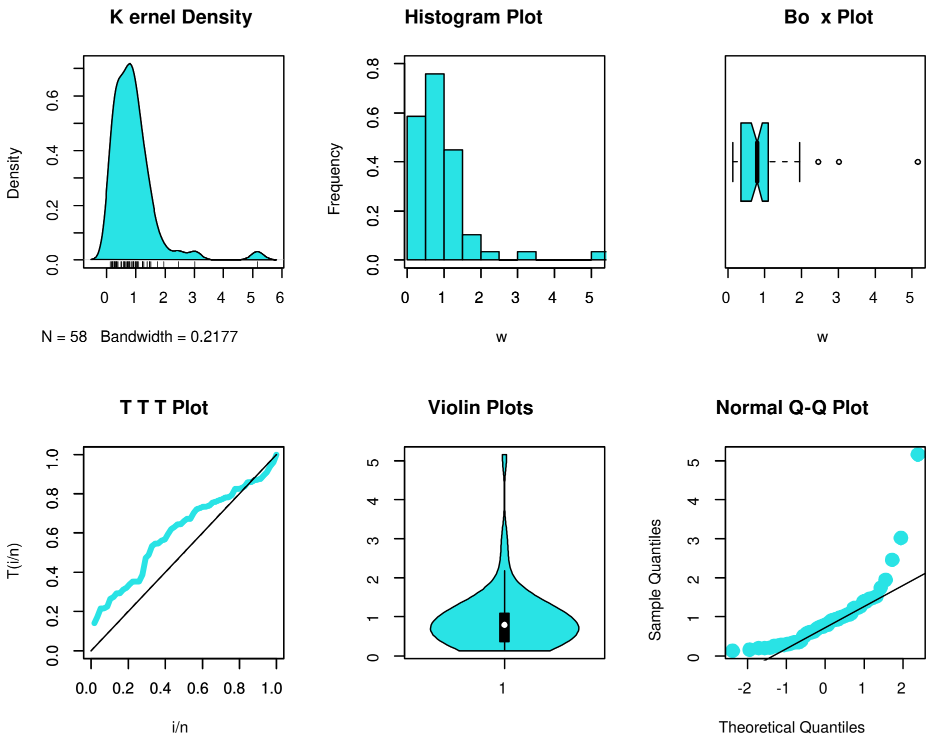

Figure 5. Non-parametric plots for accelerated condition.

By implementing the technique outlined in Section 2, PFFC samples are obtained. The original data under both normal use and accelerated stress conditions are separated into groups of a specific size under the use of CSs. Refer to the details presented in Table 6.

Table 6. PFFC under CS-PALT LED failure data.

Normal use condition: $(k_1, n_1, m_1) = (2, 29, 15)$.
R_1 = (3, 1, 1, 2, 0, 1, 2, 1, 0, 2, 0, 1, 0, 0, 0). 0.18, 0.19, 0.36, 0.45, 0.47, 0.57, 0.63, 0.70, 0.71, 0.76, 0.79, 0.85, 1.01, 3.76, 8.97.
Accelerated stress condition: $(k_2, n_2, m_2) = (2, 29, 18)$.
R_2 = (1, 1, 2, 0, 1, 0, 2, 0, 2, 0, 2, 0, 0, 1, 0, 0, 0, 0). 0.13, 0.20, 0.21, 0.26, 0.31, 0.35, 0.50, 0.58, 0.60, 0.63, 0.75, 0.78, 0.80, 0.94, 1.22, 1.95, 2.46, 3.02.

The ML and two parametric Bootstrap point estimates as well as the associated ACIs are obtained and listed in Table 7. By moving to Bayes estimates, since no previous knowledge of the unknown population parameters is provided, the non-informative (or vague) gamma priors are adequate in this situation. In this instance, the hyper-parameters are set to zero $(a_i = b_i = 0, i = 1, 2)$. As previously mentioned, the Gibbs algorithm relies on Metropolis to produce 12,000 MCMC samples using the $\hat{\beta}_{ML}$, $\hat{\delta}_{ML}$ and $\hat{\rho}_{ML}$ as initial values at the beginning of the algorithm. Additionally, the Bayes estimates are computed and recorded in Table 7. Finally, we can say that the estimated PHFD offers a superb fit for the provided data and that Bayes estimates performs better than MLEs and bootstrap.

Table 7. Estimates of (β, δ, ρ) and its corresponding 95% CI using PFFC under CS-PALT.

Parameter		$(.)_{ML}$	$(.)_{Boot-p}$	$(.)_{Boot-t}$	$(.)_{BS}$
β	Estimate	1.70985	1.74865	1.66473	1.54899
	95% CI	(0.8371, 3.4926)	(0.9436, 3.3772)	(0.8945, 2.9246)	(0.9932, 2.6745)
δ	Estimate	0.15323	0.16225	0.13482	0.12557
	95% CI	(0.0295, 0.7953)	(0.0365, 0.8766)	(0.0334, 0.7452)	(0.0215, 0.6935)
ρ	Estimate	0.28209	0.26942	0.21641	0.19984
	95% CI	(0.1701, 0.4679)	(0.1432, 0.5211)	(0.1165, 0.4263)	(0.0989, 0.3762)

8. Conclusions

This paper highlights the statistical inference issue for a system of CS-PALTs under PFFC when the testing products' lifetimes follow the PHFD. By reducing the amount of time and test units required, and hence the cost, this combination makes our study more useful and applicable in the industrial and technical domains. Several techniques have been developed throughout the study to estimate the relevant parameters, the acceleration factor, and the corresponding confidence intervals. Using the observed FIM, the MLEs are obtained as classical estimation, and the related ACIs are established. Also, two parametric Bootstrap estimates (Boot-p and Boot-t) for the relevant parameters are provided for comparison. Due to the difficulties of obtaining Bayes estimates in closed form, the point and interval estimates for the Bayesian approach are created with the use of the MCMC technique. The effectiveness of the suggested methods is examined by in-depth Monte Carlo simulations. The results clearly show that the Boot-t and Bayes estimates outperform the traditional likelihood and Boot-p estimates in terms of performance and accuracy. In the end, a single collection of actual engineering data is examined for more illustration. The study has shown that the PHFD has offered good flexibility for modeling the life test of the light-emitting diode practically. This study is innovative in that it shows that variable sample sizes k_1 and k_2 can be taken into account in each group when a type-II progressively first failure censored sample is employed. This is completely consistent with real-world examples when conducting life tests. Even though progressively first failure type-II censoring and PHFD have received most of our attention in this study, the same approach can be applied to various distribution and censoring methods. The design of the best censoring schemes, the inference of competing risk models with more failure factors, and the statistical prediction of the subsequent order statistics based on the PALTs from PHFD are just a few of the numerous additional tasks that need to be completed in this area. Finally, we advise adopting the MCMC method based on partially accelerated life testing with the progressive first failure type-II censored on data from life testing, reliability modeling, and medical analysis.

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