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Article

Spatial Behavior of Solutions in Linear Thermoelasticity with Voids and Three Delay Times

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Abstract: This brief contribution aims to complement a study of well-posedness started by the same authors in 2020, showing—for that same mathematical model—the existence of a domain of influence of external data. The object of investigation, we recall, is a linear thermoelastic model with a porous matrix modeled on the basis of the Cowin–Nunziato theory, and for which the heat exchange phenomena are intended to obey a time-differential heat transfer law with three delay times. We therefore consider, without reporting it explicitly, the (suitably adapted) initial-boundary value problem formulated at that time, as well as some analytical techniques employed to handle it in order to address the uniqueness and continuous dependence questions. Here, a domain of influence theorem is proven, showing the spatial behavior of the solution in a cylindrical domain, by activating the hypotheses that make the model thermodynamically consistent. The theorem, in detail, demonstrates that for a finite time $t > 0$, the assigned external (thermomechanical) actions generate no disturbance outside a bounded domain contained within the cylindrical region of interest. The length of the work is deliberately kept to a minimum, having opted where possible for bibliographic citations in favor of greater reading fluency.

Keywords: linear thermoelasticity; voids; delay times; domain of influence

MSC: 74A15; 74F05; 74F10

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1. Introduction

First of all, it is important to underline that this communication represents a natural extension of the study of well-posedness started in [1]; for this reason, the reading of the present paper should not ignore, and indeed should be strictly connected to, the contents of [1] also because—in order to avoid unnecessary duplications—we will not report here the equations underlying the mathematical model object of investigation, nor will we specify the conventions used, limiting ourselves to their citation. Furthermore, we immediately point out that, in the present investigation, the reference time-differential heat conduction law is assumed as follows:

$$\begin{aligned} &\tau_q^2 \ddot{q}_i(\mathbf{x},t)/2 + \tau_q \dot{q}_i(\mathbf{x},t) + q_i(\mathbf{x},t) \\ &\quad = -\tau_T k_{ij}(\mathbf{x}) \dot{T}_{,j}(\mathbf{x},t) - \left[k_{ij}(\mathbf{x}) + \tau_\alpha K_{ij}(\mathbf{x})\right] T_{,j}(\mathbf{x},t) - K_{ij}(\mathbf{x}) \alpha_{,j}(\mathbf{x},t), \end{aligned} \tag{1}$$

where $q_i(\mathbf{x},t)$ are the components of the heat flux vector, depending on space ($\mathbf{x}$) and time (t), $T(\mathbf{x},t)$ is the temperature variation from the constant reference value $T_0(>0)$, and $\alpha(\mathbf{x},t)$ (see, for instance, Green and Naghdi [2]) is the thermal displacement such that $\dot{\alpha}(\mathbf{x},t) = T(\mathbf{x},t)$; we remark that superposed dots and subscripts preceded by a comma denote partial differentiation with respect to time and space variables, respectively. Moreover, $k_{ij}(\mathbf{x})$ and $K_{ij}(\mathbf{x})$ are the components of the thermal conductivity and conductivity rate tensors, respectively, while τ_q, τ_T and τ_α (all assumed strictly positive) are the heat flux, temperature gradient, and thermal displacement gradient delay times, respectively. The restrictions able to guarantee the thermodynamic consistency of models based on this heat transfer law are derived in [3]. Nevertheless, it is worth specifying that other possible selections in terms of Taylor series expansion orders are possible, proposing, for example, the following reference equation:

$$\tau_q \dot{q}_i(\mathbf{x},t) + q_i(\mathbf{x},t) = -\tau_T k_{ij}(\mathbf{x})\dot{T}_{,j}(\mathbf{x},t) - \left[k_{ij}(\mathbf{x}) + \tau_\alpha K_{ij}(\mathbf{x})\right] T_{,j}(\mathbf{x},t) - K_{ij}(\mathbf{x})\alpha_{,j}(\mathbf{x},t), \quad (2)$$

the thermodynamic (second law) compatibility constraints of which have been recently analyzed in [4].

For bibliographic completeness, we very briefly recall that in 1995, Tzou [5] proposed a dual-phase-lag model in order to generalize the classical Fourier law of heat conduction, involving two delay times τ_q and τ_T. Based on Green and Naghdi's intuition, which included among the constitutive variables not only the temperature gradient but also the thermal displacement gradient, in 2007, Roy Choudhuri [6] introduced a three-phase-lag model, which in turn generalized the previous one by Tzou and involved also a third relaxation time τ_α. We must remember that, unlike the original models subject to some criticism in the literature, for the corresponding time-differential versions (obtained by resorting to suitable Taylor series expansions), a large number of results have been published, corroborating their well-posedness. In the context of linear thermoelasticity, and leaving aside the relevant high-expansion orders and wave propagation issues, we mention only as examples the following studies of well-posedness questions:

- Two delay times with compact [7] and porous [8] elastic matrices;
- Three delay times with compact [9] and porous [1] elastic matrices.

The present work complements, together with [1], the study of thermoelastic materials with voids and three delay times, providing a framework superimposable to the one described in [8] for media with a porous elastic matrix and only two delay times. The work plan is the following. In Section 2, we cite from [1] (without reporting them) the partial differential equations underlying the model studied, together with some important mathematical notations used. Instead, what is made explicit is the choice of external loads/supplies and initial/boundary conditions, peculiar with respect to the study of the spatial behavior of the solutions; Section 3 is devoted to the description of the mathematical process employed, while in Section 4 we give the domain of influence theorem, with a related proof sketch. In Section 5, we draw the main conclusions of the work.

It is worth remembering that the (joint) objective of the present work together with [1], which replicates the scheme of [8] for a more general thermoelasticity model, is the investigation of the well-posedness issue of the model itself, combined with the study of the spatial behavior of the solution, in the sense that the identification of explicit solutions for the above-proposed models (i.e., initial-boundary value problems) is not among the goals of such studies. In a near future, it would be interesting to make advances in this sense, as for example in [10], where a general solution is presented for the Lord–Shulman thermoelasticity model in transversely isotropic solids, proving its completeness (such general solution is also detailed for various combinations of (thermo)elastodynamic and elastostatic theories for transversely isotropic and isotropic media) or even in [11], and, very recently, in [12], or [13], in which numerical techniques are also employed.

Nonetheless, we must highlight that the study of the spatial behavior of solutions, which here in detail translates into the proof of the existence of a domain of influence for the external data, has always been a topic of great importance. In this regard, we would like to cite, without any claim to completeness and limiting the field to mathematical models close to the one investigated here, the contributions by Dhaliwal and Wang [14], where the Cowin–Nunziato theory [15] is taken into account to describe the mechanical behavior of elastic solids with small pores, highlighting potential applications for geological materials like rock and soil, as well as to manufactured porous materials. In [14], it is also clearly explained what is meant by the domain of influence: *... in the context of theory considered, the solution to a mixed initial-boundary value problem(s) vanishes outside a bounded domain Ω_t for a finite time $t > 0$. Ω_t ... is called the domain of influence of the data at time t associated with the problem*.

Proceeding in chronological order, we would like to remember, for instance, the work of Ciarletta and Ieşan [16], together with the references by Ignaczak [17] and Ignaczak and coworkers [18] cited therein; also, in [18], we underline the existence of a domain of influence, which is linked to the presence of a relaxation time; or even the works by Hetnarski and Ignaczak [19], Quintanilla and Racke [20] (In which one can read: *... the spatial behavior of solutions is analyzed in a semi-infinite cylinder* (framework also applicable to our analysis, see note at the end of Section 4) *and a result on the domain of influence is obtained*); and, more recently, Ostoja–Starzewski and Quintanilla [21], where the spatial behavior of solutions is investigated, highlighting an interesting parallel with the Moore–Gibson–Thompson (MGT) equation (see also [4,22]). More generally, as Fernández and Quintanilla state in [23]: *Mathematical studies about the spatial behavior have been proposed for elliptic, hyperbolic and parabolic equations [...]. The list of contributions in this theory is huge*.

2. Materials & Methods: Definition of the Problem

Since the present investigation, as mentioned, represents an advancement—in terms of spatial behavior and the existence of a domain of influence of the solution—of the study proposed in [1], we will not report here the equations that define the thermoelastic model in question, but rather we just mention them. Specifically, we deal with an anisotropic inhomogeneous linear thermoelastic material with voids; the presence of pores is modeled following the classical Cowin–Nunziato theory [15] and, regarding the part related to heat exchanges, the presence of three distinct relaxation times is taken into account. All the conventions defined in [1] (including those of summation and differentiation) must be considered valid and, in addition, we specify that the Greek subscripts will range over the set $\{1,2\}$. As for the definition of the model, our reference is therefore to Equations (1)–(6) and (8), Section 2 in [1], as well as to the related meanings and notations. In addition, we specify also that the preliminary analytical handling of the model does not undergo any changes with respect to [1], and so we refer here (and consider valid) also to Equations (10)–(15), again from Section 2. Instead, the selection of external loads/supplies and initial/boundary conditions deserves a separate discussion, dealing here with a domain of influence problem, and this is what we describe below.

We need to choose, for the specific purpose, null body forces and an external rate of heat supply, i.e., $f_i = l = s = 0$, as well as null initial conditions, i.e., $u_i^0 = \dot{u}_i^0 = \varphi^0 = \dot{\varphi}^0 = T^0 = q_i^0 = \dot{q}_i^0 = 0$: this clearly reverberates in the cancellation of the terms $F_i(\mathbf{x},t)$, $L(\mathbf{x},t)$, $S(\mathbf{x},t)$, and $\Omega_i(\mathbf{x},t)$.

As far as boundary conditions (BCs) are concerned, and for greater clarity, it may be useful to refer for instance to Figure 1, p. 110, in [24], considering valid, for the present case, the following notations: the right cylinder is called C, its lower base (lying in the Ox_1x_2 plane) is indicated with D_0, and its height with H. Exactly as in [8], it is worth mentioning that the domain of influence theorem that will be proven remains valid regardless of the shape of the region under investigation; nonetheless, for purely computational needs, we take into account the above-described right cylinder, the boundary of which is assumed to be sufficiently regular in order to allow the application of the divergence theorem.

Summarizing, a Cartesian reference system $Ox_1x_2x_3$ is considered, such that the coordinate x_3 (which varies orthogonally to the bases of the cylinder) ranges between 0 and H, and the lower base D_0 continues to lie in the coordinate plane Ox_1x_2. We also call D_{x_3} the plane cross-section at distance x_3 from D_0 and, correspondingly, C_{x_3} the portion of C of height $(H - x_3)$ between the cross-sections D_{x_3} and D_H. We set the following:

a. Trivial (side) BCs:

$$
\begin{aligned}
&t^*_{\alpha i}(x_1,x_2,x_3,t)n_\alpha = 0 \quad \text{or} \quad u^*_i(x_1,x_2,x_3,t) = 0,\\
&h^*_\alpha(x_1,x_2,x_3,t)n_\alpha = 0 \quad \text{or} \quad \varphi^*(x_1,x_2,x_3,t) = 0,\\
&q^*_\alpha(x_1,x_2,x_3,t)n_\alpha = 0 \quad \text{or} \quad \alpha^*(x_1,x_2,x_3,t) = 0 \qquad \text{if } (x_1,x_2) \in \partial D_{x_3},\ x_3 \in (0,H),
\end{aligned} \tag{3}
$$

where n_α ($\alpha = 1,2$) are the components of the outward unit vector normal to the side surface of the cylinder.

b. Trivial (upper base) BCs:

$$
\begin{aligned}
&t^*_{3i}(x_1,x_2,H,t) = 0 \quad \text{or} \quad u^*_i(x_1,x_2,H,t) = 0,\\
&h^*_3(x_1,x_2,H,t) = 0 \quad \text{or} \quad \varphi^*(x_1,x_2,H,t) = 0,\\
&q^*_3(x_1,x_2,H,t) = 0 \quad \text{or} \quad \alpha^*(x_1,x_2,H,t) = 0 \qquad \text{if } (x_1,x_2) \in D_H.
\end{aligned} \tag{4}
$$

c. Non-zero (lower base, D_0) BCs, intended as an assigned thermomechanical signal coming from the outside:

$$
\begin{aligned}
&t^*_{3i}(x_1,x_2,0,t) = \widehat{t}^*_i(x_1,x_2,t) \quad \text{or} \quad u^*_i(x_1,x_2,0,t) = \widehat{u}^*_i(x_1,x_2,t),\\
&h^*_3(x_1,x_2,0,t) = \widehat{h}^*(x_1,x_2,t) \quad \text{or} \quad \varphi^*(x_1,x_2,0,t) = \widehat{\varphi}^*(x_1,x_2,t),\\
&q^*_3(x_1,x_2,0,t) = \widehat{q}^*(x_1,x_2,t) \quad \text{or} \quad \alpha^*(x_1,x_2,0,t) = \widehat{\alpha}^*(x_1,x_2,t) \qquad \text{if } (x_1,x_2) \in D_0,
\end{aligned} \tag{5}
$$

for which the depth of propagation in C (i.e., the spatial behavior of the solution varying the distance x_3 from the perturbed base D_0) will be evaluated in the following. It is worth highlighting that the functions $\widehat{t}^*_i$, $\widehat{h}^*$, $\widehat{q}^*$ and $\widehat{u}^*_i$, $\widehat{\varphi}^*$, $\widehat{\alpha}^*$, assumed sufficiently regular for our purposes, evidently reverberate their presence in the measure $\mathscr{I}_\delta(x_3,t)$ of the solution of the initial-boundary value problem, which will be defined later in the work.

3. Materials & Methods: Mathematical Handling

Under the hypotheses just mentioned, let us multiply Equations $(13)_1$ and $(13)_2$ in [1] by $\partial u^*_i/\partial t$ and $\partial\varphi^*/\partial t$, respectively, integrate over the volume C_{x_3}, apply the divergence theorem, and sum up the results obtained. We receive

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_{C_{x_3}}\left[\rho\frac{\partial u^*_i}{\partial t}\frac{\partial u^*_i}{\partial t} + \rho\varkappa\left(\frac{\partial\varphi^*}{\partial t}\right)^2\right]dv\\
&\quad = -\int_{C_{x_3}}\left(t^*_{ji}\frac{\partial e^*_{ij}}{\partial t} + h^*_i\frac{\partial\varphi^*_{,i}}{\partial t} - g^*\frac{\partial\varphi^*}{\partial t}\right)dv - \int_{D_{x_3}}\left(t^*_{3i}\frac{\partial u^*_i}{\partial t} + h^*_3\frac{\partial\varphi^*}{\partial t}\right)da.
\end{aligned}
$$

Using then Equations $(13)_3$ and $(14)_1$–$(14)_4$ from [1], once more in connection with the divergence theorem, and defining, in agreement with Equation (28) in [8], or even Equation (3.10) in [25], the energy density associated with strain and void volume distortion

$$
W^*(\mathbf{x},t) = \left[C_{ijrs}e^*_{ij}e^*_{rs} + A_{ij}\varphi^*_{,i}\varphi^*_{,j} + \xi(\varphi^*)^2\right]/2 + B_{ij}e^*_{ij}\varphi^* + D_{ijk}e^*_{ij}\varphi^*_{,k} + d_i\varphi^*\varphi^*_{,i} \qquad (\geq 0)
$$

we obtain

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_{C_{x_3}}\left[2W^* + \rho\frac{\partial u^*_i}{\partial t}\frac{\partial u^*_i}{\partial t} + \rho\varkappa\left(\frac{\partial\varphi^*}{\partial t}\right)^2 + a\left(\frac{\partial\alpha^*}{\partial t}\right)^2\right]dv\\
&\quad = -\int_{D_{x_3}}\left(t^*_{3i}\frac{\partial u^*_i}{\partial t} + h^*_3\frac{\partial\varphi^*}{\partial t} + \frac{q^*_3}{T_0}\frac{\partial\alpha^*}{\partial t}\right)da - \int_{C_{x_3}}\frac{q^*_i}{T_0}\frac{\partial\beta^*_i}{\partial t}dv.
\end{aligned} \tag{6}
$$

We invoke now Equations (10), (11), and $(14)_5$ in [1], multiply our Equation (6) by $e^{-\delta t}$, where $\delta \in \mathbb{R}^+$, and recall the definitions (25) in [1], namely:

$$
\gamma_{ij} = k_{ij} + \tau_\alpha K_{ij}, \qquad \kappa_{ij} = \tau_T k_{ij} - \tau_q\gamma_{ij}/2, \qquad \varkappa_{ij} = k_{ij} + (\tau_\alpha - \tau_q)K_{ij}.
$$

In addition, we set here for convenience

$$\Gamma_{ij} = (\tau_T + \tau_q)k_{ij} + \tau_q(\tau_\alpha - 3\tau_q/2)K_{ij}$$

and remember that, in view of [3], p. 228, the requirement of compatibility of the model in question with thermodynamics implies that the tensors $\varkappa_{ij}$, κ_{ij} (and so Γ_{ij}) are positive semi-definite. After very long but straightforward calculations, we are led to the following equality

$$\begin{aligned}
&\frac{1}{T_0}\int_{C_{x_3}} e^{-\delta t} q_i^* \frac{\partial \beta_i^*}{\partial t} dv = \frac{\tau_q}{T_0}\int_{C_{x_3}} e^{-\delta t}\left(\kappa_{ij} + \frac{\delta}{4}\tau_T\tau_q k_{ij}\right)\beta_i\beta_j dv + \frac{\tau_T\tau_q^2}{4T_0}\frac{d}{dt}\int_{C_{x_3}} e^{-\delta t} k_{ij}\beta_i\beta_j dv \\
&+\frac{1}{T_0}\int_{C_{x_3}} e^{-\delta t}\left(\varkappa_{ij} + \frac{\delta}{2}\Gamma_{ij} + \frac{\delta^2}{4}\tau_q^2\gamma_{ij}\right)\bar{\beta}_i\bar{\beta}_j dv + \frac{1}{2T_0}\frac{d}{dt}\int_{C_{x_3}} e^{-\delta t}\left(\Gamma_{ij} + \delta\tau_q^2\gamma_{ij}\right)\bar{\beta}_i\bar{\beta}_j dv \\
&+\frac{\tau_q^2}{4T_0}\frac{d^2}{dt^2}\int_{C_{x_3}} e^{-\delta t}\gamma_{ij}\bar{\beta}_i\bar{\beta}_j dv + \frac{\delta}{2T_0}\left(1 + \tau_q\delta + \frac{\tau_q^2}{2}\delta^2\right)\int_{C_{x_3}} e^{-\delta t} K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dv \\
&+\frac{1}{2T_0}\left(1 + 2\tau_q\delta + \frac{3\tau_q^2}{2}\delta^2\right)\frac{d}{dt}\int_{C_{x_3}} e^{-\delta t} K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dv + \frac{\tau_q}{4T_0}(2 + 3\tau_q\delta)\frac{d^2}{dt^2}\int_{C_{x_3}} e^{-\delta t} K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dv \\
&+\frac{\tau_q^2}{4T_0}\frac{d^3}{dt^3}\int_{C_{x_3}} e^{-\delta t} K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dv
\end{aligned} \tag{7}$$

which actually represents a generalization of the relation (48), p. 1595, in [8]; we emphasize that all the terms in (round) brackets in (7) are non-negative under the assumptions of thermodynamic consistency of the model, see [3].

From (6), we multiply by $e^{-\delta t}$, and (7) then we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{C_{x_3}} \frac{e^{-\delta t}}{2}\left[2W^* + \rho\frac{\partial u_i^*}{\partial t}\frac{\partial u_i^*}{\partial t} + \rho\varkappa\left(\frac{\partial \varphi^*}{\partial t}\right)^2 + a\left(\frac{\partial \alpha^*}{\partial t}\right)^2\right] dv \\
&+\int_{C_{x_3}} \frac{\delta e^{-\delta t}}{2}\left[2W^* + \rho\frac{\partial u_i^*}{\partial t}\frac{\partial u_i^*}{\partial t} + \rho\varkappa\left(\frac{\partial \varphi^*}{\partial t}\right)^2 + a\left(\frac{\partial \alpha^*}{\partial t}\right)^2\right] dv \\
&+\frac{\tau_q}{T_0}\int_{C_{x_3}} e^{-\delta t}\left(\kappa_{ij} + \frac{\delta}{4}\tau_T\tau_q k_{ij}\right)\beta_i\beta_j dv + \frac{\tau_T\tau_q^2}{4T_0}\frac{d}{dt}\int_{C_{x_3}} e^{-\delta t} k_{ij}\beta_i\beta_j dv \\
&+\frac{1}{T_0}\int_{C_{x_3}} e^{-\delta t}\left(\varkappa_{ij} + \frac{\delta}{2}\Gamma_{ij} + \frac{\delta^2}{4}\tau_q^2\gamma_{ij}\right)\bar{\beta}_i\bar{\beta}_j dv + \frac{1}{2T_0}\frac{d}{dt}\int_{C_{x_3}} e^{-\delta t}\left(\Gamma_{ij} + \delta\tau_q^2\gamma_{ij}\right)\bar{\beta}_i\bar{\beta}_j dv \\
&+\frac{\tau_q^2}{4T_0}\frac{d^2}{dt^2}\int_{C_{x_3}} e^{-\delta t}\gamma_{ij}\bar{\beta}_i\bar{\beta}_j dv + \frac{\delta}{2T_0}\left(1 + \tau_q\delta + \frac{\tau_q^2}{2}\delta^2\right)\int_{C_{x_3}} e^{-\delta t} K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dv \\
&+\frac{1}{2T_0}\left(1 + 2\tau_q\delta + \frac{3\tau_q^2}{2}\delta^2\right)\frac{d}{dt}\int_{C_{x_3}} e^{-\delta t} K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dv + \frac{\tau_q}{4T_0}(2 + 3\tau_q\delta)\frac{d^2}{dt^2}\int_{C_{x_3}} e^{-\delta t} K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dv \\
&+\frac{\tau_q^2}{4T_0}\frac{d^3}{dt^3}\int_{C_{x_3}} e^{-\delta t} K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dv = -\int_{D_{x_3}} e^{-\delta t}\left(t_{3i}^*\frac{\partial u_i^*}{\partial t} + h_3^*\frac{\partial \varphi^*}{\partial t} + \frac{q_3^*}{T_0}\frac{\partial \alpha^*}{\partial t}\right) da.
\end{aligned} \tag{8}$$

We stress that the left-hand side of Equation (8) is non-negative, and, in view of the null initial conditions, it can be trivially integrated three times in t. For every x_3 going from 0 to H, and for every t greater than or equal to 0, the following measure $\mathscr{I}_\delta(x_3, t)$ of the solution of our initial-boundary value problem can be defined:

$$0 \le \mathscr{I}_\delta(x_3, t) = -\int_0^t\int_0^s\int_0^z\int_{D_{x_3}} e^{-\delta\xi}\left(t_{3i}^*\frac{\partial u_i^*}{\partial \xi} + h_3^*\frac{\partial \varphi^*}{\partial \xi} + \frac{q_3^*}{T_0}\frac{\partial \alpha^*}{\partial \xi}\right) da\, d\xi\, dz\, ds. \tag{9}$$

An immediate evaluation of the partial derivative $\partial \mathscr{I}_\delta(x_3,t)/\partial x_3$ leads to

$$
\begin{aligned}
-\frac{\partial \mathscr{I}_\delta(x_3,t)}{\partial x_3} ={}& \int_0^t\int_0^s\int_{D_{x_3}} \frac{e^{-\delta z}}{2}\left[2W^* + \rho\frac{\partial u_i^*}{\partial z}\frac{\partial u_i^*}{\partial z} + \rho\varkappa\left(\frac{\partial \varphi^*}{\partial z}\right)^2 + a\left(\frac{\partial \alpha^*}{\partial z}\right)^2\right] da dz ds \\
&+\int_0^t\int_0^s\int_0^z\int_{D_{x_3}} \frac{\delta e^{-\delta\xi}}{2}\left[2W^* + \rho\frac{\partial u_i^*}{\partial \xi}\frac{\partial u_i^*}{\partial \xi} + \rho\varkappa\left(\frac{\partial \varphi^*}{\partial \xi}\right)^2 + a\left(\frac{\partial \alpha^*}{\partial \xi}\right)^2\right] da d\xi dz ds \\
&+\frac{\tau_q}{T_0}\int_0^t\int_0^s\int_0^z\int_{D_{x_3}} e^{-\delta\xi}\left(\kappa_{ij} + \frac{\delta}{4}\tau_T\tau_q k_{ij}\right)\beta_i\beta_j da d\xi dz ds + \frac{\tau_T\tau_q^2}{4T_0}\int_0^t\int_0^s\int_{D_{x_3}} e^{-\delta z}k_{ij}\beta_i\beta_j da dz ds \\
&+\frac{1}{T_0}\int_0^t\int_0^s\int_0^z\int_{D_{x_3}} e^{-\delta\xi}\left(\varkappa_{ij} + \frac{\delta}{2}\Gamma_{ij} + \frac{\delta^2}{4}\tau_q^2\gamma_{ij}\right)\bar\beta_i\bar\beta_j da d\xi dz ds \\
&+\frac{1}{2T_0}\int_0^t\int_0^s\int_{D_{x_3}} e^{-\delta z}\left(\Gamma_{ij} + \delta\tau_q^2\gamma_{ij}\right)\bar\beta_i\bar\beta_j da dz ds + \frac{\tau_q^2}{4T_0}\int_0^t\int_{D_{x_3}} e^{-\delta s}\gamma_{ij}\bar\beta_i\bar\beta_j da ds \\
&+\frac{\delta}{2T_0}\left(1 + \tau_q\delta + \frac{\tau_q^2}{2}\delta^2\right)\int_0^t\int_0^s\int_0^z\int_{D_{x_3}} e^{-\delta\xi}K_{ij}\bar{\bar\beta}_i\bar{\bar\beta}_j da d\xi dz ds \\
&+\frac{1}{2T_0}\left(1 + 2\tau_q\delta + \frac{3\tau_q^2}{2}\delta^2\right)\int_0^t\int_0^s\int_{D_{x_3}} e^{-\delta z}K_{ij}\bar{\bar\beta}_i\bar{\bar\beta}_j da dz ds \\
&+\frac{\tau_q}{4T_0}(2 + 3\tau_q\delta)\int_0^t\int_{D_{x_3}} e^{-\delta s}K_{ij}\bar{\bar\beta}_i\bar{\bar\beta}_j da ds + \frac{\tau_q^2}{4T_0}\int_{D_{x_3}} e^{-\delta t}K_{ij}\bar{\bar\beta}_i\bar{\bar\beta}_j da
\end{aligned}
\tag{10}
$$

from which we deduce that $\mathscr{I}_\delta(x_3,t)$ is not increasing with respect to the coordinate x_3.

We then estimate also the partial derivative $\partial \mathscr{I}_\delta(x_3,t)/\partial t$, making use of the arithmetic-geometric mean inequality:

$$
\begin{aligned}
\left|\frac{\partial \mathscr{I}_\delta}{\partial t}(x_3,t)\right| \le{}& \int_0^t\int_0^s\int_{D_{x_3}} \frac{e^{-\delta z}}{2}\left[\frac{\varepsilon_1}{\rho}t_{3i}^*t_{3i}^* + \frac{\varepsilon_2}{\rho\varkappa}h_3^*h_3^* + \frac{\varepsilon_3}{aT_0^2}q_3^*q_3^*\right. \\
&\left.+\frac{\rho}{\varepsilon_1}\frac{\partial u_i^*}{\partial z}\frac{\partial u_i^*}{\partial z} + \frac{\rho\varkappa}{\varepsilon_2}\left(\frac{\partial \varphi^*}{\partial z}\right)^2 + \frac{a}{\varepsilon_3}\left(\frac{\partial \alpha^*}{\partial z}\right)^2\right] da dz ds, \qquad \forall \varepsilon_1,\varepsilon_2,\varepsilon_3 \in \mathbb{R}^+.
\end{aligned}
\tag{11}
$$

From now on, we will be even more concise in explaining the procedure followed, referring for technical details to [8], p. 1597, and the following: the major change to account for in terms of notations involved is that the tensor $\beta_{ij}(\mathbf{x})$ in [8] becomes $M_{ij}(\mathbf{x})$ here for the sake of avoiding ambiguity. It is thus possible to arrive at the following estimates:

$$
t_{3i}^*t_{3i}^* \le t_{ij}^*t_{ij}^* \le 2(1+\varepsilon_4)S^*W^* + \left(1 + \frac{1}{\varepsilon_4}\right)\max(M_{rs}M_{rs})\left(\frac{\partial \alpha^*}{\partial t}\right)^2, \qquad \forall \varepsilon_4 \in \mathbb{R}^+, \tag{12}
$$

where S^*, along with related defined quantities, is given by Equation (56) in [8];

$$
h_3^*h_3^* \le h_i^*h_i^* \le 2(1+\varepsilon_5)G^*W^* + \left(1 + \frac{1}{\varepsilon_5}\right)\max(a_k a_k)\left(\frac{\partial \alpha^*}{\partial t}\right)^2, \qquad \forall \varepsilon_5 \in \mathbb{R}^+, \tag{13}
$$

where G^* is defined this time by Equation (59) in [8]. The estimate of $q_3^*q_3^*$ is instead different from [8], and so we make it explicit. Starting from $(14)_5$ in [1], we employ the Cauchy–Schwarz inequality and obtain

$$
\begin{aligned}
q_i^*q_i^* &= q_i^*\left(K_{ij}\bar{\bar\beta}_j + \gamma_{ij}\bar\beta_j + \tau_T k_{ij}\beta_j\right) \\
&\le \left[(K_{rs}K_{rs})^{1/2}\left(\bar{\bar\beta}_k\bar{\bar\beta}_k\right)^{1/2} + (\gamma_{rs}\gamma_{rs})^{1/2}\left(\bar\beta_k\bar\beta_k\right)^{1/2} + \tau_T(k_{rs}k_{rs})^{1/2}(\beta_k\beta_k)^{1/2}\right]^2
\end{aligned}
$$

and therefore

$$q_3^* q_3^* \leq q_i^* q_i^* \leq 3\Big[(K_{rs}K_{rs})\Big(\bar{\bar{\beta}}_k \bar{\bar{\beta}}_k\Big) + (\gamma_{rs}\gamma_{rs})\big(\bar{\beta}_k\bar{\beta}_k\big) + \tau_T^2 (k_{rs}k_{rs})(\beta_k\beta_k) \Big], \tag{14}$$

which is Equation (29) in [24], p. 117.

At this stage, in view of our Equations (11)–(14), we obtain

$$\begin{aligned}
\left| \frac{\partial \mathscr{I}_\delta}{\partial t}(x_3,t) \right| \leq & \frac{1}{2}\int_0^t \int_0^s \int_{D_{x_3}} e^{-\delta z} \Bigg\{ 2\left[\frac{\varepsilon_1}{\rho}(1+\varepsilon_4)S^* + \frac{\varepsilon_2}{\rho\varkappa}(1+\varepsilon_5)G^* \right] W^* \\
& + \frac{\rho}{\varepsilon_1}\frac{\partial u_i^*}{\partial z}\frac{\partial u_i^*}{\partial z} + \frac{\rho\varkappa}{\varepsilon_2}\left(\frac{\partial \varphi^*}{\partial z}\right)^2 \\
& + \left[\frac{\varepsilon_1}{\rho}\left(1+\frac{1}{\varepsilon_4}\right)\max(M_{rs}M_{rs}) + \frac{\varepsilon_2}{\rho\varkappa}\left(1+\frac{1}{\varepsilon_5}\right)\max(a_k a_k) + \frac{a}{\varepsilon_3}\right]\left(\frac{\partial \alpha^*}{\partial z}\right)^2 \\
& + \frac{3\varepsilon_3}{aT_0^2}\Big[(K_{rs}K_{rs})\Big(\bar{\bar{\beta}}_k \bar{\bar{\beta}}_k\Big) + (\gamma_{rs}\gamma_{rs})\big(\bar{\beta}_k\bar{\beta}_k\big) + \tau_T^2 (k_{rs}k_{rs})(\beta_k\beta_k) \Big] \Bigg\} dadzds, \\
& \forall \varepsilon_i \in \mathbb{R}^+, \quad i = 1,2,\ldots,5.
\end{aligned} \tag{15}$$

Moreover, from (10), we deduce

$$\begin{aligned}
\frac{\partial \mathscr{I}_\delta}{\partial x_3}(x_3,t) \leq & -\int_0^t \int_0^s \int_{D_{x_3}} \frac{e^{-\delta z}}{2}\left[2W^* + \rho \frac{\partial u_i^*}{\partial z}\frac{\partial u_i^*}{\partial z} + \rho\varkappa\left(\frac{\partial \varphi^*}{\partial z}\right)^2 + a\left(\frac{\partial \alpha^*}{\partial z}\right)^2 \right] dadzds \\
& -\frac{1}{2T_0}\int_0^t \int_0^s \int_{D_{x_3}} e^{-\delta z}\left(1 + 2\tau_q\delta + \frac{3}{2}\tau_q^2\delta^2\right) K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j dadzds \\
& -\frac{1}{2T_0}\int_0^t \int_0^s \int_{D_{x_3}} e^{-\delta z}\left\{ \left[\tau_T + (1+\tau_q\delta)\tau_q\right]k_{ij} + \tau_q\left[(1+\tau_q\delta)\tau_\alpha - \frac{3}{2}\tau_q\right]K_{ij} \right\}\bar{\beta}_i\bar{\beta}_j dadzds \\
& -\frac{\tau_T\tau_q^2}{4T_0}\int_0^t \int_0^s \int_{D_{x_3}} e^{-\delta z} k_{ij}\beta_i\beta_j dadzds
\end{aligned} \tag{16}$$

which actually represents, setting $\delta = 0$, the extension to the porous case of Equation (34) in [24], p. 118. Identifying the smallest (positive) eigenvalues of the tensors k_{ij} and K_{ij} with k_m and K_m, respectively (see again p. 117 in [24]), and by a direct comparison between Equations (15) and (16), through straightforward calculations, we are led to define

$$\begin{aligned}
\Psi = \max\Bigg\{ & \frac{2}{2+4\tau_q\delta+3\tau_q^2\delta^2}\sup_{D_{x_3}}\left(\frac{K_{rs}K_{rs}}{aK_m}\right), \frac{1}{\tau_q\left[2(1+\tau_q\delta)\tau_\alpha - 3\tau_q\right]}\sup_{D_{x_3}}\left(\frac{\gamma_{rs}\gamma_{rs}}{aK_m}\right), \\
& \frac{1}{2\left[\tau_T + (1+\tau_q\delta)\tau_q\right]}\sup_{D_{x_3}}\left(\frac{\gamma_{rs}\gamma_{rs}}{ak_m}\right), \frac{2\tau_T}{\tau_q^2}\sup_{D_{x_3}}\left(\frac{k_{rs}k_{rs}}{ak_m}\right) \Bigg\}.
\end{aligned}$$

4. Spatial Behavior Result: Domain of Influence Theorem

From a comparison between spatial and temporal partial derivatives of the measure $\mathscr{I}_\delta(x_3,t)$, it is appropriate to proceed by equating the following coefficients:

$$\begin{aligned}
& \frac{\varepsilon_1}{\rho}(1+\varepsilon_4)S^* + \frac{\varepsilon_2}{\rho\varkappa}(1+\varepsilon_5)G^* = \frac{1}{\varepsilon_1} = \frac{1}{\varepsilon_2} \\
& = \frac{\varepsilon_1}{a\rho}\left(1+\frac{1}{\varepsilon_4}\right)\max(M_{rs}M_{rs}) + \frac{\varepsilon_2}{a\rho\varkappa}\left(1+\frac{1}{\varepsilon_5}\right)\max(a_k a_k) + \frac{1}{\varepsilon_3} = \frac{3\varepsilon_3}{T_0}\Psi = \sigma.
\end{aligned} \tag{17}$$

Necessarily, it has to be $\varepsilon_1 = \varepsilon_2 (= \epsilon)$, while, as in [8], p. 1599, it is convenient to set $\varepsilon_4 = \varepsilon_5 = \tilde{\varepsilon}$. Through such assumptions, Equation (17) is simplified in

$$\sigma = \frac{\epsilon}{\rho\varkappa}(1+\tilde{\epsilon})(\varkappa S^* + G^*) = \frac{1}{\epsilon}$$
$$= \frac{\epsilon}{a\rho\varkappa}\left(1+\frac{1}{\tilde{\epsilon}}\right)[\varkappa \max(M_{rs}M_{rs}) + \max(a_k a_k)] + \frac{1}{\varepsilon_3} = \frac{3\varepsilon_3}{T_0}\Psi. \tag{18}$$

Using, again, a standard procedure, we receive an ϵ as in Equation (66) in [8], while for ε_3 we obtain

$$\varepsilon_3 = \frac{T_0}{3\Psi}\sqrt{\frac{(1+\tilde{\epsilon})(\varkappa S^* + G^*)}{\rho\varkappa}} \tag{19}$$

and set $\Delta = \varkappa \max(M_{rs}M_{rs}) + \max(a_k a_k)$ in such a way as to arrive at the following second degree polynomial equation in the single variable $\tilde{\epsilon}$:

$$\tilde{\epsilon}^2 + \left[1 - \frac{T_0\Delta + 3a\rho\varkappa\Psi}{aT_0(\varkappa S^* + G^*)}\right]\tilde{\epsilon} - \frac{\Delta}{a(\varkappa S^* + G^*)} = 0 \tag{20}$$

admitting only one acceptable real, strictly positive solution, which represents the value of $\varepsilon_4 = \varepsilon_5 = \tilde{\epsilon}$.

Finally, we can proceed for instance as in [24] (or even [8]) through the integration of a suitable differential inequality (shown below) exhibiting the existence of a domain of influence of the assigned data.

Theorem 1. *(Domain of influence of the assigned data) Let* $\mathscr{S} = \{u_i, \varphi, \alpha, e_{ij}, \beta_i, t_{ij}, h_i, g, \eta, q_i\}$ *be the (unique, see [1]) solution of the initial-boundary value problem* $\mathscr{P}$ *defined in [1], p. 4. Assume* $\rho, \varkappa, a \in \mathbb{R}^+$, $W^*(\mathbf{x}, t)$ *a positive definite quadratic form and, moreover, the requirements of compatibility with thermodynamics of the time differential three-phase-lag model [3], p. 228, be valid. Also, assume that the right cylinder C is loaded from the outside (through its lower base* D_0*) by external thermomechanical actions (5). Then, there exists a constant* $\sigma \in \mathbb{R}^+$ *(a speed, from the point of view of dimensions), such that* $\mathscr{S} = 0$ *for every* $\mathbf{x} \in C$ *such that* $x_3 \geq \sigma t$*; i.e., the effects in the cylinder C due to the external data insisting on the lower base* D_0 *vanish at distances from* D_0 *greater than or equal to* σt*.*

Proof. Taking into account Equation (15) and the related following estimates, it is sufficient to consider the right-hand side coefficients equal to σ (see Equations (17)–(20)) in order to obtain

$$\begin{aligned}\left|\frac{\partial \mathscr{I}_\delta}{\partial t}(x_3, t)\right| \leq \frac{\sigma}{2}\int_0^t\int_0^s\int_{D_{x_3}} e^{-\delta z}\Bigg\{ & 2W^* + \rho\frac{\partial u_i^*}{\partial z}\frac{\partial u_i^*}{\partial z} + \rho\varkappa\left(\frac{\partial \varphi^*}{\partial z}\right)^2 + a\left(\frac{\partial \alpha^*}{\partial z}\right)^2 \\ & + \frac{1}{T_0}\left(1 + 2\tau_q\delta + \frac{3}{2}\tau_q^2\delta^2\right)K_{ij}\bar{\bar{\beta}}_i\bar{\bar{\beta}}_j + \frac{\tau_q}{T_0}\left[(1+\tau_q\delta)\tau_\alpha - \frac{3}{2}\tau_q\right]K_{ij}\bar{\beta}_i\bar{\beta}_j \\ & + \frac{1}{T_0}[\tau_T + (1+\tau_q\delta)\tau_q]k_{ij}\bar{\beta}_i\bar{\beta}_j + \frac{\tau_T\tau_q^2}{2T_0}k_{ij}\beta_i\beta_j\Bigg\} dadzds.\end{aligned}$$

From Equation (16), we are led to the following differential inequality, valid for each couple $(x_3, t) \in (0, H) \times (0, \infty)$

$$\left|\frac{\partial \mathscr{I}_\delta}{\partial t}\right| + \sigma\frac{\partial \mathscr{I}_\delta}{\partial x_3} \leq 0 \tag{21}$$

that can be easily integrated in a standard way (repeating the whole procedure is considered not particularly meaningful, see [8] or [24] for more details). The conclusion is that, for each instant t, $\mathscr{I}_\delta(x_3, t) = 0$, $\forall x_3 \geq \sigma t$, i.e., the effects of the external signal are not felt at distances from D_0 greater than or equal to σt. The existence of a domain of influence of the assigned external actions remains then proven. □

We also specify (refer to Appendix, p. 119 in [24]) that such an influence domain is preserved, also taking into account a semi-infinite cylindrical region C, i.e., letting H tend to infinity.

5. Conclusions

The main purpose of this brief work is to contribute to the completeness of the analysis started with the study of the well-posedness question in [1] for a model of linear thermoelasticity in which the porous skeleton is described through the Cowin–Nunziato theory, while the heat transfer phenomena obey a time-differential law with three relaxation times. We were able to highlight how the presence of a domain of influence of the solution remains even when the effects of a thermoelastic porous matrix (taken into account in [8], but dealing with only two delay times) are combined with those of a time-differential model with three relaxation times (taken into account in [24], but dealing with a compact elastic matrix). A theorem synthetically proven shows that for a finite time $t > 0$, the assigned data generate no disturbance outside a bounded domain within the cylindrical region C.

Aware that the proposed issue is based on a strongly mathematical theorization of heat exchange mechanisms, we believe anyway that this completion was due in order to provide an exhaustive framework for the model under consideration.

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