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On the General Sum Distance Spectra of Digraphs

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Abstract: Let G be a strongly connected digraph, and $d_G(v_i, v_j)$ denote the distance from the vertex v_i to vertex v_j and be defined as the length of the shortest directed path from v_i to v_j in G. The sum distance between vertices v_i and v_j in G is defined as $sd_G(v_i, v_j) = d_G(v_i, v_j) + d_G(v_j, v_i)$. The sum distance matrix of G is the $n \times n$ matrix $SD(G) = (sd_G(v_i, v_j))_{v_i, v_j \in V(G)}$. For vertex $v_i \in V(G)$, the sum transmission of v_i in G, denoted by $ST_G(v_i)$ or ST_i, is the row sum of the sum distance matrix $SD(G)$ corresponding to vertex v_i. Let $ST(G) = \text{diag}(ST_1, ST_2, \ldots, ST_n)$ be the diagonal matrix with the vertex sum transmissions of G in the diagonal and zeroes elsewhere. For any real number $0 \le \alpha \le 1$, the general sum distance matrix of G is defined as$SD_\alpha(G) = \alpha ST(G) + (1-\alpha)SD(G)$. The eigenvalues of $SD_\alpha(G)$ are called the general sum distance eigenvalues of G, the spectral radius of $SD_\alpha(G)$, i.e., the largest eigenvalue of $SD_\alpha(G)$, is called the general sum distance spectral radius of G, denoted by $\mu^\alpha(G)$. In this paper, we first give some spectral properties of $SD_\alpha(G)$. We also characterize the digraph minimizes the general sum distance spectral radius among all strongly connected r-partite digraphs. Moreover, for digraphs that are not sum transmission regular, we give a lower bound on the difference between the maximum vertex sum transmission and the general sum distance spectral radius.

Keywords: strongly connected digraph; general sum distance matrix; spectral radius

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1. Introduction

Let $G = (V(G), E(G))$ be a digraph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. If there is an arc from the vertex v_i to vertex v_j, then v_i is said to be adjacent to v_j, and we denote this arc by writing (v_i, v_j). For the arc (v_i, v_j), the first vertex v_i is its tail, and the second vertex v_j is its head. For any vertex $v_i \in V(G)$, the outdegree of v_i is the number of arcs of which v_i is the tail. A digraph G is simple if it has no loops and multiple arcs. A digraph G is strongly connected if for every pair of vertices $v_i, v_j \in V(G)$, there is a directed path from v_i to v_j and one from v_j to v_i. Throughout this paper, we consider finite, simple strongly connected digraphs.

A digraph is a r-partite digraph if its vertices can be partitioned into r arcless sets. Let $T_{n,r}$ denote the complete r-partite digraph of order n, whose partition sets are of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. Let $\overleftrightarrow{K_n}$ denote the complete digraph of order n in which for two arbitrary distinct vertices $v_i, v_j \in V(\overleftrightarrow{K_n})$, there are arcs (v_i, v_j) and (v_j, v_i) in $E(\overleftrightarrow{K_n})$.

For a strongly connected digraph G, the distance from the vertex v_i to vertex v_j, denoted by $d_G(v_i, v_j)$, is defined as the length of the shortest directed path from v_i to v_j in G. The diameter of the strongly connected digraph G, denoted by $diam(G)$, is the maximum $d_G(v_i, v_j)$ over all ordered pairs of vertices v_i, v_j.

Let $D(G) = (d_G(v_i, v_j))_{n \times n}$ be the distance matrix of G, where $d_G(v_i, v_j)$ is the distance from v_i to v_j. For vertex $v_i \in V(G)$, the transmission of v_i in G, denoted by $Tr_G(v_i)$ or Tr_i, is defined as the sum of distances from v_i to all other vertices in G, that is, $Tr_G(v_i) = Tr_i =$

$\sum_{j=1}^{n} d_{ij}$. Let $Tr(G) = \mathrm{diag}(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix with vertex transmissions of G in the diagonal and zeroes elsewhere. Then, the distance signless Laplacian matrix of G is the matrix $D^Q(G) = Tr(G) + D(G)$. For any real number $0 \leq \alpha \leq 1$, the general distance matrix of G is defined as the matrix $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$.

Spectral graph theory is a fast-growing branch of algebraic graph theory. One of the central issues in spectral graph theory is as follows: For a graph matrix, determine the maximization or minimization of spectral invariants over various families of graphs. Recently, the spectral radius of the distance matrix, the related distance signless Laplacian matrix, and the general distance matrix of digraphs have received increasing attention, see [1–6]. In particular, Lin et al. [6] characterized the extremal digraphs with minimum distance spectral radius among all digraphs with given vertex connectivity. Xi and Wang [4] determined the strongly connected digraphs minimizing distance spectral radius among all strongly connected digraphs with a given diameter d, for $d = 1, 2, 3, 4, 5, 6, 7, n-1$. Li et al. [2] characterized the digraph minimizing the distance signless Laplacian spectral radius among all strongly connected digraphs with given vertex connectivity. Xi et al. [7] characterized the extremal digraph achieving the minimum distance signless Laplacian spectral radius among all strongly connected digraphs with given arc connectivity. Xi et al. [8] proposed to study the generalized distance spectral radius of strongly connected digraphs, and they determined the digraphs which attain the minimum D_α spectral radius among all strongly connected digraphs with given parameters such as dichromatic number, vertex connectivity, or arc connectivity.

The matrices related to simple, undirected graphs are symmetric matrices, which have well-defined spectral properties. However, unless the digraph G is symmetric, it is not the case that $d_G(v_i, v_j) = d_G(v_j, v_i)$ for all vertices v_i, v_j of G; that is, the symmetric property does not hold for directed distance. In order to produce a symmetric matrix from a digraph, Chartrand and Tian [9] proposed one metric on strongly connected digraphs: sum distance defined as $sd_G(v_i, v_j) = d_G(v_i, v_j) + d_G(v_j, v_i)$; then, one can obtain a symmetric matrix on a strongly connected digraph, which is called the sum distance matrix.

Let G be a strongly connected digraph of order n, $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set, and $E(G)$ be the arc set. The sum distance matrix of G is the $n \times n$ matrix $SD(G) = (sd_G(v_i, v_j))_{v_i, v_j \in V(G)}$, where $sd_G(v_i, v_j) = d_G(v_i, v_j) + d_G(v_j, v_i)$. For vertex $v_i \in V(G)$, the sum transmission of v_i in G, denoted by $ST_G(v_i)$ or ST_i, is the row sum of the sum distance matrix $SD(G)$ corresponding to vertex v_i, which is $ST_G(v_i) = ST_i = \sum_{j=1}^{n} sd_G(v_i, v_j)$. A strongly connected digraph G is r-sum transmission regular if $ST_G(v_i) = r$ for each $v_i \in V(G)$; otherwise, G is not sum transmission regular. Let $ST(G) = \mathrm{diag}(ST_1, ST_2, \ldots, ST_n)$ be the diagonal matrix with vertex sum transmissions of G in the diagonal and zeroes elsewhere. The matrix $SL(G) = SD(G) - ST(G)$ is called the sum distance Laplacian matrix of G, and the matrix $SQ(G) = SD(G) + ST(G)$ is called the sum distance signless Laplacian matrix of G. Note that all the matrices $SD(G)$, $SL(G)$ and $SQ(G)$ are symmetric. Therefore, they have real eigenvalues. The eigenvalues of $SD(G)$ are called the sum distance eigenvalues of G, denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The eigenvalues of $SL(G)$ are called the sum distance Laplacian eigenvalues of G, denoted by $\nu_1(G) \geq \nu_2(G) \geq \cdots \geq \nu_n(G)$. The eigenvalues of $SQ(G)$ are called the sum distance signless Laplacian eigenvalues of G, denoted by $\gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_n(G)$.

For any real number $0 \leq \alpha \leq 1$, similar to [8,10], Xu and Zhou [11] proposed to study the general sum distance matrix of G:

$$SD_\alpha(G) = \alpha ST(G) + (1-\alpha)SD(G).$$

Obviously, $SD(G) = SD_0(G)$, $ST(G) = SD_1(G)$, $SQ(G) = 2SD_{\frac{1}{2}}(G)$, and $SD_\alpha(G) - SD_\beta(G) = (\alpha - \beta)SL(G)$. The matrix $SD_\alpha(G)$ enables a unified study of $SD(G)$ and $SQ(G)$. The eigenvalues of $SD_\alpha(G)$ are called the general sum distance eigenvalues of G, denoted by $\mu_1^\alpha(G) \geq \mu_2^\alpha(G) \geq \cdots \geq \mu_n^\alpha(G)$. The spectral radius of $SD_\alpha(G)$, i.e., the largest eigenvalue of $SD_\alpha(G)$, is called the general sum distance spectral radius of G, denoted by $\mu^\alpha(G)$. For a strongly connected digraph G, $SD_\alpha(G)$ is a non-negative irreducible matrix. Based on the Perron Frobenius Theorem [12], $\mu^\alpha(G)$ is a simple eigenvalue of $SD_\alpha(G)$, and there is a positive unit eigenvector corresponding to $\mu^\alpha(G)$. The positive unit eigenvector corresponding to $\mu^\alpha(G)$ is called the Perron vector of $SD_\alpha(G)$.

In this paper, we first give some spectral properties of $SD_\alpha(G)$. We also characterize the digraph minimizes the general sum distance spectral radius among all strongly connected r-partite digraphs. Moreover, for digraphs that are not sum transmission regular, we give a lower bound on the difference between the maximum vertex sum transmission and the general sum distance spectral radius.

2. Preliminaries and Basic Properties of $SD_\alpha(G)$

For $i = 1, 2, \ldots, n$, $\rho_i(A)$ denotes the i-th largest eigenvalue of Hermitian matrix A. In the following, we give Weyl's inequalities [12] for eigenvalues of Hermitian matrices, and the equality case was first established by So in [13].

Lemma 1 ([12] Theorem WS)**.** *Let A and B be two Hermitian matrices of order n, and also let $1 \leq i, j \leq n$. Then,*
(1) $\rho_i(A) + \rho_j(B) \leq \rho_{i+j-n}(A+B)$, if $i + j \geq n + 1$;
(2) $\rho_i(A) + \rho_j(B) \geq \rho_{i+j-1}(A+B)$, if $i + j \leq n + 1$.
Moreover, either of the equality holds if and only if there exists a unit vector that is an eigenvector to each of the three eigenvalues involved.

Lemma 2 ([14] Interlacing Theorem)**.** *Let A be a real symmetric matrix of order n, and B be a principal submatrix of A with order s. Then,*

$$\rho_{i+n-s}(A) \leq \rho_i(B) \leq \rho_i(A) \text{ for } 1 \leq i \leq s.$$

Let M be a real matrix of order n described in the following block form:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{pmatrix},$$

where the diagonal blocks M_{ii} are $n_i \times n_i$ matrices for any $i \in \{1, 2, \ldots, t\}$ and $n = n_1 + \cdots + n_t$. For any $i, j \in \{1, 2, \ldots, t\}$, let b_{ij} denote the average row sum of M_{ij}, i.e., b_{ij} is the sum of all entries in M_{ij} divided by the number of rows. Then, $B(M) = (b_{ij})$ is called the quotient matrix of M. In addition, if for each pair i, j, M_{ij} has a constant row sum, then $B(M)$ is called the equitable quotient matrix of M.

Lemma 3 ([15])**.** *Let $M = (m_{ij})_{n \times n}$ be defined as above, $B = (b_{ij})$ be the equitable quotient matrix of M. Then, the spectrum of B is contained in the spectrum of A. Moreover, if M is a non-negative matrix, then $\rho_1(B) = \rho_1(M)$, where $\rho_1(B)$ is the spectral radius of B .*

Proposition 1. *Let $0 \leq \alpha < \beta \leq 1$, G be a strongly connected digraph of order n with $SD_\alpha(G) = SD_\alpha$ and $SD_\beta(G) = SD_\beta$. Then, for any $1 \leq i \leq n$,*

$$\rho_i(SD_\beta) \geq \rho_i(SD_\alpha).$$

Proof. Notice that $\rho_n(SL(G)) = 0$, and $SD_\beta - SD_\alpha = (\beta - \alpha)SL(G)$. Set $A = SD_\alpha$ and $B = (\beta - \alpha)SL(G)$, applying the (1) of Lemma 1, one obtains

$$\begin{aligned}\rho_i(SD_\beta) = \rho_{i+n-n}(SD_\beta) &\geq \rho_i(SD_\alpha) + (\beta - \alpha)\rho_n(SL(G)) \\ &= \rho_i(SD_\alpha).\end{aligned}$$

Hence, the inequality holds. □

Proposition 2. *Let G be a strongly connected digraph of order $n \geq 3$ with $(v_i, v_j) \notin E(G)$, where $v_i, v_j \in V(G)$. Then, for any $1 > \alpha \geq \frac{1}{2}$,*

$$\mu_k^\alpha(G) \geq \mu_k^\alpha(G + (v_i, v_j)), \text{ for } 1 \leq k \leq n.$$

Proof. Taking $M = SD_\alpha(G) - SD_\alpha(G + (v_i, v_j))$. Then, the diagonal (v_k, v_k)-entry of M for $v_k \in V(G)$ is $\alpha ST_G(v_k) - \alpha ST_{G+(v_i,v_j)}(v_k) \geq 0$, and the non-diagonal (v_k, v_s) entry of M is $(1 - \alpha)(sd_G(v_k, v_s) - sd_{G+(v_i,v_j)}(v_k, v_s)) \geq 0$. Moreover, for $1 > \alpha \geq \frac{1}{2}$, it is easy to know that M is a diagonally dominant matrix. Therefore, we have M as a diagonally dominant with non-negative diagonal entries when $1 > \alpha \geq \frac{1}{2}$, so it is a positive semi-definite matrix, which implies that the least eigenvalue is at least zero. Hence, for $1 \leq k \leq n$, by Lemma 1, we obtain

$$\mu_k^\alpha(G) \geq \mu_k^\alpha(G + (v_i, v_j)) + \rho_n(M) \geq \mu_k^\alpha(G + (v_i, v_j)).$$

Thus, we achieve the desired result. □

For the complete digraph $\overleftrightarrow{K_n}$, we have $SD_\alpha(\overleftrightarrow{K_n}) = \alpha(2n - 2)I_n + 2(1 - \alpha)(J_n - I_n) = (2\alpha n - 2)I_n + 2(1 - \alpha)J_n$, where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ matrix in which every entry is 1. Thus, $\sigma(SD_\alpha(\overleftrightarrow{K_n})) = \{2n - 2, (2\alpha n - 2)^{[n-1]}\}$, where $\sigma(SD_\alpha(\overleftrightarrow{K_n}))$ denotes the spectrum of the matrix $SD_\alpha(\overleftrightarrow{K_n})$, and $(2\alpha n - 2)^{[n-1]}$ denotes $2\alpha n - 2$ is an eigenvalue of multiplicity $n - 1$.

Proposition 3. *Let G be a strongly connected digraph of order $n \geq 3$. Then, for any $1 > \alpha \geq \frac{1}{2}$,*

$$\mu_2^\alpha(G) \geq 2\alpha n - 2,$$

with equality if and only if $G \cong \overleftrightarrow{K_n}$.

Proof. By Proposition 2, we obtain $\mu_2^\alpha(G) \geq \mu_2^\alpha(\overleftrightarrow{K_n}) = 2\alpha n - 2$. Suppose that $\mu_2^\alpha(G) = 2\alpha n - 2$ and $G \ncong \overleftrightarrow{K_n}$. Then G is (isomorphic to) a subdigraph of $G' = \overleftrightarrow{K_n} - e$ for some $e \in E(\overleftrightarrow{K_n})$. Without loss of generality, we assume that $e = (v_1, v_2)$. Let $V_1 = \{v_1, v_2\}$, $V_2 = V(G) \backslash V_1$. Then the equitable quotient matrix of $SD_\alpha(G')$ corresponding to the partition $V(G') = V_1 \cup V_2$ is

$$B = \begin{pmatrix} 2\alpha n - 4\alpha + 3 & 2(1 - \alpha)(n - 2) \\ 4(1 - \alpha) & 2n + 4\alpha - 6 \end{pmatrix}.$$

According to Lemma 3, $\rho_i(B)$ is an eigenvalue of $SD_\alpha(G')$ for $i = 1, 2$. Let $g(x) = det(xI_2 - B)$. Then,

$$\begin{aligned}g(x) &= x^2 - (2n + 2\alpha n - 3)x + 4\alpha n^2 - 4\alpha n - 2n + 4\alpha - 2 \\ &= (x - (2n - 1))(x - (2\alpha n - 2)) + 2(1 - \alpha)(n - 2).\end{aligned}$$

Hence, $g(2n-1) > 0$ and $g(2\alpha n - 2) > 0$. Since $g(x)$ is strictly decreasing when $x \leq n + \alpha n - \frac{3}{2}$ and strictly increasing when $x \geq n + \alpha n - \frac{3}{2}$, $n + \alpha n - \frac{3}{2} < \rho_1(B) < 2n - 1$ and $2\alpha n - 2 < \rho_2(B) < n + \alpha n - \frac{3}{2}$. Furthermore, $\mu_2^{\alpha}(G') \geq \rho_2(B) > 2\alpha n - 2$. Based on Proposition 2, $\mu_2^{\alpha}(G) \geq \mu_2^{\alpha}(G') \geq \rho_2(B) > 2\alpha n - 2$, a contradiction. Thus, for any $1 > \alpha \geq \frac{1}{2}$,

$$\mu_2^{\alpha}(G) \geq 2\alpha n - 2,$$

with equality if and only if $G \cong \overleftrightarrow{K_n}$. $\square$

For a strongly connected digraph G of order n, $SD_{\alpha}(G) = SD_{\alpha}$ is the general sum distance matrix of G, $ST_G(v_i) = ST_i = \sum\limits_{j=1}^{n} sd_G(v_i, v_j)$ is the sum transmission of v_i in G. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be a real column vector; then,

$$\mathbf{X}^T SD_{\alpha} \mathbf{X} = \alpha \sum_{v_i \in V(G)} ST_i x_i^2 + 2(1-\alpha) \sum_{\{v_i, v_j\} \in V(G)} sd_G(v_i, v_j) x_i x_j. \tag{1}$$

$$\mathbf{X}^T SD_{\alpha} \mathbf{X} = \sum_{\{v_i, v_j\} \in V(G)} sd_G(v_i, v_j) \Big(\alpha (x_i^2 + x_j^2) + 2(1-\alpha) x_i x_j \Big). \tag{2}$$

$$\mathbf{X}^T SD_{\alpha} \mathbf{X} = (2\alpha - 1) \sum_{v_i \in V(G)} ST_i x_i^2 + (1-\alpha) \sum_{\{v_i, v_j\} \in V(G)} sd_G(v_i, v_j) (x_i + x_j)^2. \tag{3}$$

3. The General Sum Distance Spectral Radius of Strongly Connected Digraphs

In this section, we study the general sum distance spectral radius of strongly connected digraphs.

Lemma 4 ([12]). *Let $P = (p_{ij})$ be an $n \times n$ non-negative matrix with spectral radius $\rho(P)$, and let $R_i(P)$ be the i-th row sum of P, i.e., $R_i(P) = \sum\limits_{j=1}^{n} p_{ij}$ $(1 \leq i \leq n)$. Then,*

$$\min\{R_i(P) : 1 \leq i \leq n\} \leq \rho(P) \leq \max\{R_i(P) : 1 \leq i \leq n\}.$$

Moreover, if P is irreducible, then any equality holds if and only if $R_1(P) = R_2(P) = \cdots = R_n(P)$.

From Lemma 4, we have the following theorem:

Theorem 1. *Let G be a strongly connected digraph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then,*

$$\min_{v_i \in V(G)} \left\{ \alpha ST_i + (1-\alpha) \frac{T_i}{ST_i} \right\} \leq \mu^{\alpha}(G) \leq \max_{v_i \in V(G)} \left\{ \alpha ST_i + (1-\alpha) \frac{T_i}{ST_i} \right\},$$

where $T_i = \sum\limits_{j=1}^{n} sd_G(v_i, v_j) ST_j$. Moreover, if $\frac{1}{2} < \alpha < 1$, then either one equality holds if and only if G is sum transmission regular.

Proof. Since $ST(G) = \mathrm{diag}(ST_1, ST_2, \ldots, ST_n)$ is the diagonal matrix with vertex sum transmissions of G in the diagonal and zeroes elsewhere, with a simple calculation, we obtain the i-th row sum of $ST(G)^{-1} SD_{\alpha}(G) ST(G)$ as

$$R_i(ST(G)^{-1} SD_{\alpha}(G) ST(G)) = \alpha ST_i + \frac{1-\alpha}{ST_i} \sum_{j=1}^{n} sd_G(v_i, v_j) ST_j = \alpha ST_i + \frac{1-\alpha}{ST_i} T_i.$$

Take $P = ST(G)^{-1}SD_\alpha(G)ST(G)$. Using Lemma 4, the required result follows.

For $\frac{1}{2} < \alpha < 1$, suppose that either of the equalities holds; then, Lemma 4 implies that the row sums of $ST(G)^{-1}SD_\alpha(G)ST(G)$ are all equal. That is, for any vertices $v_i, v_j \in V(G)$,

$$\alpha ST_i + \frac{1-\alpha}{ST_i}T_i = \alpha ST_j + \frac{1-\alpha}{ST_j}T_j.$$

Let $ST_{\max}$ and $ST_{\min}$ denote the maximum and minimum vertex sum transmissions of G, respectively. Without a loss of generality, assume that $ST_1 = ST_{\max}$ and $ST_n = ST_{\min}$. One can easily see that $T_1 = \sum\limits_{j=1}^{n} sd_G(v_1, v_j)ST_j \geq ST_{\min}\sum\limits_{j=1}^{n} sd_G(v_1, v_j) = ST_{\min}ST_{\max}$ and $T_n = \sum\limits_{j=1}^{n} sd_G(v_n, v_j)ST_j \leq ST_{\max}\sum\limits_{j=1}^{n} sd_G(v_n, v_j) = ST_{\max}ST_{\min}$. Thus, we obtain

$$\begin{aligned}\alpha ST_{\max} + (1-\alpha)ST_{\min} &\leq \alpha ST_1 + \frac{1-\alpha}{ST_1}T_1 = \alpha ST_n + \frac{1-\alpha}{ST_n}T_n \\ &\leq \alpha ST_{\min} + (1-\alpha)ST_{\max},\end{aligned}$$

which implies that $ST_{\max} = ST_{\min}$ for $\frac{1}{2} < \alpha < 1$. Therefore, G is the sum transmission regular.

Conversely, if G is an r-sum-transmission-regular digraph, then $\mu^\alpha(G) = r$. On the other hand, through a simple calculation, we obtain

$$\alpha ST_i + \frac{1-\alpha}{ST_i}T_i = r$$

for any $v_i \in V(G)$. Therefore, both equalities hold. □

Lemma 5. *Let G be a strongly connected digraph with $(v_i, v_j) \notin E(G)$, where $v_i, v_j \in V(G)$. Then, for any $1 > \alpha \geq 0$,*

$$\mu^\alpha(G) > \mu^\alpha(G + (v_i, v_j)).$$

Recall that $T_{n,r}$ denotes the complete r-partite digraph of order n, whose partition sets are of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. Note that $T_{n,2} = \overleftrightarrow{K}_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. It is known that $T_{n,r}$ has maximum number of arcs among all r-partite digraphs of order n. Then, we have $T_{n,r}$, which has the minimum $\mu^\alpha(G)$ among all strongly connected r-partite digraphs of order n. We will consider the case $r = 2$ first.

Theorem 2. *Let G be a strongly connected bipartite digraph with order n. Then, for any $1 > \alpha \geq 0$,*

$$\mu^\alpha(G) \geq \mu^\alpha(T_{n,2}),$$

with equality if and only if $G \cong T_{n,2}$.

Proof. Suppose that G is a strongly connected bipartite digraph of order n with minimum $\mu^\alpha(G)$ among all strongly connected bipartite digraphs of order n; then, based on Lemma 5, G is a complete bipartite digraph. Let V_1 and V_2 be the partitions of the vertex set of G, where $|V_1| = n_1$, $|V_2| = n_2$, $n_1 \geq n_2$ and $n_1 + n_2 = n$. Let $\mathbf{X}$ be the Perron vector of $\mu^\alpha(G)$. One can easily infer that the entries of $\mathbf{X}$ corresponding to vertices in the same partition set have the same value, say, x_i for $V_i, i = 1, 2$. Thus, $\mathbf{X}$ can be written as $\mathbf{X} = (\underbrace{x_1, x_1, \ldots, x_1}_{n_1}, \underbrace{x_2, x_2, \ldots, x_2}_{n_2})$. From $SD_\alpha(G)\mathbf{X} = \mu^\alpha(G)\mathbf{X}$, and we have

$$\mu^\alpha(G)x_1 = \alpha(2n + 2n_1 - 4)x_1 + 4(1-\alpha)(n_1 - 1)x_1 + 2(1-\alpha)n_2x_2,$$

$$\mu^{\alpha}(G)x_2 = \alpha(2n+2n_2-4)x_2 + 4(1-\alpha)(n_2-1)x_2 + 2(1-\alpha)n_1x_1.$$

Combining the above two equations, we have

$$\mu^{\alpha}(G) = \frac{4n+\alpha n-8+\sqrt{(20\alpha^2-64\alpha+48)n_1^2-(20\alpha^2 n-64\alpha n+48n)n_1+(4-3\alpha)^2n^2}}{2}.$$

Since $20\alpha^2-64\alpha+48=(4\alpha-8)(5\alpha-6)>0$ for $0\le\alpha<1$, $(20\alpha^2-64\alpha+48)n_1^2-(20\alpha^2 n-64\alpha n+48n)n_1+(4-3\alpha)^2n^2$ is decreasing for $1\le n_1\le\frac{n}{2}$ and increasing for $\frac{n}{2}\le n_1<n$. Then $\mu^{\alpha}(G)$ is minimum whenever $n_1=\lceil\frac{n}{2}\rceil$, that is $G=T_{n,2}=\overleftrightarrow{K}_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$. Therefore, we have $\mu^{\alpha}(G)\ge\mu^{\alpha}(T_{n,2})$, with equality if and only if $G\cong T_{n,2}$. $\square$

In general, for $r\ge 2$, we have the following theorem:

Theorem 3. *Let G be a strongly connected r-partite digraph of order n, where $r\ge 2$. Then, for any $1>\alpha\ge 0$,*

$$\mu^{\alpha}(G)\ge\mu^{\alpha}(T_{n,r}),$$

with equality if and only if $G\cong T_{n,r}$.

Proof. Suppose that G is a strongly connected r-partite digraph of order n with minimum $\mu^{\alpha}(G)$ among all strongly connected r-partite digraphs of order n; then, based on Lemma 5, we have G, which is a complete r-partite digraph. Let $V_1,V_2,\ldots,V_r$ be the partition sets of $V(G)$, where $|V_1|=n_1,|V_2|=n_2,\ldots,|V_r|=n_r$ and $n_1+n_2+\cdots+n_r=n$. Let $\mathbf{X}$ be the Perron vector of $\mu^{\alpha}(G)$. One can easily infer that the entries of $\mathbf{X}$ corresponding to vertices in the same partition set have the same value, say, y_i for $V_i, i=1,2,\ldots,r$. From $SD_{\alpha}(G)\mathbf{X}=\mu^{\alpha}(G)\mathbf{X}$, for any $v_k\in V_k$, we obtain

$$\mu^{\alpha}(G)y_k = \alpha(2n+2n_k-4)y_k + 4(1-\alpha)(n_k-1)y_k + 2(1-\alpha)\sum_{i=1,i\neq k}^{r} n_iy_i, 1\le k\le r.$$

Furthermore,

$$\begin{aligned}\mu^{\alpha}(G)y_k &= \alpha(2n+2n_k-4)y_k + 4(1-\alpha)(n_k-1)y_k + 2(1-\alpha)\sum_{i=1,i\neq k}^{r} n_iy_i\\ &= \alpha(2n+2n_k-4)y_k + 4(1-\alpha)(n_k-1)y_k - 2(1-\alpha)n_ky_k + 2(1-\alpha)\sum_{i=1}^{r} n_iy_i\\ &= (2\alpha n+2n_k-4)y_k + 2(1-\alpha)\sum_{i=1}^{r} n_iy_i.\end{aligned}$$

From the above equation, we have $\mu^{\alpha}(G)>2\alpha n+2n_k-4$. Let $\sum_{i=1}^{r} n_iy_i=S$. Then, we obtain $(\mu^{\alpha}(G)-2\alpha n-2n_k+4)y_k=2(1-\alpha)S$, which implies

$$\frac{n_ky_k}{2(1-\alpha)}=\frac{n_kS}{\mu^{\alpha}(G)-2\alpha n-2n_k+4}.$$

Hence,

$$\frac{1}{2(1-\alpha)}\sum_{k=1}^{r} n_ky_k = S\sum_{k=1}^{r}\frac{n_k}{\mu^{\alpha}(G)-2\alpha n-2n_k+4},$$

that is

$$\frac{1}{2(1-\alpha)}=\sum_{k=1}^{r}\frac{n_k}{\mu^{\alpha}(G)-2\alpha n-2n_k+4}. \tag{4}$$

Let
$$f(x) = \frac{x}{\mu^{\alpha}(G) - 2\alpha n - 2x + 4},$$
one can easily see that
$$f''(x) = \frac{4(\mu^{\alpha}(G) - 2\alpha n + 4)}{(\mu^{\alpha}(G) - 2\alpha n - 2x + 4)^3} > 0,$$
for $x > 0$. That is $f'(x)$ is increasing for $x > 0$. Then, we obtain that
$$\min\left\{ \sum_{k=1}^{r} f(z_k) : z_1 + z_2 + \cdots + z_r = n, z_k > 0 \text{ is an integer for each } k = 1, 2, \ldots, r \right\}, \quad (5)$$
is attained if and only if $z_k = \lfloor \frac{n}{r} \rfloor$ or $z_k = \lceil \frac{n}{r} \rceil$ for any $k = 1, 2, \ldots, r$. In fact, the minimum of (5) can be attained, as there are finitely many vectors $(z_1, z_2, \ldots, z_r)$ satisfying the constraints. Suppose that the minimum of (5) is attained for some $z_1, z_2, \ldots, z_r$, and through symmetry, we can assume that $z_1 \geq z_2 \geq \cdots \geq z_r$. If $z_1 - z_r \leq 1$, the calculation is completed. In the following, we assume $z_1 - z_r \geq 2$ for a contradiction. Taking
$$z_1' = z_1 - 1, z_2' = z_2, \ldots, z_{r-1}' = z_{r-1}, z_r' = z_r + 1.$$

According to the mean value theorem, there exist $\xi_1 \in (z_1 - 1, z_1)$ and $\xi_r \in (z_r, z_r + 1)$ such that
$$\sum_{k=1}^{r} f(z_k) - \sum_{k=1}^{r} f(z_k') = f(z_1) - f(z_1') + f(z_r) - f(z_r') = f'(\xi_1) - f'(\xi_r).$$

Since $f'(x)$ is increasing for $x > 0$ and $\xi_1 > \xi_r$, $f'(\xi_1) - f'(\xi_r) > 0$. Therefore,
$$\sum_{k=1}^{r} f(z_k) - \sum_{k=1}^{r} f(z_k') > 0,$$
that is
$$\sum_{k=1}^{r} f(z_k) > \sum_{k=1}^{r} f(z_k'),$$
which is contrary to the assumption that $\sum_{k=1}^{r} f(z_k)$ is minimum. Therefore, $z_k = \lfloor \frac{n}{r} \rfloor$ or $z_k = \lceil \frac{n}{r} \rceil$ for any $k = 1, 2, \ldots, r$.

Let $\mu^{\alpha}(T_{n,r}) = \mu$ and $s_1, s_2, \ldots, s_r$ be the sizes of the partition sets of $T_{n,r}$, that is $s_k = \lfloor \frac{n}{r} \rfloor$ or $s_k = \lceil \frac{n}{r} \rceil$ for any $k = 1, 2, \ldots, r$ and $s_1 + s_2 + \cdots + s_r = n$. Using Equation (4), we obtain
$$\frac{1}{2(1-\alpha)} = \sum_{k=1}^{r} \frac{s_k}{\mu - 2\alpha n - 2s_k + 4}.$$

On the other hand,
$$\begin{aligned} \sum_{k=1}^{r} \frac{s_k}{\mu - 2\alpha n - 2s_k + 4} = \frac{1}{2(1-\alpha)} &= \sum_{k=1}^{r} \frac{n_k}{\mu^{\alpha}(G) - 2\alpha n - 2n_k + 4} \\ &\geq \sum_{k=1}^{r} \frac{s_k}{\mu^{\alpha}(G) - 2\alpha n - 2s_k + 4}. \end{aligned}$$

Thus, $\mu^{\alpha}(G) \geq \mu = \mu^{\alpha}(T_{n,r})$, with equality if and only if $n_k = \lfloor \frac{n}{r} \rfloor$ or $n_k = \lceil \frac{n}{r} \rceil$ for any $k = 1, 2, \ldots, r$. $\square$

4. The General Sum Distance Spectral Radius and Maximum Sum Transmission of Digraphs

Lemma 6 ([16]). *If $a, b > 0$, then $a(x-y)^2 + by^2 \geq \frac{ab}{a+b}x^2$ with equality if and only if $y = \frac{a}{a+b}x$.*

Let G be a strongly connected digraph of order n. As $\mu^\alpha(G) \leq ST_1$ with equality if and only if G is sum transmission regular, where $ST_1 = \max_{1\leq i\leq n}\{ST_i\}$. For a strongly connected non-sum-transmission-regular digraph G of order n, $\mu^\alpha(G) < ST_1$. However, we want to know how small $ST_1 - \mu^\alpha(G)$ can be when G is a non-sum-transmission-regular digraph. In the following, we will give a lower bound on $ST_1 - \mu^\alpha(G)$.

Theorem 4. *Let $G = (V(G), E(G))$ be a strongly connected digraph which is not sum transmission regular, $\{ST_1, ST_2, \ldots, ST_n\}$ be the sum transmission sequence with $ST_1 \geq ST_2 \geq \cdots \geq ST_n$. Then,*

$$ST_1 - \mu^\alpha(G) > \frac{(1-\alpha)nST_1(nST_1 - W)}{2W(nST_1 - 2W) + (1-\alpha)n^2 ST_1},$$

where $W = \sum_{i=1}^{n}\sum_{j=1}^{n} d_G(v_i, v_j)$.

Proof. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $SD_\alpha(G)$ corresponding to $\mu^\alpha(G)$, where x_i corresponds to the vertex v_i. Clearly, $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Suppose that v_s, v_t are two vertices satisfying $x_s = \max_{1\leq i\leq n} x_i$ and $x_t = \min_{1\leq i\leq n} x_i$, respectively. Taking $W = \sum_{i=1}^{n}\sum_{j=1}^{n} d_G(v_i, v_j)$. Since G is not a sum-transmission-regular digraph, we have $x_s > x_t$, and thus

$$\begin{aligned}
\mu^\alpha(G) &= \mathbf{X}^T SD_\alpha \mathbf{X} \\
&= \alpha \sum_{i=1}^{n} ST_i x_i^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} sd_G(v_i, v_j)x_i x_j \\
&< \alpha \sum_{i=1}^{n} ST_i x_s^2 + (1-\alpha)\sum_{i=1}^{n}\sum_{j=1}^{n} sd_G(v_i, v_j)x_s^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} sd_G(v_i, v_j)x_s^2 = 2Wx_s^2,
\end{aligned}$$

which implies that $x_s^2 > \frac{\mu^\alpha(G)}{2W}$. Furthermore, with (1), we obtain

$$\begin{aligned}ST_1 - \mu^{\alpha}(G) &= ST_1 - \mathbf{X}^T SD_{\alpha}\mathbf{X} \\ &= ST_1 - \alpha \sum_{v_i \in V(G)} ST_i x_i^2 - 2(1-\alpha) \sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j) x_i x_j \\ &= ST_1 \sum_{i=1}^{n} x_i^2 - \alpha \sum_{v_i \in V(G)} ST_i x_i^2 + (1-\alpha) \sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j)(x_i - x_j)^2 \\ &\quad - (1-\alpha) \sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j)(x_i^2 + x_j^2) \\ &= \sum_{i=1}^{n} ST_1 x_i^2 - \alpha \sum_{v_i \in V(G)} ST_i x_i^2 - (1-\alpha) \sum_{v_i \in V(G)} ST_i x_i^2 \\ &\quad + (1-\alpha) \sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j)(x_i - x_j)^2 \\ &= \sum_{i=1}^{n} (ST_1 - ST_i) x_i^2 + (1-\alpha) \sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j)(x_i - x_j)^2 \\ &\geq \sum_{i=1}^{n} (ST_1 - ST_i) x_t^2 + (1-\alpha) \sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j)(x_i - x_j)^2 \\ &= (nST_1 - 2W) x_t^2 + (1-\alpha) \sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j)(x_i - x_j)^2.\end{aligned}$$

In the following, we will estimate $\sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j)(x_i - x_j)^2$. Suppose $Q = v_1 v_2 \ldots v_{l+1}$ be the shortest directed path from v_s and v_t, where $v_1 = v_s$, $v_{l+1} = v_t$, and $l \geq 1$, then

$$\begin{aligned}\sum_{\{v_i,v_j\} \in V(G)} sd_G(v_i,v_j)(x_i - x_j)^2 \geq &\sum_{\{v_i,v_j\} \in V(Q)} sd_G(v_i,v_j)(x_i - x_j)^2 \\ &+ \sum_{v_i \in V(G) \setminus V(Q)} \sum_{v_j \in V(Q)} sd_G(v_i,v_j)(x_i - x_j)^2.\end{aligned}$$

For convenience, set

$$C = \sum_{\{v_i,v_j\} \in V(Q)} sd_G(v_i,v_j)(x_i - x_j)^2,$$

and

$$P = \sum_{v_i \in V(G) \setminus V(Q)} \sum_{v_j \in V(Q)} sd_G(v_i,v_j)(x_i - x_j)^2.$$

For any $v_i \in V(G) \setminus V(Q)$, using the Cauchy–Schwarz inequality, we have

$$sd_G(v_i,v_s)(x_i - x_s)^2 + sd_G(v_i,v_t)(x_i - x_t)^2 \geq 2(x_i - x_s)^2 + 2(x_i - x_t)^2 \geq (x_s - x_t)^2,$$

and then

$$\begin{aligned}P &\geq \sum_{v_i \in V(G) \setminus V(Q)} (sd_G(v_i,v_s)(x_i - x_s)^2 + sd_G(v_i,v_t)(x_i - x_t)^2) \\ &\geq \sum_{v_i \in V(G) \setminus V(Q)} (x_s - x_t)^2 \\ &= (n - l - 1)(x_s - x_t)^2.\end{aligned}$$

Moreover, based on the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
C &\geq sd_G(v_1, v_{l+1})(x_1 - x_{l+1})^2 + \sum_{i=1}^{l-1}\Big(sd_G(v_1, v_{i+1})(x_1 - x_{i+1})^2 + sd_G(v_{i+1}, v_{l+1})(x_{i+1} - x_{l+1})^2\Big) \\
&\geq (l+1)(x_1 - x_{l+1})^2 + \sum_{i=1}^{l-1} \min\{i+1, l-i+1\}\frac{1}{2}(x_1 - x_{l+1})^2 \\
&= (l+1)(x_s - x_t)^2 + \sum_{i=1}^{l-1} \min\{i+1, l-i+1\}\frac{1}{2}(x_s - x_t)^2.
\end{aligned}
$$

Then, we consider the following three cases:

Case 1: $l = 1$. In this case, we have

$$
\begin{aligned}
\sum_{\{v_i, v_j\} \in V(G)} sd_G(v_i, v_j)(x_i - x_j)^2 &\geq P + C \\
&\geq (n-2)(x_s - x_t)^2 + 2(x_s - x_t)^2 \\
&= n(x_s - x_t)^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
ST_1 - \mu^\alpha(G) &\geq (nST_1 - 2W)x_t^2 + (1-\alpha)\sum_{\{v_i, v_j\} \in V(G)} sd_G(v_i, v_j)(x_i - x_j)^2 \\
&\geq (nST_1 - 2W)x_t^2 + (1-\alpha)n(x_s - x_t)^2 \\
&\geq \frac{(1-\alpha)n(nST_1 - 2W)}{(nST_1 - 2W) + (1-\alpha)n}x_s^2. \text{ (using Lemma 6)}
\end{aligned}
$$

Note that $x_s^2 > \frac{\mu^\alpha(G)}{2W}$; then,

$$
\begin{aligned}
ST_1 - \mu^\alpha(G) &\geq \frac{(1-\alpha)n(nST_1 - 2W)}{(nST_1 - 2W) + (1-\alpha)n}x_s^2 \\
&> \frac{(1-\alpha)n(nST_1 - 2W)}{(nST_1 - 2W) + (1-\alpha)n} \cdot \frac{\mu^\alpha(G)}{2W} \\
&= \frac{(1-\alpha)n(nST_1 - 2W)ST_1}{2W((nST_1 - 2W) + (1-\alpha)n)} - \frac{(1-\alpha)n(nST_1 - 2W)(ST_1 - \mu^\alpha(G))}{2W((nST_1 - 2W) + (1-\alpha)n)},
\end{aligned}
$$

which implies

$$
ST_1 - \mu^\alpha(G) > \frac{(1-\alpha)nST_1(nST_1 - 2W)}{2W(nST_1 - 2W) + (1-\alpha)n^2 ST_1}.
$$

Case 2: $l \geq 2$ and l is even. Then,

$$
\begin{aligned}
C &\geq (l+1)(x_s - x_t)^2 + \sum_{i=1}^{l-1} \min\{i+1, l-i+1\}\frac{1}{2}(x_s - x_t)^2 \\
&\geq (l+1)(x_s - x_t)^2 + \frac{(x_s - x_t)^2}{2}\left[(2 + 3 + \cdots + \frac{l+2}{2}) + (2 + 3 + \cdots + \frac{l}{2})\right] \\
&= \frac{l^2 + 12l + 4}{8}(x_s - x_t)^2,
\end{aligned}
$$

and

$$\begin{aligned}\sum_{\{v_i,v_j\}\in V(G)} sd_G(v_i,v_j)(x_i-x_j)^2 &\geq P+C\\ &\geq (n-l-1)(x_s-x_t)^2+\frac{l^2+12l+4}{8}(x_s-x_t)^2\\ &= \frac{l^2+8n+4l-4}{8}(x_s-x_t)^2.\end{aligned}$$

Thus,

$$\begin{aligned}ST_1-\mu^\alpha(G) &\geq (nST_1-2W)x_t^2+(1-\alpha)\sum_{\{v_i,v_j\}\in V(G)} sd_G(v_i,v_j)(x_i-x_j)^2\\ &\geq (nST_1-2W)x_t^2+(1-\alpha)\frac{l^2+8n+4l-4}{8}(x_s-x_t)^2\\ &\geq \frac{(1-\alpha)(nST_1-2W)(l^2+4l+8n-4)}{8(nST_1-2W)+(1-\alpha)(l^2+4l+8n-4)}x_s^2 \text{ (using Lemma 6).}\end{aligned}$$

Since $x_s^2 > \frac{\mu^\alpha(G)}{2W}$,

$$ST_1-\mu^\alpha(G) > \frac{(1-\alpha)(nST_1-2W)(l^2+4l+8n-4)}{8(nST_1-2W)+(1-\alpha)(l^2+4l+8n-4)}\cdot\frac{\mu^\alpha(G)}{2W},$$

which means

$$ST_1-\mu^\alpha(G) > \frac{(1-\alpha)(nST_1-2W)(l^2+4l+8n-4)ST_1}{16W(nST_1-2W)+(1-\alpha)nST_1(l^2+4l+8n-4)}.$$

Let $f(x) = \frac{(1-\alpha)(nST_1-2W)(8n+x)ST_1}{16W(nST_1-2W)+(1-\alpha)nST_1(8n+x)}$. One can easily see that $f(x)$ is strictly increasing for $x \geq 2$. Therefore,

$$\begin{aligned}ST_1-\mu^\alpha(G) &> \frac{(1-\alpha)(nST_1-2W)(8n+8)ST_1}{16W(nST_1-2W)+(1-\alpha)nST_1(8n+8)}\\ &= \frac{(1-\alpha)(nST_1-2W)(n+1)ST_1}{2W(nST_1-2W)+(1-\alpha)nST_1(n+1)}\\ &> \frac{(1-\alpha)nST_1(nST_1-2W)}{2W(nST_1-2W)+(1-\alpha)n^2ST_1}.\end{aligned}$$

Case 3: $l \geq 2$ and l is odd. Then,

$$\begin{aligned}C &\geq (l+1)(x_s-x_t)^2+\sum_{i=1}^{l-1}\min\{i+1,l-i+1\}\frac{1}{2}(x_s-x_t)^2\\ &\geq (l+1)(x_s-x_t)^2+\frac{(x_s-x_t)^2}{2}\left[2\times(2+3+\cdots+\frac{l+1}{2})\right]\\ &= \frac{l^2+12l+3}{8}(x_s-x_t)^2,\end{aligned}$$

and

$$
\begin{aligned}
\sum_{\{v_i,v_j\}\in V(G)} sd_G(v_i,v_j)(x_i-x_j)^2 &\geq P+C \\
&\geq (n-l-1)(x_s-x_t)^2+\frac{l^2+12l+3}{8}(x_s-x_t)^2 \\
&= \frac{l^2+8n+4l-5}{8}(x_s-x_t)^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
ST_1-\mu^\alpha(G) &\geq (nST_1-2W)x_t^2+(1-\alpha)\sum_{\{v_i,v_j\}\in V(G)} sd_G(v_i,v_j)(x_i-x_j)^2 \\
&\geq (nST_1-2W)x_t^2+(1-\alpha)\frac{l^2+8n+4l-5}{8}(x_s-x_t)^2 \\
&\geq \frac{(1-\alpha)(nST_1-2W)(l^2+4l+8n-5)}{8(nST_1-2W)+(1-\alpha)(l^2+4l+8n-5)}x_s^2 \text{ (using Lemma 6).}
\end{aligned}
$$

Since $x_s^2 > \frac{\mu^\alpha(G)}{2W}$,

$$
ST_1-\mu^\alpha(G) > \frac{(1-\alpha)(nST_1-2W)(l^2+4l+8n-5)}{8(nST_1-2W)+(1-\alpha)(l^2+4l+8n-5)}\cdot\frac{\mu^\alpha(G)}{2W},
$$

which means

$$
ST_1-\mu^\alpha(G) > \frac{(1-\alpha)(nST_1-2W)(l^2+4l+8n-5)ST_1}{16W(nST_1-2W)+(1-\alpha)nST_1(l^2+4l+8n-5)}.
$$

Since $f(x)=\frac{(1-\alpha)(nST_1-2W)(8n+x)ST_1}{16W(nST_1-2W)+(1-\alpha)nST_1(8n+x)}$ is strictly increasing for $x\geq 2$; therefore,

$$
\begin{aligned}
ST_1-\mu^\alpha(G) &> \frac{(1-\alpha)(nST_1-2W)(8n+16)ST_1}{16W(nST_1-2W)+(1-\alpha)nST_1(8n+16)} \\
&= \frac{(1-\alpha)(nST_1-2W)(n+2)ST_1}{2W(nST_1-2W)+(1-\alpha)nST_1(n+2)} \\
&> \frac{(1-\alpha)nST_1(nST_1-2W)}{2W(nST_1-2W)+(1-\alpha)n^2ST_1}.
\end{aligned}
$$

Taking the above three cases together, we achieve the desired result. □

5. Conclusions

In this paper, we first gave some spectral properties of $SD_\alpha(G)$. Moreover, we also characterized the digraph that has the minimum general sum distance spectral radius among all strongly connected r-partite digraphs. Finally, for a strongly connected non-sum-transmission-regular digraph G of order n, we obtained a lower bound on $ST_1-\mu^\alpha(G)$, where $ST_1=\max\limits_{1\leq i\leq n}\{ST_i\}$.

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