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Article

Topological Indices of Graphs from Vector Spaces

Krishnamoorthy Mageshwaran [1], Nazeek Alessa [2,*], Singaravelu Gopinath [3] and Karuppusamy Loganathan [4,*]

1 Department of Mathematics, Rajalakshmi Engineering College, Chennai 602105, Tamil Nadu, India
2 Department of Mathematical Sciences, College of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
3 Department of Mathematics, Sri Sairam Institute of Technology, Chennai 60004, Tamil Nadu, India
4 Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur 303007, Rajasthan, India
* Correspondence: nazekaa@yahoo.com (N.A.); loganathankaruppusamy304@gmail.com (K.L.)

Abstract: Topological indices are numbers that are applied to a graph and can be used to describe specific graph properties through algebraic structures. Algebraic graph theory is a helpful tool in a range of chemistry domains. Because it helps explain how the different symmetries of molecules and crystals affect their structure and dynamics, it is a powerful theoretical approach for forecasting both the common and uncommon characteristics of molecules. A topological index converts the chemical structure into a number and contributes a lot in chemical graph theory. In this article, we compute the Wiener index, Zagreb indexes, Wiener polynomial, Hyper-Wiener index, ABC index and eccentricity-based topological index of a nonzero component union graph from vector space.

Keywords: topological indices; vector space; nonzero component union graph

MSC: 05C10; 05C25; 05C75; 05C78

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1. Introduction

In this paper, $\Re = (V, E)$ is a simple and connected graph with a vertex set V and an edge set E. The order and size of $\Re$ is established as the amount of elements in V and the number of elements in E correspondingly. If every different pair of vertices is adjacent in $\Re$, then the graph $\Re$ is complete, and it is described by K_n. The count of the edges from E that are incident with the vertex $\ell \in V$ is the degree of ℓ, and it is represented by $d(\ell)$. The maximum (minimum) degree of $\Re$ is defined as $\Delta = max\{d(\ell)\}(\delta = min\{d(\ell)\})$ for all $\ell \in V$. The distance between ℓ and $\wp$ in V is the distance of the shortest path between them; it is indicated by $d(\ell, \wp)$. The eccentricity $(ecc(\ell))$ of ℓ is the largest distance between a vertex ℓ and each of the vertices of $\Re$. The total length from any vertex ℓ of $\Re$ is calculated as $D(\ell) = \sum_{\wp \in V(\Re)} d(\ell, \wp)$. The maximum distance among all the vertices in $\Re$ is the diameter of $\Re$, symbolized by $diam(\Re)$. The amount of unordered vertex pairs in $\Re$ that are accurately t distances apart is specified as $d(\Re, t)$. We consult [1] for any graph theory keywords that are undefined.

A number, polynomial or matrix can be produced corresponding to a given graph in several ways. A topological graph index is an integer associated with a graph that fully uses the graph's topology and it is invariant under graph isomorphism. In the present years, plenty of attention has been given to studies about the special properties of a topological index; one can refer to [2–12].

Das [13–15] introduced the new graphs on vector spaces. From the above initialization, some more authors studied this in [14,16]. In particular, Das recently defined and investigated two vector space graphs, namely the nonzero component and the nonzero component union graphs [13,14].

Some authors studied the basic graph theoretical properties of nonzero component union graphs. Further, [16,17] proved the results related to the embedding of the graph defined in [13,14].

Across this article, Λ is a φ dimension vector space ($dim(\Lambda) = \varphi$) over the field $\mathbb{F}$ of order δ ($O(\mathbb{F}) = \delta$). Let $\mathcal{B} = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_\varphi\}$ as a basis of Λ. Any vector $\ell \in \Lambda$ is represented as $\ell = a_1\vartheta_1 + a_2\vartheta_2 + \ldots + a_\varphi\vartheta_\varphi$, where $a_i \in \mathbb{F}$ (simply denoted $\ell = (a_1, a_2, \ldots, a_\varphi)$). The skeleton of any vector in $\Lambda^* = \Lambda \setminus \{0\}$ based on $\mathcal{B}$ is described by $S_\mathcal{B}(\ell) = \{\vartheta_i : a_i \neq 0, 1 \leq i \leq \varphi\}$. Moreover, a vector ℓ with skeleton $\jmath$ means $|S_\mathcal{B}(\ell)| = \jmath$. The nonzero component union graph of Λ based on the basis $\mathcal{B}$ is defined as a simple graph with $V = \Lambda^*$ and different nonzero vectors ℓ and $\wp$ in V are adjacent if and only if $S_\mathcal{B}(\ell) \cup S_\mathcal{B}(\wp) = \mathcal{B}$. This graph is denoted by $\Gamma(\Lambda_\mathcal{B})$.

The graphs associated to algebra are also a significant contributor in several disciplines, including chemistry, engineering, medicine and business.

2. Fundamentals

The following definitions and theorems were used to determine the topological indices for the graphs under discussion. We put up a few current findings that will be discussed in this section.

Theorem 1 ([13])**.** *$diam(\Gamma(\Lambda_\mathcal{B})) = 2$ and $\Gamma(\Lambda_\mathcal{B})$ are connected.*

Theorem 2 ([13])**.** *Let $O(\mathbb{F}) = \delta$ and $dim(\Lambda) = \varphi$. Then, $\Gamma(\Lambda_\mathcal{B})$ is complete if and only if Λ has either ($\varphi = 1$) or ($\varphi = 2$ and $\delta = 2$).*

Theorem 3 ([13])**.** *Let $\Gamma(\Lambda_\mathcal{B})$ be weakly perfect because its chromatic and clique number are equal to $\varphi + (\delta - 1)^\varphi$.*

Theorem 4 ([13])**.** *Let $\Gamma(\Lambda_\mathcal{B})$ be the nonzero component union graph of Λ based on $\mathcal{B} = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_\varphi\}$. Let $\ell = c_1\vartheta_{i_1} + c_2\vartheta_{i_2} + \ldots + c_\jmath\vartheta_{i_\jmath}$ be a vertex in $\Gamma(\Lambda_\mathcal{B})$ with $c_i \neq 0$ where $1 \leq i \leq \jmath$. Then, $deg(\ell) = \begin{cases} (\delta-1)^{\varphi-\jmath}\delta^\jmath & \text{if } 1 \leq \jmath < \varphi; \\ \delta^\varphi - 2 & \text{if } \jmath = \varphi. \end{cases}$*

Theorem 5 ([13])**.** *Let $O(\mathbb{F}) = \delta$ and $dim(\Lambda) = \varphi$. Then, order and size of $\Gamma(\Lambda_\mathcal{B})$ is $\delta^\varphi - 1$ and $\dfrac{(\delta-1)^\varphi[(\delta+1)^\varphi - 3]}{2}$, respectively.*

Theorem 6 ([13])**.** *If the minimum and maximum degree of $\Gamma(\Lambda_\mathcal{B})$ is $\delta = \delta(\delta-1)^{\varphi-1}$ and $\Delta = \delta^\varphi - 2$.*

Theorem 7 ([13])**.** *$I = \{\vartheta \in \Gamma(\Lambda_\mathcal{B}) : S(\vartheta_\mathcal{B}) \subseteq \{\vartheta_1, \vartheta_2, \ldots, \vartheta_{\varphi-1}\}\} \in \Gamma(\Lambda_\mathcal{B})$ be a maximal independent set. Moreover, if $O(\mathbb{F}) = \delta$, $|I| = \delta^{\varphi-1} - 1$.*

Let us finish this section by listing some topological indexes of any graph $\Re$ that are also going to be considered in the current manuscript. One can refer to [2,9–12,18]

- The *Wiener index*
$$W(\Re) = \frac{1}{2} \sum_{\ell,\wp \in V(\Re)} d(\ell, \wp).$$
- The *first Zagreb index*
$$M_1(\Re) = \sum_{\ell \in V(\Re)} (d(\ell))^2.$$

- The *second Zagreb index*
$$M_2(\Re) = \sum_{\ell\wp\in E(\Re)} d(\ell)d(\wp).$$
- The *Wiener polynomial*
$$W(\Re;x) = \sum_{\ell,\wp\in V(\Re)} x^{d(\ell,\wp)} = \sum_{\jmath=1}^{diam(\Re)} d(\Re,\jmath)x^{\jmath}.$$
The amount of unordered vertex pairs in $\Re$ that are accurately $\jmath$ distances apart is specified as $d(\Re,\jmath)$.
- The *Hyper*-Wiener index
$$WW(\Re) = \frac{1}{2}W(\Re) + \frac{1}{2}\sum_{\{\ell,\wp\}\subseteq V(\Re)} d(\ell,\wp)^2 = \frac{1}{2}\sum_{\jmath=1}^{diam(\Re)} \jmath(\jmath+1)d(\Re,\jmath).$$
- The *eccentricity index*
$$\xi(\Re) = \sum_{\ell\in V(\Re)} d(\ell)ecc(\ell).$$
- The *total eccentricity index*
$$\zeta(\Re) = \sum_{\ell\in V(\Re)} ecc(\ell).$$
- The *new version (eccentricity based) of Zagreb index*
$$M_1^*(\Re) = \sum_{\ell\wp\in E(\Re)} (ecc(\ell) + ecc(\wp)).$$
$$M_1^{**}(\Re) = \sum_{\ell\in V(\Re)} (ecc(\ell))^2$$
$$M_2^*(\Re) = \sum_{\ell\wp\in E(\Re)} ecc(\ell)ecc(\wp).$$
- *Average eccentricity index*
$$aveg(\Re) = \frac{1}{n}\sum_{\ell\in V(\Re)} ecc(\ell).$$
Here, n is total count of vertices in $\Re$.
- *Eccentric distance sum index*
$$\xi^{DS}(\Re) = \sum_{\ell\in V(\Re)} ecc(\ell)D(\ell|\Re).$$
where $D(\ell|\Re) = \sum_{\wp\in V(\Re)} d(\ell,\wp)$.
- The *ABC index*
$$ABC(\Gamma(\Lambda_{\mathcal{B}})) = \sum_{\ell\wp\in E(\Re)} \sqrt{\frac{d(\ell)+d(\wp)-2}{d(\ell)d(\wp)}}.$$

3. Properties of Nonzero Component Union Graph of Vector Space

Let Λ be a vector space with $O(\mathbb{F}) = \delta$ and $dim(\Lambda) = \varphi \geq 1$. By Theorem 2, $\Gamma(\Lambda_{\mathcal{B}})$ is complete if and only if $\varphi = 1$ or ($\varphi = 2$ and $\delta = 2$). In this specific case, it is simple to identify the topological index. For the purpose of determining the aforementioned topological index, we only take into account the non-complete nonzero component union graphs of vector spaces.

Remark 1. *Let Λ be a vector space with $O(\mathbb{F}) = \beta$, $dim(\Lambda) = \varphi$ $(\varphi \geq 1)$ and ℓ be an arbitrary vertex with skeleton $\jmath$. The distance between ℓ and $\wp \in V(\Gamma(\Lambda_{\mathcal{B}}))$ is 1 or 2.*

The count of vertices in distance 1 from ℓ is the degree of the vertex ℓ. By Theorem 4, the count of vertices at a distance of 2 from ℓ is

$$|V(\Gamma(\Lambda_{\mathcal{B}}))| - d(\ell) - 1 = \begin{cases} \beta^{\varphi} - 2 - (\beta-1)^{\varphi-\jmath}\beta^{\jmath}, & \text{if } 1 \leq \jmath \leq \varphi - 1 \\ 0, & \text{if } \varphi = \jmath. \end{cases}$$

Remark 2. *Let Λ be a vector space with $O(\mathbb{F}) = \beta$, $dim(\Lambda) = \varphi$ $(\varphi \geq 1)$ and ℓ be an arbitrary vertex with skeleton $\jmath$. If the vertex ℓ is adjacent to any other vertex $\wp \in \Gamma(\Lambda_{\mathcal{B}})$, then $|S_{\mathcal{B}}(\wp)| \geq \varphi - \jmath$. Moreover, ℓ has $C_{\jmath}^{t+\jmath-\varphi}(\beta-1)^{t}$ neighborhood vertices with skeleton t where $\varphi - \jmath \leq t \leq \varphi$.*

Remark 3. *We know that diameter of $\Gamma(\Lambda_{\mathcal{B}})$ is 2. Then,*

$$\begin{aligned} d(\Gamma(\Lambda_{\mathcal{B}}),1) &= (\beta-1)^{\varphi}(\beta^{\varphi}-2) + \frac{1}{2}\Big(\sum_{\jmath=1}^{\varphi-1} C_{\varphi}^{\jmath}(\beta-1)^{\jmath}\beta^{\jmath}(\beta-1)^{\varphi-\jmath}\Big) \\ d(\Gamma(\Lambda_{\mathcal{B}}),2) &= \frac{1}{2}\sum_{\jmath=1}^{\varphi-1} C_{\varphi}^{\jmath}(\beta-1)^{\jmath}\big(\beta^{\varphi} - 2 - (\beta-1)^{\varphi-\jmath}\beta^{\jmath}\big) \end{aligned}$$

Theorem 8. *If $\Gamma(\Lambda_{\mathcal{B}})$ is not complete, then the eccentricity of the vertex $\ell \in \Gamma(\Lambda_{\mathcal{B}})$ with skeleton $\jmath$ is*

$$ecc(\ell) = \begin{cases} 2, & \text{if } 1 \leq \jmath \leq \varphi - 1 \\ 1, & \text{if } \varphi = \jmath. \end{cases}$$

Proof. Let ℓ be an arbitrary vector with skeleton $\jmath$. If $\jmath = \varphi$, then by Theorem 4 $ecc(\ell) = 1$. If $1 \leq \jmath \leq \varphi - 1$, then by Theorem 4, ℓ has non-adjacent vertices in $\Gamma(\Lambda_{\mathcal{B}})$. Because by Theorem 1, $ecc(\ell) = 2$. □

Theorem 9. *If $\Gamma(\Lambda_{\mathcal{B}})$ is not complete, then distance number of the vertex ℓ with skeleton $\jmath$ is*

$$D(\ell|\Re) = \begin{cases} 2(\beta^{\varphi} - 2) - (\beta-1)^{\varphi-\jmath}\beta^{\jmath}, & \text{if } 1 \leq \jmath \leq \varphi - 1 \\ \beta^{\varphi} - 2, & \text{if } \varphi = \jmath. \end{cases}$$

Proof. Let ℓ be an arbitrary vector with skeleton $\jmath$.

If $1 \leq \jmath \leq \varphi - 1$, because by Theorem 4 $d(\ell)$ is $(\beta-1)^{\varphi-\jmath}\beta^{\jmath}$ and by Theorem 1 gives

$$\begin{aligned} D(\ell|\Re) &= (\beta-1)^{\varphi-\jmath}\beta^{\jmath} + 2(\beta^{\varphi} - 2 - (\beta-1)^{\varphi-\jmath}\beta^{\jmath}) \\ &= (\beta-1)^{\varphi-\jmath}\beta^{\jmath} + 2(\beta^{\varphi} - 2) - 2((\beta-1)^{\varphi-\jmath}\beta^{\jmath}) \\ D(\ell|\Re) &= 2(\beta^{\varphi} - 2) - (\beta-1)^{\varphi-\jmath}\beta^{\jmath} \end{aligned}$$

If $\jmath = \varphi$, by Theorem 4

$$D(\ell|\Re) = \beta^{\varphi} - 2.$$

□

Remark 4. *Let Λ be a vector space with $O(\mathbb{F}) = \beta$, $dim(\Lambda) = \varphi$ $(\varphi \geq 1)$. By Theorem 5 $|E(\Gamma(\Lambda_{\mathcal{B}}))| = \dfrac{(\beta-1)^{\varphi}[(\beta+1)^{\varphi}-3]}{2}$. Because ([13], p. 4) collection of an element with skeleton φ in $\Gamma(\Lambda_{\mathcal{B}})$ is the complete sub-graph of $\Gamma(\Lambda_{\mathcal{B}})$.*

Hence, the count of edges with both end vertices are having skeleton φ is $\frac{(\beta-1)^{\varphi}\left((\beta-1)^{\varphi}-1\right)}{2}$.

In addition, the count of edges with exactly one end vertex having skeleton φ is

$$\Big((\beta^{\varphi}-2)-(\beta-1)^{\varphi}+1\Big)(\beta-1)^{\varphi}=\Big(\beta^{\varphi}-(\beta-1)^{\varphi}-1\Big)(\beta-1)^{\varphi}$$

Finally, the count of edges on both ends are skeleton less than φ is

$$\frac{(\beta-1)^{\varphi}[(\beta+1)^{\varphi}-3]}{2}-\Big((\beta^{\varphi}-2)-(\beta-1)^{\varphi}-1\Big)(\beta-1)^{\varphi}-\frac{(\beta-1)^{\varphi}\big((\beta-1)^{\varphi}-1\big)}{2}$$
$$=(\beta-1)^{\varphi}\Big(\frac{(\beta+1)^{\varphi}+(\beta-1)^{\varphi}-2}{2}-\beta^{\varphi}+1\Big).$$

4. Topological Index of Nonzero Component Union Graph of Vector Space

In this segment, we found the Wiener index, Wiener polynomial, Hyper-Wiener index and Zagreb index of $\Gamma(\Lambda_{\mathcal{B}})$.

Theorem 10. *Let $\Re=\Gamma(\Lambda_{\mathcal{B}})$, then*

$$W(\Gamma(\Lambda_{\mathcal{B}}))=(\beta^{\varphi}-2)\Big(\beta^{\varphi}-\frac{(\beta-1)^{\varphi}-2}{2}\Big)-(\beta-1)^{\varphi}\Big((\beta+1)^{\varphi}-\beta^{\varphi}-1\Big)$$

Proof. Remark 1 gives the following result.

$$\begin{aligned}
W(\Gamma(\Lambda_{\mathcal{B}})) &= \frac{1}{2}\Big[\Big(\sum_{j=1}^{\varphi-1}C_{\varphi}^{j}(\beta-1)^{j}(\beta-1)^{\varphi-j}\beta^{j}\Big)\\
&\quad+2\Big[\sum_{j=1}^{\varphi-1}C_{\varphi}^{j}(\beta-1)^{j}\big(\beta^{\varphi}-2-(\beta-1)^{\varphi-j}\beta^{j}\big)\Big]\\
&\quad+(\beta-1)^{\varphi}(\beta^{\varphi}-2)\Big]\\
W(\Gamma(\Lambda_{\mathcal{B}})) &= (\beta^{\varphi}-2)\Big(\beta^{\varphi}-\frac{(\beta-1)^{\varphi}-2}{2}\Big)-(\beta-1)^{\varphi}\Big((\beta+1)^{\varphi}-\beta^{\varphi}-1\Big).
\end{aligned}$$

□

Now, consider the following Figure 1 to visualize $W(\Gamma(\Lambda_{\mathcal{B}}))$ with φ dimension over the field of order β.

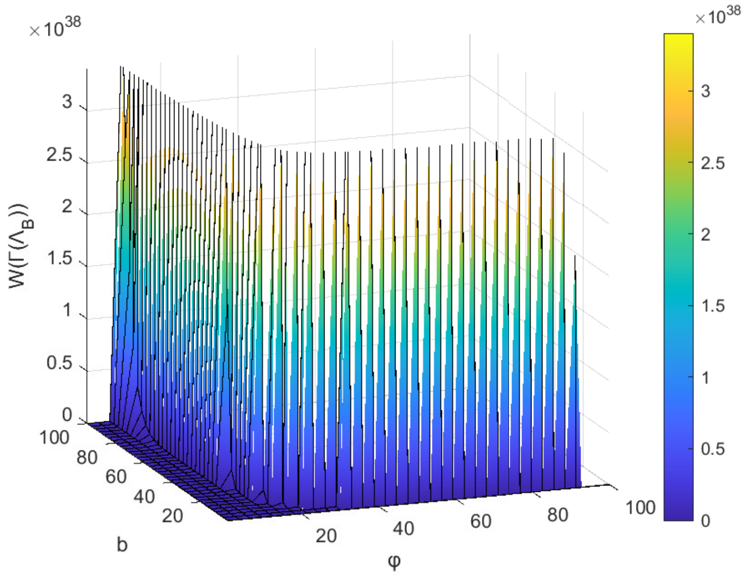

Figure 1. Wiener index of $\Gamma(\Lambda_{\mathcal{B}})$.

Theorem 11. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then the first Zagreb index*

$$M_1(\Gamma(\Lambda_{\mathcal{B}})) = (\delta-1)^{\varphi}\Big((\delta^2+\delta-1)^{\varphi} - (\delta-1)^{\varphi} - 4(\delta^{\varphi}-1)\Big)$$

Proof. Let $\ell = c_1\vartheta_{i_1} + c_2\vartheta_{i_2} + \ldots + c_j\vartheta_{i_j}$ be a vertex in $\Gamma(\Lambda_{\mathcal{B}})$ with $c_1, c_2, \ldots, c_j \neq 0$. Then, by Theorem 4, the degree of the vertex ℓ is $d(\ell) = \begin{cases} (\delta-1)^{\varphi-j}\delta^j & \text{if } 1 \leq j \leq \varphi; \\ \delta^{\varphi}-2 & \text{if } j = \varphi. \end{cases}$ Therefore, the first Zagreb index

$$\begin{aligned} M_1(\Gamma(\Lambda_{\mathcal{B}})) &= \sum_{j=1}^{\varphi-1} C_{\varphi}^{j}(\delta-1)^{j}\big((\delta-1)^{\varphi-j}\delta^{j}\big)^2 + (\delta-1)^{\varphi}(\delta^{\varphi}-2)^2 \\ &= (\delta-1)^{2\varphi}\sum_{j=1}^{\varphi-1} C_{\varphi}^{j}\Big(\frac{\delta^2}{\delta-1}\Big)^{j} + (\delta-1)^{\varphi}(\delta^{\varphi}-2)^2 \\ &= (\delta-1)^{2\varphi}\Big(\frac{\delta^2}{\delta-1}+1\Big)^{\varphi} - \Big(\frac{\delta^2}{\delta-1}\Big)^{\varphi} - 1 + (\delta-1)^{\varphi}(\delta^{\varphi}-2)^2 \\ M_1(\Gamma(\Lambda_{\mathcal{B}})) &= (\delta-1)^{\varphi}\Big((\delta^2+\delta-1)^{\varphi} - (\delta-1)^{\varphi} - 4(\delta^{\varphi}-1)\Big). \end{aligned}$$

□

Now, consider the following Figure 2 to visualize $M_1(\Gamma(\Lambda_{\mathcal{B}}))$ with φ dimension over the field of order δ.

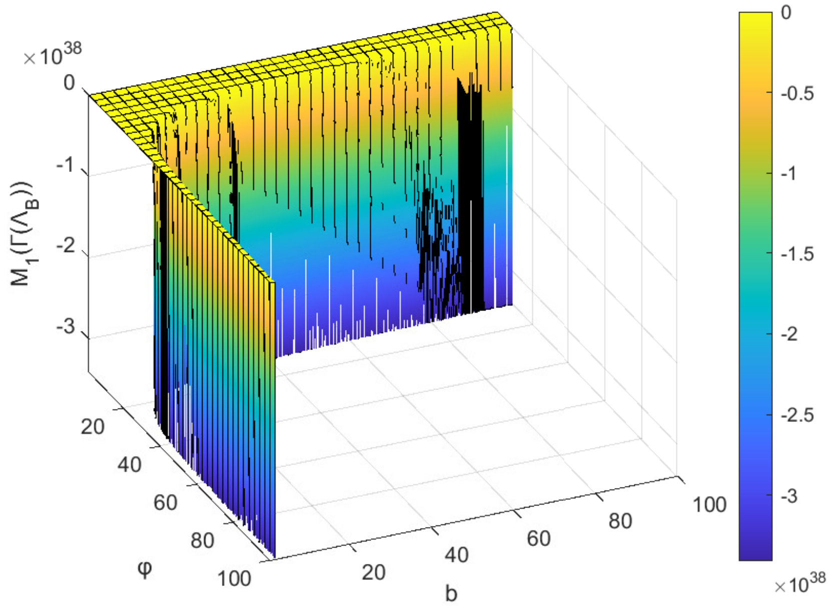

Figure 2. First Zagreb index of $\Gamma(\Lambda_{\mathcal{B}})$.

Because by using Remark 2, we have the following.

Theorem 12. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then the second Zagreb index*

$$\begin{aligned} M_2(\Gamma(\Lambda_{\mathcal{B}})) &= \frac{1}{2}\Big[\sum_{j=1}^{\varphi-1} C_{\varphi}^{j}(\delta-1)^{\varphi}\delta^{j}\Big(\sum_{t=\varphi-j}^{\varphi-1} C_{\varphi}^{|t+j-\varphi|}(\delta-1)^{\varphi}\delta^{t} + (\delta-1)^{\varphi}(\delta^{\varphi}-2)\Big) \\ &\quad +(\delta-1)^{\varphi}(\delta^{\varphi}-2)\Big(\sum_{j=1}^{\varphi-1} C_{\varphi}^{j}(\delta-1)^{\varphi}\delta^{j} + \big((\delta-1)^{\varphi}-1\big)(\delta^{\varphi}-2)\Big)\Big] \end{aligned}$$

Theorem 13. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then the Wiener polynomial*

$$W(\Gamma(\Lambda_{\mathcal{B}});x) = \frac{1}{2}(\beta-1)^{\varphi}\big((\beta+1)^{\varphi}-3\big)x + \frac{1}{2}\big((\beta^{\varphi}-2)\beta^{\varphi}-(\beta-1)^{\varphi}(\beta+1)^{\varphi}\big)x^2$$

Proof. By Remark 3, we have the following result:

$$\begin{aligned}
W(\Gamma(\Lambda_{\mathcal{B}});x) &= \sum_{\jmath=1}^{diam(\Gamma(\Lambda_{\mathcal{B}}))} d(\Gamma(\Lambda_{\mathcal{B}}),\jmath)x^{\jmath} \\
&= d(\Gamma(\Lambda_{\mathcal{B}}),1)x + d(\Gamma(\Lambda_{\mathcal{B}}),2)x^2 \\
&= \frac{x}{2}\Big[\Big(\sum_{\jmath=1}^{\varphi-1} C_{\varphi}^{\jmath}(\beta-1)^{\jmath}(\beta-1)^{\varphi-\jmath}\beta^{\jmath}\Big) + (\beta-1)^{\varphi}(\beta^{\varphi}-2)\Big] \\
&\quad + \frac{x^2}{2}\Big(\sum_{\jmath=1}^{\varphi-1} C_{\varphi}^{\jmath}(\beta-1)^{\jmath}\big(\beta^{\varphi}-2-(\beta-1)^{\varphi-\jmath}\beta^{\jmath}\big)\Big) \\
&= \frac{1}{2}\Big[(\beta-1)^{\varphi}\Big((\beta+1)^{\varphi}-3\Big)x + \big((\beta^{\varphi}-2)\beta^{\varphi}-(\beta-1)^{\varphi}(\beta+1)^{\varphi}\big)x^2\Big]
\end{aligned}$$

□

Theorem 14. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then the Hyper-Wiener index*

$$WW(\Gamma(\Lambda_{\mathcal{B}}) = 3(\beta^{\varphi}-2)\big(\beta^{\varphi} - \frac{2(\beta-1)^{\varphi}}{3} - 1\big) - 2(\beta-1)^{\varphi}((\beta+1)^{\varphi}-\beta^{\varphi}-1)$$

Proof. By Remark 3, we have the following result:

$$\begin{aligned}
WW(\Gamma(\Lambda_{\mathcal{B}}) &= \frac{1}{2}\sum_{\jmath=1}^{diam(\Gamma(\Lambda_{\mathcal{B}}))} \jmath(\jmath+1)d(\Gamma(\Lambda_{\mathcal{B}}),\jmath) \\
&= \frac{1}{2}\big(2d(\Gamma(\Lambda_{\mathcal{B}}),1) + 6d(\Gamma(\Lambda_{\mathcal{B}}),2)\big) \\
&= \frac{1}{2}\Big[2\Big(\sum_{\jmath=1}^{\varphi-1} C_{\varphi}^{\jmath}(\beta-1)^{\jmath}(\beta-1)^{\varphi-\jmath}\beta^{\jmath}\Big) + 2(\beta-1)^{\varphi}(\beta^{\varphi}-2) \\
&\quad + 6\Big(\sum_{\jmath=1}^{\varphi-1} C_{\varphi}^{\jmath}(\beta-1)^{\jmath}\big((\beta^{\varphi}-2)-(\beta-1)^{\varphi-\jmath}\beta^{\jmath}\big)\Big)\Big] \\
&= 3(\beta^{\varphi}-2)(\beta^{\varphi}-(\beta-1)^{\varphi}-1) - 2(\beta-1)^{\varphi}((\beta+1)^{\varphi}-1-\beta^{\varphi}) \\
&\quad + (\beta-1)^{\varphi}(\beta^{\varphi}-2) \\
&= 3(\beta^{\varphi}-2)(\beta^{\varphi}-(\beta-1)^{\varphi}-1+\frac{(\beta-1)^{\varphi}}{3}) \\
&\quad - 2(\beta-1)^{\varphi}((\beta+1)^{\varphi}-1-\beta^{\varphi}).
\end{aligned}$$

□

Now, consider the following Figure 3 to visualize $WW(\Gamma(\Lambda_{\mathcal{B}})$ with φ dimension over the field of order β.

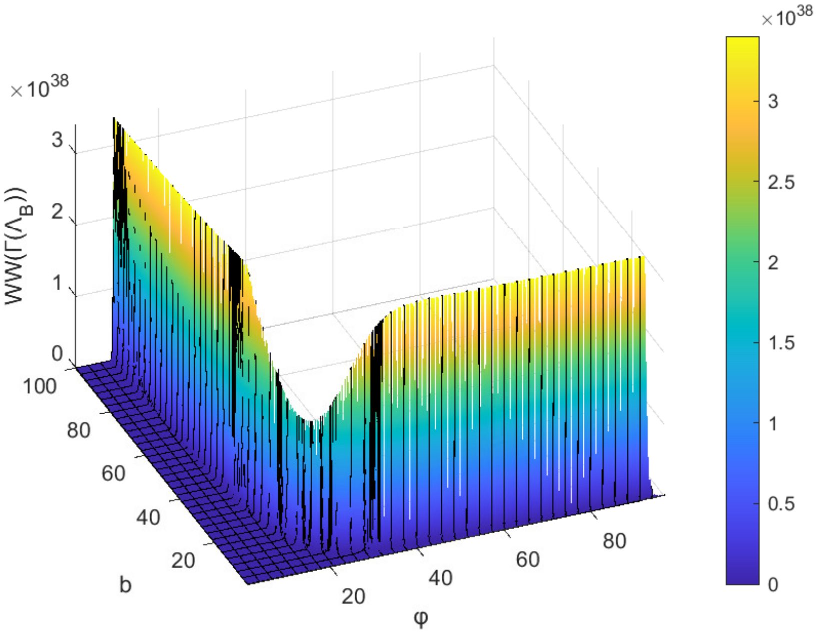

Figure 3. Hyper-Wiener index of $\Gamma(\Lambda_{\mathcal{B}})$.

5. Eccentricity Topological Indices

This section contains the vector space's eccentricity-based topological index of the nonzero component union graph with $dim(\Lambda) = \varphi$ and $O(\mathbb{F}) = \beta$.

Theorem 15. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then the eccentricity index.*

$$\xi(\Gamma(\Lambda_{\mathcal{B}})) = (\beta - 1)^{\varphi}\left[2(\beta + 1)^{\varphi} - \beta^{\varphi} - 4\right]$$

Proof. By Theorem 8, we obtain the following relation:

$$\begin{aligned}
\xi(\Gamma(\Lambda_{\mathcal{B}})) &= \sum_{\ell \in V(\Gamma(\Lambda_{\mathcal{B}}))} d(\ell) ecc(\ell) \\
&= 2\sum_{\jmath=1}^{\varphi-1} (\beta - 1)^{\varphi-\jmath} C_{\varphi}^{\jmath} \beta^{\jmath} (\beta - 1)^{\jmath} + (\beta - 1)^{\varphi}(\beta^{\varphi} - 2) \\
&= 2(\beta - 1)^{\varphi} \sum_{\jmath=1}^{\varphi-1} C_{\varphi}^{\jmath} \beta^{\jmath} + (\beta - 1)^{\varphi}(\beta^{\varphi} - 2) \\
\xi(\Gamma(\Lambda_{\mathcal{B}})) &= (\beta - 1)^{\varphi}\left[2(\beta + 1)^{\varphi} - \beta^{\varphi} - 4\right].
\end{aligned}$$

□

Now, consider the following Figure 4 to visualize the eccentric-connectivity index of $\Gamma(\Lambda_{\mathcal{B}})$.

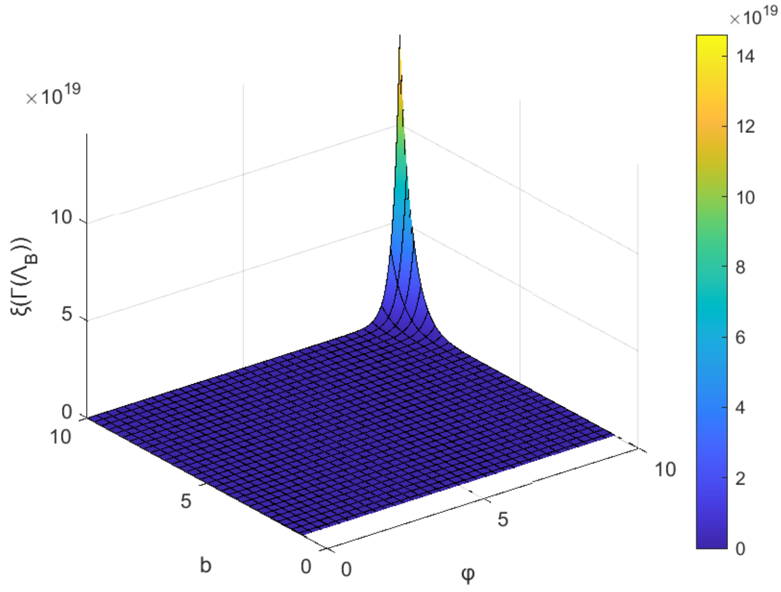

Figure 4. Eccentricity index of $\Gamma(\Lambda_{\mathcal{B}})$.

Theorem 16. *Let* $\Re = \Gamma(\Lambda_{\mathcal{B}})$, *then the total eccentricity index.*

$$\zeta(\Gamma(\Lambda_{\mathcal{B}})) = 2(\mathfrak{b}^{\varphi} - 1) - (\mathfrak{b} - 1)^{\varphi}$$

Proof. By Theorem 8, we obtain the following relation:

$$\begin{aligned}\zeta(\Gamma(\Lambda_{\mathcal{B}})) &= \sum_{\ell \in V(\Gamma(\Lambda_{\mathcal{B}}))} ecc(\ell) \\ &= 2\left[(\mathfrak{b}^{\varphi} - 1) - (\mathfrak{b} - 1)^{\varphi}\right] + (\mathfrak{b} - 1)^{\varphi} \\ \zeta(\Gamma(\Lambda_{\mathcal{B}})) &= 2(\mathfrak{b}^{\varphi} - 1) - (\mathfrak{b} - 1)^{\varphi}.\end{aligned}$$

□

Now, consider the following Figure 5 to visualize the total eccentricity index of $\Gamma(\Lambda_{\mathcal{B}})$.

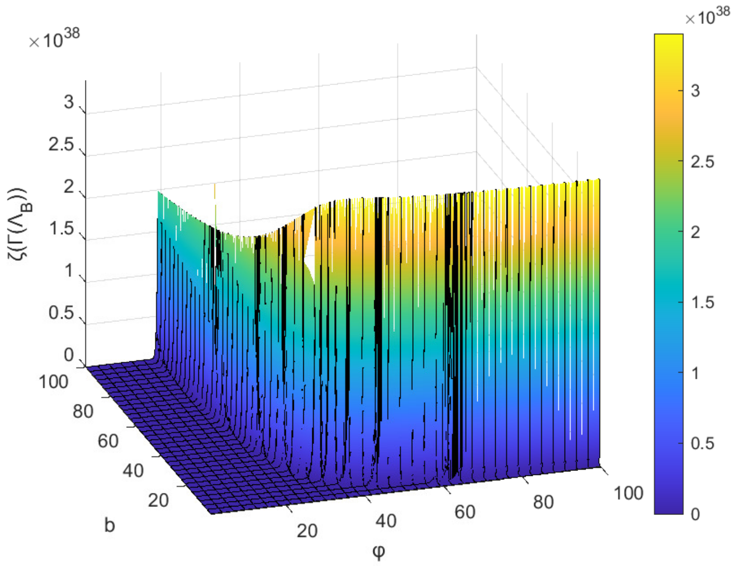

Figure 5. Total eccentricity index of $\Gamma(\Lambda_{\mathcal{B}})$.

Theorem 17. *If $dim(\Lambda) = \varphi$ and $O(\mathbb{F}) = \beta$, then*

$$M_1^*(\Gamma(\Lambda_{\mathcal{B}})) = (\beta-1)^{\varphi}\Big(2(\beta+1)^{\varphi} - (\beta-1)^{\varphi} - \beta^{\varphi} - 2\Big)$$

Proof. By Theorem 8 and Remark 4, we obtain

$$\begin{aligned} M_1^*(\Gamma(\Lambda_{\mathcal{B}})) &= (\beta-1)^{\varphi}\big((\beta-1)^{\varphi}-1\big) + 3\Big(\Big((\beta^{\varphi}-2) - (\beta-1)^{\varphi}+1\Big)(\beta-1)^{\varphi}\Big) \\ &\quad +4\Big((\beta-1)^{\varphi}\Big(\frac{(\beta+1)^{\varphi}+(\beta-1)^{\varphi}-2}{2} - \beta^{\varphi}+1\Big)\Big) \\ M_1^*(\Gamma(\Lambda_{\mathcal{B}})) &= (\beta-1)^{\varphi}\Big(2(\beta+1)^{\varphi} - (\beta-1)^{\varphi} - \beta^{\varphi} - 2\Big). \end{aligned}$$

□

Theorem 18. *If $dim(\Lambda) = \varphi$ and $O(\mathbb{F}) = \beta$, then*

$$M_1^{**}(\Gamma(\Lambda_{\mathcal{B}})) = 4\beta^{\varphi} - 3(\beta-1)^{\varphi} - 4$$

Proof. By Theorem 8, we obtain the number of vertices having eccentricity 1 is $(\beta-1)^{\varphi}$ and the number of vertices having eccentricity 2 is $\beta^{\varphi} - (\beta-1)^{\varphi} - 1$.

$$\begin{aligned} M_1^{**}(\Gamma(\Lambda_{\mathcal{B}})) &= 4\big(\beta^{\varphi} - (\beta-1)^{\varphi} - 1\big) + (\beta-1)^{\varphi} \\ M_1^{**}(\Gamma(\Lambda_{\mathcal{B}})) &= 4\beta^{\varphi} - 3(\beta-1)^{\varphi} - 4. \end{aligned}$$

□

Theorem 19. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then*

$$M_2^*(\Gamma(\Lambda_{\mathcal{B}})) = (\beta-1)^{\varphi}\Big(2(\beta+1)^{\varphi} - 2\beta^{\varphi} - 1\Big)$$

Proof. By Theorem 8 and Remark 4, we obtain

$$\begin{aligned} M_2^*(\Gamma(\Lambda_{\mathcal{B}})) &= (\beta-1)^{\varphi}\big((\beta-1)^{\varphi}-1\big) + 2\Big(\Big((\beta^{\varphi}-2) - (\beta-1)^{\varphi}+1\Big)(\beta-1)^{\varphi}\Big) \\ &\quad +4\Big((\beta-1)^{\varphi}\Big(\frac{(\beta+1)^{\varphi}+(\beta-1)^{\varphi}-2}{2} - \beta^{\varphi}+1\Big)\Big) \\ M_2^*(\Gamma(\Lambda_{\mathcal{B}})) &= (\beta-1)^{\varphi}\Big(2(\beta+1)^{\varphi} - 2\beta^{\varphi} - 1\Big) \end{aligned}$$

□

Theorem 20. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then average eccentricity index*

$$aveg(\Gamma(\Lambda_{\mathcal{B}})) = 2 - \frac{(\beta-1)^{\varphi}}{\beta^{\varphi}-1}$$

Proof. By Theorem 8, we obtain the number of vertices having eccentricity 1 is $(\beta-1)^{\varphi}$ and the number of vertices having eccentricity 2 is $\beta^{\varphi} - (\beta-1)^{\varphi} - 1$.

$$\begin{aligned} aveg(\Gamma(\Lambda_{\mathcal{B}})) &= \frac{1}{\beta^{\varphi}-1}\sum_{\ell \in V(\Gamma(\Lambda_{\mathcal{B}}))} ecc(\ell) \\ &= \frac{2\big((\beta^{\varphi}-1) - (\beta-1)^{\varphi}\big) + (\beta-1)^{\varphi}}{\beta^{\varphi}-1} \\ aveg(\Gamma(\Lambda_{\mathcal{B}})) &= 2 - \frac{(\beta-1)^{\varphi}}{\beta^{\varphi}-1} \end{aligned}$$

□

By Theorems 8 and 9, we obtain

Theorem 21. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then eccentric distance sum index*

$$\xi^{DS}(\Gamma(\Lambda_{\mathcal{B}})) = 4(\beta^{\varphi} - 2)\big(\beta^{\varphi} - \frac{3}{4} - (\beta - 1)^{\varphi}\big) - 2(\beta - 1)^{\varphi}\big((\beta + 1)^{\varphi} - 1 - \beta^{\varphi}\big)$$

Proof.

$$\begin{aligned}
\xi^{DS}(\Gamma(\Lambda_{\mathcal{B}})) &= \sum_{\ell \in V(\Gamma(\Lambda_{\mathcal{B}}))} ecc(\ell) D(\wp | V(\Gamma(\Lambda_{\mathcal{B}}))) \\
&= \sum_{J=1}^{\varphi-1} 2C_{\varphi}^{J} (\beta - 1)^{J} \big(2(\beta^{\varphi} - 2) - (\beta - 1)^{\varphi - J}\beta^{J}\big) + \beta^{\varphi} - 2 \\
&= 4(\beta^{\varphi} - 2)\big(\sum_{J=1}^{\varphi-1} 2C_{\varphi}^{J}(\beta - 1)^{J}\big) - \big(2\sum_{J=1}^{\varphi-1} 2C_{\varphi}^{J}(\beta - 1)^{\varphi}\beta^{J}\big) + \beta^{\varphi} - 2 \\
\xi^{DS}(\Gamma(\Lambda_{\mathcal{B}})) &= 4(\beta^{\varphi} - 2)\big(\beta^{\varphi} - \frac{3}{4} - (\beta - 1)^{\varphi}\big) - 2(\beta - 1)^{\varphi}\big((\beta + 1)^{\varphi} - 1 - \beta^{\varphi}\big).
\end{aligned}$$

□

Because by the Remark 2, we have the following result.

Theorem 22. *Let $\Re = \Gamma(\Lambda_{\mathcal{B}})$, then ABC index*

$$\begin{aligned}
ABC(\Gamma(\Lambda_{\mathcal{B}})) &= \frac{1}{2}\Big\{\Big[\sum_{J=1}^{\varphi-1}\Big(\sum_{t=\varphi-J}^{\varphi-1} C_{J}^{|t+J-\varphi|}(\beta - 1)^{t}\sqrt{\frac{(\beta-1)^{\varphi-J}\beta^{J} + (\beta-1)^{\varphi-t}\beta^{t} - 2}{(\beta-1)^{\varphi-J}\beta^{J}(\beta-1)^{\varphi-t}\beta^{t}}} \\
&+(\beta-1)^{\varphi}\sqrt{\frac{(\beta-1)^{\varphi-J}\beta^{J} + (\beta^{\varphi}-2) - 2}{(\beta-1)^{\varphi-J}\beta^{J}(\beta^{\varphi}-2)}}\Big)\Big] \\
&+\Big(\sum_{J=1}^{\varphi-1}(\beta-1)^{J}\sqrt{\frac{(\beta-1)^{\varphi-J}\beta^{J} + (\beta^{\varphi}-2) - 2}{(\beta-1)^{\varphi-J}\beta^{J}\beta^{\varphi} - 2}}\Big) + \\
&(\beta-1)^{\varphi}\big((\beta-1)^{\varphi} - 1\big)\sqrt{\frac{2(\beta^{\varphi}-2) - 2}{(\beta^{\varphi}-2)^{2}}}\Big\}
\end{aligned}$$

Example 1. *Let us consider the following Figure 6 illustrating the $\Gamma(\Lambda_{\mathcal{B}})$ of the dimension $\varphi = 4$ over the field of order $\beta = 2$.*

Now, the following Table 1 gives the information about some topological index of nonzero component union graph of vector space.

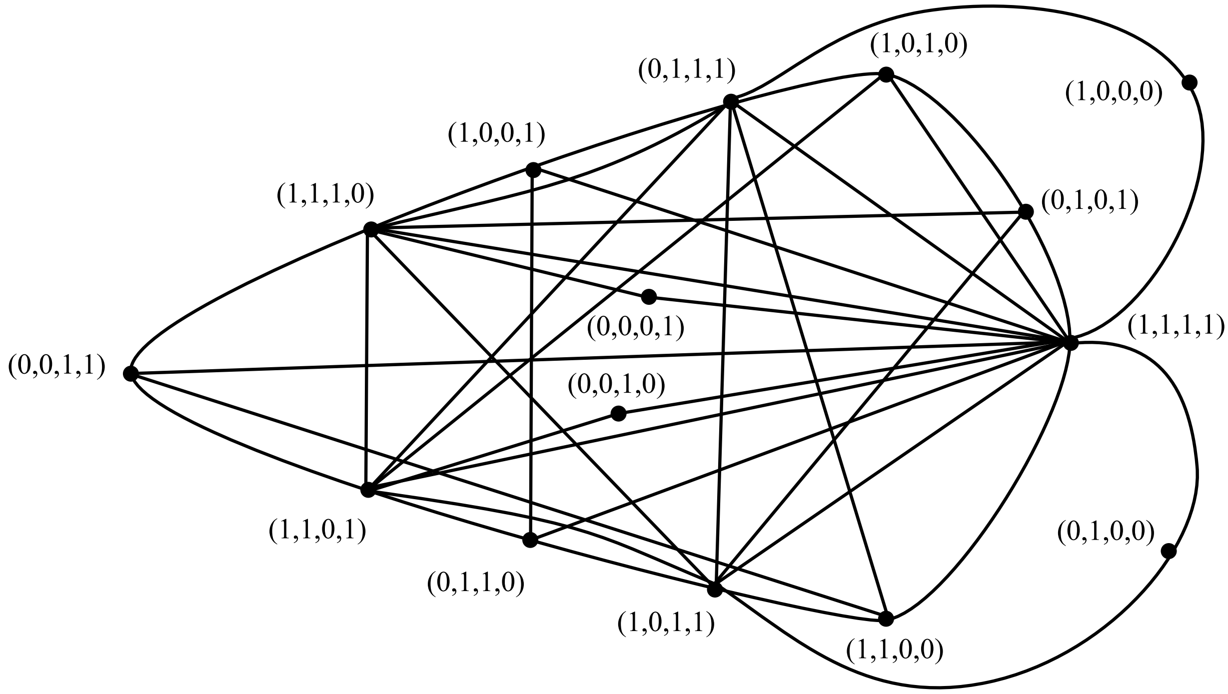

Figure 6. $\Gamma(\Lambda_{\mathcal{B}})$ with $\varphi = 4$ and $\mathscr{b} = 2$.

Table 1. Topological index of $\Gamma(\Lambda_{\mathcal{B}})$.

$\mathscr{b}$	Index	φ = 2	φ = 3	φ = 4
2	$W(\Gamma(\Lambda_{\mathcal{B}}))$	2	27	153
	$WW(\Gamma(\Lambda_{\mathcal{B}}))$	6	78	747
	$\xi(\Gamma(\Lambda_{\mathcal{B}}))$	10	42	142
	$\zeta(\Gamma(\Lambda_{\mathcal{B}}))$	5	13	29
	$aveg(\Gamma(\Lambda_{\mathcal{B}}))$	1.666	1.857	1.933
	$\xi^{DS}(\Gamma(\Lambda_{\mathcal{B}}))$	10	114	670
3	$W(\Gamma(\Lambda_{\mathcal{B}}))$	32	312	3030
	$WW(\Gamma(\Lambda_{\mathcal{B}}))$	64	974	10,864
	$\xi(\Gamma(\Lambda_{\mathcal{B}}))$	76	776	6832
	$\zeta(\Gamma(\Lambda_{\mathcal{B}}))$	12	44	144
	$aveg(\Gamma(\Lambda_{\mathcal{B}}))$	1.5	1.692	1.8
	$\xi^{DS}(\Gamma(\Lambda_{\mathcal{B}}))$	71	1249	14,735

6. Conclusions

In this paper, we found the topological indices of nonzero component union graphs from vector spaces Λ with order n over the field $\mathbb{F}$ with order $\mathscr{b}$, and we give the general formula and a comparison table for finding a different topological index to the number of graphs constructed from the vector space. Depending on the respective quantitative data, these resulting indices are graphically contrasted. Future scholars can build on our research of the indices for these structures to find and investigate other algebraic structures and their features.

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