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On the Properties of λ-Prolongations and λ-Symmetries

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Abstract: In this paper, (1) We show that if there are not enough symmetries and λ-symmetries, some first integrals can still be obtained. And we give two examples to illustrate this theorem. (2) We prove that when X is a λ-symmetry of differential equation field Γ, by multiplying Γ a function μ defineded on $J^{n-1}M$, the vector fields $\mu\Gamma$ can pass to quotient manifold Q by a group action of λ-symmetry X. (3) If there are some λ-symmetries of equation considered, we show that the vector fields from their linear combination are symmetries of the equation under some conditions. And if we have vector field X defined on $J^{n-1}M$ with first-order λ-prolongation Y and first-order standard prolongations Z of X defined on J^nM, we prove that gY cannot be first-order λ-prolonged vector field of vector field gX if g is not a constant function. (4) We provide a complete set of functionally independent $(n-1)$ order invariants for $V^{(n-1)}$ which are $n-1$th prolongation of λ-symmetry of V and get an explicit $n-1$ order reduced equation of explicit n order ordinary equation considered. (5) Assume there is a set of vector fields $X_i, i = 1, \ldots, n$ that are in involution, We claim that under some conditions, their λ-prolongations also in involution.

Keywords: volume form; λ-symmetry; jet bundle

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1. Introduction

Classical symmetry theory for general systems of differential equations was created by Sophus Lie more than 100 years ago. It plays an important role in analyzing differential equations. The geometric theory of symmetries makes it possible to understand and generalize the standard procedures of the explicit integration of ODE and to obtain new results. They are often used to reduce the order of ordinary differential equations and to reduce the number of independent variables of partial differential equations. The method of symmetry reduction for ODEs can be roughly described as follows: in J^nM, which is the jet bundle of order n over the manifold M of independent and dependent variables, if ODEs $\mathcal{E}$ of order $n > 1$ have a vector field X (on M) as a symmetry (generator), then $\mathcal{E}$ is invariant under $Y = X^{(n)}$, which is the prolongation of X to J^nM. Thus, $\mathcal{E}$ can be expressed in terms of the differential invariants of Y, which can be recursively generated starting from those of orders 0 and 1 thanks to the "invariants by differentiation" property (see [1–7]).

Although the Lie symmetry theory is a good method for analyzing ordinary (and partial) differential equations [4,5], not every technique can be based on symmetry analysis [8–11], thus people need generalizations of classical Lie methods. In past decades, λ-symmetry was introduced by Muriel and Romero [12] based on a new method of prolonging vector fields known as λ-prolongation, which strictly include symmetries. Generally speaking, λ-symmetries may not be symmetries, as they do not map solutions into solutions. The relationship between λ-symmetries and the first integral of ordinary differential equations has been studied extensively [13–19]. Moreover, λ-symmetries have been extended to σ-symmetries [20,21] and to μ-symmetries [22,23], which are different from other

analytical methods (such as the Kudryashov approach; see [24]) and can be used to derive conservation laws of partial differential equations (PDEs). And the analogous theory of symmetries of stochastic differential equations (SDEs) has also emerged in recent years (see [25–30]). In a review of the group analysis methods, symmetries for discrete equations were shown as well. Among the most complete and noteworthy works on this subject at the moment, one can refer to Refs. [31,32]. In [18], the authors considered that if there are not enough symmetries for ODE (4), one can use λ-symmetries to obtain some first integrals. Here, we generalize this situation and show that if there are not enough symmetries and λ-symmetries, some first integrals can still be obtained (see Theorem 2). Moreover, we prove that when X is a λ-symmetry of the differential equation field Γ of (4), by multiplying Γ, a function μ defined on $J^{n-1}M$, the vector fields $\mu\Gamma$ can pass to quotient manifold Q by a group action of λ-symmetry X, i.e., $L_X(\mu\Gamma) = \bar{\lambda}X$ (see Theorem 3). If there are some λ-symmetries of (4), we show that the vector fields from their linear combination are symmetries of (4) under some conditions (see Theorem 4). And if we have vector field X defined on $J^{n-1}M$ with first-order λ-prolongation Y and first-order standard prolongations Z of X defined on J^nM, we prove that gY cannot be a first-order λ-prolonged vector field of vector field gX if g is not a constant function (see Theorem 5). We provide a complete set of functionally independent $(n-1)$-order invariants for $V^{(n-1)}$, which are $(n-1)$th prolongations of λ-symmetry of V (see Theorem 6) and obtain an explicit $n-1$-order-reduced equation of explicit n-order ordinary Equation (4). Compared with [12], we deal with an n-order explicit equation system that has a λ-symmetry V, and we just need to consider the invariants of $(n-1)$-order prolongation $V^{(n-1)}$ of V, but in [12], the authors dealt with implicit equation with a λ-symmetry V, which need to consider the invariants of n-order prolongation $V^{(n-1)}$ of V. At last, we consider that there is a set of vector fields $X_i, i = 1, \ldots, n$ that are in involution and claim that under some conditions their λ-prolongations also in involution (see Theorem 7).

This paper is organized as follows. In Section 2, we recall some basic notations we need from the Lie group theory and give some definitions of the λ-symmetries of vector fields, differential forms, distributions and λ-prolongations. In Section 3, we obtain the first integrals of a system of ordinary differential equations by using λ-symmetries. In Section 4, we obtain a quotient manifold and discuss the relationship between λ-symmetry and symmetry. In Section 5, we obtain an explicit $n-1$-order-reduced equation of the explicit n-order ordinary equation by using a λ-symmetry. In Section 6, we discuss the involution of λ-prolonged sets of vector fields.

2. Preliminaries and Geometric Properties for λ-Symmetries

Firstly, we give some basic notions that may be used later.

2.1. Equation, Solutions and Symmetries

In this paper, we only consider ordinary differential equations that have the independent variable $x \in R$ and the dependent one(s) $y \in U = R$ or $y_a \in U \subset R^m$ in the multidimensional case. Let $M = X \times U$ be a phase bundle and (J^kM, π^k, M), or J^kM for short, the associated jet bundle of order k.

A differential equation $\mathcal{E}$ of order n is a map $F : J^nM \to R$ and is identified with the solution manifold $S(\mathcal{E}) = F^{-1}(0) \subset J^nM$. In the case of l-dimensional systems $\mathcal{E}$, we have l maps $F^j : J^nM \to R$ and a solution manifold $S(\mathcal{E}) = F^{-1}(\mathbf{0}) = (F^1)^{-1}(0) \cap \cdots \cap (F^l)^{-1}(0) \subset J^kM$. If differential equation $\mathcal{E}$ has m-dimensional dependent variables, then J^nM has dimension $(m(n+1)+1)$, thus the dimension of a solution manifold of a system of l independent equations is $(m(n+1)+1-l)$.

For a smooth function vector $y = f(x)$, we can obtain an induced function $pr^{(n)}f(x)$, which is called the n-th prolongation of f and defined by the equations

$$\frac{d^\alpha y_a}{dx_\alpha} = \frac{d^\alpha f^a}{dx^\alpha}, a = 1, \ldots, m, \alpha = 0, \ldots, n.$$

Thus, $pr^{(n)}f(x)$ is a function vector from R^1 to the space J^nM.

For the differential equation(s) $\mathcal{E}$ under study, a function vector $f : R^1 \to U$ is a solution if and only if its n-th prolongation lies entirely in the solution manifold $S(\mathcal{E})$.

Now, consider a vector field Y on J^nM. We say that $\mathcal{E}$ is invariant under Y if and only if its solution manifold has the property $Y : S(\mathcal{E}) \to TS(\mathcal{E})$. This is also equivalent to condition $[Y(\mathcal{E})]_{S(\mathcal{E})} = 0$.

Let Y be the prolongation of a (Lie-point) vector field X on M, i.e., $Y = X^{(n)}$, then X is called a symmetry for $\mathcal{E}$ (more precisely, X would be a symmetry generator. We will adopt this standard abuse of notation for ease of language). Thus, X is a symmetry if $[X^{(n)}(\mathcal{E})]_{S(\mathcal{E})} = 0$.

2.2. Local Coordinates

We will consider local coordinates $(x, y^a), a = 1, \ldots, m$ in $M = X \times U$, and correspondingly local coordinates $(x, y_a^{(k)})$ (with $k = 0, \ldots, n$), where $y_a^{(k)} := (\partial^k y_a / \partial x^k)$, in J^nM.

A general vector field defined on J^nM will be written in local coordinates (in this paper, we will use the Einstein summation convention in some places. The notation $y^{(k)}$ denotes the k-order derivative of y) as

$$Y = \xi(x, y_1, \ldots, y_m)\frac{\partial}{\partial x} + \psi_k^a(x, y_1, \ldots, y_m, y_1^{(1)}, \ldots, y_m^{(1)}, \ldots, y_m^{(n)})\frac{\partial}{\partial y_a^{(k)}}, \tag{1}$$

which is the prolongation of vector field

$$X = \xi(x, y_1, \ldots, y_m)\frac{\partial}{\partial x} + \phi^a(x, y_1, \ldots, y_m)\frac{\partial}{\partial y_a} \tag{2}$$

if and only if the ψ_k^a satisfies the (standard) prolongation formula

$$\psi_{k+1}^a = D_x\psi_k^a - y_a^{(k+1)}D_x\xi, \quad \psi_0^a = \phi^a.$$

We use D_x to denote the total derivative with respect to x, i.e.,

$$D_x = \frac{\partial}{\partial x} + y_a^{(1)}\frac{\partial}{\partial y_a} + y_a^{(2)}\frac{\partial}{\partial y_a^{(1)}} \cdots .$$

2.3. λ-Prolongations

After the work of Muriel and Romero [12,33], people also considered λ-symmetries of ODEs. A vector field Y written in local coordinates in the form (1) is a λ-prolonged vector field if its coefficients satisfy

$$\psi_{k+1}^a = D_x\psi_k^a - y_a^{(k+1)}D_x\xi + \lambda(\psi_k^a - y_a^{(k+1)}\xi) \tag{3}$$

with λ being a smooth function $\lambda : J^1M \to R$. We also say that Y is the nth λ-prolongation of X (see (2)) if $\psi_0^a = \phi^a$, i.e., if X is the restriction of Y to M.

We say that X is a λ-symmetry for $\mathcal{E}$ if the λ-prolongation Y of X leaves an equation (or system) $\mathcal{E}$ invariant.

2.4. Explicit n-Order Equation

Consider the m-dimension system of nth-order differential equations

$$\frac{d^n y_a}{dx^n} = f^a(x, y_a, \ldots, y_a^{(n-1)}) \tag{4}$$

where $(x, y_1, \ldots, y_m) \in M \subset R^{m+1}$ denote $(y_a) = (y_1, \ldots, y_m), a = 1, \ldots, m$. $x, y_a, y_a^{(1)}, \ldots, y_a^{(n-1)}$ for $a = 1, \ldots, m$ as natural coordinates on the $(n-1)$th jet bundle $J^{n-1}M$. Put $\sigma_1^a = dy_a - y_a^{(1)}\,dx, \sigma_j^a = dy_a^{(j-1)} - y_a^{(j)}dx$. Then, $\{\sigma_1^a, \ldots, \sigma_{n-1}^a\}_{a=1}^m$ is the natural basis for the contact forms on $J^{n-1}M$, which are the forms on $J^{n-1}M$ that are zero on curves or surfaces lifted from J^0M. Let $f^a \in \wedge^0 J^{n-1}M$, then put $\sigma_n^a = dy_a^{(n-1)} - f^a dx$. Then, the σ_n^as are the natural force forms for (4). Put

$$\Omega = \sigma_1^1 \wedge \cdots \wedge \sigma_1^m \wedge \cdots \wedge \sigma_n^1 \cdots \wedge \sigma_n^m,$$

meaning the mn-fold wedge product, then Ω is the natural characteristic form for (4). Let

$$\Gamma = \frac{\partial}{\partial x} + y_a^{(1)}\frac{\partial}{\partial y_a} + y_a^{(2)}\frac{\partial}{\partial y_a^{(1)}} + \cdots + f^a\frac{\partial}{\partial y_a^{(n-1)}}, \tag{5}$$

then Γ is an $(n-1)$th-order differential equation field. It is the characteristic vector field of (4), tangent to the lifted solution curves, and represents differentiation along the solution curves. For any p-form ω and vector field X, we denote $i_X\omega$ as the interior product of X and ω [34] (the interior product of the vector field with a differential form reduces the degree of the latter by one), so it is a vector field in the kernel of Ω normalized by $i_\Gamma dx = 1$.

Proposition 1. *We can obtain that* $i_\Gamma\sigma_j^a = (dy_a^{(j-1)} - y_a^{(j)}dx)(\frac{\partial}{\partial x} + y_a^{(1)}\frac{\partial}{\partial y_a} + y_a^{(2)}\frac{\partial}{\partial y_a^{(1)}} + \cdots + f^a\frac{\partial}{\partial y_a^{(n-1)}}) = 0, a = 1, \ldots, m, j = 1, \ldots, n$, *so* $i_\Gamma\Omega = 0$ *and* $i_\Gamma\Omega_V = \Omega$, *where* $\Omega_V = dx \wedge dy_1 \wedge dy_2 \wedge \cdots \wedge dy_m \wedge dy_1^{(1)} \wedge dy_2^{(1)} \wedge \cdots \wedge dy_m^{(n-1)}$ *is the volume form of* $J^{n-1}M$, *here* $J^{n-1}M$ *is the* $(n-1)$*th-order jet bundle.*

Through the whole paper, all calculations involving differential forms such as Cartan's identity, one can see [34].

Let $C(\Delta) := \{(x, y_a, y_a^{(1)}, \ldots, y_a^{(n)}) | y_a^{(n)} = f(x, y_a, y_a^{(1)}, \ldots, y_a^{(n-1)})\}$, We know that a λ-symmetry, or more precisely a Lie-point λ-symmetry generator of (4) is a vector field X on M such that its n-order λ-prolongation Y satisfies that

$$\left[Y\left(\frac{d^n y_i}{dx^n} - f^i(x, y_a, \ldots, y_a^{(n-1)})\right)\right]_{C(\Delta)} = 0, \quad i = 1, \ldots, m. \tag{6}$$

Lemma 1.

(1) *We suppose that, for some* $\lambda \in C^\infty(J^1M)$, *the vector field* $X = \xi(x, y_a)\frac{\partial}{\partial x} + \eta(x, y_a)\frac{\partial}{\partial y_a}$ *is a* λ*-symmetry of Equation* (4). *Then,*

$$[Y, \Gamma] = \lambda \cdot Y + \mu \cdot \Gamma$$

for some $\mu \in C^\infty(J^1M)$, *where* Y *is the* $n-1$*-order* λ*-prolongation of* X, *and* $[\cdot, \cdot]$ *denotes the Lie bracket.*

(2) *Conversely, if*

$$X = \xi(x, y_a)\frac{\partial}{\partial x} + \eta^a(x, y_a)\frac{\partial}{\partial y_a} + \eta_1^a(x, y_a, y_a^{(1)})\frac{\partial}{\partial y_a^{(1)}} + \cdots + \eta_{n-1}^a(x, y_a, y_a^{(1)}, \ldots, y_a^{(n-1)})\frac{\partial}{\partial y_a^{(n-1)}}$$

is a vector field defined on $J^{n-1}M$ *such that*

$$[X, \Gamma] = \lambda X + \mu\Gamma$$

for some $\lambda, \mu \in C^\infty(J^1M)$*, then the vector field*

$$v = \xi(x, y_a)\frac{\partial}{\partial x} + \eta^a(x, y_a)\frac{\partial}{\partial y_a},$$

defined on M*, is a* λ*-symmetry of the equation* (4)*, and* X *is the* $n-1$*-order* λ*-prolongation of* v*.*

Proof. See [18]. □

Definition 1. *A differential p-form is simple or decomposable if it is the wedge product of p 1-forms.*

Definition 2. *We call a set of vector fields* $\mathcal{D}$ *defined on manifold* $\mathcal{M}$ *a distribution if it is a vector subspace of* $T_p\mathcal{M}$*,* $\forall p \in \mathcal{M}$*, and for any vector fields* $X, Y \in \mathcal{D}$*,* $[X, Y] \in \mathcal{D}$*.*

Definition 3. *We say a distribution* $\mathcal{D}$ *is a Frobenius integral if it is rank-invariant, which means that for any point* $p \in \mathcal{M}$*, dim*$\mathcal{D}$ *is a constant.*

Definition 4. *We say a p-form* ω *is a Frobenius integrable if its kernel is a Frobenius integrable and of maximal dimension everywhere.*

Based on the properties of λ-symmetries for a system, we give three more general definitions of λ-symmetries for vector fields, distributions and differential forms.

Definition 5. *Let* X *and* Y *be vector fields,* $\mathcal{D}$ *be a distribution, and* Θ *be a simple differential form defined on manifold* $J^{n-1}M$*.*

1. X *is called a* λ*-symmetry of* Y *if there exist functions* $\lambda \in C^\infty(J^1M), \mu \in C^\infty(J^1M)$ *such that* $L_X Y = [X, Y] = \lambda X + \mu Y$*.*

2. X *is called a* λ*-symmetry of* $\mathcal{D}$ *if* $L_X Y - \lambda X \in \mathcal{D}, \forall Y \in \mathcal{D}$ *for some function* $\lambda \in C^\infty(J^1M)$*.*

3. X *is called a* λ*-symmetry of* Θ *if* X *is a* λ*-symmetry of ker*Θ *for some function* $\lambda \in C^\infty(J^1M)$*, where ker*$\Theta$ *is the kernel or characteristic space of* Θ *defined by ker*$\Theta = \{Y : i_Y\Theta = 0\}$*.*

Remark 1. *A vector field defined on* $J^{n-1}M$ *may not be an* $(n-1)$*-order prolongation of any vector field defined on* M*.*

3. First Integral of System of Ordinary Differential Equation

Lemma 2. *Given a natural number* $N, 3 \leq N \leq mn$*, let span*$\{X_N, \ldots, X_{mn}, \Gamma\}$ *be a Frobenius integral and* X_j *be a* λ_j*-symmetry of span*$\{X_{j+1}, \ldots, X_{mn}, \Gamma\}$*,* $j = 2, \ldots, N-1$ *such that* $X_1, \ldots, X_n$ *and* Γ *are linearly independent everywhere. Then,* $i_{X_j} \cdots i_{X_{mn}}\Omega$ *is a Frobenius integral* $j = 2, \ldots, N$*.*

Proof. Since X_j is a λ_j-symmetry of span$\{X_{j+1}, \ldots, X_{mn}, \Gamma\}$, the Lie bracket on span$\{X_j, \ldots, X_{mn}, \Gamma\}$ = ker$(i_{X_j} \cdots i_{X_{mn}}\Omega)$ is closed. Then, $i_{X_j} \cdots i_{X_{mn}}\Omega$ is Frobenius integral $j = 2, \ldots, N-1$. □

Lemma 3. *If* $X_j, j = 1, \ldots, mn$*, are vector fields and* $X_1, \ldots, X_{mn}, \Gamma$ *are linearly independent, then* $i_{X_1} \cdots i_{X_{mn}}\Omega \neq 0$*.*

Proof. Since $X_1, \ldots, X_{mn}, \Gamma$ are linearly independent, we have $i_{X_j} \neq 0$. If $i_{X_1} \cdots i_{X_{mn}}\Omega = 0$, then ker$(i_{X_2} \cdots i_{X_{mn}}\Omega)$ =span$\{X_1, \ldots, X_{mn}, \Gamma\}$. On each point, p of $M^{(n-1)}$, $T_pM^{(n-1)}$ =span$\{X_1, \ldots, X_{mn}, \Gamma\}|_p$, thus $i_{X_2} \cdots i_{X_{mn}}\Omega = 0$, so we know ker$(i_{X_3} \cdots i_{X_{mn}}\Omega)$ ⊃span$\{X_2, \ldots, X_{mn}, \Gamma\}$. On the other hand, $i_{X_1} \cdots i_{X_{mn}}\Omega = 0 = i_{X_2}i_{X_1}i_{X_3} \cdots i_{X_{mn}}$, ker$(i_{X_1}i_{X_3} \cdots i_{X_{mn}}\Omega)$ =span$\{X_1, \ldots, X_{mn}, \Gamma\}$, thus $i_{X_2} \cdots i_{X_{mn}}\Omega = 0$, so we know ker$(i_{X_3} \cdots i_{X_{mn}}\Omega)$ ⊃span$\{X_1, X_3, \ldots, X_{mn}, \Gamma\}$. Hence, ker$(i_{X_3} \cdots i_{X_{mn}}\Omega)$ =span$\{X_1, \ldots, X_{mn}, \Gamma\}$, and we obtain $i_{X_3} \cdots i_{X_{mn}}\Omega = 0$. Clearly the process is inductive, and finally, we can obtain that $i_{X_{mn}}\Omega = 0$, which contradicts the assumption, and the proof is complete. □

In [18], Zhang showed that λ-symmetries can be used to obtain some first integrals if there are not enough symmetries, i.e., the following Theorem 1.

Theorem 1 ([18]). *Given natural numbers N and k , $3 \leq N \leq mn, 1 \leq k \leq N-1$, suppose that*
(1) $span\{X_N, \ldots, X_{mn}, \Gamma\}$ is Frobenius integrable;
(2) X_j are λ-symmetries of $span\{X_{j+1}, \ldots, X_{N-1}, X_N, \ldots, X_{mn}, \Gamma\}, j = k+1, \ldots, N-1$;
(3) X_l is a symmetry of $span\{X_1, \ldots, X_{l-1}, X_{l+1}, \ldots, X_{mn}, \Gamma\}, l = 1, \ldots, k$;
(4) $X_1, \ldots, X_{mn}$ and Γ are linearly independent everywhere.
Put

$$\sigma^l = i_{X_1} \cdots i_{X_{l-1}} i_{X_{l+1}} \cdots i_{X_{mn}} \Omega, \quad l = 1, \ldots, k.$$

Then

$$\omega^l = \frac{\sigma^l}{i_{X_l} \sigma^l}, \quad l = 1, \ldots, k,$$

are closed, and locally provide k functionally independent first integrals of (4).

Proof. From Lemma 2, we know that σ^l is a Frobenius integrable 1-form, because X_l is a symmetry of $\text{span}\{X_1, \ldots, \hat{X}_l, \ldots, X_{k+1}, \ldots, X_{mn}, \Gamma\} = \ker\sigma^i, l = 1, \ldots, k$. This implies that X_i is a symmetry of σ^i. By [35], we know that for a Frobenius integrable 1-form θ, which is nowhere zero, if it has a symmetry $Z \notin \ker\theta$, then locally, there is an integrating factor $\frac{1}{i_Z \Omega}$ such that $\frac{\theta}{i_Z \Omega}$ is a closed 1-form; therefore, ω^l. So, by the converse of the Poincaré Lemma, locally, $\omega^l = dI_l$, where $I_1, \ldots, I_k$ are functions. Next, we prove that $I_l, l = 1, \ldots, k$ are functionally independent first integrals of (4). Due to $\ker\omega^l \neq \ker\omega^j$ as $l \neq j$, where $l, j = 1, \ldots, k$, we obtain that $\omega^l, l = 1, \ldots, k$ are linearly independent, which implies that $I_l, l = 1, \ldots, k$ are linearly independent. Moreover, by $i_\Gamma \Omega = 0$, we know that $\Gamma(I_l) = i_\Gamma dI_l = \frac{i_\Gamma i_{X_1} \cdots i_{X_{l-1}} i_{X_{l+1}} \cdots i_{X_{mn}} \Omega}{i_{X_l} i_{X_1} \cdots i_{X_{l-1}} i_{X_{l+1}} \cdots i_{X_{mn}} \Omega} = 0$, so $I_l, l = 1, \ldots, k$ are functionally independent first integrals of (4). □

Here, we generalize the theorem above in [18] and show that if there are not enough symmetries and λ-symmetries, some first integrals can still be obtained.

Theorem 2. *Given natural numbers N_1, N_2 and k , $3 \leq N_2 < N_1 \leq mn, 1 \leq k \leq N-1$, suppose that*
(1) $Span\{X_{N_1}, \ldots, X_{mn}, \Gamma\}$ is Frobenius integrable;
(2) Vector fields $X_j, N_2 \leq j \leq N_1 - 1$ satisfy

$$[X_j, X_k] \in span\{X_{N_2}, \ldots, X_{mn}, \Gamma\}, \quad N_2 \leq k \leq mn;$$

(3) X_i are λ-symmetries of $span\{X_{i+1}, \ldots, X_{mn}, \Gamma\}, i = k+1, \ldots, N_2 - 1$;
(3) X_l is a symmetry of $span\{X_1, \ldots, X_{l-1}, X_{l+1}, \ldots, X_{mn}, \Gamma\}, l = 1, \ldots, k$;
(4) $X_1, \ldots, X_{mn}$ and Γ are linearly independent everywhere.
Put

$$\sigma^l = i_{X_1} \cdots i_{X_{l-1}} i_{X_{l+1}} \cdots i_{X_{mn}} \Omega, \quad l = 1, \ldots, k.$$

Then

$$\omega^l = \frac{\sigma^l}{i_{X_l} \sigma^l}, \quad l = 1, \ldots, k,$$

are closed, and locally provide k functionally independent first integrals of (4).

Proof. Similar to Theorem 1. □

Here, we give two examples to illustrate the utility of Theorem 2.

Example 1. *Consider the following second-order equation*

$$y'' = -\left(\frac{x^2}{4y^3} + y + \frac{1}{2y}\right). \tag{7}$$

In [12], the authors pointed out this equation has no nontrivial symmetries, and they give a λ*-symmetry* $X = y\frac{\partial}{\partial y}$, $\lambda = x/y^2$. *Now we know that* $\Gamma = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} - \left(\frac{x^2}{4y^3} + y + \frac{1}{2y}\right)\frac{\partial}{\partial y'}$, $\Omega = dy \wedge dy' - y'dx \wedge dp - \left(\frac{x^2}{4y^3} + y + \frac{1}{2y}\right)dx \wedge dy$. *We can calculate* $W = cy\frac{\partial}{\partial y} + c(y' + \frac{x}{y})\frac{\partial}{\partial y'}$ *is a* λ*-symmetry of (7), for* $\lambda = \frac{x}{y^2}$, c *is some constant. It is easy to verify that* $Y = 2\frac{\partial}{\partial x} - \left(\frac{1}{2y}\right)\frac{\partial}{\partial y'}$ *is a symmetry of span* $\{W, \Gamma\}$. *Then, by Theorem 1 or 2,*

$$\omega = \frac{i_W\Omega}{i_Y i_W\Omega} = d\left(x + arctan\left(\frac{y'}{y} + \frac{x}{2y^2}\right)\right)$$

is closed, and $I = x + arctan\left(\frac{y'}{y} + \frac{x}{2y^2}\right)$ *is a first integral.*

Remark 2. *Although [18] showed that Theorem 2 can be used to obtain the first integral for Example 1, here, we point out we can also use Theorem 2 to obtain the first integral for this equation.*

Example 2. *Consider system*

$$\begin{cases} y_1' = xy_1 + x^2y_2 + x^3y_3 \\ y_2' = 2xy_1 + 2x^2y_2 + 2x^3y_3 \\ y_3' = xy_1 + x^2y_2 + x^3y_3 \end{cases} \tag{8}$$

so we can obtain $\Gamma = \frac{\partial}{\partial x} + (xy_1 + x^2y_2 + x^3y_3)\frac{\partial}{\partial y_1} + (2xy_1 + 2x^2y_2 + 2x^3y_3)\frac{\partial}{\partial y_2} + (xy_1 + x^2 y_2 + x^3y_3)\frac{\partial}{\partial y_3}$, *and* $\Omega = i_\Gamma dx \wedge dy_1 \wedge dy_2 \wedge dy_3$. *We can verify that the vector field set* $\{X_1 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, X_2 = \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\}$ *satisfies*

$$\begin{aligned} [X_1, \Gamma] &= X_1\Gamma - \Gamma\, X_1 \\ &= (x + x^2)X_1 + (x + x^2)X_2, \\ [X_2, \Gamma] &= X_2\Gamma - \Gamma\, X_2 \\ &= (x^2 + x^3)X_1 + (x^2 + x^3)X_2, \\ [X_1, X_2] &= 0. \end{aligned}$$

Thus, X_1 *and* X_2 *are not the* λ*-symmetries of (8). We can also verify that* $X_3 = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}$ *is a symmetry of* $\{X_1, X_2, \Gamma\}$; *in fact,* $[X_3, X_2] = [X_3, X_1] = 0, [X_3, \Gamma] = X_3\Gamma - \Gamma X_3 = (x + x^3)$ $X_1 + (x + x^3)X_2$. *Then, by using Theorem 2, we can obtain a first integral of (8). Calculate*

$$i_{X_2}i_{X_1}i_\Gamma(dx \wedge dy_1 \wedge dy_2 \wedge dy_3) = dy_3 - dy_2 + dy_1,$$

moreover, $i_{X_3}i_{X_2}i_{X_1}i_\Gamma(dx \wedge dy_1 \wedge dy_2 \wedge dy_3) = 2$, *so* $\frac{i_{X_2}i_{X_1}i_\Gamma(dx \wedge dy_1 \wedge dy_2 \wedge dy_3)}{i_{X_3}i_{X_2}i_{X_1}i_\Gamma(dx \wedge dy_1 \wedge dy_2 \wedge dy_3)} = d\frac{y_1 - y_2 + y_3}{2}$, *and it is obvious* $\frac{d\frac{y_1 - y_2 + y_3}{2}}{dx} = \frac{1}{2}(xy_1 + x^2y_2 + x^3y_3 - (2xy_1 + 2x^2y_2 + 2x^3y_3) + xy_1 + x^2y_2$ $+x^3y_3) = 0$. *Thus,* $\frac{y_1 - y_2 + y_3}{2}$ *is a first integral of (8).*

4. Quotient Manifold, Relationship between λ-Symmetry and Symmetry

In [4], Olver stated that if X as a vector field defined on $J^{n-1}M$ has the property $[X, \Gamma] = \sigma X$, then X is a symmetry of System (4), where Γ is the equation field defined as (5), and σ is defined on $J^{n-1}M$. Thus, a quotient manifold Q by a group action of symmetry X can be obtained, and we call the equation field Γ to pass to it. Here, we prove that when

X is a λ-symmetry of Γ, by multiplying Γ, a function μ defined on $J^{n-1}M$, we can obtain a vector field that passes to quotient manifold Q, i.e., we obtain a vector field $\mu\Gamma$ that passes to Q. Moreover, we also discuss the relationship between λ-symmetries and symmetries.

Theorem 3. *Let Γ be the $(n-1)$th-order equation field of (4). Let X be a λ-symmetry of (4), and let Γ, X be two linearly independent vector fields. Assume Q is the quotient of $J^{n-1}M$ by the one-parameter group generated by X. Then, there exists a function μ defined on $J^{n-1}M$ such that $\mu\Gamma$ passes to Q.*

Proof. A vector field W passes to Q if and only if $L_X W = \rho X$ for some function ρ. On the other hand, we have $L_X\Gamma = \lambda\Gamma + \bar{\lambda}X$, so we look for a multiple $\mu\Gamma$ of Γ such that $L_X(\mu\Gamma) = \bar{\lambda}X$. Because $L_X(\mu\Gamma) = (L_X\mu)\Gamma + \mu L_X\Gamma = (L_X\mu + \lambda\mu)\Gamma + \bar{\lambda}X = (X(\mu) + \lambda\mu)\Gamma + \bar{\lambda}X$ and Γ, X are two linearly independent vector fields, μ must satisfy $X(\mu) + \mu\lambda = 0$. If we attempt $\mu = (i_\Gamma\omega)^{-1}$ in which ω satisfies $i_\Gamma\omega \neq 0$ for some appropriate 1-form ω, we find that if $i_\Gamma(L_X\omega) = 0$, $i_\Gamma\omega \neq 0$ and $\omega(X) = 0$ then $\mu = (i_\Gamma\omega)^{-1}$ satisfies the condition $X(\mu) + \mu\lambda = 0$, because

$$\begin{aligned}
0 = i_\Gamma(L_X\omega) &= i_\Gamma(i_X d\omega + d i_X\omega)\\
&= i_\Gamma i_X d\omega + i_\Gamma d i_X\omega\\
&= d\omega(X,\Gamma) + \Gamma(\omega(X))\\
&= X(\omega(\Gamma)) - \omega([X,Y])\\
&= X(\omega(\Gamma)) - \lambda\omega(\Gamma) - \bar{\lambda}\omega(X)\\
&= X(\omega(\Gamma)) - \lambda\omega(\Gamma)\\
&= -(i_\Gamma\omega)^2\left(\frac{-X(\omega(\Gamma)) + \lambda\omega(\Gamma)}{(i_\Gamma\omega)^2}\right)\\
&= -(i_\Gamma\omega)^2(X((i_\Gamma\omega)^{-1}) + \lambda(i_\Gamma\omega)^{-1}).
\end{aligned}$$

For example, if g is a differential invariant of X, so that $X(g) = 0$, with $\Gamma(g) \neq 0$, then we take $\omega = dg$. Clearly, $i_\Gamma(L_X\omega) = i_\Gamma(i_X d\omega + d i_X\omega) = i_\Gamma(i_X(d(dg)) + d(i_X dg)) = i_\Gamma(d(i_X dg))$ $= i_\Gamma(d(X(g))) = 0$, so we know $\hat{\Gamma}/\Gamma(g)$ passes to the quotient manifold Q. □

Example 3. *As for ordinary equation*

$$\begin{cases} \frac{du}{dx} = e^v \\ \frac{dv}{dx} = 1 \end{cases}$$

we have characteristic vector fields $\Gamma = \frac{\partial}{\partial x} + e^v\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$. Given vector fields $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial v}$ and $Z = (1+u^2)\frac{\partial}{\partial u}$, let Q be the quotient of R^3 by the one-parameter group generated by X. We can compute that

$$[\Gamma, X] = \Gamma X - X\Gamma = -\Gamma + X, \quad [X, Z] = 0,$$

thus, Γ does not pass to Q. That is to say X is a λ-symmetry of Γ, and Z passes to Q. According to Theorem 3, let $\mu = e^{-v}$, then $[e^{-v}\Gamma, X] = e^{-v}X$, so $e^{-v}\Gamma$ passes to Q.

Recall that for vector fields X, Y defined on a manifold $\mathcal{M}$, we say X is a symmetry of Y if we have

$$[X, Y] = gY,$$

where g is a function defined on $\mathcal{M}$.

Lemma 4. *Let X, Γ be defined as Theorem 3 and g be some function defined on $J^{n-1}M$. Then, gX is a symmetry of (4) if and only if*

$$\lambda g = \Gamma g.$$

Proof. Because

$$\begin{aligned}[gX,\Gamma] &= gX\Gamma - \Gamma(gX)\\ &= gX\Gamma - \Gamma(g)X - g\Gamma\ X\\ &= g[X,\Gamma] - \Gamma(g)X\\ &= g(\lambda X + \bar{\lambda}\Gamma) - \Gamma(g)X\\ &= (\lambda g - \Gamma(g))X + g\bar{\lambda}\Gamma,\end{aligned}$$

we know that if $\lambda g = \Gamma g$, then gX is a symmetry of (4) and vice versa. □

If there are l λ-symmetries, similar to Lemma 4, we show that under some conditions, the vector fields from their linear combination are symmetries.

Theorem 4. *Let X_i be the λ-symmetries of (4), $i = 1,\dots,l$, $Y_i := A_{ij}X_j, i,j = 1,\dots,l$, A_{ij} be functions defined on $J^{n-1}M$, Γ be the equation field as Lemma 3. Then, Y_i are the symmetries of (4) if and only if*

$$A_{ij}\lambda_j = \Gamma(A_{ij}). \tag{9}$$

where λ_j are functions defined on J^1M. Moreover, if g is the common first integral of $X_i, i = 1,\dots l$, then Y_i is the symmetry of $g\Gamma$.

Proof. These follow from explicit computation. In fact,

$$\begin{aligned}[Y_i,\Gamma] &= [A_{ij}X_j,\Gamma]\\ &= A_{ij}X_j\Gamma - \Gamma(A_{ij})X_j - A_{ij}\Gamma\ X_j\\ &= A_{ij}[X_j,\Gamma] - \Gamma(A_{ij})X_j\\ &= A_{ij}(\lambda_j X_j + \bar{\lambda}_j\Gamma) - \Gamma(A_{ij})X_j\\ &= (A_{ij}\lambda_j - \Gamma(A_{ij}))X_j + A_{ij}\bar{\lambda}_j\Gamma,\end{aligned}$$

which implies (9). Also,

$$\begin{aligned}[Y_i,g\Gamma] &= [A_{ij}X_j,g\Gamma]\\ &= A_{ij}gX_j\Gamma - g\Gamma(A_{ij})X_j - gA_{ij}\Gamma\ X_j\\ &= gA_{ij}[X_j,\Gamma] - g\Gamma(A_{ij})X_j\\ &= gA_{ij}(\lambda_j X_j + \bar{\lambda}_j\Gamma) - g\Gamma(A_{ij})X_j\\ &= g(A_{ij}\lambda_j - \Gamma(A_{ij}))X_j + A_{ij}\bar{\lambda}_j(g\Gamma),\end{aligned}$$

which completes the proof. □

If we have vector field X defined on $J^{n-1}M$ with first-order λ-prolongation Y defined on J^nM, then we want to find the connection between first-order standard prolongations Z of X and their first-order λ-prolongation Y. It clearly suffices to discuss the situation $n = 1$.

Lemma 5. *Let X be a vector field in M and Y be its first-order λ-prolongation. Then, gY coincides with the standard prolongation Z of vector field $W := gX$ with g smooth function on M satisfying*

$$D_x g = g\lambda. \tag{10}$$

Proof. We will work in local coordinates and write

$$X = \xi\frac{\partial}{\partial x} + \phi^a\frac{\partial}{\partial y_a}, \quad Y = X + \psi^a\frac{\partial}{\partial y_a^{(1)}},$$

with

$$\psi^a = (D_x\phi^a - y_a^{(1)}D_x\xi) + \lambda(\phi^a - y_a^{(1)}\xi),$$

and

$$W = gX, \quad Z = W + \Theta^a \frac{\partial}{\partial y_a^{(1)}},$$

where (standard prolongation formula)

$$\begin{aligned}\Theta^a &= D_x(g\phi^a) - y_a^{(1)}D_x(g\xi)\\ &= D_x(g)\phi^a + gD_x(\phi^a) - y_a^{(1)}\xi D_x(g) - y_a^{(1)}gD_x(\xi).\end{aligned}$$

Now, requiring that $Z = gY$ amounts to requiring that $\Theta^a = g\psi^a$, this is written as

$$g(D_x\phi^a - y_a^{(1)}D_x\xi) + g\lambda(\phi^a - y_a^{(1)}\xi) = D_x(g)\phi^a + gD_x(\phi^a) - y_a^{(1)}\xi D_x(g) - y_a^{(1)}gD_x(\xi).$$

Eliminating the equal terms on both sides yields

$$(g\lambda - D_xg)(\phi^a - y_a^{(1)}\xi) = 0,$$

which implies (10). □

Example 4. *Let us consider $M = R^2$ with coordinates (x, y). We now take the vector fields $X_1 = y\frac{\partial}{\partial u}$ and $\lambda = y_x, g = e^y$, then we can compute that*

$$D_xg = g\lambda = y_xe^y.$$

The 1-order λ-prolongation Y of X_1 and standard prolongation Z of gX_1 are equal. And $Y = (y_x + y_xy)\frac{\partial}{\partial}, Z = y_x(e^yy + e^y)\frac{\partial}{\partial y}$, clearly, $gY = Z$.

Inspired by Lemma 5, we prove that gY cannot be a first-order λ-prolonged vector field of the vector field gX if g is not a constant function.

Theorem 5. *Let $X = \xi(x, y^a)\frac{\partial}{\partial x} + \phi^a\frac{\partial}{\partial y^a}$ be vector field defined on M and $Y = X + \psi_1^a\frac{\partial}{\partial y^{a(1)}}$ be a first-order λ-prolonged vector field of vector field X. There is no nontrivial function g defined on M that makes gY be the first-order λ-prolonged vector field of vector field gX.*

Proof. In fact,

$$\begin{aligned}(D_x + \lambda)(g\phi^a) - y_a^{(1)}(D_x + \lambda)(g\xi) &= (D_xg)\phi^a + gD_x\phi^a + \lambda g\phi^a - y_a^{(1)}((D_xg)\xi + g(D_x\xi) + \lambda g\xi)\\ &= (D_xg)(\phi^a - y_a^{(1)}\xi) + g((D_x + \lambda)\phi^a - y_a^{(1)}(D_x + \lambda)\xi)\\ &= (D_xg)(\phi^a - y_a^{(1)}\xi) + g\psi_1^a,\end{aligned}$$

which implies that $g\psi_1^a = (D_x + \lambda)(g\phi^a) - y_a^{(1)}(D_x + \lambda)(g\xi)$ if and only if $D_xg = 0$. So, gY is the 1-order λ-prolonged vector field of vector field gX if and only if $D_xg = 0$, i.e., $g = C, C$ is a constant. □

5. Reduction of System of Explicit Ordinary Differential Equations

In this section, we consider the reduction of the system of explicit ordinary differential Equations (4). Let V be a vector field defined on M.

Definition 6. *A function ζ defined on J^jM is a j-th-order invariant of V if $V^{(j)}(\zeta) = 0$, where $V^{(j)}$ is the j-order λ-prolongation of V.*

Clearly, a j-th-order invariant is a $(j + 1)$-th-order invariant.

Lemma 6. *If $V^{(k)}$ is a k-order λ-prolonged vector field, then it satisfies*

$$[V^{(k)}, D_x] = \lambda V^{(k)} - [(D_x\xi) + \lambda\xi]D_x.$$

Proof. This follows from explicit computation. In fact,

$$D_x = \frac{\partial}{\partial x} + y_a^{(1)}\frac{\partial}{\partial y_a} + y_a^{(2)}\frac{\partial}{\partial y_a^{(1)}} + \cdots + y_a^{(n+1)}\frac{\partial}{\partial y_a^{(n)}} + \cdots,$$

by (1) and (3), we have immediately

$$\begin{aligned}
[V^{(k)}, D_x] &= V^{(k)}D_x - D_xV^{(k)} \\
&= (\psi_{k+1}^a - D_x\psi_k^a)\frac{\partial}{\partial y_a^{(k)}} - (D_x\xi)\frac{\partial}{\partial x} \\
&= (-y_a^{(k+1)}D_x\xi + \lambda(\psi_k^a - y_a^{(k+1)}\xi))\frac{\partial}{\partial y_a^{(k)}} - (D_x\xi)\frac{\partial}{\partial x} \\
&= -(D_x\xi)D_x + \lambda(V^{(k)} - \xi\frac{\partial}{\partial x}) - \lambda\xi(D_x - \frac{\partial}{\partial x}) \\
&= -[(D_x\xi) + \lambda\xi]D_x + \lambda V^{(k)}. \qquad (11)
\end{aligned}$$

□

Lemma 7. *Assume that if ξ and ζ are j-th-order invariants of a (k)-order λ-prolonged vector field $V^{(k)}$, then $\Theta := D_x(\zeta)/D_x(\eta)$ is a $(j+1)$-th-order invariant of $(k+1)$-order λ-prolonged vector field $V^{(k+1)}$.*

Proof. It is obvious that Θ is a function defined on $J^{k+1}M$. To show that it is invariant under the $(k+1)$-order λ-prolonged vector field $V^{(k+1)}$, we just proceed by straightforward computation. First of all, by

$$V^{(k+1)}(\Theta) = \frac{(V^{(k+1)}(D_x\zeta))\cdot(D_x\eta) - (D_x\zeta)\cdot(V^{(k+1)}(D_x\eta))}{(D_x\eta)^2} := \frac{\chi}{(D_x\eta)^2},$$

it is clear that we just have to show that the numerator χ vanishes. On the other hand, we have (recalling that by assumption $V^{(k)}(\zeta) = V^{(k)}(\eta) = 0$, and using Lemma 6)

$$\begin{aligned}
\chi &= (V^{(k+1)}(D_x\zeta))\cdot(D_x\eta) - (D_x\zeta)\cdot(V^{(k+1)}(D_x\eta)) \\
&= (D_x(V^{(k+1)}\zeta) + [V^{(k+1)}, D_x]\zeta)\cdot(D_x\eta) - (D_x\zeta)\cdot((D_x(V^{(k+1)}\eta)) + [V^{(k+1)}, D_x]\eta) \\
&= ([V^{(k+1)}, D_x]\zeta)\cdot(D_x\eta) - (D_x\zeta)\cdot([V^{(k+1)}, D_x]\eta) \\
&= ((\lambda V^{(k+1)} - [(D_x\xi) + \lambda\xi]D_x)\zeta)\cdot(D_x\eta) - (D_x\zeta)\cdot((\lambda V^{(k+1)} - [(D_x\xi) + \lambda\xi]D_x)\eta) \\
&= 0.
\end{aligned}$$

This shows indeed $\chi = 0$ and hence the lemma. □

Remark 3. *Let V_i^k be k-th λ_i-prolonged vector fields $i = 1, \ldots, l$, and let η, ζ be independent common differential invariants of order k for all of them. Then, by using the same method as Lemma 7, we can know that*

$$\Theta := D_x(\zeta)/D_x(\eta)$$

is a common differential invariant of order $k+1$ for all of them.

Suppose we have find a zero-th-order invariant u and m independent first-order invariants $z_1, \ldots, z_m$ for λ-symmetry V. Define $z_a^{(0)} = z_a, a = 1, \ldots, m$ and $z_a^{(j)} = D_x(z_a^{(j-1)})/D_x u$ for $j = 1, \ldots, n-2$ so that $z_a^{(j)}$ are $(j+1)$th invariants of V.

Remark 4. *Because* $\Gamma = \frac{\partial}{\partial x} + y_a^{(1)}\frac{\partial}{\partial y_a} + \cdots + y_a^{(n-1)}\frac{\partial}{\partial y_a^{(n-2)}} + f^a\frac{\partial}{\partial y_a^{(n-1)}} = \frac{\partial}{\partial x} + y_a^{(1)}\frac{\partial}{\partial y_a} + \cdots + y_a^{(n-1)}\frac{\partial}{\partial y_a^{(n-2)}} + y_a^{(n)}\frac{\partial}{\partial y_a^{(n-1)}}$ *and* $z_a^{(j)}$ *is defined on* $J^{j+1}M, j = 1, \ldots, n-2, z_a^{(j)} = D_x(z_a^{(j-1)})/D_x u = \Gamma(z_a^{(j-1)})/\Gamma(u), j = 1, \ldots, n-2$.

Theorem 6. *Suppose* u *and* $z_a, a = 1, \ldots, m$ *are functionally independent zero-order and first-order invariants, respectively, of the* λ*-symmetry* V*, that is* $V(u) = V^{(1)}(z_a) = 0, a = 1, \ldots, m$*, and assume that* $\Gamma(z_a)$ *and* $\Gamma(u), a = 1, \ldots, m$ *are nowhere zero. Put* $z_a^{(0)} =$ *zlinebreak and* $z_a^{(j)} = \Gamma(z_a^{(j-1)})/\Gamma(u)$ *for* $j = 1, \ldots, n-2$*. Then,* $u, z_a, z_a^{(1)}, \ldots, z_a^{(n-2)}, a = 1, \ldots, m$ *provide a complete set of functionally independent* $(n-1)$*th-order invariant for* V^{n-1}.

Proof. We use induction by assumption, $[V^{(n-1)}, \Gamma] = \lambda\Gamma + \bar{\lambda}V^{(n-1)}$, and suppose that $z_a^{(j)}$ is an invariant of $V^{(n-1)}$. Because $V(u) = V^{(1)}(z^a) = 0$, we can obtain $V^{(n-1)}(u) = 0$, $V^{(n-1)}(z_a^{(j)}) = V^{(j+1)}(z_a^{(j)}) = 0, j = 0, \ldots, n-2$. Then, we prove that $z_a^{(j+1)}$ is an invariant of $V^{(n-1)}$,

$$
\begin{aligned}
(\Gamma(u))^2 V^{(n-1)}(z_a^{(j+1)}) &= (\Gamma(u))^2 V^{(n-1)}\left(\frac{\Gamma(z_a^{(j)})}{\Gamma(u)}\right)\\
&= (\Gamma(u))^2 \frac{\Gamma(u)V^{(n-1)}(\Gamma(z_a^{(j)})) - \Gamma(z_a^{(j)})V^{(n-1)}(\Gamma(u))}{(\Gamma(u))^2}\\
&= \Gamma(u)V^{(n-1)}(\Gamma(z_a^{(j)})) - \Gamma(z_a^{(j)})V^{(n-1)}(\Gamma(u))\\
&= \Gamma(u)(V^{(n-1)}(\Gamma(z_a^{(j)})) - \Gamma(V^{(n-1)}(z_a^{(j)}))) - \Gamma(z_a^{(j)})(V^{(n-1)}(\Gamma(u)) - \Gamma(V^{(n-1)}(u)))\\
&= \Gamma(u)[V^{(n-1)}, \Gamma](z_a^{(j)}) - \Gamma(z_a^{(j)})[V^{(n-1)}, \Gamma](u)\\
&= \Gamma(u)(\lambda\Gamma(z_a^{(j)}) + \bar{\lambda}V^{(n-1)}(z_a^{(j)})) - \Gamma(z_a^{(j)})(\lambda\Gamma(u) + \bar{\lambda}V^{(n-1)}(u))\\
&= \Gamma(u)\lambda\Gamma(z_a^{(j)}) - \Gamma(z_a^{(j)})\lambda\Gamma(u)\\
&= 0,
\end{aligned}
$$

hence $V^{(n-1)}(z_a^{(j)}) = 0$ for $j = 0, \ldots, n-2, a = 1, \ldots, m$.

To see the functional independence, we note how Γ acting on a function effectively increases the order of derivatives, which it depends on by one. Because u is a zero-order invariant with no $y_a^{(1)}, a = 1, \ldots, m$ dependence and $z_a, a = 1, \ldots, m$ must have nowhere zero $y_a^{(1)}$ dependence, u and $z_a, a = 1, \ldots, m$ are functionally independent. Similarly, we know $\Gamma(z_a)$ will have nowhere zero $y_a^{(2)}$ and only $z_a^{(1)}$ depends on $y_a^{(2)}, a = 1, \ldots, m$, for instance, only $z_1^{(1)}$ depends on $y_1^{(2)}$. From here, the result follows by induction. □

Proposition 2. *Denote* $\hat{f}^a = \Gamma(z_a^{(n-2)})/\Gamma(u), a = 1, \ldots, m$*, then* $\hat{f}^a$ *is also a first integral of* $V^{(n-1)}$.

Proof.

$$
\begin{aligned}
(\Gamma(u))^2 V^{(n-1)}(\hat{f}^a) &= (\Gamma(u))^2 V^{(n-1)}\left(\frac{\Gamma(z_a^{(n-2)})}{\Gamma(u)}\right)\\
&= (\Gamma(u))^2 \frac{\Gamma(u)V^{(n-1)}(\Gamma(z_a^{(n-2)})) - \Gamma(z_a^{(n-2)})V^{(n-1)}(\Gamma(u))}{(\Gamma(u))^2}\\
&= \Gamma(u)V^{(n-1)}(\Gamma(z_a^{(n-2)})) - \Gamma(z_a^{(n-2)})V^{(n-1)}(\Gamma(u))\\
&= \Gamma(u)(V^{(n-1)}(\Gamma(z^{(n-2)})) - \Gamma(V^{(n-1)}(z_a^{(n-2)}))) - \Gamma(z_a^{(n-2)})(V^{(n-1)}(\Gamma(u)) - \Gamma(V^{(n-1)}(u)))\\
&= \Gamma(u)[V^{(n-1)},\Gamma](z_a^{(n-2)}) - \Gamma(z_a^{(n-2)})[V^{(n-1)},\Gamma](u)\\
&= \Gamma(u)(\lambda\Gamma(z_a^{(n-2)}) + \bar{\lambda}V^{(n-1)}(z_a^{(n-2)})) - \Gamma(z_a^{(n-2)})(\lambda\Gamma(u) + \bar{\lambda}V^{(n-1)}(u))\\
&= \Gamma(u)\lambda\Gamma(z_a^{(n-2)}) - \Gamma(z_a^{(n-2)})\lambda\Gamma(u)\\
&= 0.
\end{aligned}
$$

□

By Theorem 6, $V^{(n-1)}$ has a complete set of functionally independent invariants $\{u, z_a, z_a^{(1)}, \ldots, z_a^{(n-2)}\}$. And on the set $S(\Delta) := \{(x, y_a, y_a^{(1)}, \ldots, y_a^{(n)}) | y_a^{(n)} = f^a(x, y_a, y_a^{(1)}, \ldots, y_a^{(n-1)}), a = 1, \ldots, m\}$, $z_a^{(n-1)} = D_x(z_a^{(n-2)})/D_x u = d^{n-1}z_a/du^{n-1} = \Gamma(z_a^{(n-2)})/\Gamma(u) = \hat{f}^a$, which is also a first integral of $V^{(n-1)}$, so it depends on functions $u, z_a, z_a^{(1)}, \ldots, z_a^{(n-2)}$, i.e., $\hat{f}^a = \hat{f}^a(u, z_a, z_a^{(1)}, \ldots, z_a^{(n-2)})$. Thus, we obtain the $(n-1)$th-order equation

$$d^{n-1}z_a/du^{n-1} = \hat{f}(u, z_a, z_a^{(1)}, \ldots, z_a^{(n-2)}). \tag{12}$$

We can recover the general solution of the original equation from the general solution of (12) and the corresponding system of the first-order auxiliary equation

$$z_a = z_a(x, y_a, y_a^{(1)}), \quad a = 1, \ldots, m. \tag{13}$$

In fact, as for Equation (12), if we obtain the solution $z_a(u)$, together with (13), we only need to solve system of the first-order equation $z_a(u(x, y_a)) = z_a(x, y_a, y_a^{(1)})$ to obtain the solution $y_a(x)$ of (4).

Remark 5. *In [12], the authors studied ordinary differential equations (4) with $m = 1$. Here, we discuss the system of ordinary differential equations (4) with $m \geq 1$, and our method states that if we deal with an n-order explicit equation that has a λ-symmetry V, then we just need to consider the invariants of $(n-1)$-order prolongation $V^{(n-1)}$ of V. In [12], the authors only dealt with implicit equation with a λ-symmetry V, which needs to consider the invariants of n-order prolongation $V^{(n-1)}$ of V.*

On the other hand, putting $\theta^{aj} = dz_a^{(j)} - z_a^{(j+1)}du$ for $j = 0, \ldots, n-3, a = 1, \ldots, m$ and $\theta^{an-2} = dz_a^{(n-2)} - \hat{f}^a du$, we have $m(n-1)$ independent 1-forms.

Proposition 3. *On $S(\Delta)$, $i_\Gamma \theta^{aj} = 0, i_{V^{(n-1)}}\theta^{aj} = 0, j = 0, \ldots, n-2, a = 1, \ldots, m$ and $i_{V^{(n-1)}}\Omega = \alpha \wedge \theta^{a0} \wedge \cdots \wedge \theta^{an-2}$, α is a some function defined on $J^{n-1}M$.*

Proof. Because $z_a^{(j+1)} = \frac{\Gamma(z_a^{(j)})}{\Gamma(u)} = \frac{dz_a^{(j)}(\Gamma)}{du(\Gamma)}, j = 0, \ldots, n-3, \hat{f}^a = \frac{\Gamma(z_a^{(n-2)})}{\Gamma(u)} = \frac{dz_a^{(n-2)}(\Gamma)}{du(\Gamma)}$, $V^{(n-1)}(z_a^{(j)}) = 0, V^{(n-1)}(u) = 0$. We have $(dz_a^{(j)} - z_a^{(j+1)}du)(\Gamma) = i_\Gamma\theta^{aj} = 0, j = 0, \ldots,$ n-3, $(dz_a^{(n-2)} - \hat{f}^a du)(\Gamma) = i_\Gamma\theta^{an-2} = 0$, and $i_{V^{(n-1)}}\theta^{aj} = V^{n-1}(z_a^{(j)}) - z_a^{(j+1)}V^{(n-1)}(u) = 0$, $j = 0, \ldots, n-3, i_{V^{(n-1)}}\theta^{an-2} = V^{n-1}(z_a^{(n-2)}) - \hat{f}^a V^{(n-1)}(u) = 0$, thus $i_{V^{(n-1)}}(\theta^{a0} \wedge \cdots \wedge$

$\theta^{an-2}) = i_\Gamma(\theta^{a0} \wedge \cdots \wedge \theta^{an-2}) = 0$, moreover, we know that $\ker i_{V^{(n-1)}}\Omega = \text{span}\{V^{(n-1)}, \Gamma\}$, hence $i_{V^{(n-1)}}\Omega = \alpha \wedge \theta^{a0} \wedge \cdots \wedge \theta^{an-2}$. □

Example 5. *Let $M = R^2$ with coordinate (x, y), and consider the equation*

$$y'' = y + xy'. \tag{14}$$

Then, $\Gamma = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + (y + xy')\frac{\partial}{\partial y'}$, and we can also compute that $X = \frac{\partial}{\partial y}$ is a λ-symmetry of (14), $\lambda = x$. In fact, using the λ-prolongation formula (3), we obtain the 1-order λ-prolongation Y of X, i.e., $Y = X + (D_x + x)1\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} + x\frac{\partial}{\partial y'}$. Then,

$$\begin{aligned}
[Y, \Gamma] &= Y\Gamma - \Gamma Y = Y\Gamma - \Gamma\, Y \\
&= \frac{\partial}{\partial y'} + x\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial y'} - \frac{\partial}{\partial y'} \\
&= x(\frac{\partial}{\partial y} + x\frac{\partial}{\partial y'}) \\
&= xY.
\end{aligned}$$

$X(x) = 0, Y(y' + x^2 - xy) = 0$, $u := x, z := y' + x^2 - xy$ are 0-order and 1-order invariants of X, respectively. So, we obtain an explicit order equation

$$z' = \frac{\Gamma(z)}{\Gamma(u)} = 2x = 2u, \tag{15}$$

thus, we can obtain a solution $z = u^2 + C, C$ as a constant. Together with $z = y' + x^2 - xy$, we have $y' + x^2 - xy = x^2 + C$, i.e.,

$$y' = xy + C.$$

Hence, we have reduced Equation (14) to the first-order Equation (15).

6. Involution of λ-Prolonged Sets of Vector Fields

Assume there is a set of vector fields $X_i, i = 1, \ldots, n$ that are in involution, and all of them have the form $X_i = \phi_i^a \frac{\partial}{\partial y_a}, i = 1, \ldots, n$. We will give a theorem that claims under what conditions are their λ-prolongations also in involution.

Theorem 7. *Assume vector fields $X_i = \phi_i^a \frac{\partial}{\partial y_a}, i = 1, \ldots, n$ defined on M are involution, with $[X_i, X_j] = \mu_{ij}^k X_k$ for μ_{ij}^k smooth functions on M. Then, their λ-prolongations $Y_i, i = 1, \ldots, n$ satisfy the same involution relations, i.e., $[Y_i, Y_j] = \mu_{ij}^k Y_k$, if*

(1) $Y_i(\lambda_j) = 0$ for $i, j = 1, \ldots, n, i \neq j$;

(2) $D_x(\mu_{ij}^k) + \lambda_i\mu_{ij}^k + \lambda_j\mu_{ij}^k + \lambda_k\mu_{ij}^k = 0$, for $i, j, k = 1, \ldots, n$.

Proof. This follows from an explicit computation. We proceed by induction on the order of the prolongation. Denoting by Z_i the $(q-1)$th λ_i-prolongation of X_i, we have $Y_i = \sum_{k=0}^{q} \psi_{i,k}^a \frac{\partial}{\partial y_a^{(k)}} + Z_i + \psi_{i,q}^a \frac{\partial}{\partial y_a^{(q)}}$, thus

$$\begin{aligned}
[Y_i, Y_j] &= [Y_i, Z_j + \psi^a_{j,q}\frac{\partial}{\partial y^{(q)}_a}] \\
&= [Y_i, Z_j] + [Y_i, \psi^a_{j,q}\frac{\partial}{\partial y^{(q)}_a}] \\
&= [Z_i + \psi^a_{i,q}\frac{\partial}{\partial y^{(q)}_a}, Z_j] + [Y_i, \psi^a_{j,q}\frac{\partial}{\partial y^{(q)}_a}] \\
&= [Z_i, Z_j] - Z_j(\psi^a_{i,q})\frac{\partial}{\partial y^{(q)}_a} + Y_i(\psi^a_{j,q})\frac{\partial}{\partial y^{(q)}_a} - \psi^a_{j,q}\frac{\partial \psi^a_{i,q}}{\partial y^{(q)}_a}\frac{\partial}{\partial y^{(q)}_a} \\
&= [Z_i, Z_j] + (Y_i(\psi^a_{j,q}) - Y_j(\psi^a_{i,q}))\frac{\partial}{\partial y^{(q)}_a}.
\end{aligned}$$

Denote $F^a_{ij,q} = Y_i(\psi^a_{j,q}) - Y_j(\psi^a_{i,q})$, because by induction $[Z_i, Z_j] = \mu^k_{ij} Z_k$ (i.e., the involution relations are satisfied for $(q-1)$th prolongations), the requirement that $[Y_i, Y_j] = \mu^k_{ij} Y_k = \mu^k_{ij} Z_k + \mu^k_{ij}\psi^a_{k,q}\frac{\partial}{\partial y^{(q)}_a}$ is equivalent to the requirement that

$$F^q_{ij,q} = \mu^k_{ij}\psi^a_{k,q}. \tag{16}$$

$F^a_{ij,k}$ needs to be rewritten for easier comparison with the rhs of (16), so by $X_i = \phi^a_i \frac{\partial}{\partial y_a}$, we can write (16) as

$$F^q_{ij,q} = \mu^k_{ij}(D_x\psi^a_{k,q-1} + \lambda_k\psi^a_{k,q-1}). \tag{17}$$

Using Lemma 6, Condition (1) and $Y_i\psi^a_{j,q-1} = Z_i\psi^a_{j,q-1}$, with standard computation, we obtain

$$\begin{aligned}
F^a_{ij,q} &= Y_i(\psi^a_{j,q}) - Y_j(\psi^a_{i,q}) \\
&= Y_i((D_x + \lambda_j)\psi^a_{j,q-1}) - Y_j((D_x + \lambda_i)\psi^a_{i,q-1}) \\
&= Y_iD_x\psi^a_{j,q-1} - Y_jD_x\psi^a_{i,q-1} + \psi^a_{j,q-1}Y_i(\lambda_j) - \psi^a_{i,q-1}Y_j(\lambda_i) + \lambda_jY_i(\psi^a_{j,q-1}) - \lambda_iY_j(\psi^a_{i,q-1}) \\
&= D_xY_i\psi^a_{j,q-1} + \lambda_iY_i\psi^a_{j,q-1} - D_xY_j\psi^a_{i,q-1} - \lambda_jY_j\psi^a_{i,q-1} + \lambda_jY_i(\psi^a_{j,q-1}) - \lambda_iY_j(\psi^a_{i,q-1}) \\
&= D_x(Y_i\psi^a_{j,q-1} - Y_j\psi^a_{i,q-1}) + \lambda_i(Y_i\psi^a_{j,q-1} - Y_j\psi^a_{i,q-1}) + \lambda_j(Y_i\psi^a_{j,q-1} - Y_j\psi^a_{i,q-1}) \\
&= (D_x + \lambda_i + \lambda_j)(Y_i\psi^a_{j,q-1} - Y_j\psi^a_{i,q-1}) \\
&= (D_x + \lambda_i + \lambda_j)(Z_i\psi^a_{j,q-1} - Z_j\psi^a_{i,q-1}) \\
&= (D_x + \lambda_i + \lambda_j)(\mu^k_{ij}\psi^a_{k,q-1}) \\
&= (\lambda_i + \lambda_j)(\mu^k_{ij}\psi^a_{k,q-1}) + D_x(\mu^k_{ij})\psi^a_{k,q-1} + \mu^k_{ij}D_x\psi^a_{k,q-1} \\
&= (D_x(\mu^k_{ij}) + \lambda_i\mu^k_{ij} + \lambda_j\mu^k_{ij})\psi^a_{k,q-1} + \mu^k_{ij}D_x\psi^a_{k,q-1}.
\end{aligned}$$

Comparing this with (17), we must require

$$F^a_{ij,q} = (D_x(\mu^k_{ij}) + \lambda_i\mu^k_{ij} + \lambda_j\mu^k_{ij})\psi^a_{k,q-1} + \mu^k_{ij}D_x\psi^a_{k,q-1} = \mu^k_{ij}(D_x\psi^a_{k,q-1} + \lambda_k\psi^a_{k,q-1}).$$

Eliminating equal terms on both sides, we obtain

$$(D_x(\mu^k_{ij}) + \lambda_i\mu^k_{ij} + \lambda_j\mu^k_{ij} + \lambda_k\mu^k_{ij})\psi^a_{k,q-1} = 0,$$

which holds under Condition (2), so the involution properties holding up to order $q-1$ are also holding at order q. □

Remark 6. *Noting that in Condition (2) of Theorem 7, $\lambda_k\mu^k_{ij}$ is not the Einstein summation.*

Remark 7. *If vector fields* $X_i = \phi_i^a \frac{\partial}{\partial y_a}, i = 1, \ldots, n$ *defined on* M *commute, i.e.,* $[X_i, X_j] = 0$, $\mu_{ij}^k = 0$, *and their* λ_i*-prolongations* $Y_i, i = 1, \ldots, n$ *satisfy Condition (1) of Theorem 7, then* Y_i *commute* $i = 1, \ldots, n$.

Remark 8. *If* λ_i *is a common first integral of* $X_j, j \neq i, j = 1, \ldots, n$. *It is natural* $Y_i(\lambda_j) = 0$ *for* $i, j = 1, \ldots, n, i \neq j$.

Example 6. *Let us consider* $X = R$, *with coordinate* x *and* $U = R^2$ *with coordinates* (u, v). *We now take the vector fields*

$$X_1 = u\frac{\partial}{\partial u}, \quad X_2 = -u\frac{\partial}{\partial v},$$

these are in involution, in fact,

$$[X_1, X_2] = X_1X_2 - X_2X_1 = -u\frac{\partial}{\partial v} = X_2,$$

thus $\mu_{12}^1 = \mu_{21}^1 = 0, \mu_{12}^2 = 1, \mu_{21}^2 = -1, \mu_{ii}^k = 0, i, k = 1, 2$. *We take* $\lambda_1 = 2x, \lambda_2 = -x$, *so*

$$D_x(\mu_{ij}^k) + \lambda_i\mu_{ij}^k + \lambda_j\mu_{ij}^k + \lambda_k\mu_{ij}^k = 0,$$

and

$$Y_1 = u\frac{\partial}{\partial u} + (u_x + 2xu)\frac{\partial}{\partial u_x}, Y_1(\lambda_2) = -Y_1(x) = 0,$$
$$Y_2 = -u\frac{\partial}{\partial v} - (u_x - xu)\frac{\partial}{\partial v_x}, Y_2(\lambda_1) = 2Y_2(x) = 0.$$

Hence, by Theorem 7, we know $[Y_1, Y_2] \in span\{Y_1, Y_2\}$. *In fact,*

$$\begin{aligned}[Y_1, Y_2] &= [u\frac{\partial}{\partial u} + (u_x + 2xu)\frac{\partial}{\partial u_x}, -u\frac{\partial}{\partial v} - (u_x - xu)\frac{\partial}{\partial v_x}] \\ &= -u\frac{\partial}{\partial v} + xu\frac{\partial}{\partial v_x} - u_x\frac{\partial}{\partial v_x} \\ &= Y_2.\end{aligned}$$

Example 7. *Let us consider* $X = R$, *with coordinate* x *and* $U = R^2$ *with coordinates* (u, v). *We now take the vector fields*

$$X_1 = u\frac{\partial}{\partial u}, \quad X_2 = v\frac{\partial}{\partial v},$$

these are in involution, i.e., $[X_1, X_2] = 0$, *by Remark 8. Just take* $\lambda_1 = v, \lambda_2 = u$. *We know that* $X_1(v) = X_2(u) = 0$, *so for n-order* λ_1*-prolongations* Y_1 *and* λ_2*-prolongations* Y_2 *of* X_1, X_2, *we can obtain* $[Y_1, Y_2] = 0$. *For instance,* Y_1, Y_2 *are* 1*-order* λ_1*-prolongation and* 1*-order* λ_2*-prolongation of* X_1, X_2, *i.e.,*

$$Y_1 = u\frac{\partial}{\partial u} + (u_x + vu)\frac{\partial}{\partial u_x},$$
$$Y_2 = v\frac{\partial}{\partial v} + (v_x + vu)\frac{\partial}{\partial v_x}.$$

It is clear that $[Y_1, Y_2] = 0$. *Moreover, we can also compute the* 2*-order* λ_1*-prolongations and* 2*-order* λ_2*-prolongation of* X_1, X_2, *i.e.,*

$$\tilde{Y}_1 = u\frac{\partial}{\partial u} + (u_x + vu)\frac{\partial}{\partial u_x} + (2u_xv + u_{xx} + v^2u)\frac{\partial}{\partial u_{xx}},$$
$$\tilde{Y}_2 = v\frac{\partial}{\partial v} + (v_x + vu)\frac{\partial}{\partial v_x} + (2v_xu + v_{xx} + vu^2)\frac{\partial}{\partial v_{xx}},$$

it straightly follows that $[\tilde{Y}_1, \tilde{Y}_2] = 0$.

7. Conclusions

In this paper, we generalize a theorem in [18] and show that if there are not enough symmetries and λ-symmetries, some first integrals can still be obtained. Moreover, we give two examples to illustrate this theorem. Secondly, we prove that when X is a λ-symmetry of differential equation field Γ of (4), by multiplying Γ, a function μ defined on $J^{n-1}M$, and the vector fields $\mu\Gamma$ can pass to quotient manifold Q by a group action of λ-symmetry X. Thirdly, if there are some λ-symmetries of (4), we show that the vector fields from their linear combination are symmetries of (4) under some conditions. And if we have vector field X defined on $J^{n-1}M$ with first-order λ-prolongation Y and first-order standard prolongations Z of X defined on J^nM, we prove that gY cannot be first-order λ-prolonged vector field of vector field gX if g is not a constant function. Fifth, we provide a complete set of functionally independent $(n-1)$-order invariants for $V^{(n-1)}$, which are $n-1$th prolongations of λ-symmetry of V, and obtain an explicit $n-1$-order-reduced equation of explicit n-order ordinary Equation (4). Finally, assume there is a set of vector fields $X_i, i = 1, \ldots, n$ that are in involution and all of them have the form $X_i = \phi_i^a \frac{\partial}{\partial y_a}, i = 1, \ldots, n$. We give Theorem 7 to claim under what conditions are their λ-prolongations also in involution.

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