mathematics

MDPI

Article

Quasi-Metrics for Possibility Results: Intergenerational Preferences and Continuity

Asier Estevan [1], Roberto Maura [2] and Óscar Valero [3,4,*]

[1] Departmento de Estadística, Informática y Matemáticas, Institute INARBE, Public University of Navarre, Iruñea-Pamplona, Navarre, 31006 Pamplona, Spain
[2] Department of Economics, London School of Economics, London WC2A 2AE, UK
[3] Departament de Ciències Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122 Palma de Mallorca, Spain
[4] Institut d' Investigació Sanitària Illes Balears (IdISBa), Hospital Universitari Son Espases, 07120 Palma de Mallorca, Spain
* Correspondence: o.valero@uib.es

Abstract: In this paper, we provide the counterparts of a few celebrated impossibility theorems for continuous social intergenerational preferences according to P. Diamond, L.G. Svensson and T. Sakai. In particular, we give a topology that must be refined for continuous preferences to satisfy anonymity and strong monotonicity. Furthermore, we suggest quasi-pseudo-metrics as an appropriate quantitative tool for reconciling topology and social intergenerational preferences. Thus, we develop a metric-type method which is able to guarantee the possibility counterparts of the aforesaid impossibility theorems and, in addition, it is able to give numerical quantifications of the improvement of welfare. Finally, a refinement of the previous method is presented in such a way that metrics are involved.

Keywords: quasi-pseudo-metrics; Pareto; anonymity; distributive fairness semiconvexity; social welfare (pre)orders; possibility result

MSC: 1C99; 06A06; 06F30

Citation: Estevan, A.; Maura, R.; Valero, Ó. Quasi-Metrics for Possibility Results: Intergenerational Preferences and Continuit. *Mathematics* **2023**, *11*, 395. https://doi.org/10.3390/math11020395

Academic Editor: Alfio Giarlotta

Received: 11 November 2022
Revised: 8 January 2023
Accepted: 10 January 2023
Published: 12 January 2023

1. Introduction

The intergenerational distribution problem has been studied in depth since the beginning of the twentieth century. In 1907, Henry Sidgwick stated that every rational distributional criterion (social intergenerational preferences) with an infinite horizon must satisfy the finite anonymity [1]. Later on, in 1960, Tjalling Koopmands added to this intergenerational equity requirement the continuity and the impatience axiom [2]. Then, Peter Diamond showed that the former conditions conflict the continuity requirement in his celebrated impossibility theorem [3]. Concretely, the aforementioned theorem states a conflict between finite anonymity, impatience (Pareto efficiency) and the continuity with respect to the topology induced by the so-called supremum metric. The finding of Diamond caused several authors to try to discern, on the one hand, whether there exists any distributional criterion satisfying the finite anonymity and impatience at the same time and, on the other hand, whether both conditions can be compatible with continuity with respect to any topology that is T_0. In this direction, Lars-Gunnar Svensson firstly proved the existence of an intergenerational distributional criterion which fulfills simultaneously equity and Pareto efficiency [4]. Secondly, he explored the role of continuity and, thus, he provided an example of the intergenerational distributional criterion which satisfies equity, Pareto efficiency and, in addition, continuity. However, this time, the continuity axiom was considered with respect to a topology finer than the topology induced by the supremum metric.

Svensson did not answer completely the question about what topologies can be considered in order to make continuous the intergenerational distributional criterion when the equity and Pareto efficiency requirements are also under consideration. Motivated, in part, by Sevensson's partial answer to the posed question, Kuntal Banerjee and Tapan Mitra addressed the problem of identifying those topologies that are compatible with equity and Pareto efficiency in [5]. To this end, they provided a necessary condition which is expressed in terms of a simplex condition that must be satisfied by the metric inducing the topology. In this case, the considered topologies came from a collection of metrics that belong to a class whose properties are commonly used in the literature, and they appear to be natural from a social decision-making viewpoint. Of course, the supremum metric and the metric that induces the topology explored by Svensson belong to the aforesaid class. Banerjee and Mitra prove that, among the topologies induced by the metrics in such a class, the topology considered by Svensson is the coarsest one for which an intergenerational distributional criterion can be continuous, as well as equity and Pareto efficiency being satisfied.

In the exposed studies, the authors considered the intergenerational equity and Pareto efficiency expressed by the so-called anonymity and strong Pareto axioms, respectively. The first requirement, anonymity, is an ethical criterion which expresses that every generation must be treated equally regardless of how far they are in time. The second one, strong Pareto, exhibits sensitivity to changes in the welfare levels of each generation. So it seems natural to wonder whether it is possible to express both requirements by means of another criterion that brings compatibility with continuity.

Regarding the strong Pareto axiom, in [6], Marc Fleurbaey and Phillippe Michel considered the so-called weak Pareto axiom in order to express the intergenerational efficiency and showed that a stronger version of Diamond's impossibility theorem can be deduced. Hence, they proved that anonymity, weak Pareto and continuity with respect to the topology induced by the supremum metric are also incompatible.

Toyotaka Sakai introduced a new concept of equity in [7]. Specifically, Sakai proved that anonymity is not able to capture all aspects of intergenerational equity because this requirement expresses that present-biased and future-biased intergenerational distributions must be treated equally and it is not sensitive to balanced distributions. Motivated by this fact, he introduced the distributive fairness semiconvexity axiom, which expresses that balanced distributions are preferable to the aforementioned biased intergenerational distributions. Moreover, Sakai proved again the incompatibility of anonymity, distributive fairness semiconvexity and continuity induced by the supremum metric. Furthermore, a distributive fairness version of Svensson's possibility result was provided by Sakai when the strong Pareto requirement was replaced by (strong) distributive fairness semiconvexity. It must be stressed that the intergenerational preference constructed by Svensson violates the strong distributive fairness semiconvexity. So the impossibility result due to Sakai is only based on intergenerational ethical requirements because no Pareto axioms are assumed.

In [8], Sakai introduced a new requirement that he called sensitivity to the present, which is able to capture in some sense anonymity and distributive fairness semiconvexity in such a way that it is sensitive to changes of the utility or welfare of present generations. Concretely, Sakai showed that such an axiom can be derived independently from the strong Pareto requirement and from the distributive fairness semiconvexity requirement; in addition, a generalization of Diamond's and Sakai's impossibility theorems was obtained, showing that sensitivity to the present is incompatible with anonymity and continuity with respect to the supremum metric. These criteria have continued to be used in more recent studies [9–11].

Motivated by the exposed facts, in this paper, we focus our efforts on studying how the intergenerational distributional criteria and the topology can be made compatible. The start point is those topologies finer than the corresponding upper topology, which is the smallest one among those that make the social intergenerational preferences continuous. Moreover, we provide one topology (by means of the grading principle) such that any preference satisfying anonymity and strong monotonicity is now continuous.

We use that in order to provide possibility counterparts of the above mentioned impossibility theorems of Diamond, Svensson and Sakai. Our methodology is in accordance with the classification of Banerjee and Mitra, of the metrics belonging to the class considered in [5]. However, the new method presents two advantages with respect to the approach given in the aforesaid reference. On the one hand, the new result allows us to decide the continuity of the preference even if the topology under consideration is not metrizable. On the other hand, Banerjee and Mitra only provide a necessary condition. Hence, one can find preferences that enjoy anonymity and strong Pareto requirements and, in addition, they fulfill the simplex condition in [5] but they are not continuous. An example of this type of preference is provided.

The fact that the upper topology is not metrizable (notice that it is not Hausdorff) suggests to us that the appropriate quantitative tool for reconciling topology and social intergenerational preferences is exactly provided by quasi-pseudo-metrics, which are able to encode the order relation that induces the intergenerational preference. Observe that quasi-pseudo-metrics have already been successfully applied to model risk measures in finance (see [12–14]) and to the representability problem of rational preferences (see [15–18]). This generalized metric notion helps us to provide two things: the numerical quantification about the increase in welfare and the arrow of such an increase. Note that a metric would be able to yield information on the increase but it, however, will not give the aforementioned arrow.

Based on the fact that every preorder, and thus, every social intergenerational preference, can be encoded by means of a quasi-pseudo-metric (see, for instance, [19]) we develop a method to induce a quasi-pseudo-metric that always makes the preference continuous with respect to its induced topology, the Alexandroff topology generated by the preorder (by the grading principle, for the general case, when dealing with strong monotonicity and anonymity), which is finer than the upper topology. Thus, such a method is again able to guarantee the possibility counterparts of the celebrate impossibility theorems of Diamond, Svensson and Sakai and, in addition, it is able to give numerical quantifications of the improvement of welfare.

Since in economics analysis it is convenient to represent preferences through real valued functions [20,21], the so-called utility functions, we also show that our method makes always the preferences semi-continuous multi-utility representable in the sense of [22].

Finally, in order to keep close to the classical way of measuring in the literature, a refinement of the previous method is presented in such a way that metrics are involved.

2. Preliminaries on Preorders and Intergenerational Preferences

In this section, we recall the basics on order theory and intergenerational preferences in decision-making theory that will be useful in our subsequent discussion.

According to [23], a *preorder* on a non-empty set X is a reflexive and transitive binary relation $\precsim$ on X. A preorder is called a *preference* in [24]. A complete preorder is a *rational preference* in [20]. Complete preorders are also known as *total preorders* in [23]. Although the preorders have usually been assumed as rational preferences in the literature, the notion of preorder has turned to be very useful in many fields of economics (for a deeper treatment of the topic, we refer the reader to [22] and references therein).

From now on, given a non-empty set X endowed with a preference $\precsim$, as usual, we will denote by $x \sim y$ the fact that ($x \precsim y$ and $y \precsim x$). Moreover, $x \prec y$ will denote the fact that $x \precsim y$ and, in addition, not $y \precsim x$. When x and y are incomparable, we will write $x \bowtie y$. Thus, $x \bowtie y$ if and only if $\neg(x \precsim y)$ as well as $\neg(y \precsim x)$.

Following [23], for any $y \in X$ and any relation $\precsim$ on X, the *contour sets* (*lower* and *upper*) are defined as follows:

(1) $L^{\precsim}(y) = \{x \in X \colon x \precsim y\}$ (lower counter set),
(2) $U^{\precsim}(y) = \{x \in X \colon y \precsim x\}$ (upper counter set).

On account of [19], a subset G of a non-empty set X is said to be an *up-set* (or *upward closed*) with respect a preorder $\precsim$ on X provided that $y \in G$ whenever $x, y \in X$ with $x \in G$ and $x \precsim y$. Dually, a subset G is said to be a *down-set* (or *downward closed*) with respect a preorder $\precsim$ on X provided that $y \in G$ whenever $x, y \in X$ with $x \in G$ and $y \precsim x$.

According to [25], a rational preference $\precsim$ on X is called *representable* if there is a real-valued function $u\colon X \to \mathbb{R}$ that is order-preserving so that for every $x, y \in X$, it holds that

$$x \precsim y \iff u(x) \leq u(y).$$

The map u is said to be a *utility function* for $\precsim$.

According to [23], a rational preference $\precsim$ on X is said to be *separable* (Debreu separable in [23]) if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$, there exists $d \in D$ such that $x \precsim d \precsim y$. In the case of rational preferences, it is representable if and only if it is separable.

When the preorder is not the total, then a representation can also be proposed. Hence, according to [26] (see also [27]), a preorder is *Richter–Peleg-representable* if there is a function $u\colon X \to \mathbb{R}$ that is strictly isotonic so that for every $x, y \in X$, it holds that

$$x \precsim y \implies u(x) \leq u(y) \text{ and } x \prec y \implies u(x) < u(y).$$

The map u is said to be a *Richter-Peleg utility function* for $\precsim$.

Obviously, a Richter–Peleg representation does not characterize the preorder, i.e., the preorder cannot be retrieved, in general, from the Richter–Peleg utility function. Motivated by this fact, the notion of a multi-utility representation was introduced in [22]. In particular, a preorder $\precsim$ on a set X is said to have a *multi-utility representation* if there exists a family $\mathcal{U}$ of isotonic real-valued functions (*weak-utilities*) such that for all points $x, y \in X$, the following equivalence holds:

$$x \precsim y \Leftrightarrow \forall u \in \mathcal{U} \ (u(x) \leq u(y)) \tag{1}$$

Observe that the members of a multi-utility representation $\mathcal{U}$ are isotonic, but they do not need to be strict isotonic in general. This fact makes the multi-utility representation different from Ritger–Peleg utility representation. It must be pointed out that a rational preference admits a multi-utility representation even when it is not separable and, thus, it does not admit a utility representation.

The advantage of the multi-utility representation with respect to the above discussed type of representations is twofold. On the one hand, it always exists (see Proposition 1 in [22]). On the other hand, it fully characterizes the preorder.

When discussing about intergenerational distribution criteria, the following axioms can be assumed to be satisfied for those preorders that are applied to rank the different alternatives. In the literature, a few alternative sets are considered and, usually, all of them are subsets of the set $l_\infty = \{(x_n)_{n\in\mathbb{N}} : x_i \text{ with } \sup_{i\in\mathbb{N}} x_i < \infty\}$.

Let us recall that the most usual alternative sets are

$$l_\infty^+ = \{(x_n)_{n\in\mathbb{N}} \in l_\infty : x_i \geq 0 \text{ for all } i \in \mathbb{N}\}$$

and

$$l_\infty^{[0,1]} = \{(x_n)_{n\in\mathbb{N}} \in l_\infty : 0 \leq x_i \leq 1 \text{ for all } i \in \mathbb{N}\}.$$

A sign of the interest aroused by such sets is given by the fact that many references have considered them as appropriate framework for their respective studies. Hence, the alternative sets l_∞^+ and $l_\infty^{[0,1]}$ have been considered in [6,7,28,29] and [3–5,8], respectively. Moreover, the whole space l_∞ has been considered in [30–32].

From now on, an alternative set will be any subset X of l_∞, i.e., $X \subseteq l_\infty$. Next, we recall the below concepts which will play a crucial role in order to state possibility results later on. For a fuller treatment of such notions we refer the reader, for instance, to [5–7].

A *finite permutation* is a bijection $\pi\colon \mathbb{N} \to \mathbb{N}$ such that there is $t_0 \in \mathbb{N}$ satisfying $t = \pi(t), \forall t > t_0$. In the sequel, Π_∞ will denote the set of all such π.

A preorder $\precsim$ on X is said to satisfy the *anonymity axiom* if and only if $x \sim \pi(x)$ for all $x \in X$ and for all $\pi \in \Pi$. Anonymity expresses that every generation must be treated equally regardless how far they are in time. However, as exposed in the Introduction, such an axiom does not capture all aspects of intergenerational equity because it is not sensitive to balanced distributions. In order to avoid this handicap, the *distributive fairness semiconvexity axiom* has been considered. This axiom expresses that balanced distributions are preferable to the aforementioned biased intergenerational distributions, and it can be stated as follows.

A preorder $\precsim$ on X is said to satisfy the distributive fairness semiconvexity axiom if and only if for all $x \in X$ and for all $\pi \in \Pi$, we have that there exists $s \in (0,1)$ such that $sx + (1-s)\pi(x) \succ x, \pi(x)$ whenever $x \neq \pi(x)$. Moreover, a stronger version of the previous axioms can be expressed via the *strong distributive fairness semiconvexity*, which states that a preorder $\precsim$ on X satisfies the strong distributive fairness semiconvexity axiom if and only if for all $x \in X$ and for all $\pi \in \Pi$ we have that $sx + (1-s)\pi(x) \succ x, \pi(x)$ for all $s \in (0,1)$ whenever $x \neq \pi(x)$.

An axiom which captures sensitivity to changes in the the welfare levels of each generation is called the *weak monotonicity axiom* or *weak Pareto axiom*. It can be stated in the following way.

A preorder $\precsim$ on X is said to be *weak monotone* or *weak Pareto* if and only if, for all $x, y \in X$, $x \prec y$ provided that $x_t < y_t$ for all $t \in \mathbb{N}$. A stronger version of weak monotonicity axiom is the *strong monotonicity axiom* or *strong Pareto axiom*. Thus, a preorder $\precsim$ is said to be *strong monotone* or *strong Pareto* if and only if, for all $x, y \in X$, $x \prec y$ provided that $x_t \leq y_t$ for all $t \in \mathbb{N}$ and, in addition, $x \neq y$. Clearly, every strong Pareto preorder is always weak Pareto.

Sensitivity to the present is an axiom which is able to capture, in some sense, anonymity and distributive fairness semiconvexity in such a way that the preorder is sensitive to changes in the utility or welfare of present generations. Formally, a preorder $\precsim$ on X satisfies sensitivity to the present provided that, for each $x \in X$, there are $y, z \in X$ and $t \in \mathbb{N}$ such that $(z^t, {}^{t+1}x) \prec (y^t, {}^{t+1}x)$, where, for each $w \in X$, $(w^t, {}^{t+1}x)_i = w_i$ for all $i \in \mathbb{N}$ with $i \leq t$ and, in addition, $(w^t, {}^{t+1}x)_i = x_i$ for all $i \in \mathbb{N}$ with $t+1 \leq i$.

In the remainder of the paper, a preorder on X fulfilling any equity requirement (anonymity, distributive fairness semiconvexity or sensitivity to the present) and any monotony (strong or weak) will be called an *ethical social welfare preorder*. An *ethical social welfare preorder* that is a rational preference (complete preorder) will be called *ethical social welfare order* (ethical preference in [4]). It is worthy to mention that ethical social welfare preorders and ethical social welfare orders have been shown to exist in [4,7].

2.1. The Continuity of Preferences: Topologies for Possibilities Results

In this section, we study the way through which the intergenerational preferences and the topology can be made compatible. Since two notions of continuity have been taken into account in the intergenerational distribution problems, we include a characterization of both types of continuities, and they are independent of any equity or Pareto requirement. Moreover, we remark which topology is the smallest one among those that make the preorder continuous in both senses. This is our start point.

Partial answers to those problems have been given by means of the so-called impossibilities results, which state that there does not exist any ethical social welfare (pre)order which is continuous with respect to the topology under consideration (mainly the product topology or the supremum topology on $X \subseteq l_\infty$). Then, we pass from metrics to quasi-metrics in order to obtain possibility counterparts of the aforementioned impossibility theorems of Diamond, Svensson and Sakai.

First, we recall a few pertinent notions from topology that will be very useful in order to achieve our target.

According to [19], a preorder can be always induced on a topological space (Y, τ). Such a preorder $\precsim_\tau$ is called the *specialization preorder* induced by τ and it is defined as follows:

$$x \precsim_\tau y \Leftrightarrow \text{ every open subset containing } x \text{ also contains } y.$$

It is not hard to check that $x \precsim_\tau y \Leftrightarrow x \in cl_\tau(\{y\})$, where by $cl_\tau(\{y\})$ we denote the closure of $\{y\}$ with respect to τ.

It is clear that the specialization preorder allows us to achieve a preorder from every topology. It is known too that every preorder can be obtained as a specialization preorder of some topology [19]. However, the correspondence is not bijective, since there are in general many topologies on a set X which induce a given preorder $\precsim$ as their specialization preorder. Among the aforesaid topologies, we find the upper topology and the Alexandroff topology. The first one is the coarsest topology, and the second one is the finest topology that induces the preorder $\precsim$ as their specialization preorder. Notice that there are many other topologies between them and that, in general, the Alexandroff and the upper topologies do not coincide. An example that shows that the upper topology and the Alexandroff topology are not the same in general can be found in [33], Example 1.

Let us recall that, given a preorder $\precsim$ on a non-empty set X, the *upper topology* $\tau_U^{\precsim}$ is defined as that which has the lower contour set $L^{\precsim}(x)$ closed ($x \in X$), that is, $\tau_U^{\precsim}$ is the topology arisen from the subbase $\{Y \setminus L^{\precsim}(x)\}_{x\in X}$. Observe that a preorder $\precsim^{-1}$ can be induced from a preorder $\precsim$ on X as follows: $x \precsim^{-1} y \Leftrightarrow y \precsim x$. The preorder $\precsim^{-1}$ is called the dual preorder or the opposite of $\precsim$. Clearly, $L^{\precsim^{-1}}(y) = U^{\precsim}(y)$ for all $y \in Y$. Taking this into account, we will denote by $\tau_L^{\precsim}$ the upper topology on Y induced by $\precsim^{-1}$. Notice that such a topology matches up with the *lower topology* induced by $\precsim$ on X, that is, the topology whose subbase is $\{Y \setminus U^{\precsim}(y)\}_{y\in X}$.

Usually, intergenerational preferences are assumed to satisfy the notion that two intertemporal distributions that are not very different must be have similar welfare levels. This is accomplished by assuming that the preorder under consideration is continuous. Let us recall the two usual notions of continuity.

A preorder $\precsim$ on a topological space (Y, τ) is said to be *τ-continuous* if, for all $y \in Y$, the lower contour $L^{\precsim}(x)$ and the upper contour $U^{\precsim}(x)$ are closed with respect to τ (see, for instance, [3]). However, a weak form of continuity is stated in the literature, the so-called lower continuity (among others, see [30]). Thus, a preorder on a topological space is said to be *lower τ-continuous* provided that, for all $y \in Y$, the lower contour $L^{\precsim}(x)$ is closed with respect to τ.

From now on, given a preorder $\precsim$ on Y and $x_1, \ldots, x_n \in Y$, we will set

$$\downarrow_{\precsim} \{x_1, \ldots, x_n\} = \{z \text{ such that there exists } i \in \{1, \ldots, n\} \text{ with } z \precsim x_i\}.$$

Dually, $\uparrow_{\precsim} \{x_1, \ldots, x_n\}$ can be defined.

The following starting point is well-known, but we prefer to include it for the sake of completeness (see, for instance, [34]).

Proposition 1. *Let $\precsim$ be a preorder on a topological space (Y, τ). The following assertions are equivalent:*

(1) *$\precsim$ is τ-continuous.*

(2) *The topology τ is finer than the coarsest topology, including $\tau_U^{\precsim}$ and $\tau_L^{\precsim}$.*

The next result characterizes the lower continuity of a preorder $\precsim$ with respect to a topology τ.

Proposition 2. *Let $\precsim$ be a preorder on a topological space (Y, τ). The following assertions are equivalent:*

(1) *$\precsim$ is lower τ-continuous.*

(2) *The topology τ is finer than $\tau_U^{\precsim}$.*

The preceding characterization can be found in [33], Corollary 1. In view of Propositions 1 and 2, it makes no sense to work with a topology which does not refine the aforesaid ones.

Notice that these results turn out to be key when the continuity of ethical social welfare preorders and orders is discussed. This fact will be exploited in the the next subsection, where we introduce the possibility results, i.e., propositions that reconcile social welfare (pre)orders and the topology making them continuous.

Going back to the specialization preorder, let us recall that, given a preorder $\precsim$ on Y, the *Alexandroff topology* $\tau_A^{\precsim}$ is formed by all up-sets with respect to $\precsim$. That is, $\tau_A^{\precsim} = \{U \subseteq X$ such that if $x, y \in X$ with $x \in U$ and $x \precsim y$ then $y \in U\}$. Observe that the lower sets are closed sets with respect to $\tau_A^{\precsim}$.

From the preceding characterizations, we obtain the following ones which give sufficient conditions to make continuous a preorder.

Corollary 1. *Let $\precsim$ be a preorder on a topological space (Y, τ). If τ is finer than $\tau_A^{\precsim}$ and $\tau_A^{\precsim^{-1}}$, then $\precsim$ is τ-continuous.*

Proof. Since $\tau_U^{\precsim} \subseteq \tau_A^{\precsim}$ and $\tau_L^{\precsim} \subseteq \tau_A^{\precsim^{-1}}$ we conclude, from Proposition 1, that $\precsim$ is τ-continuous. □

Corollary 2. *Let $\precsim$ be a preorder on a topological space (Y, τ). If τ is finer than $\tau_A^{\precsim}$, then $\precsim$ is lower τ-continuous.*

Proof. Since $\tau_U^{\precsim} \subseteq \tau_A^{\precsim}$ we conclude, from Proposition 2, that $\precsim$ is lower τ-continuous. □

The next example shows that the converse of Corollaries 1 and 2 does not hold in general. In order to introduce such an example, notice that a sequence $(x_n)_{n\in\mathbb{N}}$ in Y converges to $x \in Y$ with respect to $\tau_A^{\precsim}$ if and only if there exists $n_0 \in \mathbb{N}$ such that $x \precsim x_n$ for all $n \geq n_0$.

Example 1. *Consider the preorder $\precsim$ on $l_\infty^{[0,1]}$ defined by*

$$y \precsim x \Leftrightarrow y_t \leq x_t \text{ for all } t \in \mathbb{N}.$$

Then $\precsim$ is τ_{d_s}-continuos and, thus, lower τ_{d_s}-continuos, where d_s stands for the restriction of the supremum metric on l_∞ to $l_\infty^{[0,1]}$, i.e., $d_s(x, y) = \sup_{t\in\mathbb{N}} |x_t - y_t|$ for all $x, y \in l_\infty$.

Next we show that $\tau_A^{\precsim} \not\subseteq \tau_{d_s}$. Indeed, set $x = (0, 1, 0, \frac{1}{2}, \frac{2}{2}, 0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, 0, \ldots)$ and $l = (1, 1, 0, \frac{1}{2}, \frac{2}{2}, 0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, 0, \ldots)$. Now the sequence $(y_n)_{n\in\mathbb{N}}$ is defined as follows:

$$\begin{aligned}
y_1 &= x = (0, 1, 0, \tfrac{1}{2}, \tfrac{2}{2}, 0, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, 0, \ldots), \\
y_2 &= (\tfrac{2}{2}, 1, 0, 0, \tfrac{1}{2}, 0, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, 0, \ldots), \\
y_3 &= (\tfrac{3}{3}, 1, 0, \tfrac{1}{2}, \tfrac{2}{2}, 0, 0, \tfrac{1}{3}, \tfrac{2}{3}, 0, \ldots), \\
&\ldots \\
y_n &= (\tfrac{n}{n}, 1, 0, \tfrac{1}{2}, \tfrac{2}{2}, 0, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, 0, \ldots, 0, 0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 0, \ldots)
\end{aligned}$$

Clearly the sequence $(y_n)_{n\in\mathbb{N}}$ converges to $l = (1, 1, 0, \frac{1}{2}, \frac{2}{2}, 0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, 0, \ldots)$ on τ_s, since $d_s(l, y_n) = \frac{1}{n}$. However the sequence fails to converge in $\tau_A^{\precsim}$, since $l \not\precsim y_t$ for any $t \in \mathbb{N}$.

As shown before, in economics analysis, it is convenient to represent preorders through real-valued functions [20,21]. We end this subsection giving conditions so that a preorder admits a semi-continuous multi-utility representation [22].

Let us recall that, given a topological space (Y, τ), a function $f : Y \to \mathbb{R}$ which is continuous from (Y, τ) into $(\mathbb{R}, \tau_U^{\leq})$ is said to be *lower semi-continuous*.

According to [22], Proposition 2, every (pre)order $\precsim$ on a topological space (Y, τ) which is lower τ-continuous always has a multi-utility representation $\mathcal{U}$ of isotonic real-valued functions such that every member belonging to $\mathcal{U}$ is a lower semi-continuous function.

In light of Propositions 1 and 2, we conclude by stating that every (pre)order $\precsim$ on a topological space (Y, τ) admits a semi-continuous multi-utility representation provided that τ is finer than $\tau_{\mathcal{U}}^{\precsim}$.

2.2. The Impossibility Theorems

As already mentioned, Diamond showed in his celebrated impossibility theorem a conflict between the fact that a preorder satisfies the finite anonymity, strong monotonicity and the continuity with respect to the topology induced by the supremum metric τ_{d_s}, with $d_s(x, y) = \sup_{i\in\mathbb{N}} |x_t - y_t|$ for all $x, y \in l_\infty$, [3]. The aforesaid theorem can be stated as follows.

Theorem 1. *There is no ethical social welfare (pre)order $\precsim$ on $l_\infty^{[0,1]}$ which satisfies anonymity, strong monotonicity and τ_{d_s}-continuity.*

Diamond's impossibility theorem was extended to the case of preorders fulfilling weak monotonicity by Fleurbaey and Michel in [6]. Concretely, they proved the next result.

Theorem 2. *There is no ethical social welfare (pre)order $\precsim$ on l_∞^+ which satisfies anonymity, weak monotonicity and τ_{d_s}-continuity.*

In [7], Sakai introduced the distributive fairness semiconvexity in order to overcome the lack of sensitivity of anonymity to balanced distributions. He proved again the incompatibility of anonymity, distributive fairness semiconvexity and continuity induced by the supremum metric. Specifically, the next result was obtained.

Theorem 3. *There is no ethical social welfare (pre)order $\precsim$ on l_∞^+ which satisfies anonymity, distributive fairness semiconvexity and τ_{d_s}-continuity.*

It is obvious that if there is no preorder on l_∞^+ satisfying anonymity, distributive fairness semiconvexity and τ_{d_s}-continuity, then there is no any preorder fulfilling anonymity, strong distributive fairness semiconvexity and τ_{d_s}-continuity.

Later on, Sakai introduced the sensitivity to the present axiom in order to capture, in some sense, anonymity and distributive fairness semiconvexity at the same time. Again, an incompatibility was shown in such a way that the following impossibility result, which generalizes Diamond's and Sakai's impossibility theorems, was proved.

Theorem 4. *There is no ethical social welfare (pre)order $\precsim$ on $l_\infty^{[0,1]}$ which satisfies anonymity, sensitivity to the present and τ_{d_s}-continuity.*

From the preceding results in which the sets l_∞^+ and $l_\infty^{[0,1]}$ are fixed as the alternative set, one could infer the same impossibility results considering l_∞ as the alternative set.

3. The Possibilities Results

Banerjee and Mitra addressed the problem of identifying those topologies that make an ethical social welfare order continuous when anonymity and strong monotonicity are assumed [5]. They provided a necessary condition which is expressed in terms of a simplex condition that must be satisfied by the metric inducing the topology. To this end, they consider a class $\triangle$ of metrics which satisfy four properties that we do not discuss here because they are not relevant to our work. For a broad discussion of such properties, we refer the reader to [5].

Although the considered class imposes constraints about the metrics, the most usual metrics applied to the intergenerational distribution problem belong to $\triangle$. Concretely, the following celebrated metrics on l_∞ are d_c, d_s, d_p, d_1, d_q, where

$$d_c(x,y) = \sum_{i=1}^{\infty} \frac{|x_t - y_t|}{2^i} \quad .$$

$$d_p(x,y) = \min\{1, (\sum_{i=1}^{\infty} |x_t - y_t|^p)^{\frac{1}{p}}\} \text{ with } p \in]1,\infty[.$$

$$d_q(x,y) = \min\{1, \sum_{i=1}^{\infty} (|x_t - y_t|^q)\} \text{ with } q \in]0,1[.$$

Notice that $\tau_{d_c} \subseteq \tau_{d_s} \subseteq \tau_{d_p} \subseteq \tau_{d_1} \subseteq \tau_{d_q}$.

The Banerjee and Mitra result can be stated as follows.

Proposition 3. *Let $d \in \triangle$ and let $\precsim$ be an ethical social welfare preorder on $l_\infty^{[0,1]}$ which satisfies anonymity and strong monotonicity. If $\precsim$ is lower τ_d-continuous, then the metric d satisfies $d(\mathbf{0}, S) > 0$ with $d(x,S) = \inf_{y \in S} d(x,y)$, $S = \{x \in X : \sum_{t=1}^{\infty} x_t = 1\}$ and $\mathbf{0} = (0,0,\ldots,0,\ldots)$.*

From Proposition 3, Banerjee and Mitra deduced that there is no ethical welfare order satisfying anonymity, strong monotonicity and, in addition, lower τ_{d_c}-continuity, τ_{d_s}-continuity and τ_{d_p}-continuity. Notice that the metrics d_c, d_s, d_p do not hold the simplex condition "$d(\mathbf{0}, S) > 0$".

Note that we can restate Proposition 3 interchanging in its statement the lower τ_d-continuity of $\precsim$ by the fact that τ is finer than $\tau_U^{\precsim}$. So if a metric belonging to $\triangle$ violates the simplex condition, then $\tau_U^{\precsim} \not\subseteq \tau_d$ necessarily.

It must be stressed that our approach presents two advantages with respect to the approach given by Banerjee and Mitra. On the one hand, it allows us to decide the continuity of the ethical welfare order even if the topology under consideration is not metrizable and the alternative space is l_∞ instead of $l_\infty^{[0,1]}$. Observe that a few properties that a metric in the class $\triangle$ must satisfy are not true when the intergenerational distributions are not in l_∞^+. Moreover, every ethical social welfare order $\precsim$ will be continuous with respect to the topology τ_d induced by a metric (belonging to $\triangle$ or not) on l_∞ if and only if $\tau_U^{\precsim} \subseteq \tau_d$. On the other hand, contrary to Propositions 1 and 2, Banerjee and Mitra only provide a necessary condition, and they do not prove the converse of Proposition 3. Instead, they provide an example of ethical social welfare orders on $l_\infty^{[0,1]}$ which is (lower) τ_{d_1}-continuous (which satisfies the simplex condition). The aforesaid example is given by the extension of the overtaking type criterion due to Svensson [4].

Svensson proved that every preorder that refines the *grading principle* can be extended in such a way that the extension fulfills anonymity and strong monotonicity in [4]. The aforementioned grading principle is the preorder $\precsim_m$ defined on l_∞ as follows:

$$x \precsim_m y \iff x \leq \pi(y) \text{ for some } \pi \in \Pi_\infty.$$

However, Example 2 shows that the converse of Proposition 3 does not hold in general.

Example 2. *Let $\precsim^{\frac{1}{2}}$ be the preorder on l_∞ defined by*

$$x \precsim^{\frac{1}{2}} y \iff \begin{cases} x \precsim_m y, \\ \text{or} \\ \sigma(x) > \sigma(y), \end{cases},$$

where $\sigma(x)$ denotes the number of coordinates of x which are lower than $\frac{1}{2}$. Notice that the preorder $\precsim^{\frac{1}{2}}$ is related to the satisfaction of the basic needs criterion introduced by G. Chichilnisky in [35] (see, also, [36]).

Clearly, $\precsim^{\frac{1}{2}}$ refines the preorder $\precsim_m$. It is not hard to check that $\precsim^{\frac{1}{2}}$ satisfies anonymity and strong monotonocity (see Proposition 4 below). By [4], $\precsim^{\frac{1}{2}}$ can be extended in such a way that the extension fulfills anonymity and strong monotonicity (see the paragraph before Proposition 4). Set $\preceq$ as the ethical social welfare order on l_∞ that extends $\precsim^{\frac{1}{2}}$.

Now, we define the sequence $(x_n)_{n\in\mathbb{N}}$ in l_∞ by

$$x_n = (\frac{1}{2} - \frac{1}{2^n}, \frac{1}{2} - \frac{1}{2^n}, 0, 0, \ldots, 0, \ldots)$$

for each $n \in \mathbb{N}$. It is clear that $(x_n)_{n\in\mathbb{N}}$ converges to $Z = (\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots)$ with respect to the topology τ_{d_q} since we have that $d_p(l, y_n) = \frac{2^{\frac{1}{p}}}{2^n}$ for all $n \in \mathbb{N}$.

Set $y = (\frac{1}{2}, 0, \ldots, 0, \ldots)$. It is clear that $x_n \prec^{\frac{1}{2}} y$ and, thus, $x_n \prec y$. Moreover, $y \prec^{\frac{1}{2}} \mathbf{\frac{1}{2}}$ and, thus, $y \prec \mathbf{\frac{1}{2}}$, whence $\mathbf{\frac{1}{2}} \in X \setminus L_{\preceq}(y)$, whereas $x_n \notin X \setminus L_{\preceq}(y)$. Therefore $(x_n)_{n\in\mathbb{N}}$ fails to converge with respect to $\tau_U^{\preceq}$. It follows that τ_{d_q} is not finer than $\tau_U^{\preceq}$. Thus, we conclude that the preorder $\preceq$ is not lower τ_{d_q}-continuous.

In light of the preceding facts, although, as mentioned above, Propositions 1 and 2 characterize the topologies for which an ethical social welfare order is continuous, next we explore the possibility of giving a method, based on Corollary 2, that ensures the continuity of any extension of an ethical social welfare preorder satisfying anonymity and strong monotonicity on l_∞ (not only on $l_\infty^{[0,1]}$). The possibility of extending ethical social welfare preorder in such a way the every extension preserves the continuity has attracted the attention of several authors (see [37,38], among others).

Next, we go one step further than Svensson and we show that every ethical social welfare order on l_∞ satisfying anonymity and strong monotonicity is continuous with respect to every topology finer than the Alexandroff topology induced by the grading principle.

Before stating the announced property, we point out, on account of [39], Proposition 1, that every ethical social welfare order that satisfies anonymity and strong monotonicity refines the grading principle.

Proposition 4. *The relation $\precsim_m$ is the smallest ethical social welfare preorder defined on l_∞ satisfying anonymity and strong monotonicity, where $\precsim_m$ is defined as follows:*

$$x \precsim_m y \iff x \leq \pi(y) \text{ for some } \pi \in \Pi_\infty.$$

The following interesting property was proved in [33], Lemma 1, and it will be crucial in order to guarantee the continuity of any extension of the grading principle $\precsim_m$.

Lemma 1. *Let $\precsim_1$ and $\precsim_2$ be two preorders on a nonempty set Y, and let $\tau_A^{\precsim_1}$ and $\tau_A^{\precsim_2}$ be their corresponding Alexandroff topologies. The following assertions are equivalent:*

1. $\precsim_1 \subseteq \precsim_2$ ($\precsim_2$ *refines* $\precsim_1$).
2. $\tau_A^{\precsim_2} \subseteq \tau_A^{\precsim_1}$.

It must be pointed out that the upper topology does not fulfill the preceding property such as it is shown in [33], Example 4.

From Corollary 2 and Lemma 1, we obtain a method that gives the continuity of any ethical social welfare order on l_∞ satisfying anonymity and strong monotonicity.

Proposition 5. *Let τ be a topology on l_∞. If the Alexandroff topology τ_A^m associated with $\precsim_m$ is contained in τ, then any ethical social welfare order satisfying anonymity and strong monotonicity is lower τ-continuous.*

Proof. By Proposition 4, we have that any ethical social welfare order satisfying anonymity and strong monotonicity refines the grading principle $\precsim_m$. Lemma 1 gives that $\tau_A^{\precsim} \subseteq \tau_A^{\precsim_m} \subseteq \tau$. Corollary 2 provides the lower τ-continuity. □

In view of Proposition 5, it is worth mentioning that, although any ethical social welfare order $\precsim$ is lower τ-continuous when $\tau_A^{\precsim_m} \subseteq \tau$, in general, there does not exist a lower semicontinuous utility function that represents it. Remember that, according to [23], for the existence of this utility function, the ethical social welfare order must be perfectly separable. However, an extension of a preorder that satisfies anonymity and strong monotonicity fails to be separable (in the Debreu sense) in general. Every ethical social welfare order admits a lower semi-continuous multi-utility representation provided that τ is finer than $\tau_A^{\precsim_m}$ and, thus, finer than $\tau_U^{\precsim}$.

Since Proposition 3 allows us to discard the topologies induced by the metrics d_c, d_s and d_p as an appropriate topology for making lower continuous an ethical social welfare order that fulfills anonymity and strong monotonicity, it seems natural to wonder whether our exposed theory is in accordance with the aforementioned result and, thus, we can infer the same conclusion in our new framework. The next result gives a positive answer to the posed question.

Proposition 6. *Let $\precsim_m$ be the grading principle on l_∞. The upper topology $\tau_U^{\precsim_m}$ is not coarser than the topology τ_{d_p}. Therefore, the Alexandroff topology $\tau_A^{\precsim_m}$ is also not coarser than τ_{d_p}.*

Proof. As proved in [4], Proposition 2, $\precsim_m$ can be extended in such a way that the extension is a total preorder and fulfills anonymity and strong monotonicity. Set $\preceq$ as such an extension. Thus, $\preceq$ is a ethical social welfare order on l_∞. Consider the sequence $(y_n)_{n\in\mathbb{N}}$ in l_∞ introduced in Example 1 and given as follows:

$$
\begin{aligned}
y_1 = x &= (0,1,0,\tfrac{1}{2},\tfrac{2}{2},0,\tfrac{1}{3},\tfrac{2}{3},\tfrac{3}{3},0,\dots),\\
y_2 &= (\tfrac{2}{2},1,0,0,\tfrac{1}{2},0,\tfrac{1}{3},\tfrac{2}{3},\tfrac{3}{3},0,\dots),\\
y_3 &= (\tfrac{3}{3},1,0,\tfrac{1}{2},\tfrac{2}{2},0,0,\tfrac{1}{3},\tfrac{2}{3},0,\dots),\\
&\dots\\
y_n &= (\tfrac{n}{n},1,0,\tfrac{1}{2},\tfrac{2}{2},0,\tfrac{1}{3},\tfrac{2}{3},\tfrac{3}{3},0,\dots,0,0,\tfrac{1}{n},\dots,\tfrac{n-1}{n},0,\dots)
\end{aligned}
$$

Set $x = (0,1,0,\frac{1}{2},\frac{2}{2},0,\frac{1}{3},\frac{2}{3},\frac{3}{3},0,\dots)$ and $l = (1,1,0,\frac{1}{2},\frac{2}{2},0,\frac{1}{3},\frac{2}{3},\frac{3}{3},0,\dots)$.

It is clear that $y_n \in L^{\precsim_m}(x)$ for all $n \in \mathbb{N}$. Thus $y_n \in L^{\preceq}(x)$ for all $n \in \mathbb{N}$. Moreover, the sequence $(y_n)_{n\in\mathbb{N}}$ converges to $l = (1,1,0,\frac{1}{2},\frac{2}{2},0,\frac{1}{3},\frac{2}{3},\frac{3}{3},0,\dots)$ with respect to τ_{d_p}, since $d_p(l,y_n) = \frac{n^{\frac{1}{p}}}{n}$ for all $n \in \mathbb{N}$.

Nevertheless, the sequence $(y_n)_{n\in\mathbb{N}}$ fails to converge with respect to $\tau_U^{\preceq}$ to l, since $y_n \notin X \setminus L^{\preceq}(x)$ for all $n \in \mathbb{N}$ but $l \in X \setminus L^{\preceq}(x)$ because $x \prec_m l$ and, thus, $x \prec l$. Consequently, $\tau_U^{\preceq}$ is not coarser than the topology τ_{d_p}. Since $\tau_U^{\preceq} \subseteq \tau_A^{\preceq}$, we have that $\tau_A^{\preceq}$ is also not coarser than τ_{d_p} as claimed. □

Proposition 6 explains the reason for which the impossibility results Theorem 1 and Theorem 2 hold.

Observe that, on the one hand, the overtaking type criterion introduced in the proof of Proposition 5 is an example of the ethical social welfare order on l_∞ that satisfies the aforementioned requirements and it is, in addition, lower τ_{d_1}-continuous. It must be stressed that the same fact on $l_\infty^{[0,1]}$ was proved in [5]. It was also shown that τ_{d_1} is the smallest topology, among the induced by the metrics in $\triangle$, for which there exists an ethical social welfare ordering satisfying anonymity, strong monotonicity and being lower continuous. On the other hand, we present an example of an ethical social welfare order on l_∞ that satisfies anonymity and strong monotonicity but it is not lower τ_{d_q}-continuous. So in the general l_∞ framework, $\tau_U^{\precsim}$ is the smallest topology for which there exists an ethical social welfare ordering $\precsim$, satisfying anonymity and strong monotonicity that is lower continuous. Nonetheless, if we restrict ourselves to topologies induced by the metrics in $\triangle$, then again τ_{d_1} is the smallest topology that achieves this end.

We end the subsection recovering Example 2, but now we modify it in order to construct an example of ethical social welfare order on l_∞ that fails to be lower τ_{d_q}-continuous. To this end, let us introduce a new axiom that we call *negativity*. We shall say that a preorder on a nonempty set l_∞ satisfies the negativity if, given $(x, y \in l_\infty)$, then $x \prec y$ provided that $\sigma(x) > \sigma(y)$, where $\sigma(x)$ denotes the number of negative coordinates of x. Then, the preorder $\precsim^+$ on l_∞ defined by $x \precsim^+ y$ if and only if $x \precsim_m y$ or $\sigma(x) > \sigma(y)$, is the smallest preorder satisfying anonymity, strong monotonocity and negativity. Again, as proved in [4], $\precsim^+$ can be extended in such a way that the extension $\preceq$ fulfills anonymity and strong monotonicity. However, similar to Example 2, it can be proved that the preorder $\preceq$ is not lower τ_{d_q}-continuous.

Hence, we infer that the upper topology is not, in general, coarser than τ_{d_q}. Therefore, it is not possible in general to guarantee the continuity of an ethical social welfare (pre)order on l_∞ neither with respect to τ_{d_q} nor with respect to τ_{d_1}.

Finally, we remark that the negativity axiom could be interpreted from an economical viewpoint as follows: the negative values can be understood as extreme and generalized cases (that affect all generations) of war, famine, natural disasters, etc. In the case of anonymous data, negative values could suggest losses, debts, bankruptcies, etc.

4. Quasi-Pseudo-Metrics: A Quantitative Tool for Reconciling Order, Topology and Preferences

The manifested difficulty to reconcile the order and topology when this last is induced by a metric motivates us to leave such structures. The fact, on the one hand, that Propositions 1 and 2 show that in order to make a preorder continuous, it is necessary to take into account the upper topology generated by such a preorder. On the other hand, the fact that the upper topology is not metrizable (notice that it is not Hausdorff) suggests that the appropriate quantitative tool for reconciling topology and order is exactly provided by quasi-pseudo-metrics, which are able to encode the preorder.

Following [40] (see also [19]), a quasi-pseudo-metric on a nonempty set Y is a function $d : Y \times Y \to \mathbb{R}^+$ such that for all $x, y, z \in Y$:

(i) $d(x, x) = 0$,
(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

Each quasi-pseudo-metric d on a set X induces a topology τ_d on Y which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A quasi-pseudo-metric space is a pair (Y, d) such that Y is a nonempty set and d is a quasi-metric on Y.

Notice that the topology τ_d is T_0 if and only if $d(x, y) = d(y, x) = 0$ for all $x, y \in Y$.

Observe that a pseudo-metric d on a nonempty set Y is a quasi-pseudo-metric which enjoys additionally the following properties for all $x, y \in Y$:

(iii) $d(x, x) = 0 \Rightarrow x = y$,
(iv) $d(x, y) = d(y, x)$.

A metric is a pseudo-metric d on a nonempty set Y which, in addition, fulfills for all $x, y \in Y$ the property below:

(v) $d(x, y) = 0 \Rightarrow x = y$,

If d is a quasi-pseudo-metric on a set Y, then the function d^s defined on $Y \times Y$ by $d^s(x, y) = \max\{d(y, x), d(x, y)\}$ for all $x, y \in Y$ is a pseudo-metric on Y.

Every quasi-pseudo-metric space d on Y induces a preorder $\precsim_d$ which is defined on Y as follows: $x \precsim_d y \Leftrightarrow d(x, y) = 0$.

An illustrative example of quasi-pseudo-metric spaces is given by the pair $(\mathbb{R}, d_L)$, where $d_L(x,y) = \max\{x-y, 0\}$ for all $x, y \in \mathbb{R}$. Observe that τ_{d_L} is the upper topology $\tau_U^{\leq}$ on $\mathbb{R}$, where $\leq$ stands for the usual preorder on $\mathbb{R}$. Note that $d_L^s(x,y) = |y-x|$ for all $x, y \in \mathbb{R}$.

Following [41], every preorder $\precsim$ can be encoded by means of a quasi-pseudo-metric. Indeed, if $\precsim$ is a preorder on X, then the function $d_{\precsim} \colon X \times X \to \mathbb{R}^+$ given by

$$d_{\precsim}(x,y) = \begin{cases} 0, & x \precsim y \\ 1, & otherwise \end{cases}$$

is a quasi-pseudo-metric on X.

Obviously, $x \precsim_{d_{\precsim}} y \Leftrightarrow d_{\precsim}(x,y) = 0 \Leftrightarrow x \precsim y$ and, in addition, we have that $\tau_{d_{\precsim}} = \tau_A^{\precsim}$ and that $\tau_U^{\precsim}, \tau_L^{\precsim} \subseteq \tau_{d^s_{\precsim}}$. Therefore, Corollaries 1 and 2 give, respectively, the $\tau_{d^s_{\precsim}}$-continuity and the lower $\tau_{d_{\precsim}}$-continuity of $\precsim$.

It must be stressed that (pseudo-)metrics are not able to encode any preorder except the equality order $\precsim_=$, that is, $x \precsim_= y \Leftrightarrow x = y$.

In view of the discussed facts, the use of quasi-pseudo-metrics makes it possible to reconcile "metric methods" of measure and order. In the particular case of the intergenerational distribution problem, these generalized metrics help us to provide both things, the numerical quantifications about the increase in welfare and the arrow of such an increase. Note that a metric would be able to give information on the increase but it will not give the aforementioned arrow.

The preceding method of "metrization" is able to guarantee, in contrast to Propositions 1 and 2, the possibility counterparts of the celebrate impossibility propositions of Diamond, Svensson and Sakai introduced in Section 2.2 in an appropriate metric approach. Specifically, we obtain them combining the preceding quasi-pseudo-metrization and Corollaries 1 and 2.

Theorem 5. *There exists an ethical social welfare order $\precsim$ on l_∞ which satisfies anonymity, strong monotonicity, strong distributive fairness semi convexity and $\tau_{d^s_{\precsim}}$-continuity and, thus, lower $\tau_{d_{\precsim}}$-continuity.*

Proof. First, notice that it is already proved in [7] that there exists an ethical social welfare order $\precsim$ on l_∞ which satisfies anonymity, strong monotonicity and strong distributive fairness semi-convexity. Thus, now it is enough to observe that $\tau_U^{\precsim}, \tau_L^{\precsim} \subseteq \tau_{d^s_{\precsim}}$ and $\tau_U^{\precsim} \subseteq \tau_{d_{\precsim}}$. □

It must be stressed that the metrics considered in the statement of impossibility results in Section 3 are well-known and, in addition, they are often considered in applications. Of course the quasi-pseudo-metric $d_{\precsim}$ considered in statement of Theorem 5 only has the merit of being able to encode the preorder $\precsim$. Motivated by this fact, Section 5 is devoted to developed quasi-pseudo-metrics which make use of metrics in order to measure, and, at the same time, they are able, as required, to induce a topology which is finer than the upper topology $\tau_U^{\precsim}$. Notice that the aforementioned topology is not metrizable, and this fact makes it such that quasi-pseudo-metrics play a relevant role in order to reconcile topology and order in this framework.

Returning to the discussion made in Subsection 2.2 about the continuity of any extension of an ethical social welfare preorder satisfying anonymity and strong monotonicity on l_∞, we have the following.

Proposition 7. *Let $\precsim_m$ be the smallest preorder on X satisfying anonymity and strong monotonicity on l_∞. Any other ethical social welfare (pre)order satisfying anonymity and strong monotonicity is $\tau_{d^s_{\precsim_m}}$-continuous on l_∞ and, thus, lower $\tau_{d_{\precsim_m}}$-continuous.*

Proof. The desired result follows from Corollary 1 and Lemma 1. □

In light of the above facts and the fact that a preorder is lower τ-continuous with respect to a topology only in the case that such a topology refines the upper topology induced by the the preorder, it seems natural to restrict attention to the use of quasi-pseudo-metrics as a quantitative tool that allows us, at the same time, to obtain a numerical quantification of the improvement of welfare and of the closeness between intergenerational distributions.

5. Order, Topology and Preferences: Going Back to Metrics

In Section 4, we show that the use of quasi-pseudo-metrics reconciles "metric methods" of measuring and order requirements of ethical social welfare preorders. In order to follow in the footsteps of classical studies, a refinement of the method that encodes the preorder in such a way that classical metrics are involved is needed. The economical interpretations of their quantifications are also exposed.

In the remainder of this section, we introduce a collection of techniques which generate quasi-pseudo-metrics from a given preorder and a metric on a nonempty set. The aforesaid quasi-pseudo-metrics generate either the Alexandroff topology induced by the preorder or a topology finer than it. So, the below techniques provide the lower continuity of the preorder.

In order to state the mentioned techniques, let us recall that, following [19], a quasi-metric on a nonempty set Y is a quasi-pseudo-metric on Y such that, for all $x, y \in Y$, the following property holds:

(vi) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$.

A quasi-metric is called T_1 provided that, for all $x, y \in Y$, the next property is true:

(vii) $d(x,y) = 0 \Leftrightarrow x = y$.

Notice that the topology τ_d is T_0 when the quasi-pseudo-metric is just a quasi-metric and, in addition, such a topology is T_1 when the quasi-metric is T_1.

Taking this into account, we have the next result. Before stating it, let us recall that a pseudo-metric space (Y,d) is 1-bounded whenever $d(x,y) \leq 1$ for all $x, y \in Y$.

Proposition 8. *Let (Y,d) be a 1-bounded pseudo-metric space and let $\precsim$ be a preorder on Y. The function $d^1_{\precsim} : X \times X \to \mathbb{R}^+$ defined by*

$$d^1_{\precsim}(x,y) = \begin{cases} d(x,y), & x \precsim y \\ 1, & otherwise \end{cases}.$$

is a quasi-pseudo-metric such that $\tau_{d^1_{\precsim}}$ is finer than $\tau^{\precsim}_A$. Therefore, $\precsim$ is lower $\tau_{d_{\precsim}}$- continuous. If d is a metric on Y, then $d^1_{\precsim}$ is a T_1 quasi-metric.

Proof. The function $d^1_{\precsim}$ is a quasi-pseudo-metric. To see that, notice that in the case $x \precsim y \precsim z$, the triangular inequality $d^1_{\precsim}(x,z) \leq d^1_{\precsim}(x,y) + d^1_{\precsim}(y,z)$ is satisfied due the fact that d is a metric. In any other case, either $d^1_{\precsim}(x,y) = 1$ or $d^1_{\precsim}(y,z) = 1$ and, hence, the triangle inequality is satisfied too. Since $d^1_{\precsim}(x,x) = 0 \Leftrightarrow d(x,x) = 0$ for any $x \in Y$, we conclude that it is actually a quasi-pseudo-metric.

Of course, if d is a metric on Y, then $d^1_{\precsim}(x,y) = 0 \Leftrightarrow d(x,y) = 0$ for any $x, y \in Y$. It follows that $d^1_{\precsim}$ is actually a T_1 quasi-metric.

Observe that that $\tau_{d^1_{\precsim}}$ is finer than $\tau^{\precsim}_A$. To this end, let $O \in \tau^{\precsim}_A$ and $x \in O$. Then $O = \bigcup_{x \in O} U^{\precsim}(x)$. Fix $r < 1$. Then $B_{d^1_{\precsim}}(x,r) \subseteq U^{\precsim}(x) \subseteq O$. Hence, we conclude that $\tau^{\precsim}_A \subseteq \tau_{d^1_{\precsim}}$. By Corollary 2, we have the lower $\tau_{d^1_{\precsim}}$- continuity. □

Regarding intergenerational distributions, the quasi-pseudo-metric $d^1_{\precsim}$ introduced in the previous result is able to quantify the increase in welfare (when $x \precsim y$) by means of the use of a metric. Moreover, it differentiates this case from the rest of the cases, assigning the retrogress ($y \prec x$) and the incomparability ($x \bowtie y$) with 1 as a quantification.

A slight modification of the technique introduced in Proposition 8 gives the next one.

Proposition 9. *Let (Y, d) be a pseudo-metric space and let $\precsim$ be a preorder on Y. Then, the function $d^2_{\precsim} \colon X \times X \to \mathbb{R}^+$, defined by*

$$d^2_{\precsim}(x,y) = \begin{cases} \frac{d(x,y)}{2}, & x \precsim y \\ \frac{1}{2} + \frac{d(x,y)}{2}, & otherwise \end{cases}.$$

is a quasi-pseudo-metric such that $\tau_{d^2_{\precsim}}$ is finer than $\tau^{\precsim}_{A}$. Therefore, $\precsim$ is lower $\tau_{d^2_{\precsim}}$- continuous. If d is a metric on Y, then $d^1_{\precsim}$ is a T_1 quasi-metric.

Proof. Next, we show that $d^2_{\precsim}$ is a quasi-metric. Indeed, the triangular inequality $d^2_{\precsim}(x,z) \leq d^2_{\precsim}(x,y) + d^2_{\precsim}(y,z)$ holds whenever $x \precsim y \precsim z$, since d is a metric. In any other case, either $d^2_{\precsim}(x,y) = \frac{1}{2} + \frac{d(x,y)}{2}$ or $d^2_{\precsim}(y,z) = \frac{1}{2} + \frac{d(y,z)}{2}$ and, hence, $\frac{1}{2} + \frac{d(x,y)}{2} + \frac{d(y,z)}{2} \leq d^2_{\precsim}(x,y) + d^2_{\precsim}(y,z)$. Since $d(x,y) \leq d(x,y) + d(y,z)$, we deduce that $d^2_{\precsim}(x,z) \leq \frac{1}{2} + \frac{d(x,y)}{2} + \frac{d(y,z)}{2}$ and, hence, $d^2_{\precsim}(x,z) \leq d^2_{\precsim}(x,y) + d^2_{\precsim}(y,z)$.

The same arguments to those given in the proof of Proposition 8 apply in order to show that $d^2_{\precsim}(x,x) = 0 \Leftrightarrow d(x,x) = 0$ for all $x \in Y$ and that $d^2_{\precsim}(x,y) = 0 \Leftrightarrow x = y$ whenever d is a metric on Y and, in addition, that $\tau_{d^2_{\precsim}}$ is finer than $\tau^{\precsim}_{A}$. Therefore, $\precsim$ is lower $\tau_{d^2_{\precsim}}$- continuous. $\square$

In the same way that $d^1_{\precsim}$, when intergenerational distributions are under consideration, the quasi-pseudo-metric $d^2_{\precsim}$ is able to quantify, by means of a metric, the increase in welfare (when $x \precsim y$). Moreover, it differentiates this case from the rest of the cases, the retrogress ($y \prec x$) and the incomparability ($x \bowtie y$). However, this time, it assigns a lower value for the former case.

Notice that, among the possible metrics, those belonging to the Banerjee and Mitra class $\triangle$ can be considered in the statement of Propositions 8 and 9.

It must be stressed that modifications of the preceding technique can be obtained, proceeding as follows:

$$d^2_{\precsim}(x,y) = \begin{cases} \frac{k \cdot d(x,y)}{n}, & x \precsim y \\ \frac{k}{n} + \frac{(n-k) \cdot d(x,y)}{n}, & otherwise \end{cases}.$$

for some $n \in \mathbb{R}_+$ and $k \in [0, n]$.

The next result introduces a technique which is related to the methods shown in [15,16,42].

Proposition 10. *Let $\precsim$ be a preorder on Y. If $u \colon (X, \leq) \to (0,1)$ is a weak utility for $\precsim$, then the function $d^3_{\precsim} \colon X \times X \to \mathbb{R}^+$ defined by*

$$d^3_{\precsim}(x,y) = \begin{cases} 0, & x \precsim y \\ 1 + |u(x) - u(y)|, & y \prec x \\ 1, & otherwise \end{cases}.$$

is a quasi-pseudo-metric such that $\tau_{d^3_{\precsim}} = \tau^{\precsim}_{A}$. Therefore, $\precsim$ is lower $\tau_{d^3_{\precsim}}$- continuous.

Proof. It is trivial that $d^3_{\precsim}(x,x) = 0$, for any $x \in Y$. Let us see that the triangular inequality is satisfied, i.e., that

$$d^3_{\precsim}(x,z) \leq d^3_{\precsim}(x,y) + d^3_{\precsim}(y,z)$$

for any $x, y, z \in X$. For this proposition, we set $d(x,y) = |u(x) - u(y)|$ for all $x, y \in Y$ and distinguish the following possible cases.

Case 1. $x \precsim z$. Then the inequality is trivially satisfied.

Case 2. $x \bowtie z$. Then $d^3_{\precsim}(x,z) = 1$. Notice that the case $x \precsim y \precsim z$ is impossible. Then the following cases may hold:

(i) If $x \bowtie y$ or $y \bowtie z$, then the inequality is satisfied because we have either $d^3_{\precsim}(x,y) = 1$ or $d^3_{\precsim}(y,z) = 1$.

(ii) If $x \precsim y$, then we have that $\neg(y \precsim z)$. In fact, we have that $z \prec y$; otherwise, we would be either in case (i) or in the impossible case $x \precsim y \precsim z$. Therefore, we obtain that $1 \leq 1 + d(y,z)$ and, thus, the inequality is satisfied.

(iii) If $\neg(x \precsim y)$, then we have that $y \prec x$; otherwise, we would be in case (i) above. Hence, we have that either $y \precsim z$ or $z \prec y$. Observe that $y \bowtie z$ matches up with the case (i). Thus, if $y \precsim z$, then we obtain $d^3_{\precsim}(x,y) = 1 + d(x,y)$, $d^3_{\precsim}(y,z) = 0$ and, therefore, the inequality holds because $1 \leq 1 + d(x,y)$. Finally, if $z \prec y$, then we obtain $z \prec y \prec x$, which contradicts the hypothesis $x \bowtie z$.

Case 3. $z \prec x$. Then $d^3_{\precsim}(x,z) = 1 + d(x,z)$ and the following cases may hold:

(i) If $x \bowtie y$ as well as $y \bowtie z$, then $d^3_{\precsim}(x,y) = d^3_{\precsim}(y,z) = 1$ and, thus, the inequality is satisfied because $1 + d(x,y) \leq 2$.

(ii) If $x \bowtie y$ or $y \bowtie z$, then we have the following cases:

- (ii_1) If $z \bowtie y$, then $y \prec x$. In this case, the inequality is satisfied because $d^3_{\precsim}(x,y) = 1 + d(x,y)$, $d^3_{\precsim}(y,z) = 1$ and, thus, $1 + d(x,z) \leq 2 + d(x,y)$.
- (ii_2) If $x \bowtie y$, then $z \prec y$. In this case, the inequality is satisfied too since $d^3_{\precsim}(x,y) = 1$, $d^3_{\precsim}(y,z) = 1 + d(y,z)$ and, thus, $1 + d(x,z) \leq 2 + d(y,z)$.

(iii) If neither $x \bowtie y$ nor $y \bowtie z$ hold, then we have the following cases:

- (iii_1) If $z \prec y \prec x$, then $1 + d(x,z) = d^3_{\precsim}(x,z) \leq 1 + d(x,y) + d(y,z) \leq d^3_{\precsim}(x,y) + d^3_{\precsim}(y,z)$.
- (iii_2) If $y \precsim z \prec x$, then $d^3_{\precsim}(x,z) = 1 + d(x,z) \leq 1 + d(x,y) = d^3_{\precsim}(x,y) + d^3_{\precsim}(y,z)$ with $d^3_{\precsim}(y,z) = 0$.
- (iii_3) If $z \prec x \precsim y$, then $d^3_{\precsim}(x,z) = 1 + d(x,z) \leq 1 + d(y,z) = d^3_{\precsim}(x,y) + d^3_{\precsim}(y,z)$ with $d^3_{\precsim}(x,y) = 0$.

Therefore, taking into account all of the above studied cases, we conclude that $d^3_{\precsim}$ satisfies the triangular inequality and, hence, it is actually a quasi-pseudo-metric.

Finally, it remains to prove that $\tau_{d^3_{\precsim}} \subseteq \tau^{\precsim}_A$. The fact that $\tau^{\precsim}_A \subseteq \tau_{d^3_{\precsim}}$ can be deduced following the same arguments applied to the proof of Proposition 8. Next we show that $\tau_{d^3_{\precsim}} \subseteq \tau^{\precsim}_A$. Thus, consider $A \in \tau_{d^3_{\precsim}}$. Then, for each $x \in A$, there exists $0 < \varepsilon < 1$ such that $B_{d^3_{\precsim}}(x,\varepsilon) \subseteq A$. Clearly, $U^{\precsim}(x) \subseteq B_{d^3_{\precsim}}(x,\varepsilon) \subseteq A$. So $A \in \tau^{\precsim}_A$. Thus, we conclude that $\tau_{d^3_{\precsim}} \subseteq \tau^{\precsim}_A$. $\square$

Similar to $d^1_{\precsim}$ and $d^2_{\precsim}$, the quasi-pseudo-metric $d^3_{\precsim}$ quantifies, by means of a metric, the increase of welfare $x \precsim y$ when intergenerational distributions are under consideration. Moreover, it differentiates this case from the rest of the cases, the retrogress ($y \prec x$) and the incomparability ($x \bowtie y$). However, now it assigns a greater and constant value 1 when we want to measure the distance between incomparable elements and even a bigger value in case of regression.

The quasi-pseudo-metric $d^2_{\precsim}$ introduced in Proposition 9 can be modified in such a way that its quantifications can be understood in the spirit of the quasi-pseudo-metric $d^3_{\precsim}$ of Proposition 10 such as the next result shows.

Proposition 11. *Let $\precsim$ be a preorder on Y. If $u\colon (X, \leq) \to (0,1)$ is a weak-utility for $\precsim$, then the function $d^4_{\precsim}\colon X \times X \to \mathbb{R}^+$ defined by*

$$d^4_{\precsim}(x,y) = \begin{cases} \frac{u(y)-u(x)}{2}, & x \precsim y, \\ \frac{1}{2} + \frac{u(x)-u(y)}{2}, & y \prec x, \\ \frac{1}{2}, & otherwise, \end{cases}$$

is a quasi-pseudo-metric such that $\tau_{d^4_{\precsim}}$ is finer than $\tau^{\precsim}_{A}$. Therefore, $\precsim$ is lower $\tau_{d^4_{\precsim}}$- continuous.

Proof. The proof is similar to the proof of Proposition 10. □

Finally, we obtain the following interesting sequence.

Corollary 3. *Any ethical social welfare preorder satisfying anonymity and strong monotonicity is lower $\tau_{d^i_{\precsim_m}}$-continuous with $i = 1,2,3,4$.*

Proof. By Propositions 8–10, $d^i_{\precsim_m}$ is a quasi-metric whose topology $\tau_{d^i_{\precsim_m}}$ is finer than or equal to $\tau^{\precsim_m}_{A}$ for all $i = 1,2,3,4$.

By Proposition 6, we have that every ethical social welfare preorder $\preceq$ satisfying anonymity and strong monotonicity is an extension of $\precsim_m$. Thus, by Lemma 1, we obtain that $\tau^{\precsim}_{A} \subseteq \tau^{\precsim_m}_{A} \subseteq \tau_{d^i_{\precsim_m}}$ for all $i = 1,2,3,4$. This concludes the proof. □

6. Conclusions

Summarizing, in the present paper, we studied the compatibility between preorders and topologies. We provided a topology that guarantees the continuity of any preference satisfying anonymity and strong monotonicity. This topology is defined by means of the grading principle and the corresponding Alexandroff topology.

We showed that our approach presents two advantages with respect to the approach given by Banerjee and Mitra. On the one hand, the new result allows us to decide the continuity of the preference even if the topology under consideration is not metrizable. On the other hand, Banerjee and Mitra only provide a necessary condition. In this direction, we provided an example of social intergenerational preference that enjoys anonymity and strong Pareto requirements; in addition, it fulfills the simplex condition of Banerjee and Mitra, but it is not continuous.

As a matter of the above exposed facts and the fact that the upper topology is not metrizable, we suggest quasi-pseudo-metrics as an appropriate quantitative tool for reconciling topology and social intergenerational preferences. Concretely, we showed that such a generalized metric notion is able to encode the order relation that induces the intergenerational preference. Thus, it provides numerical quantifications about the increase in welfare and the arrow of such an increase. Note that a metric would be able to yield information on the increase but it will not give the aforementioned arrow.

Based on the fact that every preorder, and thus every social intergenerational preference, can be encoded by means of a quasi-pseudo-metric, we developed a method to induce a quasi-pseudo-metric that makes always the preference continuous with respect to its induced topology, the Alexandroff topology generated by the preorder, which is finer than the upper topology. Such a method is able to guarantee the possibility counterparts of the celebrated impossibility theorems of Diamond, Svensson and Sakai and, in addition, it is able to give numerical quantifications of the improvement of welfare.

Finally, a refinement of the previous method is also presented in such a way that metrics are involved.

Author Contributions: A.E. contributed to formulation of research goals; methodology and verification of results; writing the initial dratf; critical review of the initial draft; acquisition of the financial support for the project leading to this publication, R.M. contributed to formulation of research goals; methodology and verification of results; Ó.V. contributed to formulation of research goals; methodology and verification of results; writing the initial draft; critical review of the initial draft; acquisition of the financial support for the project leading to this publication. All authors have read and agreed to the published version of the manuscript.

Funding: Asier Estevan acknowledges financial support from the Ministry of Science and Innovation of Spain under grant PID2021-127799NB-I00 as well as from the UPNA under grant JIUPNA19-2022. Oscar Valero acknowledges financial support from Proyecto PGC2018-095709-B-C21 financiado por MCIN/AEI/10.13039/501100011033 y FEDER "Una manera de hacer Europa" and from project BUGWRIGHT2. This last project has received funding from the European Union's Horizon 2020 research and innovation program under grant agreement No. 871260. This publication reflects only the authors views, and the European Union is not liable for any use that may be made of the information contained therein.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Sidgwick, S. *The Methods of Ethics*; Macmillan: London, UK, 1999.
2. Koopmans, T. Stationary ordinal utility and impatience. *Econometrica* **1960**, *28*, 287–309. [CrossRef]
3. Diamond, P. The evaluation on infinite utility streams. *Econometrica* **1965**, *33*, 170–177. [CrossRef]
4. Svensson, L.G. Equity Among Generations. *Econometrica* **1980**, *48*, 1251–1256. [CrossRef]
5. Banerjee, K.; Mitra, T. On the continuity of ethical social welfare orders on infinite utility streams. *Soc. Choice Welf.* **2008**, *30*, 1–12. [CrossRef]
6. Fleurbaey, M.; Michel, P. Intertemporal equity and the extension of the Ramsey criterion. *J. Math. Econ.* **2003**, *39*, 777–802. [CrossRef]
7. Sakai, T. An axiomatic approach to intergenerational equity. *Soc. Choice Welf.* **2003**, *20*, 167–176. [CrossRef]
8. Sakai, T. Intergenerational preferences and sensitivity to the present. *Econ. Bull.* **2003**, *4*, 1–5.
9. Banerjee, K.; Dubey, R.S. Do all constructive strongly monotone inter-temporal orders exhibit impatience? *J. Math. Econ.* **2014**, *52*, 66–69.
10. Sakai, T. Intergenerational equity and an explicit construction of welfare criteria. *Soc. Choice Welf.* **2010**, *35*, 393–414. [CrossRef]
11. Sakai, T. Limit representations of intergenerational equity. *Social Choice Welfare* **2016**, *47*, 481–500. [CrossRef]
12. Faugeras, O.P.; Rüschendorff, L. Risk excess measures induced by hemi-metrics. *Probab. Uncertain. Quant. Risk* **2018**, *3*, 6. [CrossRef]
13. Rachev, S.T.; Soyanov, S.V.; Fabozzi, F.J. *A Probability Metrics Approach to Financial Risk Measures*; Wiley-Blackwell: Oxford, UK, 2011. [CrossRef]
14. Soyanov, S.V.; Rachev, S.T.; Fabozzi, F.J. Metrization of stochastic dominance rules. *Int. J. Theor. Appl. Financ.* **2012**, *15*, 1250017-1.
15. Levin, V.L. Lipschitz pre-orders and Lipschitz utility functions. *Russ. Math. Surv.* **1984**, *39*, 217–218. [CrossRef]
16. Levin, V.L. Reduced cost functions and their applications. *J. Math. Econ.* **1997**, *28*, 155–186. [CrossRef]
17. Levin, V.L. Smooth feasible solutions to a dual Monge-Kantorovich problem and their application to the best approximation and mathematical economics problems. *Dokl. Math.* **2008**, *77*, 281–283. [CrossRef]
18. Levin, V.L. General preferences and utility functions. An approach based on the dual Kantarovich problem. *Dokl. Math.* **2011**, *83*, 236–237. [CrossRef]
19. Goubault-Larrecq, J. *Non-Hausdorff Topology and Domain Theory*; Cambridge University Press: New York, NY, USA, 2013. [CrossRef]
20. Mas-Colell, A.; Whinston, M.D.; Green, J.R. *Microeconomic Theory*; Oxford University Press: Oxford, UK, 1995.
21. Varian, H.R. *Microeconomics Analysis*; W.W. Norton & Company: London, UK, 1992.
22. Evren, O.; Ok, E.A. On the multi-utility representation of preference relations. *J. Math. Econ.* **2011**, *47*, 554–563.

23. Bridges, D.S.; Mehta, G.B. *Representations of Preference Orderings*; Springer: Berlin, Germany, 1995. [CrossRef]
24. Ok, E.A. *Real Analysis with Economics Applications*; Princeton University Press: Princeton, NJ, USA, 2007.
25. Metha, G.B. Preference and utility. In *Handbook of Utility Theory*; Barbetà, S., Ed.; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1998; Volume 1, pp. 1–50.
26. Peleg, B. Utility functions for partially ordered topological spaces. *Econometrica* **1970**, *38*, 93–96.
27. Richter, M. Revealed preference theory. *Econometrica* **1966**, *34*, 635–645. [CrossRef]
28. Epstein, L.G. Intergenerational preference orderings. *Soc. Choice Welf.* **1986**, *3*, 151–160. [CrossRef]
29. Campbell, D.E. Impossibility theorems and infinite horizon planning. *Soc. Choice Welf.* **1985**, *2*, 283–293. [CrossRef]
30. Sakai, T. Equitable intergenerational preferences on restricted domains. *Soc. Choice Welf.* **2006**, *27*, 41–54. [CrossRef]
31. Basu, K.; Mitra, T. Aggregating infinite utility streams with intergenerational equity: The impossibility of being paretian. *Econometrica* **2003**, *71*, 1557–1563. [CrossRef]
32. Lauwers, L. Continuity and equity with infinite horizons. *Soc. Choice Welf.* **1997**, *14*, 345–356. [CrossRef]
33. Bosi, G.; Estevan, A.; Raventós-Pujol, A. Topologies for semicontinuous Richter-Peleg multi-utilities. *Theory Decis.* **2020**, *88*, 457–470. [CrossRef]
34. Dugundji, J. *Topology*; Allyn and Bacon: Boston, MA, USA, 1966. [CrossRef]
35. Chichilnisky, G. Economic development and efficiency criteria in the satisfaction of basic needs. *Appl. Math. Model.* **1977**, *1*, 290–297. [CrossRef]
36. Chichilnisky, G. An axiomatic approach to sustainable development. *Soc. Choice Welf.* **1996**, *13*, 231–257.
37. Herden, G. On the existence of utility functions II. *Math. Soc. Sci.* **1989**, *18*, 107–117. [CrossRef]
38. Mashburn, J.D. A note on reordering ordered topological spaces and the existence of continuous, strictly increasing functions. *Topol. Proc.* **1995**, *20*, 207–250. [CrossRef]
39. Asheim, G.B.; Buchholf, W. Justifying Sustainability. *J. Environ. Econ. Manag.* **2001**, *41*, 252–268. [CrossRef]
40. Künzi, H.P.A. Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology. In *Handbook of the History of General Topology*; Aull, C.E., Lowen R., Eds.; Kluwer: Dordrecht, The Netherlands, 2001; Volume 3, pp. 853–968.
41. Bonsangue, M.M.; van Breugel, F.; Rutten, J.J.M.M. Alexandroff and Scott topologies for generalized ultrametric spaces. *Ann. N. Y. Acad. Sci.* **1996**, *806*, 49–68. [CrossRef]
42. Levin, V.L. Some applications of set-valued mappings in mathematical economics. *J. Math. Econ.* **1991**, *20*, 69–87.