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Fixed Point Approach: Ulam Stability Results of Functional Equation in Non-Archimedean Fuzzy φ-2-Normed Spaces and Non-Archimedean Banach Spaces

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Abstract: In this work, we introduce a new type of generalized mixed-type quadratic-additive functional equation and obtain its general solution. The main goal of this work is to investigate Ulam stability of this mixed type of quadratic-additive functional equation in the setting of non-Archimedean fuzzy φ-2-normed space and non-Archimedean Banach space using the direct and fixed point approaches by taking into our account two cases: even mapping and odd mapping.

Keywords: non-Archimedean φ-2-normed space; fixed point; quadratic-additive functional equation

MSC: 39B52; 39B72; 39B82

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1. Introduction and Preliminaries

The study of stability problems for functional equations is one of the essential research areas in mathematics, which originated in issues related to applied mathematics. The first question concerning the stability of homomorphisms was given by Ulam [1] as follows.

Given a group $(G, *)$, a metric group $(G', \cdot)$ with the metric d, and a mapping f from G and G', does $\delta > 0$ exist such that

$$d(f(x * y), f(x) \cdot f(y)) \leq \delta$$

for all $x, y \in G$? If such a mapping exists, then does a homomorphism $g : G \to G'$ exist such that

$$d(f(x), g(x)) \leq \epsilon$$

for all $x \in G$? Ulam defined such a problem in 1940 and solved it the following year for the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

by the method of Hyers [2]. The consequence of Hyers becomes stretched out by Aoki [3] with the aid of assuming the unbounded Cauchy contrasts. The additive mapping demonstrated by Rassias [4]. Rassias's result is summed up by Găvruţa [5].

From that point forward, numerous stability problems for different functional equations have been explored in [6–15]. Later, the stability issues for different types of functional equations were investigated in [16,17]. Particularly, while the possibility of intuitionistic fuzzy normed space was introduced in [18] and further studied in [19,20] to manage some summability issues.

The additive functional equation and quadratic functional equation are denoted by

$$f(x+y) = f(x) + f(y)$$

and

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

respectively. Every solution to the functional equations containing additive and quadratic terms is particularly called as an additive function and a quadratic function, respectively.

During the most recent thirty years, the hypothesis of non-Archimedean spaces has acquired the interest of physicists for their research specifically in issues coming from quantum physical science, p-adic strings and superstrings [21]. In [22], Hensel presented a normed space that does not have the Archimedean property. Although numerous outcomes in the traditional normed space hypothesis have a non-Archimedean counterpart, their confirmations are basically unique and require a completely new sort of instinct [21,23–26].

In this current work, the authors present a new mixed type quadratic-additive functional equation

$$\psi\left(\sum_{1\le a\le n} s_a\right) + \sum_{1\le a\le n}\psi\left(-s_a + \sum_{1\le a<b\le n} s_b\right) = (n-3)\sum_{1\le a<b\le n}\psi(s_a+s_b) - \left(n^2-5n+2\right)\sum_{a=1}^{n}\left[\frac{\psi(s_a)+\psi(-s_a)}{2}\right] - \left(n^2-5n+4\right)\sum_{a=1}^{n}\left[\frac{\psi(s_a)-\psi(-s_a)}{2}\right], \quad (1)$$

where $n > 0$ is a non-negative integer with $\mathbb{N} - \{0,1,2\}$, and obtain its general solution. The main aim of this work is to investigate the Ulam stability of equation (1) by using direct and fixed point methods in non-Archimedean fuzzy φ-2-normed space and non-Archimedean Banach space. It is clear that the mapping $\psi(s) = as^2 + bs$ is a solution of (1).

We can refer to some usual definitions, terminology and notions to achieve our main results from [16,17,19,20,22–24,27]

A map $|\cdot| : \mathbb{K} \to [0,\infty)$ is a valuation such that zero is the only one element having the zero valuation, $|k_1k_2| = |k_1|\,|k_2|$, and the inequality of the triangle holds true, that is, $|k_1+k_2| \le |k_1| + |k_2|$, for all $k_1, k_2 \in \mathbb{K}$.

We call a field $\mathbb{K}$ valued if $\mathbb{K}$ holds a valuation. Examples of valuations include the typical absolute values of $\mathbb{R}$ and $\mathbb{C}$.

Consider a valuation that satisfies a criterion that is stronger than the triangle inequality. A $|\cdot|$ is called a non-Archimedean valuation if the triangle inequality is replaced by $|k_1+k_2| \le \max\{|k_1|,|k_2|\}$, for all $k_1,k_2 \in \mathbb{K}$, and a field is called a non-Archimedean field. Evidently, $|-1| = 1 = |1|$ and $|n|$ are greater than or equal to 1, for all n in $\mathbb{N}$. The map $|\cdot|$ takes everything except 0 for 1, and $|0| = 0$ is a basic example of a non-Archimedean valuation.

Definition 1 ([22]). *By a non-Archimedean field, we denote a field $\mathbb{K}$ equipped with a valuation $|\cdot| : \mathbb{K} \to [0,\infty)$ satisfies $|p| = 0 \Leftrightarrow p = 0$, $|pq| = |p||q|$, and $|p+q| \le \max\{|p|,|q|\}$ for every $p,q \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|m| \le 1$ for every $m \in \mathbb{N}$.*

Let V be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A mapping $\|\cdot\| : V \to \mathbb{R}$ is referred to as a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|v\| = 0 \Leftrightarrow v = 0$;
(ii) $\|pv\| = |p|\|v\|, \quad (v \in V, p \in \mathbb{K})$;
(iii) $\|v_1+v_2\| \le \max\{\|v_1\|,\|v_2\|\} \quad (v_1,v_2 \in V)$ *(called as ultrametric).*

The pair $(V, \|\cdot\|)$ is known as a non-Archimedean normed space.

We know that

$$\|v_n - v_m\| \leq \max\{\|v_{j+1} - v_j\|; m \leq j \leq n-1\} \ \ (n \geq m).$$

sequence $\{v_n\}$ is called Cauchy if $\{v_{n+1} - v_n\} \to 0$ in a non-Archimedean normed space V. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

Definition 2 ([27]). *A t-norm $\diamond$ is a function $\diamond : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, non decreasing and satisfies $\lambda \diamond 1 = \lambda$ for every $\lambda \in [0,1]$.*

Definition 3. *Let V be a non-Archimedean normed space and a sequence $\{v_p\}$ in V. Then*

1. *A sequence $\{v_p\}_{p=1}^{\infty}$ in V is a Cauchy sequence if and only if $\{v_{p+1} - v_p\}_{p=1}^{\infty}$ converges to 0.*
2. *$\{v_p\}$ is called convergent if, for any $\epsilon > 0$, there is a integer $p > 0$ in $\mathbb{N}$ and $v \in V$ satisfies*

$$\|v_p - v\| \leq \epsilon \ \ \text{for all } p \geq \mathbb{N}.$$

Then we called as v is a limit of the sequence $\{v_p\}$ and represented by $\lim_{p\to\infty} v_p = v$.

3. *If every Cauchy sequence in a non-Archimedean normed space V converges, it is called a non-Archimedean Banach space.*

Example 1. *For a given prime number p, any non-zero rational number x can be uniquely written as $x = \frac{a}{b}p^n x$, where $a, b, n_x \in \mathbb{Z}$ and a, b are integers not divisible by p. Then, the function $|\cdot|_p : \mathbb{Q} \to [0, +\infty)$ defined by*

$$|x| := \begin{cases} 0, & if \quad x = 0; \\ p^{-n_x}, & if \quad x \neq 0; \end{cases}$$

is a non-Archimedean (non-trivial) valuation on $\mathbb{Q}$.

Definition 4 ([23,24]). *Let V be a real linear space with dimension greater than 1 and $F : V^2 \times [0,\infty) \to [0,1]$ satusfying the following conditions: For all $v_1, v_2, v_3 \in V$ and all $p, q \in [0,\infty)$,*

(NAF1) $F(v_1, v_2, 0) = 0$;
(NAF2) $F(v_1, v_2, p) = 1$, for all $p > 0$ if and only if v_1, v_2 are linear dependent;
(NAF3) $F(v_1, v_2, p) = N(v_2, v_1, p)$ for all $v_1, v_2 \in V$, and $p > 0$;
(NAF4) $F(v_1 + v_2, v_3, \max(p, q)) \geq \min(F(v_1, v_3, p) \diamond F(v_2, v_3, q))$;
(NAF5) $F(v_1, v_2, \cdot) : [0, \infty) \to [0,1]$ is left continuous.
(NAF6) $F(\alpha v_1, v_2, p) = F\left(v_1, v_2, \frac{p}{\varphi(\alpha)}\right)$, for all $\alpha \in \mathbb{R}$.

The triple $(V, F, \diamond)$ will be referred to as a non-Archimedean fuzzy φ-2-normed space.

Definition 5 ([23,24]). *Suppose we have a non-Archimedean fuzzy φ-2-normed space $(V, F, \diamond)$ and $\{v_n\}$ in V. Then $\{v_n\}$ is said to be convergent if there is $v \in V$ satisfying*

$$\lim_{n\to\infty} F(v_n - v, w, \epsilon) = 1,$$

for all $w \in V$ and all $\epsilon > 0$. In this case, v is the limit point of v_n. We denote it by

$$F - \lim_{n\to\infty} v_n = v.$$

Definition 6. *A sequence $\{v_n\} \in V$ is said to be Cauchy if for a given $\delta > 0$, there is $m \in \mathbb{N}$ such that*

$$F(v_{n+q} - v_n, w, \epsilon) < 1 - \delta$$

for all $w \in V$, $q, \varepsilon > 0$ and $n > m$.

Every convergent sequence in a non-Archimedean fuzzy φ-2-normed space is a Cauchy sequence.

Definition 7. *If every Cauchy sequence is convergent, a non-Archimedean fuzzy φ-2-normed space is called a non-Archimedean fuzzy φ-2-Banach space.*

Theorem 1 ([28,29])**.** *If (V, d) is a complete generalized metric space and a mapping $\Phi : V \to V$ is strictly contractive with $0 < L < 1$. Then, for each given element $v \in V$, either*

(B1) $d(\Phi^n v, \Phi^{n+1} v) = \infty$, *for all* $v \geq 0$,
or
(B2) *there is* $n_0 \in \mathbb{N}$ *satisfies*
- *(i)* $d(\Phi^n v, \Phi^{n+1} v) < \infty$, *for all* $n \geq n_0$;
- *(ii)* $\{\Phi^n v\}$ *is convergent to a fixed point* t^* *of* Φ;
- *(iii)* t^* *is the only one fixed point of* Φ *in the set* $\Delta = \{t \in V : d(\Phi^{n_0} v, t) < \infty\}$;
- *(iv)* $d(t^*, t) \leq \frac{1}{1-L} d(t, \Phi t)$, *for all* $t \in \Delta$.

2. Solution

In this section, let us consider P and R are real vector spaces.

Theorem 2. *If an odd mapping $\psi : P \to R$ satisfies the functional Equation (1), then the mapping ψ is additive.*

Proof. In the view of oddness, we have $\psi(-s) = -\psi(s)$, for all $s \in P$, then the functional Equation (1) reduced to

$$\psi\left(\sum_{1\leq a\leq n} s_a\right) + \sum_{1\leq a\leq n} \psi\left(-s_a + \sum_{1\leq a<b\leq n} s_b\right) = (n-3) \sum_{1\leq a<b\leq n} \psi(s_a + s_b) - \left(n^2 - 5n + 4\right) \sum_{1\leq a\leq n} \psi(s_a), \quad (2)$$

for all $s_1, s_2, \cdots, s_n \in P$. Now, replacing $(s_1, s_2, \cdots, s_n)$ by $(0, 0 \cdots, 0)$ in (2), we have $\psi(0) = 0$. Replacing $(s_1, s_2, \cdots, s_n)$ by $(s, s, 0, \cdots, 0)$ in (2), we get

$$\psi(2s) = 2\psi(s), \quad (3)$$

for all $s \in P$. Replacing s by $2s$ in (3), we obtain

$$\psi(2^2 s) = 2^2 \psi(s), \quad (4)$$

for all $s \in P$. In general, for any non-negative integer n, we get

$$\psi(2^n s) = 2^n \psi(s),$$

for all $s \in P$. Replacing $(s_1, s_2, \cdots, s_n)$ by $(x, y, 0, \cdots, 0)$ in (2), we have

$$\psi(x + y) = \psi(x) + \psi(y),$$

for all $x, y \in P$. Hence, the mapping ψ is additive. □

Theorem 3. *If an even mapping $\psi : P \to R$ satisfies the functional Equation (1), then the mapping ψ is quadratic.*

Proof. In the view of evenness, $\psi(-s) = \psi(s)$, for all $s \in P$, then the functional Equation (1) reduced to

$$\psi\left(\sum_{1\leq a\leq n} s_a\right) + \sum_{1\leq a\leq n} \psi\left(-s_a + \sum_{b=1;a\neq b}^{n} s_b\right) = (n-3) \sum_{1\leq a<b\leq n} \psi(s_a + s_b) - \left(n^2 - 5n + 2\right) \sum_{a=1}^{n} \psi(s_a), \tag{5}$$

for all $s_1, s_2, \cdots, s_n \in P$. Now, letting $s_1 = s_2 = \cdots = s_n = 0$ in (5), we obtain $\psi(0) = 0$. Replacing $(s_1, s_2, \cdots, s_n)$ by $(s, s, 0, \cdots, 0)$ in (5), we obtain

$$\psi(2s) = 2^2\psi(s), \tag{6}$$

for all $s \in P$. Replacing s by $2s$ in (6), we get

$$\psi(2^2 s) = 2^4\psi(s), \tag{7}$$

for all $s \in P$. In general, for any non-negative integer n, we have

$$\psi(2^n s) = 2^{2n}\psi(s),$$

for all $s \in P$. Finally, replacing $(s_1, s_2, \cdots, s_n)$ by $(x, y, 0, \cdots, 0)$ in (5), we have

$$\psi(x + y) + \psi(x - y) = 2\psi(x) + 2\psi(y),$$

for all $x, y \in P$. Hence, the mapping ψ is quadratic. □

Proposition 1. *A mapping $\psi : P \to R$ satisfies $\psi(0) = 0$ and (1) if and only if there exist a symmetric bi-additive mapping $Q : P^2 \to R$, and an additive mapping $A : P \to R$ satisfying $\psi(s) = Q(s, s) + A(s)$ for all $s \in P$.*

Proof. Let a mapping $\psi : P \to R$ with $\psi(0) = 0$ satisfy the functional Equation (1). We divide ψ into the odd part and even parts as

$$\psi_o(v) = \frac{\psi(v) - \psi(-v)}{2}, \quad \psi_e(v) = \frac{\psi(v) + \psi(-v)}{2}, \quad v \in V,$$

respectively. Clearly, $\psi(v) = \psi_e(v) + \psi_o(v)$, for all $v \in V$. It is easy to prove that ψ_o and ψ_e satisfy the functional Equation (1). Hence by Theorems 2 and 3, we conclude that ψ_o and ψ_e are additive and quadratic, respectively. So there exist a symmetric bi-additive mapping $Q : V \times V \to W$ which satisfies $\psi_e(v) = Q(v, v)$ and an additive mapping $A : V \to W$ which satisfies $\psi_o(v) = A(v)$, for all $v \in V$. Hence $\psi(v) = Q(v, v) + A(v)$ for all $v \in V$.

Conversely, suppose that there exists a mapping $Q : V \times V \to W$ which is symmetric bi-additive and a mapping $A : V \to W$ which is additive such that $\psi(v) = Q(v, v) + A(v)$ for all $v \in V$. Easily, we can show that the mappings $v \mapsto Q(v, v)$ and $A : V \to W$ fulfill the functional Equation (1). Thus the mapping $\psi : V \to W$ satisfies the functional Equation (1). □

For notational simplicity, we may define a function $\psi : P \to R$ by

$$
\begin{aligned}
&D\psi(s_1, s_2, \cdots, s_n) \\
&= \psi\left(\sum_{a=1}^{n} s_a\right) + \sum_{a=1}^{n} \psi\left(-s_a + \sum_{b=1; a\neq b}^{n} s_b\right) - (n-3) \sum_{1\leq a<b\leq n} \psi(s_a + s_b) \\
&\quad + \left(n^2 - 5n + 2\right) \sum_{a=1}^{n} \left[\frac{\psi(s_a) + \psi(-s_a)}{2}\right] + \left(n^2 - 5n + 4\right) \sum_{a=1}^{n} \left[\frac{\psi(s_a) - \psi(-s_a)}{2}\right],
\end{aligned}
$$

for all $s_1, s_2, \cdots, s_n \in P$.

3. Ulam Stability Results in Non-Archimedean Fuzzy φ-2-Normed Spaces

In the following subsections, we assume that P , $(R, F, \diamond)$ and $(Z, F', \diamond)$ are a linear vector space, real non-Archimedean fuzzy φ-2-Banach space and real non-Archimedean fuzzy φ-2-normed space, respectively.

3.1. Stability for the Even Case: Direct Method

In this subsection, we investigate the Ulam stability of the functional Equation (1) for the even case by using direct method.

Theorem 4. *Let $j = \pm 1$ be fixed and a mapping $\chi : P^n \to Z$ such that for some α with $0 < \left(\frac{\varphi(\alpha)}{\varphi(2^2)}\right)^j < 1$,*

$$
F'\left(\chi\left(2^j s, 2^j s, 0, \cdots, 0\right), w, \epsilon\right) \geq F'\left([\varphi(\alpha)]^j \chi(s, s, 0, \cdots, 0), w, \epsilon\right), \tag{8}
$$

for all $s, w \in V$ and all $\epsilon > 0$, and

$$
\lim_{i\to\infty} F'\left(\chi\left(2^{ji} s_1, 2^{ji} s_2, \cdots, 2^{ji} s_n\right), w, \left[\varphi\left(2^{2i}\right)\right]^j \epsilon\right) = 1, \tag{9}
$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$. If an even mapping $\psi : P \to R$ such that

$$
F(D\psi(s_1, s_2, \cdots, s_n), w, \epsilon) \geq F'(\chi(s_1, s_2, \cdots, s_n), w, \epsilon), \tag{10}
$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$, then the limit function

$$
Q_2(s) = F - \lim_{i\to\infty} \frac{\psi(2^{ji} s)}{2^{2ji}} \tag{11}
$$

exists for every $s \in P$ and a unique qudratic mapping $Q_2 : P \to R$ satisfying the functional Equation (1) and

$$
F(\psi(s) - Q_2(s), w, \epsilon) \geq F'\left(\chi(s, s, 0, \cdots, 0), w, 2\epsilon\left|\varphi(2^2) - \varphi(\alpha)\right|\right), \tag{12}
$$

for all $s, w \in P$ and all $\epsilon > 0$.

Proof. Take $j = 1$. Replacing $(s_1, s_2, \cdots, s_n)$ by $(s, s, 0, \cdots, 0)$ in (10), we have

$$
F(2\psi(2s) - 8\psi(s), w, \epsilon) \geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon), \tag{13}
$$

for all $s, w \in P$ and all $\epsilon > 0$. From inequality (13), we obtain

$$
F\left(\frac{\psi(2s)}{2^2} - \psi(s), w, \frac{\epsilon}{2\varphi(2^2)}\right) \geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon), \tag{14}
$$

for all $s, w \in P$ and all $\epsilon > 0$. Replacing s by $2^i s$ in (14), we get

$$F\left(\frac{\psi(2^{i+1}s)}{2^{2(i+1)}} - \frac{\psi(2^i s)}{2^{2i}}, w, \frac{\epsilon}{2\varphi(2^{2(i+1)})}\right) \geq F'\left(\chi(2^i s, 2^i s, 0, \cdots, 0), w, \epsilon\right), \quad (15)$$

for all $s, w \in P$ and all $\epsilon > 0$. Using (8), (NAF6) in (15), we get

$$F\left(\frac{\psi(2^{i+1}s)}{2^{2(i+1)}} - \frac{\psi(2^i s)}{2^{2i}}, w, \frac{\epsilon}{2\varphi(2^{2(i+1)})}\right) \geq F'\left(\chi(s, s, 0, \cdots, 0), w, \frac{\epsilon}{\varphi(\alpha^i)}\right), \quad (16)$$

for all $s, w \in P$ and all $\epsilon > 0$. Replacing ϵ by $\varphi(\alpha^i)\epsilon$ in (16), we obtain

$$F\left(\frac{\psi(2^{i+1}s)}{2^{2(i+1)}} - \frac{\psi(2^i s)}{2^{2i}}, w, \frac{\varphi(\alpha^i)\epsilon}{2\varphi(2^{2(i+1)})}\right) \geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon), \quad (17)$$

for all $s, w \in P$ and all $\epsilon > 0$. Since

$$\frac{\psi(2^i s)}{2^{2i}} - \psi(s) = \sum_{j=0}^{i-1} \frac{\psi(2^{j+1}s)}{2^{2(j+1)}} - \frac{\psi(2^j s)}{2^{2j}}, \quad (18)$$

for all $s \in P$, by (17) and (18), we have

$$F\left(\frac{\psi(2^i s)}{2^{2i}} - \psi(s), w, \sum_{j=0}^{i-1} \frac{\varphi(\alpha^j)\epsilon}{2\varphi(2^{2(j+1)})}\right) \quad (19)$$

$$\geq \min \cup_{j=0}^{i-1} \left\{ F\left(\frac{\psi(2^{j+1}s)}{2^{2(j+1)}} - \frac{\psi(2^j s)}{2^{2j}}, w, \frac{\varphi(\alpha^j)\epsilon}{2\varphi(2^{2(j+1)})}\right) \right\}$$

$$\geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon), \quad (20)$$

for all $s, w \in P$ and all $\epsilon > 0$. Replacing s by $2^k s$ in (19) and using (8) and (NAF6), we obtain

$$F\left(\frac{\psi(2^{i+k}s)}{2^{2(i+k)}} - \frac{\psi(2^k s)}{2^{2(k)}}, w, \sum_{j=k}^{i+k-1} \frac{\varphi(\alpha^j)\epsilon}{2\varphi(2^{2(j+1)})}\right) \geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon), \quad (21)$$

for all $s, w \in P$, $\epsilon > 0$ and all $i, k \geq 0$. Replacing ϵ by $\frac{\epsilon}{\sum_{j=k}^{i+k-1} \frac{\varphi(\alpha^j)}{2\varphi(2^{2(j+1)})}}$ in (21), we get

$$F\left(\frac{\psi(2^{i+k}s)}{2^{2(i+k)}} - \frac{\psi(2^k s)}{2^{2(k)}}, w, \epsilon\right) \geq F'\left(\chi(s, s, 0, \cdots, 0), w, \frac{\epsilon}{\sum_{j=k}^{i+k-1} \frac{\varphi(\alpha^j)}{2\varphi(2^{2(j+1)})}}\right), \quad (22)$$

for all $s, w \in P$, $\epsilon > 0$ and all $i, k \geq 0$. Simce, $0 < \varphi(\alpha) < \varphi(2^2)$ and $\sum_{j=0}^{\infty} \left(\frac{\varphi(\alpha)}{\varphi(2^2)}\right) < \infty$, in order to the Cauchy criteria for convergence, $\left\{\frac{\psi(2^i s)}{2^{2i}}\right\}$ is a Cauchy sequence in $(R, F', \diamond)$. Since $(R, F', \diamond)$ is a non-Archimedean fuzzy φ-2-Banach space, this sequence $\left\{\frac{\psi(2^i s)}{2^{2i}}\right\}$ converges to $Q_2(s) \in R$. Now, we can define a mapping $Q_2 : P \to R$ by

$$Q_2(s) = F - \lim_{i \to \infty} \frac{\psi(2^i s)}{2^{2i}},$$

for all $s \in P$. Taking $k = 0$ in (22), we get

$$F\left(\frac{\psi(2^i s)}{2^{2i}} - \psi(s), w, \epsilon\right) \geq F'\left(\chi(s, s, 0, \cdots, 0), w, \frac{\epsilon}{\sum_{j=0}^{i-1} \frac{\varphi(\alpha^j)}{2\varphi(2^{2(j+1)})}}\right), \tag{23}$$

for all $s, w \in P$ and all $\epsilon > 0$. Passing the limit $i \to \infty$ in (23) and using (NAF5), we get

$$F(\psi(s) - Q_2(s), w, \epsilon) \geq F'\left(\chi(s, s, 0, \cdots, 0), w, 2\epsilon\left(\varphi(2^2) - \varphi(\alpha)\right)\right),$$

for all $s, w \in P$ and all $\epsilon > 0$. To prove that Q_2 satisfies the functional Equation (1), replacing $(s_1, s_2, \cdots, s_n)$ by $(2^i s_1, 2^i s_2, \cdots, 2^i s_n)$ in (10), we obtain

$$F\left(\frac{1}{2^{2i}} D\psi(2^i s_1, 2^i s_2, \cdots, 2^i s_n), w, \epsilon\right) \geq F'\left(\chi(2^i s_1, 2^i s_2, \cdots, 2^i s_n), w, \varphi(2^{2i})\epsilon\right), \tag{24}$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$. Now,

$$\begin{aligned}
&F\left(DQ_2(s_1, s_2, \cdots, s_n), w, \epsilon\right) \\
&\geq \min\Bigg\{F\left(Q_2\left(\sum_{a=1}^{n} s_a\right) - \frac{1}{2^2}\psi\left(\sum_{a=1}^{n} 2s_a\right), w, \frac{\epsilon}{5}\right), F\left(\sum_{a=1}^{n} Q_2\left(-s_a + \sum_{b=1;a\neq b}^{n} s_b\right)\right. \\
&\left.-\frac{1}{2^2}\sum_{1\leq a\leq n}\psi\left(2\left(-s_a + \sum_{b=1;a\neq b}^{n} s_b\right)\right), w, \frac{\epsilon}{5}\right), F\left(-(n-3)\sum_{1\leq a<b\leq n} Q_2(s_a + s_b)\right. \\
&\left.+\frac{1}{2^2}(n-3)\sum_{1\leq a<b\leq n}\psi(2(s_a + s_b)), w, \frac{\epsilon}{5}\right), F\left(\left(n^2 - 5n + 2\right)\sum_{a=1}^{n} Q_2(s_a)\right. \\
&\left.-\frac{1}{2^2}\left(n^2 - 5n + 2\right)\sum_{1\leq a\leq n}\psi(2s_a), w, \frac{\epsilon}{5}\right), \\
&F\left(\frac{1}{2^2}\psi\left(\sum_{a=1}^{n} 2s_a\right) + \frac{1}{2^2}\sum_{a=1}^{n}\psi\left(2\left(-s_a + \sum_{b=1;a\neq b}^{n} s_b\right)\right)\right. \\
&\left.-\frac{1}{2^2}(n-3)\sum_{1\leq a<b\leq n}\psi(2(s_a + s_b)) + \frac{1}{2^2}\left(n^2 - 5n + 2\right)\sum_{a=1}^{n}\psi(2s_a), w, \frac{\epsilon}{5}\right)\Bigg\},
\end{aligned} \tag{25}$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$. Using (24) and (NAF5) in (25), we get

$$\begin{aligned}
F(DQ_2(s_1, s_2, \cdots, s_n), w, \epsilon)& \\
&\geq \min\left\{1, 1, 1, 1, F'\left(\chi\left(2^i s_1, 2^i s_2, \cdots, 2^i s_n\right), w, \varphi(2^{2i})\epsilon\right)\right\} \\
&\geq F'\left(\chi\left(2^i s_1, 2^i s_2, \cdots, 2^i s_n\right), w, \varphi(2^{2i})\epsilon\right),
\end{aligned} \tag{26}$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$. Taking the limit $i \to \infty$ in (26) and using (9), we obtain that

$$F(DQ_2(s_1, s_2, \cdots, s_n), w, \epsilon) = 1,$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$. Hence, the function Q_2 satisfies the functional Equation (1). Next, we want to prove that $Q_2(s)$ is the unique mapping and consider another mapping $Q_2'(s)$, which satisfies (11) and (12). Then

$$\begin{aligned} F\Big(Q_2(s)-Q_2'(s),w,\epsilon\Big) &= F\left(\frac{Q_2(2^is)}{2^{2i}}-\frac{Q_2'(2^is)}{2^{2i}},w,\epsilon\right)\\ &\geq \min\left\{F\Big(\frac{Q_2(2^is)}{2^{2i}}-\frac{\psi(2^is)}{2^{2i}},w,\frac{\epsilon}{2}\Big),F\Big(\frac{Q_2'(2^is)}{2^{2i}}-\frac{\psi(2^is)}{2^{2i}},w,\frac{\epsilon}{2}\Big)\right\}\\ &\geq F'\left(\chi\Big(2^is,2^is,0,\cdots,0\Big),w,\frac{\epsilon\varphi(2^{2i})\big(\varphi(2^2)-\varphi(\alpha)\big)}{2}\right)\\ &\geq F'\left(\chi(s,s,0,\cdots,0),w,\frac{\epsilon\varphi(2^{2i})\big(\varphi(2^2)-\varphi(\alpha)\big)}{2\varphi(\alpha^i)}\right), \end{aligned}$$

for all $s,w\in P$ and all $\epsilon>0$. Since

$$\lim_{i\to\infty}\frac{\epsilon\varphi(2^{2i})\big(\varphi(2^2)-\varphi(\alpha)\big)}{2\varphi(\alpha^i)}=\infty,$$

we obtain

$$\lim_{i\to\infty}F'\left(\chi(s,s,0,\cdots,0),w,\frac{\epsilon\varphi(2^{2i})\big(\varphi(2^2)-\varphi(\alpha)\big)}{2\varphi(\alpha^i)}\right)=1.$$

Thus, $F(Q_2(s)-Q_2'(s),w,\epsilon)=1$ for all $s,w\in P$ and all $\epsilon>0$. Thus, $Q_2(s)=Q_2'(s)$. Hence, $Q_2(s)$ is unique.

In a similar way, we can demonstrate the other part of the proof for $j=-1$. The theorem has now been proved. □

Corollary 1. *If a function $\psi:P\to R$ such that*

$$F(D\psi(s_1,s_2,\cdots,s_n),w,\epsilon)\geq F'\left(c\sum_{i=1}^{n}\|s_i\|^{\gamma},w,\epsilon\right), \tag{27}$$

for all $s_1,s_2,\cdots,s_n,w,\in P$ and all $\epsilon>0$, where c and γ are in $\mathbb{R}^+$ with $c>0$ and $\gamma\in(0,2)\cup(2,+\infty)$, then there exists a unique quadratic mapping $Q_2:P\to R$ satisfying

$$F(\psi(s)-Q_2(s),w,\epsilon)\geq F'\left(c\|s\|^{\gamma},w,\epsilon\Big|\varphi(2^2)-\varphi(2^{\gamma})\Big|\right),$$

for all $s,w\in P$ and all $\epsilon>0$.

Corollary 2. *If a function $\psi:P\to R$ such that*

$$F(D\psi(s_1,s_2,\cdots,s_n),w,\epsilon)\geq F'\left(c\Big(\sum_{i=1}^{n}\|s_i\|^{n\gamma}+\prod_{i=1}^{n}\|s_i\|^{\gamma}\Big),w,\epsilon\right), \tag{28}$$

for all $s_1,s_2,\cdots,s_n,w,\in P$ and all $\epsilon>0$, where c and γ are in $\mathbb{R}^+$ with $c>0$ and $n\gamma\in(0,2)\cup(2,+\infty)$, then there exists a unique quadratic mapping $Q_2:P\to R$ satisfying

$$F(\psi(s)-Q_2(s),w,\epsilon)\geq F'\left(c\|s\|^{n\gamma},w,\epsilon\Big|\varphi(2^2)-\varphi(2^{n\gamma})\Big|\right),$$

for all $s,w\in P$ and all $\epsilon>0$.

3.2. Stability for the Even Case: Fixed Point Method

In this subsection, we investigate the Ulam stability results of the functional Equation (1) for the even case by using Theorem 1.

Theorem 5. *Let $\psi : P \to R$ be an even mapping such that (10), for which there exists a mapping $\chi : P^n \to Z$ with*

$$\lim_{i\to\infty} F^{'}\left(\chi(\delta_j^i s_1, \delta_j^i s_2, \cdots, \delta_j^i s_n), w, \epsilon\varphi(\delta_j^{2i})\right) = 1,$$

where $\delta_j = 2$ if $j = 0$ and $\delta_j = \frac{1}{2}$ if $j = 1$,

$$F(D\psi(s_1, s_2, \cdots, s_n), w, \epsilon) \geq F^{'}(\chi(s_1, s_2, \cdots, s_n), w, \epsilon), \tag{29}$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$. If there exists $0 < L < 1$ such that

$$s \to \rho(s) = \frac{1}{2}\chi\left(\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0\right),$$

which has the property

$$F^{'}\left(\frac{1}{\delta_j^2}\rho(\delta_j s), w, \epsilon\right) \geq F^{'}(L\rho(s), w, \epsilon), \tag{30}$$

then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying the functional Equation (1) and

$$F(\psi(s) - Q_2(s), w, \epsilon) \geq F^{'}\left(\frac{L^{1-j}}{1-L}\rho(s), w, \epsilon\right) \tag{31}$$

for all $s, w \in P$ and all $\epsilon > 0$.

Proof. Consider the set

$$\Lambda = \{q/q : P \to R, q(0) = 0\}$$

and define the generalized metric d on Λ,

$$d(p,q) = \inf\left\{\theta \in (0,\infty)/F(p(s) - q(s), w, \epsilon) \geq F^{'}(\theta\rho(s), w, \epsilon),\ s, w \in P, \epsilon > 0\right\}.$$

Clearly, (Λ, d) is complete. Let us define a mapping $T : \Lambda \to \Lambda$ by

$$Tp(s) = \frac{1}{\delta_j^2} p(\delta_j s),$$

for all $s \in P$. One can show that

$$d(Tp, Tq) \leq Ld(p,q),$$

for all $p, q \in \Lambda$. i.e., T is a strictly contractive mapping on Λ with $L = \delta_j^2$. Replacing $(s_1, s_2, \cdots, s_n)$ by $(s, s, 0, \cdots, 0)$ in (10), we obtain

$$F(2\psi(2s) - 8\psi(s), w, \epsilon) \geq F^{'}(\chi(s, s, 0, \cdots, 0), w, \epsilon). \tag{32}$$

Using the condition (30) for $j = 0$ in (32), it becomes to

$$\begin{aligned} F\left(\frac{\psi(2s)}{2^2} - \psi(s), w, \frac{\epsilon}{2\varphi(2^2)}\right) &\geq F^{'}\left(\frac{1}{2}\frac{\chi(s, s, 0, \cdots, 0)}{2^2}, w, \epsilon\right) \\ &\geq F^{'}(L\rho(s), w, \epsilon). \end{aligned}$$

That is,

$$d(\psi, T\psi) \leq L = L^1 < \infty.$$

Again, replacing s by $\frac{s}{2}$ in (32), we get

$$F\Big(\psi(s) - 2^2\psi\Big(\frac{s}{2}\Big), w, \frac{\epsilon}{2}\Big) \geq F'\Big(\frac{1}{2}\chi\Big(\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0\Big), w, \epsilon\Big). \tag{33}$$

Using (30) for $j = 1$ in (33), it becomes to

$$F\Big(\psi(s) - 2^2\psi\Big(\frac{s}{2}\Big), w, \frac{\epsilon}{2}\Big) \geq F'(\rho(s), w, \epsilon).$$

That is,

$$d(\psi, T\psi) \leq 1 = L^{1-j} < \infty.$$

In both cases, we have

$$d(\psi, T\psi) \leq L^{1-j}.$$

Thus (B2)(i) holds. By (B2)(ii), it arises that there exists a fixed point Q_2 of T in Λ satisfying

$$Q_2(s) = F - \lim_{i\to\infty} \frac{\psi(\delta_j^i s)}{\delta_j^{2i}}.$$

Next, we want to prove that Q_2 satisfies the functional Equation (1). Replacing $(s_1, s_2, \cdots, s_n)$ by $(\delta_j^i s_1, \delta_j^i s_2, \cdots, \delta_j^i s_n)$ in (29), we obtain

$$F\left(\frac{1}{\delta_j^{2i}} D\psi(\delta_j^i s_1, \delta_j^i s_2, \cdots, \delta_j^i s_n), w, \epsilon\right) \geq F'\Big(\chi(\delta_j^i s_1, \delta_j^i s_2, \cdots, \delta_j^i s_n\Big), w, \varphi\Big(\delta_j^{2i}\Big)\epsilon\Big), \tag{34}$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$. Taking the limit $i \to \infty$ in (34) and using the definition of $Q_2(s)$, we have that the function Q_2 satisfies the functional Equation (1). Hence, the mapping Q_2 is quadratic.

By (B2)(iii), Q_2 is the unique fixed point of T in $\Delta = \Big\{\psi \in \Lambda : d(\psi, Q_2) < \infty\Big\}$, i.e., Q_2 is the unique function such that

$$F(\psi(s) - Q_2(s), w, \epsilon) \geq F'(\theta\rho(s), w, \epsilon),$$

for all $s, w \in P$ and all $\epsilon, \theta > 0$. Again by (B2)(iv), we obtain

$$d(\psi, Q_2) \leq \frac{1}{1-L} d(\psi, T\psi) \Rightarrow d(\psi, Q_2) \leq \frac{L^{1-j}}{1-L}.$$

This yields

$$F(\psi(s) - Q_2(s), w, \epsilon) \geq F'\left(\frac{L^{1-j}}{1-L}\rho(s), w, \epsilon\right),$$

for all $s, w \in P$ and all $\epsilon > 0$. Hence, the proof of the theorem is now completed. □

Corollary 3. *If a mapping $\psi : P \to R$ such that*

$$F(D\psi(s_1, s_2, \cdots, s_n), w, \epsilon) \geq \begin{cases} F'(c\sum_{i=1}^n \|s_i\|^\gamma, w, \epsilon) \\ F'(c(\sum_{i=1}^n \|s_i\|^{n\gamma} + \prod_{i=1}^n \|s_i\|^\gamma), w, \epsilon), \end{cases} \tag{35}$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$, where c and γ are constants with $c > 0$, then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying

$$F(\psi(s) - Q_2(s), w, \epsilon) \geq \begin{cases} F'\left(c\|s\|^\gamma, w, \epsilon\big|\varphi(2^2) - \varphi(2^\gamma)\big|\right); & \gamma \neq 2, \\ F'\left(c\|s\|^{n\gamma}, w, \epsilon\big|\varphi(2^2) - \varphi(2^{n\gamma})\big|\right); & \gamma \neq \frac{2}{n}, \end{cases}$$

for all $s, w \in P$ and all $\epsilon > 0$.

Proof. Set

$$\chi(s_1, s_2, \cdots, s_n) = \begin{cases} c\sum_{i=1}^{n} \|s_i\|^{\gamma}, \\ c(\sum_{i=1}^{n} \|s_i\|^{n\gamma} + \prod_{i=1}^{n} \|s_i\|^{\gamma}), \end{cases}$$

for all $s_1, s_2, \cdots, s_n \in P$. Now,

$$\begin{aligned} & F'\left(\chi(\delta_j^i s_1, \delta_j^i s_2, \cdots, \delta_j^i s_n), w, \delta_j^{2i}\epsilon\right) \\ & \quad = \begin{cases} F'\left(c\sum_{j=1}^{n} \|\delta_j^i s_j\|^{\gamma}, w, \epsilon\right) \\ F'\left(c\left(\sum_{j=1}^{n} \|\delta_j^i s_j\|^{n\gamma} + \prod_{j=1}^{n} \|\delta_j^i s_j\|^{\gamma}\right), w, \epsilon\right), \end{cases} \\ & \quad = \begin{cases} 1, \text{ if } (j=0 \text{ and } \gamma < 2) \text{ or } (j=1 \text{ and } \gamma > 2), \\ 1, \text{ if } (j=0 \text{ and } \gamma n < 2) \text{ or } (j=1 \text{ and } \gamma n > 2). \end{cases} \end{aligned}$$

Let

$$\rho(s) = \frac{1}{2}\chi\left(\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0\right).$$

Next, we have

$$F'\left(\frac{1}{\delta_j^2}\rho(\delta_j s), w, \epsilon\right) = \begin{cases} F'\left(\delta_j^{\gamma-2}\rho(s), w, \epsilon\right) \\ F'\left(\delta_j^{n\gamma-2}\rho(s), w, \epsilon\right), \end{cases}$$

and

$$\begin{aligned} F'(\rho(s), w, \epsilon) &= F'\left(\chi\left(\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0\right), w, 2\epsilon\right) \\ &= \begin{cases} F'\left(\frac{2c}{2^{\gamma}}\|s\|^{\gamma}, w, 2\epsilon\right), \\ F'\left(\frac{2c}{2^{n\gamma}}\|s\|^{n\gamma}, w, 2\epsilon\right). \end{cases} \end{aligned}$$

Thus, the inequality (30) holds either $L = 2^{\gamma-2}$ for $\gamma < 2$ if $j = 0$, $L = 2^{2-\gamma}$ for $\gamma > 2$ if $j = 1$, $L = 2^{n\gamma-2}$ for $\gamma < \frac{2}{n}$ if $j = 0$ and $L = 2^{2-n\gamma}$ for $\gamma > \frac{2}{n}$ if $j = 1$. From inequality (31), we obtain our needed results. □

3.3. Stability for the Odd Case: Direct Method

In this subsection, we investigate the Ulam stability of the functional Equation (1) for the odd case by using direct method.

Theorem 6. *Let $j \in \{-1, 1\}$ be fixed and let a mapping $\chi : P^n \to Z$ such that for some α with $0 < \left(\frac{\varphi(\alpha)}{\varphi(2)}\right)^i < 1$, (8) and*

$$\lim_{i\to\infty} F'\left(\chi\left(2^{ji}s_1, 2^{ji}s_2, \cdots, 2^{ji}s_n\right), w, \left[\varphi\left(2^i\right)\right]^j \epsilon\right) = 1,$$

for all $s_1, s_2, \cdots, s_n, w \in P$ and all $\epsilon > 0$. If a mapping $\psi : P \to R$ satisfies (10), then the limit

$$A_1(s) = F - \lim_{i\to\infty} \frac{\psi(2^{ji}s)}{2^{ji}},$$

exists for each $s \in P$ *and a unique additive mapping* $A_1 : P \to R$ *satisfying the functional Equation* (1) *and*

$$F(\psi(s) - A_1(s), w, \epsilon) \geq F'(\chi(s, s, 0, \cdots, 0), w, 2\epsilon|\varphi(2) - \varphi(\alpha)|),$$

for all $s, w \in P$ *and all* $\epsilon > 0$.

Proof. Take $j = 1$. Replacing $(s_1, s_2, \cdots, s_n)$ by $(s, s, 0, \cdots, 0)$ in (10), we get

$$F(2\psi(2s) - 4\psi(s), w, \epsilon) \geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon), \tag{36}$$

for all $s, w \in P$ and all $\epsilon > 0$. From (36), we obtain

$$F\left(\frac{\psi(2s)}{2} - \psi(s), w, \frac{\epsilon}{2\varphi(2)}\right) \geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon), \tag{37}$$

for all $s, w \in P$ and all $\epsilon > 0$. Replacing s by $2^i s$ in (37), we have

$$F\left(\frac{\psi(2^{i+1}s)}{2^{(i+1)}} - \frac{\psi(2^i s)}{2^i}, w, \frac{\epsilon}{2\varphi\left(2^{(i+1)}\right)}\right) \geq F'\left(\chi(2^i s, 2^i s, 0, \cdots, 0), w, \epsilon\right), \tag{38}$$

for all $s, w \in P$ and all $\epsilon > 0$. Using (8), (NAF6) in (38), we get

$$F\left(\frac{\psi(2^{i+1}s)}{2^{(i+1)}} - \frac{\psi(2^i s)}{2^i}, w, \frac{\epsilon}{2\varphi(2^{(i+1)})}\right) \geq F'\left(\chi(s, s, 0, \cdots, 0), w, \frac{\epsilon}{\varphi(\alpha^i)}\right), \tag{39}$$

for all $s, w \in P$ and all $\epsilon > 0$. Replacing ϵ by $\varphi(\alpha^i)\epsilon$ in (39), we obtain

$$F\left(\frac{\psi(2^{i+1}s)}{2^{(i+1)}} - \frac{\psi(2^i s)}{2^i}, w, \frac{\varphi(\alpha^i)\epsilon}{2\varphi(2^{(i+1)})}\right) \geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon),$$

for all $s, w \in P$ and all $\epsilon > 0$. The remaining part of proof is same as in the proof of Theorem 4. □

Corollary 4. *If a mapping* $\psi : P \to R$ *satisfies* (27), *where* c *and* γ *are real constants with* $c > 0$ *and* $\gamma \in (0, 1) \cup (1, +\infty)$, *then there exists a unique additive mapping* $A_1 : P \to R$ *such that*

$$F(\psi(s) - A_1(s), w, \epsilon) \geq F'(c\|s\|^{\gamma}, w, \epsilon|\varphi(2) - \varphi(2^{\gamma})|),$$

for all $s, w \in P$ *and all* $\epsilon > 0$.

Corollary 5. *If a mapping* $\psi : P \to R$ *satisfies* (28), *where* c *and* γ *are the real constants with* $c > 0$ *and* $n\gamma \in (0, 1) \cup (1, +\infty)$, *then there exists a unique additive mapping* $A_1 : P \to R$ *such that*

$$F(\psi(s) - A_1(s), w, \epsilon) \geq F'(c\|s\|^{n\gamma}, w, \epsilon|\varphi(2) - \varphi(2^{n\gamma})|),$$

for all $s, w \in P$ *and* $\epsilon > 0$.

3.4. Stability for the Odd Case: Fixed Point Method

In this subsection, we investigate the Ulam stability of the functional equation (1) for the odd case by using Theorem 1.

Theorem 7. *Let* $\psi : P \to R$ *be a mapping such that* (10), *for which there exists a mapping* $\chi : P^n \to Z$ *with*

$$\lim_{i\to\infty} F'\left(\chi(\delta_j^i s_1, \delta_j^i s_2, \cdots, \delta_j^i s_n), w, \epsilon\varphi(\delta_j^i)\right) = 1,$$

where $\delta_j = 2$ if $j = 0$ and $\delta_j = \frac{1}{2}$ if $j = 1$. If there exists $L > 0$ satisfying

$$s \to \rho(s) = \frac{1}{2}\chi\Big(\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0\Big),$$

which has the property

$$F'\left(\frac{1}{\delta_j}\rho(\delta_j s), w, \epsilon\right) \geq F'(L\rho(s), w, \epsilon),$$

for all $s, w \in P$ and all $\epsilon > 0$, then there exists a unique additive mapping $A_1 : P \to R$ satisfying the functional Equation (1) and

$$F(\psi(s) - A_1(s), w, \epsilon) \geq F'\left(\frac{L^{1-j}}{1-L}\rho(s), w, \epsilon\right),$$

for all $s, w \in P$ and all $\epsilon > 0$.

Proof. Consider a set

$$\Lambda = \{q/q : P \to R, q(0) = 0\}$$

and define the generalized metric d on Λ,

$$d(p,q) \quad = \quad \inf\Big\{\theta \in (0,\infty)/F(p(s) - q(s), w, \epsilon) \geq F'(\theta\rho(s), w, \epsilon),\ s, w \in P, \epsilon > 0\Big\}.$$

Clearly, (Λ, d) is complete. Let us define a mapping $T : \Lambda \to \Lambda$ by

$$Tp(s) = \frac{1}{\delta_j}p(\delta_j s),$$

for all $s \in P$. One can show that

$$d(Tp, Tq) \leq Ld(p,q)$$

for all $p, q \in \Lambda$, i.e., T is a strictly contractive mapping on Λ with $L = \delta_j$. Replacing $(s_1, s_2, \cdots, s_n)$ by $(s, s, 0, \cdots, 0)$ in (10), we get

$$F(2\psi(2s) - 4\psi(s), w, \epsilon) \geq F'(\chi(s, s, 0, \cdots, 0), w, \epsilon),$$

for all $s, w \in P$ and all $\epsilon > 0$. The remaining part of proof is the similar as in the proof of Theorem 5. □

Corollary 6. *If a mapping $\psi : P \to R$ such that (35), where c and γ are constants with $c > 0$, then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$F(\psi(s) - A_1(s), w, \epsilon) \geq \begin{cases} F'(c\|s\|^{\gamma}, w, \epsilon|\varphi(2) - \varphi(2^{\gamma})|); & \gamma \neq 1, \\ F'(c\|s\|^{n\gamma}, w, \epsilon|\varphi(2) - \varphi(2^{n\gamma})|); & \gamma \neq \frac{1}{n}, \end{cases}$$

for all $s, w \in P$ and all $\epsilon > 0$.

4. Ulam Stability in Non-Archimedean Banach Spaces

In this section, we consider that P and R are non-Archimedean normed space and non-Archimedean Banach space, respectively, and let $|2| \neq 1$.

4.1. Stability Results: Direct Method

In this subsection, we investigate the Ulam stability of the functional Equation (1) for the even case by using the direct method.

Theorem 8. *If a mapping $\chi : P^n \to [0,\infty)$ and a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and*

$$\|D\psi(s_1, s_2, \cdots, s_n)\| \le \chi(s_1, s_2, \cdots, s_n), \ \ s_1, s_2, \cdots, s_n \in P. \tag{40}$$

with

$$\lim_{l\to\infty} |2|^{2l}\chi\Big(2^{-l}s_1, 2^{-l}s_2, \cdots, 2^{-l}s_n\Big) = 0. \tag{41}$$

Then there exists a unique quadratic mapping $Q_2 : P \to Q$ satisfying

$$\Big\|\psi(s) - Q_2(s)\Big\| \le \sup_{l\in\mathbb{N}}\Big\{\frac{1}{2}|2|^{2(l-1)}\chi\Big(\frac{s}{2^l}, \frac{s}{2^l}, 0, \cdots, 0\Big)\Big\} \tag{42}$$

for all $s \in P$.

Proof. Replacing $(s_1, s_2, \cdots, s_n)$ by $(s, s, 0, \cdots, 0)$ in (40), we have

$$\Big\|2\psi(2s) - 8\psi(s)\Big\| \le \chi(s, s, 0, \cdots, 0), \tag{43}$$

for all $s \in P$. This implies

$$\Big\|\psi(s) - 2^2\psi\Big(\frac{s}{2}\Big)\Big\| \le \frac{1}{2}\chi\Big(\frac{s}{2}, \frac{s}{2}, 0, \cdots, 0\Big),$$

for all $s \in P$. Hence

$$\begin{aligned}
&\Big\|2^{2m}\psi\Big(\frac{s}{2^m}\Big) - 2^{2t}\psi\Big(\frac{s}{2^t}\Big)\Big\| \\
&\le \max\Big\{\Big\|2^{2m}\psi\Big(\frac{s}{2^m}\Big) - 2^{2(m+1)}\psi\Big(\frac{s}{2^{m+1}}\Big)\Big\|, \cdots, \Big\|2^{2(t-1)}\psi\Big(\frac{s}{2^{t-1}}\Big) - 2^{2t}\psi\Big(\frac{s}{2^t}\Big)\Big\|\Big\} \\
&\le \Big\{|2|^{2m}\Big\|\psi\Big(\frac{s}{2^m}\Big) - 2^2\psi\Big(\frac{s}{2^{m+1}}\Big)\Big\|, \cdots, |2|^{2(t-1)}\Big\|\psi\Big(\frac{s}{2^{t-1}}\Big) - 2^2\psi\Big(\frac{s}{2^t}\Big)\Big\|\Big\} \\
&\le \sup_{l\in\{m, m+1, \cdots\}}\Big\{\frac{1}{2}|2|^{2l}\chi\Big(\frac{s}{2^{l+1}}, \frac{s}{2^{l+1}}, 0, \cdots, 0\Big)\Big\}
\end{aligned} \tag{44}$$

for all $t > m > 0$ and all $s \in P$. As a result of (44) that $\Big\{2^{2l}\psi\Big(\frac{s}{2^l}\Big)\Big\}$ is a Cauchy sequence for all $s \in P$. As R is complete, the sequence $\{2^{2l}\psi\Big(\frac{s}{2^l}\Big)\}$ converges. Next, we can define a mapping $Q_2 : P \to R$ by

$$Q_2(s) := \lim_{m\to\infty} 2^{2m}\psi(2^{-m}s), \ \ s \in P.$$

Setting $m = 0$ and taking the limit $t \to \infty$ in (44), we obtain (42). As a result of (40) and (41) that

$$\begin{aligned}
\Big\|DQ_2(s_1, s_2, \cdots, s_n)\Big\| &= \lim_{l\to\infty} |2|^{2l}\Big\|D\psi(2^{-l}s_1, 2^{-l}s_2, \cdots, 2^{-l}s_n)\Big\| \\
&\le \lim_{l\to\infty} |2|^{2l}\chi\Big(2^{-l}s_1, 2^{-l}s_2, \cdots, 2^{-l}s_n\Big) = 0
\end{aligned}$$

for all $s_1, s_2, \cdots, s_n \in P$. So

$$DQ_2(s_1, s_2, \cdots, s_n) = 0.$$

From Theorem 3, the function $Q_2 : P \to R$ is quadratic.

Now, consider another quadratic mapping $R_2 : P \to R$ which satisfying (42). Then, we have

$$\begin{aligned}\left\|Q_2(s) - R_2(s)\right\| &= \left\|2^{2k} Q_2\left(\frac{s}{2^k}\right) - 2^{2k} R_2\left(\frac{s}{2^k}\right)\right\| \\ &\le \max\left\{\left\|2^{2k} Q_2\left(\frac{s}{2^k}\right) - 2^{2k}\phi\left(\frac{s}{2^k}\right)\right\|, \ \left\|2^{2k} R_2\left(\frac{s}{2^k}\right) - 2^{2k}\phi\left(\frac{s}{2^k}\right)\right\|\right\} \\ &\le \sup_{l\in\mathbb{N}}\left\{\frac{1}{2}|2|^{2(k+l-1)}\chi\left(\frac{s}{2^{k+l}}, \frac{s}{2^{k+l}}, 0, \cdots, 0\right)\right\}, \\ &\to 0 \ \text{ as } \ k \to \infty.\end{aligned}$$

So, we can infer that $Q_2(s) = R_2(s)$ for all $s \in P$. This proves that the function Q_2 is unique. Therefore, the mapping $Q_2 : P \to R$ is the unique quadratic mapping that satisfies (42). □

Corollary 7. *If a function $\psi : P \to R$ with $\psi(0) = 0$ and satisfies*

$$\left\|D\psi(s_1, s_2, \cdots, s_n)\right\| \le \lambda\left(\sum_{j=1}^{n}\|s_j\|^{\alpha}\right) \tag{45}$$

for all $s_1, s_2, \cdots, s_n \in P$, then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfies

$$\left\|\psi(s) - Q_2(s)\right\| \le \frac{\lambda}{|2|^{\alpha}}\|s\|^{\alpha}, \quad s \in P,$$

where $\alpha < 2$ and λ are in $\mathbb{R}^+$.

Corollary 8. *If a function $\psi : P \to R$ with $\psi(0) = 0$ and satisfies*

$$\left\|D\psi(s_1, s_2, \cdots, s_n)\right\| \le \alpha\left(\sum_{j=1}^{n}\|s_j\|^{n\alpha} + \prod_{j=1}^{n}\|s_j\|^{\alpha}\right) \tag{46}$$

for all $s_1, s_2, \cdots, s_n \in P$, then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying

$$\left\|\psi(s) - Q_2(s)\right\| \le \frac{\lambda}{|2|^{n\alpha}}\|s\|^{n\alpha}, \ \ s \in P,$$

where $n\alpha < 2$ and λ are in $\mathbb{R}^+$.

Theorem 9. *If a mapping $\chi : P^n \to [0, \infty)$ and a mapping $\psi : P \to R$ with $\psi(0) = 0$ and satisfying (40) and*

$$\lim_{l\to\infty}\left\{\frac{1}{|2|^{2l}}\chi\left(2^{l-1}s_1, 2^{l-1}s_2, \cdots, 2^{l-1}s_n\right)\right\} = 0, \ \ s_1, s_2, \cdots, s_n \in P.$$

Then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying

$$\left\|\psi(s) - Q_2(s)\right\| \le \sup_{l\in\mathbb{N}}\left\{\frac{1}{2}\frac{1}{|2|^{2l}}\chi\left(2^{l-1}s, 2^{l-1}s, 0, \cdots, 0\right)\right\} \tag{47}$$

for all $s \in P$.

Proof. It follows from (43) that

$$\left\|\psi(s) - 2^{-2}\psi(2s)\right\| \le \frac{1}{2|2|^2}\chi(s, s, 0, \cdots, 0),$$

for all $s \in P$. Hence

$$\begin{aligned}
&\left\| \frac{1}{2^{2m}}\psi(2^m s) - \frac{1}{2^{2t}}\psi(2^t s) \right\| \\
&\leq \max\left\{ \left\| \frac{1}{2^{2m}}\psi(2^m s) - \frac{1}{2^{2(m+1)}}\psi(2^{m+1}s) \right\|, \cdots, \left\| \frac{1}{2^{2(t-1)}}\psi(2^{t-1}s) - \frac{1}{2^{2t}}\psi(2^t s) \right\| \right\} \\
&\leq \max\left\{ \frac{1}{|2|^{2m}} \left\| \psi(2^m s) - \frac{1}{2^2}\psi(2^{(m+1)}s) \right\|, \cdots, \frac{1}{|2|^{2(t-1)}} \left\| \psi(2^{t-1}s) - \frac{1}{2^2}\psi(2^t s) \right\| \right\} \\
&\leq \sup_{l \in \{m, m+1, \cdots\}} \left\{ \frac{1}{2}\frac{1}{|2|^{2(l+1)}}\chi(2^l s, 2^l s, 0, \cdots, 0) \right\} \qquad (48)
\end{aligned}$$

for all $t > m > 0$. As a result of (48) that $\{\frac{1}{2^{2l}}\psi(2^l s)\}$ is a Cauchy sequence. Since R is complete, $\{\frac{1}{2^{2l}}\psi(2^l s)\}$ converges. Thus, we can define a mapping $Q_2 : P \to R$ by

$$Q_2(s) := \lim_{l \to \infty} \frac{1}{2^{2l}}\psi(2^l s),$$

for all $s \in P$. Putting $m = 0$ and taking the limit $t \to \infty$ in (48), we obtain (47). The remainder of the proof is similar to that of Theorem 8. □

Corollary 9. *If a function $\psi : P \to R$ with $\psi(0) = 0$ and such that (45) , then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying*

$$\left\| \psi(s) - Q_2(s) \right\| \leq \frac{\lambda}{|2|^2}\|s\|^{\alpha}$$

for all $s \in P$, where $\alpha > 2$ and λ are in $\mathbb{R}^+$.

Corollary 10. *If a function $\psi : P \to R$ with $\psi(0) = 0$ and such that (46), then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfies*

$$\left\| \psi(s) - Q_2(s) \right\| \leq \frac{\lambda}{|2|^2}\|s\|^{n\alpha}$$

for all $s \in P$, where $n\alpha > 2$ and λ are in $\mathbb{R}^+$.

Next, we investigate the Ulam stability of the functional Equation (1) for the odd case by using direct method.

Theorem 10. *Let a mapping $\chi : P^n \to [0, \infty)$ and a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (40) with*

$$\lim_{l \to \infty} |2|^l \chi\left(2^{-l}s_1, 2^{-l}s_2, \cdots, 2^{-l}s_n\right) = 0. \qquad (49)$$

Then there exists a unique additive mapping $A_1 : P \to Q$ satisfying

$$\left\| \psi(s) - A_1(s) \right\| \leq \sup_{l \in \mathbb{N}} \left\{ \frac{1}{2}|2|^{(l-1)}\chi\left(\frac{s}{2^l}, \frac{s}{2^l}, 0, \cdots, 0\right) \right\} \qquad (50)$$

for all $s \in P$.

Corollary 11. *If a mapping $\psi : P \to R$ with $\psi(0) = 0$ and satisfies (45) , then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$\left\| \psi(s) - A_1(s) \right\| \leq \frac{\lambda}{|2|^{\alpha}}\|s\|^{\alpha},$$

for all $s \in P$, where $\alpha < 1$ and λ are in $\mathbb{R}^+$.

Corollary 12. *If a mapping $\psi : P \to R$ with $\psi(0) = 0$ and satisfying (46) , then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$\Big\|\psi(s) - A_1(s)\Big\| \leq \frac{\lambda}{|2|^{n\alpha}}\|s\|^{n\alpha},$$

for all $s \in P$, where $n\alpha < 1$ and λ are in $\mathbb{R}^+$.

Theorem 11. *If a mapping $\chi : P^n \to [0, \infty)$ and a mapping $\psi : P \to R$ with $\psi(0) = 0$ and satisfying (40) and*

$$\lim_{l\to\infty}\left\{\frac{1}{|2|^l}\chi\Big(2^{l-1}s_1, 2^{l-1}s_2, \cdots, 2^{l-1}s_n\Big)\right\} = 0,$$

for all $s_1, s_2, \cdots, s_n \in P$.. Then there exists a unique additive mapping $A_1 : P \to R$ satisfying

$$\Big\|\psi(s) - A_1(s)\Big\| \leq \sup_{l\in\mathbb{N}}\left\{\frac{1}{2}\frac{1}{|2|^l}\chi\Big(2^{l-1}s, 2^{l-1}s, 0, \cdots, 0\Big)\right\} \tag{51}$$

for all $s \in P$.

Corollary 13. *If a mapping $\psi : P \to R$ with $\psi(0) = 0$ and satisfies (45) , then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$\Big\|\psi(s) - A_1(s)\Big\| \leq \frac{\lambda}{|2|}\|s\|^{\alpha}$$

for all $s \in P$, where $\alpha > 1$ and λ are in $\mathbb{R}^+$.

Corollary 14. *If a mapping $\psi : P \to R$ with $\psi(0) = 0$ and satisfies (46), then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$\Big\|\psi(s) - A_1(s)\Big\| \leq \frac{\lambda}{|2|}\|s\|^{n\alpha}$$

for all $s \in P$, where $n\alpha > 1$ and λ are in $\mathbb{R}^+$.

4.2. Stability Results: Fixed Point Method

In this subsection, we investigate the Ulam stability of the functional Equation (1) for the even case by using a fixed point method.

Theorem 12. *Let a function $\chi : P^n \to [0, \infty)$ such that there exists $0 < L < 1$ with*

$$\chi\Big(\frac{s_1}{2}, \frac{s_2}{2}, \cdots, \frac{s_n}{2}\Big) \leq \frac{L}{|2|^2}\chi(s_1, s_2, \cdots, s_n), \ \ s_1, s_2, \cdots, s_n \in P. \tag{52}$$

Let a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (40). Then there exists a unique quadratic mapping $Q_2 : P \to Q$ satisfying

$$\Big\|\psi(s) - Q_2(s)\Big\| \leq \frac{L}{2|2|^2(1-L)}\chi(s, s, 0, \cdot, 0) \tag{53}$$

for all $s \in P$.

Proof. Replacing $(s_1, s_2, \cdots, s_n)$ by $(s, s, 0, \cdots, 0)$ in (40), we get

$$\Big\|2\psi(2s) - 2^3\psi(s)\Big\| \leq \chi(s, s, 0, \cdots, 0) \tag{54}$$

for all $s \in P$. Consider the set

$$A := \{q : P \to R,\ q(0) = 0\},$$

and we can define a generalised metric d on A:

$$d(t,q) = \inf\Big\{\delta \in \mathbb{R}_+ \ :\ \Big\|t(s) - q(s)\Big\| \le \delta\chi(s,s,0,\cdots,0),\ \ \forall\, s \in P\Big\},$$

where, as usual, $\inf\emptyset = +\infty$. It is easy to show that (A,d) is complete (see [30]). Now, we prove that the mapping $S : A \to A$ satisfying

$$St(s) := 2^2 t\Big(\frac{s}{2}\Big),\ \ s \in P.$$

For any given $t,q \in A$ such that $d(t,q) = \epsilon$. Then

$$\Big\|t(s) - q(s)\Big\| \le \epsilon\chi(s,s,0,\cdots,0),$$

for all $s \in P$. Hence

$$\begin{aligned}
\Big\|St(s) - Sq(s)\Big\| &= \Big\|2^2 t\Big(\frac{s}{2}\Big) - 2^2 q\Big(\frac{s}{2}\Big)\Big\| \\
&\le \frac{1}{2}|2|^2\epsilon\chi\Big(\frac{s}{2},\frac{s}{2},0,\cdots,0\Big) \\
&\le \frac{1}{2}|2|^2\epsilon\frac{L}{|2|^2}\chi(s,s,0,\cdots,0) \\
&\le \frac{L}{2}\epsilon\chi(s,s,0,\cdots,0)
\end{aligned}$$

for all $s \in P$. So $d(t,q) = \epsilon$ implies that

$$d(St,Sq) \le \epsilon L.$$

This means that

$$d(St,Sq) \le Ld(t,q),$$

for all $t,q \in A$. It follows from (54) that

$$\begin{aligned}
\Big\|\psi(s) - 2^2\psi\Big(\frac{s}{2}\Big)\Big\| &\le \frac{1}{2}\chi\Big(\frac{s}{2},\frac{s}{2},0,\cdots,0\Big) \\
&\le \frac{1}{2}\frac{L}{|2|^2}\chi(s,s,0\cdots,0)
\end{aligned}$$

for all $s \in P$. So $d(\psi,S\psi) \le \frac{L}{2|2|^2}$. From Theorem 1, there exists a quadratic mapping $Q_2 : P \to R$ satisfying

(1) Q_2 is a fixed point of S,

$$i.e.,\ \ Q_2(s) = 2^2 Q\Big(\frac{s}{2}\Big), \tag{55}$$

for all $s \in P$. The function Q_2 is a unique fixed point of A in

$$T = \{t \in A :\ d(\psi,t) < \infty\}.$$

This yields that Q_2 is a unique function satisfying (55) such that there exists $\delta \in (0, \infty)$ such that

$$\Big\|\psi(s) - Q_2(s)\Big\| \leq \delta\chi(s, s, 0, \cdots, 0),$$

for all $s \in P$.

(2) $d(S^l\phi, Q_2) \to 0$ as $l \to \infty$. This indicates inequality

$$\lim_{l\to\infty} 2^{2l}\psi\Big(2^{-l}s\Big) = Q_2(s),$$

for all $s \in P$.

(3) $d(\psi, Q_2) \leq \frac{1}{1-L}d(\psi, S\psi)$, it implies

$$\Big\|\psi(s) - Q_2(s)\Big\| \leq \frac{L}{2|2|^2(1-L)}\chi(s, s, 0, \cdots, 0)$$

for all $s \in P$. It follows from (40) and (59) that

$$\begin{aligned} \Big\|DQ_2(s_1, s_2, \cdots, s_n)\Big\| &= \lim_{l\to\infty} |2|^{2l}\Big\|D\psi(2^{-l}s_1, 2^{-l}s_2, \cdots, 2^{-l}s_n)\Big\| \\ &\leq \lim_{l\to\infty} |2|^{2l}\chi\Big(2^{-l}s_1, 2^{-l}s_2, \cdots, 2^{-l}s_n\Big) = 0. \end{aligned}$$

Thus

$$DQ_2(s_1, s_2, \cdots, s_n) = 0,$$

for all $s_1, s_2, \cdots, s_n \in P$. By Theorem 3, the mapping $Q_2 : P \to R$ is the unique quadratic mapping. $\square$

Corollary 15. *If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (45), where $\alpha < 2$ and λ are in $\mathbb{R}^+$, then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying*

$$\Big\|\psi(s) - Q_2(s)\Big\| \leq \frac{\lambda\|s\|^\alpha}{|2|^\alpha - |2|^2},$$

for all $s \in P$.

Proof. By letting

$$\chi(s_1, s_2, \cdots, s_n) = \lambda\left(\sum_{j=1}^{n}\|s_j\|^\alpha\right),$$

the proof is based on Theorem 12. Then, we can use $L = \frac{1}{2}|2|^{2-\alpha}$ to obtain the required result. $\square$

Corollary 16. *If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (46), where $n\alpha < 2$ and λ are in $\mathbb{R}^+$, then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying*

$$\Big\|\psi(s) - Q_2(s)\Big\| \leq \frac{\lambda\|s\|^{n\alpha}}{|2|^{n\alpha} - |2|^2}$$

for all $s \in P$.

Proof. By letting

$$\chi(s_1, s_2, \cdots, s_n) = \lambda\left(\sum_{j=1}^{n}\|s_j\|^{n\alpha} + \prod_{j=1}^{n}\|s_j\|^\alpha\right),$$

the proof is based on Theorem 12. Then, we can use $L = \frac{1}{2}|2|^{2-n\alpha}$ to obtain the required result. $\square$

Theorem 13. *Let a mapping $\chi : P^n \to [0, \infty)$ such that there exists $0 < L < 1$ with*

$$\chi(s_1, s_2, \cdots, s_n) \leq |2^2| L \chi\left(2^{-1}s_1, 2^{-1}s_2, \cdots, 2^{-1}s_n\right) \tag{56}$$

for all $s_1, s_2, \cdots, s_n \in P$. If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (40), then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying

$$\left\|\psi(s) - Q_2(s)\right\| \leq \frac{1}{|2^2|(1-L)} \chi(s, s, 0, \cdots, 0) \tag{57}$$

for all $s \in P$.

Proof. It follows from (54) that

$$\left\|\psi(s) - \frac{1}{2^2}\psi(2s)\right\| \leq \frac{1}{|2|^3} \chi(s, s, 0, \cdots, 0) \tag{58}$$

for all $s \in P$. Let (A, d) denote the generalised metric space noted in Theorem 1. Let us define a mapping $S : A \to A$ by

$$St(s) := \frac{1}{2^2} t(2s)$$

for all $s \in P$. This comes from (58) that

$$d(\psi, S\psi) \leq \frac{1}{|2|^3}.$$

So

$$\left\|\psi(s) - Q_2(s)\right\| \leq \frac{1}{|2|^3(1-L)} \chi(s, s, 0, \cdots, 0),$$

for all $s \in P$. The remaining part of the proof is similar to that of Theorem 12. $\square$

Corollary 17. *If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (45), then there exists a unique quadratic mapping $Q_2 : P \to R$ such that*

$$\left\|\psi(s) - Q_2(s)\right\| \leq \frac{\lambda \|s\|^\alpha}{|2|^2 - |2|^\alpha}$$

for all $s \in P$, where $\alpha > 2$ and λ are in $\mathbb{R}^+$.

Proof. By letting

$$\chi(s_1, s_2, \cdots, s_n) = \lambda \left(\sum_{j=1}^{n} \|s_j\|^\alpha \right),$$

the proof is based on Theorem 13. Then, we can replace $L = |2|^{\alpha-2}$, we obtain our needed result. $\square$

Corollary 18. *If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (46), then there exists a unique quadratic mapping $Q_2 : P \to R$ satisfying*

$$\left\|\psi(s) - Q_2(s)\right\| \leq \frac{\lambda \|s\|^{n\alpha}}{|2|^2 - |2|^{n\alpha}}$$

for all $s \in P$, where $n\alpha > 2$ and λ are in $\mathbb{R}^+$.

Proof. By letting

$$\chi(s_1, s_2, \cdots, s_n) = \lambda\left(\sum_{j=1}^{n} \|s_j\|^{n\alpha} + \prod_{j=1}^{n} \|s_j\|^{\alpha}\right),$$

the proof is based on Theorem 13. Then, we can replace $L = |2|^{n\alpha-2}$, and we obtain our needed result. □

Next, we investigate the Ulam stability of the functional Equation (1) for the odd case by using fixed point method.

Theorem 14. *Let a mapping $\chi : P^n \to [0, \infty)$ such that there exists $0 < L < 1$ with*

$$\chi\left(\frac{s_1}{2}, \frac{s_2}{2}, \cdots, \frac{s_n}{2}\right) \le \frac{L}{|2|}\chi(s_1, s_2, \cdots, s_n), \tag{59}$$

for all $s_1, s_2, \cdots, s_n \in P$. Let a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (40). Then there exists a unique additive mapping $A_1 : P \to Q$ satisfying

$$\left\|\psi(s) - A_1(s)\right\| \le \frac{L}{|2|(1-L)}\chi(s, s, 0, \cdot, 0) \tag{60}$$

for all $s \in P$.

Corollary 19. *If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (45), where $\alpha < 1$ and λ are in $\mathbb{R}^+$, then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$\left\|\psi(s) - A_1(s)\right\| \le \frac{\lambda\|s\|^{\alpha}}{|2|^{\alpha} - |2|},$$

for all $s \in P$.

Corollary 20. *If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (46), where $n\alpha < 1$ and λ are in $\mathbb{R}^+$, then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$\left\|\psi(s) - A_1(s)\right\| \le \frac{\lambda\|s\|^{n\alpha}}{|2|^{n\alpha} - |2|}$$

for all $s \in P$.

Theorem 15. *If a mapping $\chi : P^n \to [0, \infty)$ such that there exists $0 < L < 1$ with*

$$\chi(s_1, s_2, \cdots, s_n) \le |2|L\chi\left(2^{-1}s_1, 2^{-1}s_2, \cdots, 2^{-1}s_n\right) \tag{61}$$

for all $s_1, s_2, \cdots, s_n \in P$. If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (40), then there exists a unique additive mapping $A_1 : P \to R$ satisfying

$$\left\|\psi(s) - A_1(s)\right\| \le \frac{1}{|2|(1-L)}\chi(s, s, 0, \cdots, 0) \tag{62}$$

for all $s \in P$.

Corollary 21. *If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (45), then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$\left\|\psi(s) - A_1(s)\right\| \le \frac{\lambda\|s\|^{\alpha}}{|2| - |2|^{\alpha}}$$

for all $s \in P$, where $\alpha > 1$ and λ are in $\mathbb{R}^+$.

Corollary 22. *If a mapping $\psi : P \to R$ such that $\psi(0) = 0$ and (46), then there exists a unique additive mapping $A_1 : P \to R$ satisfying*

$$\left\|\psi(s) - A_1(s)\right\| \leq \frac{\lambda\|s\|^{n\alpha}}{|2| - |2|^{n\alpha}}$$

for all $s \in P$, where $n\alpha > 1$ and λ are in $\mathbb{R}^+$.

5. Conclusions

Many mathematicians investigated the Ulam stability of many different types of additive, quadratic, and cubic functional equations in various normed spaces. In our investigations, we first defined a new kind of mixed type of quadratic-additive functional Equation (1) and derived its general solution. Mainly, the authors investigated the Ulam stability of this mixed type of quadratic-additive functional Equation (1) in the setting of on-Archimedean fuzzy φ-2-normed space and non-Archimedean Banach space using the direct and fixed point approaches by taking into our account two cases, even mapping and odd mapping.

It should be noted that the Ulam stability of this mixed type of quadratic-additive functional Equation (1) can be determined in a variety of frameworks, such as quasi-β-normed spaces, fuzzy normed spaces, random normed spaces, probabilistic normed spaces, intuitionistic fuzzy normed spaces, and so on. The findings and techniques used in this study might be valuable to other researchers who want to conduct further work in this area.

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