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# Zeros of Convex Combinations of Elementary Families of Harmonic Functions 

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#### Abstract

Brilleslyper et al. investigated how the number of zeros of a one-parameter family of harmonic trinomials varies with a real parameter. Brooks and Lee obtained a similar theorem for an analogous family of harmonic trinomials with poles. In this paper, we investigate the number of zeros of convex combinations of members of these families and show that it is possible for a convex combination of two members of a family to have more zeros than either of its constituent parts. Our main tool to prove these results is the harmonic analog of Rouchés theorem.


Keywords: harmonic; polynomials; zeros

MSC: 30C15

## 1. Introduction

A complex-valued harmonic function on a simply connected domain $D$ in $\mathbb{C}$ has the form

$$
\begin{equation*}
f(z)=h(z)+\overline{g(z)} \tag{1}
\end{equation*}
$$

for analytic functions $h$ and $g$, where $\bar{g}$ is called the co-analytic part of $f$. We are especially interested in the case in which $h$ and $g$ are polynomials. Whereas the number of zeros of a non-constant analytic polynomial is always equal to its degree, the situation for harmonic polynomials is more complicated. In particular, for $f$ as in (1), the number of zeros is not a simple function of the degrees of $h$ and $g$ and depends on the coefficients. Some of the first results about harmonic polynomials sought to determine a sharp bound on the number of zeros in terms of the degrees of $h$ and $g$. If the degree of $h$ is $n$ and the degree of $g$ is $k$, where $n \geq k$, then a reasonable expectation is that the maximum number of zeros of $f$ is $n^{2}$. Sheil-Small [1] made this conjecture, and it was later shown to be correct [2-4]. A natural extension is to consider the more restrictive case in which $1 \leq k \leq n-1$. In this case, Wilmshurst [4] conjectured that the maximum number of zeros of $f$ is $k(k-1)+3 n-2$. Khavinson and Swiatek [5] proved this conjecture for $k=1$, but Lee et al. [6] later showed it is not true in general.

Because the number of zeros of a harmonic polynomial is not a simple function of the degrees of the analytic and co-analytic parts, several researchers have turned their attention to a detailed study of specific families of harmonic functions. They have sought to determine the maximum and minimum number of zeros, how the number of zeros varies with the coefficients, and the location of these zeros. See [7-15]. We discuss several of their results in some detail because they are the motivation for the current work.

In $[9,10]$, the authors considered a one-parameter family of trinomials. Although they determined precisely how the number of zeros varies with the parameter, a corollary of their main theorem is as follows:

Corollary 1 ([9,10]). Suppose $m, k \in \mathbb{N}$ with $m>k$ and $\operatorname{gcd}(m, k)=1$, and let

$$
\begin{equation*}
f_{a}(z)=z^{m}+a \bar{z}^{k}-1, \quad a \in \mathbb{C} \backslash\{0\} . \tag{2}
\end{equation*}
$$

Then $f_{a}$ has a maximum of $m+2 k$ zeros, obtained for all sufficiently large $|a|$.
Remark 1. The proof given in [9] shows that $f_{a}$ attains this maximum of $m+2 k$ zeros for all $|a|>\frac{m}{m-k}\left(\frac{m-k}{k}\right)^{\frac{k}{m}}$. For all of the results stated below, it is similarly possible to make precise what "sufficiently large" and "sufficiently small" mean. However, the thresholds are complicated expressions that are not illuminating.

Brooks and Lee [12] carried out a similar detailed analysis of an analogous oneparameter family of harmonic functions with poles. A corollary of their main theorem is as follows:

Corollary 2. Suppose $m, k \in \mathbb{N}$ with $m>k$ and $\operatorname{gcd}(m, k)=1$, and let

$$
\begin{equation*}
r_{a}(z)=z^{m}+\frac{a}{\bar{z}^{k}}-1, \quad a \in \mathbb{C} \backslash\{0\} . \tag{3}
\end{equation*}
$$

Then

1. $\quad r_{a}$ has a maximum of $m+k$ zeros, obtained when $|a|$ is sufficiently small; and
2. $\quad r_{a}$ has a minimum of $m-k$ zeros, obtained when $|a|$ is sufficiently large.

We wish to understand the zeros of more complicated harmonic functions. We might naturally ask: if we understand the zeros of some simple families of harmonic functions, can we use this information to make conclusions about more complicated families constructed from these simple pieces? In this paper, we study convex combinations of members of families (2) and (3). Thus we study

$$
\begin{equation*}
f_{a, b, s}(z)=s\left(z^{m}+a \bar{z}^{k}-1\right)+(1-s)\left(z^{n}+b \bar{z}^{\ell}-1\right), \quad a, b>0,0 \leq s \leq 1 \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{a, b, s}(z)=s\left(z^{m}+\frac{a}{\bar{z}^{k}}-1\right)+(1-s)\left(z^{n}+\frac{b}{\bar{z}^{\ell}}-1\right), \quad a, b>0,0 \leq s \leq 1 \tag{5}
\end{equation*}
$$

Although one might conjecture that once we fix all exponents, the maximum number of zeros for a convex combination is the larger of the maximum number of zeros for the constituent pieces, we prove that there are choices of the parameters for which the convex combination has strictly more zeros than either piece. This result is rather surprising. Before we state our theorems precisely, we consider two examples of this phenomenon.

Example 1. Consider

$$
f_{a, b, \frac{4}{9}}(z)=\frac{4}{9}\left(z^{5}+a \bar{z}^{2}-1\right)+\frac{5}{9}\left(z^{4}+b \bar{z}^{3}-1\right) .
$$

If we take $a=\frac{105}{4}$ and $b=14$, we are in the situation described in Corollary 1, in which the two trinomials making up the convex combination have the maximum number of zeros of $m+2 k=9$ and $n+2 \ell=10$. These zeros are shown in Figure 1.


Figure 1. The zeros and critical curve for $z^{5}+\frac{105}{4} \bar{z}^{2}-1$ (left) and $z^{4}+14 \bar{z}^{3}-1$ (right).
However, the convex combination has 11 zeros, as shown in Figure 2.


Figure 2. The zeros and critical curve for $f_{a, b, s}$.
Example 2. Next, consider

$$
r_{a, b, \frac{4}{9}}(z)=\frac{4}{9}\left(z^{5}+\frac{a}{\bar{z}^{2}}-1\right)+\frac{5}{9}\left(z^{4}+\frac{b}{\bar{z}^{3}}-1\right) .
$$

Take $a=\frac{15 b}{8}$ and vary $b$. First, consider $b=0.1$. For such $a$ and $b$, the first part of Corollary 2 applies, and each piece making up the convex combination has $m+k=n+\ell=7$ zeros. See Figure 3. However, for $b=0.1$, the convex combination has 8 zeros. See Figure 4.

Now consider $b=14$. The second part of Corollary 2 now applies; the numbers of zeros of the constituent pieces of the convex combination are $m-k=3$ and $n-\ell=1$, respectively. See Figure 5. However, the convex combination itself has 4 zeros for $b$ (and, in fact, for all sufficiently large b), displaying different behavior compared to its constituent parts. See Figure 4.


Figure 3. The zeros and critical curve of $z^{5}+\frac{15 b}{8 \bar{z}^{2}}-1$ (left) and $z^{4}+\frac{b}{\bar{z}^{3}}-1$ (right) for $b=0.1$.


Figure 4. The zeros and critical curve of $r_{a, b, s}$ as in Example 2 for $b=0.1$ (left) and $b=14$ (right).


Figure 5. The zeros and critical curve of $z^{5}+\frac{15 b}{8 \bar{z}^{2}}-1$ (left) and $z^{4}+\frac{b}{\bar{z}^{3}}-1$ (right) for $b=14$.
Our two main theorems illustrate that these are not isolated examples; in fact, there are large subfamilies of (4) and (5) exhibiting this interesting behavior.

Theorem 1. Let

$$
f_{a, b, s}(z)=s\left(z^{m}+a \bar{z}^{k}-1\right)+(1-s)\left(z^{n}+b \bar{z}^{\ell}-1\right)
$$

Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m+k=n+\ell$ and $m>n>\ell>k$. Let $s=\frac{n}{m+n}$ and $a=\frac{m b \ell}{k n}$. Then, there exists $b_{0}$, such that for all $b>b_{0} f_{a, b, s}$ has $m+2 \ell$ zeros.

Theorem 2. Let

$$
r_{a, b, s}(z)=s\left(z^{m}+\frac{a}{\bar{z}^{k}}-1\right)+(1-s)\left(z^{n}+\frac{b}{\bar{z}^{\ell}}-1\right) .
$$

Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m+k=n+\ell$ and $m>n>\ell>k$. Let $s=\frac{n}{m+n}$ and $a=\frac{m b \ell}{k n}$. Then there exist $b_{0}$ and $b_{1}$, such that
(1) For all $b<b_{0}, r_{a, b, s}$ has $m+\ell$ zeros.
(2) For all $b>b_{1}, r_{a, b, s}$ has $m+\ell-2 k$ zeros.

## 2. Background and Set-Up

In this section, we state all the definitions and theorems needed for the proofs of the theorems, in particular, Rouché's theorem for harmonic functions with poles. We also study some basic properties of the subfamilies of (4) and (5) considered in Theorems 1 and 2.

### 2.1. Definitions and Tools

One of the most useful tools for studying zeros of analytic functions is Rouché's theorem, which allows one to count the zeros of a given function inside a simple closed curve $C$ by comparing it to a function whose zeros are understood. An analog of this theorem exists for harmonic functions with poles. In order to state it, we must first understand more about the geometry of harmonic functions and how zeros and poles are counted.

An analytic function $h$ on a domain $D$ is sense-preserving. Thus, if $C$ is a simple, closed, positively oriented contour in $D$ whose interior lies entirely in $D$, its image under $h$ remains positively oriented. A co-analytic function, on the other hand, is sense-reversing. Because a harmonic function is the sum of an analytic function and a co-analytic function, it is, in general, sense-preserving in some parts of its domain and sense-reversing in others. The complex dilatation allows us to identify these regions.

Definition 1. Let $f=h+\bar{g}$ be a harmonic function on $D$, with $h$ and $g$ analytic on $D$. The complex dilatation is the function $\omega$ defined by

$$
\begin{equation*}
\omega(z)=\frac{g^{\prime}(z)}{h^{\prime}(z)} \tag{6}
\end{equation*}
$$

The key fact about the complex dilatation is contained in the following well-known proposition.

Proposition 1. Let $f=h+\bar{g}$ be harmonic on $D$, with complex dilatation $\omega$.

1. If $|\omega(z)|<1$ throughout some region $R \subseteq D$, then $f$ is sense-preserving on $R$.
2. If $|\omega(z)|>1$ on $R \subseteq D$, then $f$ is sense-reversing on $R$.

The set of points at which $|\omega(z)|=1$ is called the critical curve for $f$.
The sense-preserving and sense-reversing regions are important because they play a role in the definitions of the order of a zero and the order of a pole. More specifically, the sign of the order of a zero or a pole depends on whether it is in the sense-preserving region (positive order) or the sense-reversing region (negative order). See [11] for a more detailed discussion of why this convention makes sense. For this paper, we take the following to be our definitions of the order of a zero and the order of a pole.

Definition 2 ([16]). Let $f=h+\bar{g}$ be a complex-valued harmonic function and suppose $f\left(z_{0}\right)=0$. Write

$$
f(z)=a_{0}+\sum_{j=r}^{\infty} a_{j}\left(z-z_{0}\right)^{j}+\overline{b_{0}+\sum_{j=s}^{\infty} b_{j}\left(z-z_{0}\right)^{j}} .
$$

If $z_{0}$ is in a sense-preserving region, we define the order of $z_{0}$ to be $r$ and if $z_{0}$ is in a sense-reversing region, we define its order to be $-s$. If $z_{0}$ is on the critical curve, we call it a singular zero and its order is not defined.

Definition 3 ([17]). Let $f$ be a harmonic function on a domain $D \subseteq \mathbb{C}$ except at a finite number of poles. Suppose that the local representation of $f$ around a pole $z_{0} \in D$ is

$$
f(z)=\sum_{j=-r}^{\infty} a_{j}\left(z-z_{0}\right)^{j}+\overline{\sum_{j=-s}^{\infty} b_{j}\left(z-z_{0}\right)^{j}}+2 A \log \left|z-z_{0}\right|
$$

for some constant $A$, and where $r$ and $s$ are finite.
(i) If $a_{-r} \neq 0$ for some $r>0$ and $r>s$, or $r=s$ with $\left|a_{-r}\right|>\left|b_{-s}\right|$, then $f$ is sense-preserving near $z_{0}$ and $f$ has a pole at $z_{0}$ of order $r$.
(ii) If $b_{-s} \neq 0$ for some $s>0$ and $r<s$, or $r=s$ with $\left|a_{-r}\right|<\left|b_{-s}\right|$, then $f$ is sense-reversing near $z_{0}$ and $f$ has a pole at $z_{0}$ of order $-s$.

With this background, we can state the main tool we need to prove our theorems, which is the harmonic analog of Rouché's theorem. The theorem allows us to relate the sums of the orders of zeros and poles of two functions within a simple closed curve $C$ provided a certain inequality is satisfied on $C$. In the statement of the theorem and throughout the rest of the paper, we let $Z_{p, C}$ denote the sum of the orders of the zeros of $p$ in $C$ and we let $P_{p, C}$ denote the sum of the orders of the poles of $p$ in $C$.

Theorem 3 (Rouché's theorem for harmonic functions). Let $p$ and $q$ be harmonic, except for a finite number of poles, in a simply connected domain $D \subseteq \mathbb{C}$. Let $C$ be a simple, closed curve contained in $D$. If $|p(z)|>|q(z)|$ at each point on $C$, and if $p$ and $q$ have no poles on $C$ and no singular zeros in $C$, then $Z_{p, C}-P_{p, C}=Z_{p+q, C}-P_{p+q, C}$.

### 2.2. The Subfamilies

Although studying families (4) and (5) for all positive $a$ and $b$ and all $0 \leq s \leq 1$ would be potentially interesting, it would also be extremely difficult. A large part of the difficulty comes from the fact that, for most choices of $a, b$, and $s$, the critical curve for the function is quite complicated. Thus, as we seek to address the question of how the maximum number of zeros of a convex combination relates to the maximum numbers of zeros for the constituent pieces, it is reasonable to begin by studying subfamilies for which the critical curve is particularly simple.

We begin by computing the complex dilatations. For (4),

$$
\begin{equation*}
\omega_{f}(z)=\frac{a k s z^{k-1}+b \ell(1-s) z^{\ell-1}}{m s z^{m-1}+n(1-s) z^{n-1}} \tag{7}
\end{equation*}
$$

and for (5),

$$
\begin{equation*}
\omega_{r}(z)=\frac{-a k s z^{-k-1}-b \ell(1-s) z^{-\ell-1}}{m s z^{m-1}+n(1-s) z^{n-1}} \tag{8}
\end{equation*}
$$

Several different conditions on the parameters will yield a complex dilatation of the form

$$
\begin{equation*}
\omega(z)=C z^{D}\left(\frac{z^{B}+A}{1+A z^{B}}\right) \tag{9}
\end{equation*}
$$

Note that $\frac{z^{B}+A}{1+A z^{B}}$ is a composition of a Möbius transformation and a power function. This function is subordinate to the Möbius transformation as it also sends circles to circles. Even so, in general, the critical curve will be complicated. If, however, we require $A$ to be 1 , then the dilatation will reduce to $\omega(z)=C z^{D}$ and the critical curve will be precisely a circle. We now derive a set of conditions on the parameters for each family leading to this desirable form.

We begin with the complex dilatation of the first family (7). Factoring aksz ${ }^{\ell-1}$ out of the numerator and $m s z^{m-1}$ out of the denominator gives

$$
\begin{equation*}
\omega_{f}(z)=\frac{a k s z^{\ell-1}}{m s z^{m-1}}\left(\frac{z^{k-\ell}+\frac{b \ell(1-s)}{a k s}}{1+\frac{n(1-s)}{m s} z^{n-m}}\right) . \tag{10}
\end{equation*}
$$

In order for the rational function in parentheses to have the form $\frac{z^{B}+A}{1+A z^{B}}$, we require

$$
k-\ell=n-m \quad \text { and } \quad \frac{b \ell(1-s)}{a k s}=\frac{n(1-s)}{m s}
$$

or, equivalently,

$$
\begin{equation*}
m+k=n+\ell \quad \text { and } \quad \frac{a k}{m}=\frac{b \ell}{n} \tag{11}
\end{equation*}
$$

If we further require $A$ to be 1 , then

$$
\frac{n(1-s)}{m s}=1
$$

or, equivalently,

$$
\begin{equation*}
s=\frac{n}{m+n} \tag{12}
\end{equation*}
$$

In order to apply Corollaries 1 and 2 , we require $m>k$ and $n>\ell$. Without loss of generality, we also assume that $m>n$. Together with the first condition in (11), we obtain a strict ordering of the exponents

$$
\begin{equation*}
m>n>\ell>k \tag{13}
\end{equation*}
$$

With this strict ordering, $m>\ell$, we rewrite the dilatation as

$$
\begin{equation*}
\omega_{f}(z)=\frac{b \ell}{n} \frac{1}{z^{n-k}} \tag{14}
\end{equation*}
$$

The following proposition summarizes the properties of the subfamily of (4) considered in Theorem 1.

Proposition 2. Let $f_{a, b, s}$ be as in (4). Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m+k=n+\ell$ and $m>n>\ell>k$. Let $s=\frac{n}{m+n}$ and $a=\frac{m b l}{k n}$.

1. $\omega_{f}(z)=\frac{b \ell}{n} \frac{1}{z^{n-k}}$
2. The critical curve is a circle, $\Gamma_{b}$, centered at the origin with radius $\left(\frac{b \ell}{n}\right)^{\frac{1}{n-k}}$.
3. The region inside the critical curve is sense-reversing.
4. The region outside the critical curve is sense-preserving.

We now consider the complex dilatation of the second family (8). We again desire this dilatation to be of the form given in (9). Factoring $-b \ell(1-s) z^{-k-1}$ out of the numerator and $m s z^{m-1}$ out of the denominator gives

$$
\begin{equation*}
\omega_{r}(z)=\frac{-b \ell(1-s) z^{-k-1}}{m s z^{m-1}}\left(\frac{z^{k-\ell}+\frac{a k s}{b \ell(1-s)}}{1+\frac{n(1-s)}{m s} z^{n-m}}\right) \tag{15}
\end{equation*}
$$

In order for the rational function in parentheses to have the form $\frac{z^{B}+A}{1+A z^{B}}$, we require

$$
k-\ell=n-m \quad \text { and } \quad \frac{a k s}{b \ell(1-s)}=\frac{n(1-s)}{m s} .
$$

The first condition is equivalent to

$$
\begin{equation*}
m+k=n+\ell \tag{16}
\end{equation*}
$$

If we further require $A$ to be 1 , then

$$
\frac{n(1-s)}{m s}=1,
$$

or, equivalently,

$$
\begin{equation*}
s=\frac{n}{m+n} \tag{17}
\end{equation*}
$$

With this form of $s$, the condition that $\frac{a k s}{b \ell(1-s)}=\frac{n(1-s)}{m s}$ is equivalent to

$$
\begin{equation*}
\frac{a k}{m}=\frac{b \ell}{n} . \tag{18}
\end{equation*}
$$

Therefore, the conditions for the subfamily of (5) are the same as the conditions for the subfamily of (4). The same strict ordering of the exponents (13) also applies to this subfamily. With these conditions, the dilatation simplifies to

$$
\begin{equation*}
\omega_{r}(z)=\frac{-b \ell}{n} \frac{1}{z^{n+\ell}} \tag{19}
\end{equation*}
$$

The following proposition summarizes the properties of the subfamily of (5) considered in Theorem 2.

Proposition 3. Let $r_{a, b, s}$ be as in (5). Let $m, k, n, \ell \in \mathbb{N}$ satisfy $m+k=n+\ell$ and $m>n>\ell>k$. Let $s=\frac{n}{m+n}$ and $a=\frac{m b l}{k n}$.

1. $\quad \omega_{r}(z)=\frac{-b \ell}{n} \frac{1}{z^{n+\ell}}$
2. The critical curve is a circle, $\Gamma_{b}$, centered at the origin with radius $\left(\frac{b \ell}{n}\right)^{\frac{1}{n+\ell}}$.
3. The region inside the critical curve is sense-reversing.
4. The region outside the critical curve is sense-preserving.

We are now ready to proceed with the proofs of Theorems 1 and 2.

## 3. Proof of Theorem 1

In order to prove Theorem 1, we determine the sum of the orders of the zeros of $f_{a, b, s}$ in $\mathbb{C}$ and within the critical curve for large values of $b$. With this information and an understanding of the sense-reversing and sense-preserving regions, we deduce the number of zeros of $f_{a, b, s}$ for large $b$. Due to the restrictions we impose on $f_{a, b, s}$ that fix $s$ and make $a$ a function of $b$ once the exponents are fixed, $f_{a, b, s}$ will be referred to as $f_{b}$ for the rest of this section.

Lemma 1. The sum of the orders of the zeros of $f_{b}$ in $\mathbb{C}$ is $m$.
Proof. We prove this lemma by proving that the sum of the orders of the zeros of $f_{b}$ inside a circle of a sufficiently large radius $R$ is $m$. We prove this latter statement using Rouché's theorem to compare the zeros of $f_{b}$ to the zeros of $p(z)=s z^{m}$. If $|z|>1$, then because $a, b, s,(1-s)>0$, and because the natural number $m$ exceeds $n, k$, and $\ell$,

$$
\begin{aligned}
\left|f_{b}(z)-p(z)\right| & =\left|s a \bar{z}^{k}+(1-s) z^{n}+(1-s) b \bar{z}^{\ell}-1\right| \\
& \leq s a|z|^{k}+(1-s)|z|^{n}+(1-s) b|z|^{\ell}+1 \\
& <s a|z|^{m-1}+(1-s)|z|^{m-1}+(1-s) b|z|^{m-1}+|z|^{m-1} \\
& =(s a+(1-s)+(1-s) b+1)|z|^{m-1} .
\end{aligned}
$$

If also $|z| \geq \frac{s a+(1-s)+(1-s) b+1}{s}$, then

$$
\left|f_{b}(z)-p(z)\right|<(s a+(1-s)+(1-s) b+1)|z|^{m-1} \leq(s|z|)|z|^{m-1}=\left|s z^{m}\right|=|p(z)| .
$$

It is easy to check that, because $0<s<1, \frac{s a+(1-s)+(1-s) b+1}{s}>1$. Thus, if $R \geq$ $\frac{s a+(1-s)+(1-s) b+1}{s}$, then for all $z \in \mathbb{C}$ with $|z|=R$

$$
\left|f_{b}(z)-p(z)\right|<|p(z)| .
$$

By Rouché's theorem, the sum of the orders of the zeros for $f_{b}$ inside the circle of radius $R$ is the same as the sum of the orders of the zeros for $p$. Since $p$ has a zero of order $m$ at the origin, the sum of the orders of the zeros of $f_{b}$ inside the circle $|z|=R$ is $m$, for all sufficiently large $R$.

Now, we find the sum of the orders of the zeros inside the critical curve.
Lemma 2. For sufficiently large $b$, the sum of the orders of the zeros of $f_{b}$ inside the critical curve is $-\ell$.

Proof. We prove this lemma using Rouché's theorem. We show that $f_{b}$ has the same sum of the orders of zeros as $p(z)=(1-s) b z^{\ell}$ inside of the critical curve; that is, we show

$$
\begin{equation*}
\left|s z^{m}+s a \bar{z}^{k}+(1-s) z^{n}-1\right|<\left|(1-s) b \bar{z}^{\ell}\right| \tag{20}
\end{equation*}
$$

for all $z$ on the critical curve. Equation (20) will follow if we show

$$
\begin{equation*}
(1-s) b|z|^{\ell}-\left(s|z|^{m}+s a|z|^{k}+(1-s)|z|^{n}+1\right) \tag{21}
\end{equation*}
$$

is greater than zero for all $z$ on the critical curve. Since points on the critical curve satisfy $|z|=\left(\frac{b \ell}{n}\right)^{\frac{1}{n-k}},(21)$ is equivalent to

$$
(1-s) b\left(\frac{b \ell}{n}\right)^{\frac{\ell}{n-k}}-s\left(\frac{b \ell}{n}\right)^{\frac{m}{n-k}}-s a\left(\frac{b \ell}{n}\right)^{\frac{k}{n-k}}-(1-s)\left(\frac{b \ell}{n}\right)^{\frac{n}{n-k}}-1
$$

By factoring out $(1-s) b\left(\frac{b \ell}{n}\right)^{\frac{\ell}{n-k}}$ and using $m-\ell=n-k$ and $\frac{s \ell}{(1-s) n}=\frac{\ell}{m}$ to simplify the second term, (21) is equivalent to

$$
(1-s) b\left(\frac{b \ell}{n}\right)^{\frac{\ell}{n-k}}\left[1-\frac{\ell}{m}-\frac{s a}{(1-s) b}\left(\frac{b \ell}{n}\right)^{\frac{k-\ell}{n-k}}-\frac{1}{b}\left(\frac{b \ell}{n}\right)^{\frac{n-\ell}{n-k}}-\frac{1}{(1-s) b}\left(\frac{b \ell}{n}\right)^{\frac{-\ell}{n-k}}\right]
$$

The third term can also be simplified using $\frac{s a}{(1-s) b}=\frac{\ell}{k}$, giving

$$
(1-s) b\left(\frac{b \ell}{n}\right)^{\frac{\ell}{n-k}}\left[1-\frac{\ell}{m}-\frac{\ell}{k}\left(\frac{b \ell}{n}\right)^{\frac{k-\ell}{n-k}}-\frac{1}{b}\left(\frac{b \ell}{n}\right)^{\frac{n-\ell}{n-k}}-\frac{1}{(1-s) b}\left(\frac{b \ell}{n}\right)^{\frac{-\ell}{n-k}}\right] .
$$

This expression is positive if and only if the expression in square brackets is positive. Due to the relation between $m, n, \ell, k$ seen in (13), we find that the sum of the exponents on $b$ in each term is negative. Indeed,

$$
\frac{-\ell}{n-k}-1<\frac{k-\ell}{n-k}=\frac{n-\ell}{n-k}-1<0
$$

Therefore, each term involving $b$ tends to 0 as $b$ tends to infinity. Thus, there exists $b_{0}$ such that for all $b>b_{0}$,

$$
\frac{\ell}{k}\left(\frac{b \ell}{n}\right)^{\frac{k-\ell}{n-k}}+\frac{1}{b}\left(\frac{b \ell}{n}\right)^{\frac{n-\ell}{n-k}}+\frac{1}{(1-s) b}\left(\frac{b \ell}{n}\right)^{\frac{-\ell}{n-k}}<1-\frac{\ell}{m}
$$

Therefore, for such $b$,

$$
0<1-\frac{\ell}{m}-\frac{\ell}{k}\left(\frac{b \ell}{n}\right)^{\frac{k-\ell}{n-k}}-\frac{1}{b}\left(\frac{b \ell}{n}\right)^{\frac{n-\ell}{n-k}}-\frac{1}{(1-s) b}\left(\frac{b \ell}{n}\right)^{\frac{-\ell}{n-k}}
$$

It follows that $p$ is the dominant term of $f_{b}$ on the critical curve, and by Rouchés theorem, the sum of the orders of the zeros of $f_{b}$ inside its critical curve is the same as the sum of the orders of the zeros of $p$. Since $p$ only has a zero of order $-\ell$ located at the origin, the sum of the orders of the zeros of $f_{b}$ inside the critical curve is $-\ell$ for sufficiently large $b$.

Now, we have everything needed to prove Theorem 1.
Proof of Theorem 1. By Lemma 1, the sum of the orders of the zeros of $f_{b}$ in $\mathbb{C}$ is $m$. Since the inside of the critical curve is the sense-reversing region and the outside is the sensepreserving region, Lemma 2 gives that the sum of the orders of the zeros of $f_{b}$ in the sense-reversing region is $-\ell$. Therefore, the sum of the orders of the zeros of $f_{b}$ outside of the critical curve is $m+\ell$, and the total number of zeros for $f_{b}$ is $m+2 \ell$.

## 4. Proof of Theorem 2

We now turn to the family $r_{a, b, s}$ as given in (5), with the additional restrictions on the exponents and parameters given in the statement of Theorem 2. Thus, as above, $s$ is now fixed, $a$ is a function of $b$, and we abbreviate $r_{a, b, s}$ by $r_{b}$. In order to count the zeros of $r_{b}$ in the complex plane, we first determine the sum of the orders of the zeros of $r_{b}$, both in the plane and within the critical circle for small and large $b$.

Lemma 3. The sum of the orders of the zeros of $r_{b}$ in $\mathbb{C}$ is $m-\ell$.
Proof. We apply Rouché's theorem to $p(z)=s z^{m}$ and $q(z)=r_{b}(z)-p(z)=(1-s) z^{n}+$ $\frac{s a}{\bar{z}^{k}}+\frac{(1-s) b}{\bar{z}^{\ell}}-1$ on a circle $C_{R}$ centered at the origin with a sufficiently large radius $R$. Recalling that $m>n$, we see that for $|z|>1$,

$$
\begin{aligned}
|q(z)| & =\left|(1-s) z^{n}+\frac{s a}{\bar{z}^{k}}+\frac{(1-s) b}{\bar{z}^{\ell}}-1\right| \\
& \leq(1-s)|z|^{n}+\frac{s a}{|z|^{k}}+\frac{(1-s) b}{|z|^{\ell}}+1 \\
& <[(1-s)+s a+(1-s) b+1]|z|^{n} \\
& \leq[(1-s)+s a+(1-s) b+1]|z|^{m-1} .
\end{aligned}
$$

If we also take $|z| \geq \frac{s a+(1-s)+(1-s) b+1}{s}$ (which, as above, is greater than 1 ), then

$$
|q(z)|<(s|z|)|z|^{m-1}=\left|s z^{m}\right|=|p(z)| .
$$

Thus if $R \geq \frac{s a+(1-s)+(1-s) b+1}{s}$, it follows that $|q(z)|<|p(z)|$ for all $|z|=R$; that is, for all $z$ on $C_{R}$. By Rouché's theorem,

$$
Z_{r_{b}, C_{R}}-P_{r_{b}, C_{R}}=Z_{p, C_{R}}-P_{p, C_{R}}=m-0=m .
$$

Because $r_{b}$ has a pole at the origin of order $-\ell$, we have $Z_{r_{b}, C_{R}}=m-\ell$.
We now count the zeros inside the critical circle, $\Gamma_{b}$, for small and large $b$. In conjunction with Lemma 3, this will allow us to calculate the total number of zeros, proving Theorem 2.

Proof of Theorem 2. First, consider $b$ to be small. In this case, we apply Rouché's theorem to $p(z)=-1$ and $q(z)=s z^{m}+(1-s) z^{n}+\frac{a s}{\bar{z}^{k}}+\frac{(1-s) b}{\bar{z}^{\ell}}$ on the critical circle, $\Gamma_{b}$. Points on this circle satisfy $|z|=\left(\frac{b \ell}{n}\right)^{\frac{1}{n+\ell}}$, so on this circle,

$$
\begin{aligned}
&|q(z)|=\left|s z^{m}+(1-s) z^{n}+\frac{a s}{\bar{z}^{k}}+\frac{(1-s) b}{\bar{z}^{\ell}}\right| \\
& \leq s|z|^{m}+(1-s)|z|^{n}+\frac{a s}{|z|^{k}}+\frac{(1-s) b}{|z|^{\ell}} \\
&= s\left(\frac{b \ell}{n}\right)^{\frac{m}{n+\ell}}+(1-s)\left(\frac{b \ell}{n}\right)^{\frac{n}{n+\ell}} \\
&+a s\left(\frac{b \ell}{n}\right)^{-\frac{k}{n+\ell}}+(1-s) b\left(\frac{b \ell}{n}\right)^{-\frac{\ell}{n+\ell}} .
\end{aligned}
$$

We show that all powers of $b$ in this last expression are positive. The powers of $b$ in the first and second terms are clearly positive. The power of $b$ in the fourth term is $1-\frac{\ell}{n+\ell}=\frac{n}{n+\ell}>0$. Finally, because $a=\frac{m b \ell}{k n}$ and $m=n+\ell-k$, the power of $b$ in the third term is

$$
1-\frac{k}{n+\ell}=\frac{n+\ell-k}{n+\ell}=\frac{m}{n+\ell}>0
$$

Because all powers of $b$ are positive, the expression approaches 0 as $b \rightarrow 0$. In particular, there exists $b_{0}$, such that this final expression is less than 1 for all $b<b_{0}$. Then for all $b<b_{0}$ and $z$ on $\Gamma_{b}$,

$$
|q(z)|=\left|(1-s) z^{n}+\frac{s a}{\bar{z}^{k}}+\frac{(1-s) b}{\bar{z}^{\ell}}-1\right|<1=|p(z)| .
$$

By Rouché's theorem,

$$
Z_{r_{b}, \Gamma_{b}}-P_{r_{b}, \Gamma_{b}}=Z_{r_{b}, \Gamma_{b}}+\ell=Z_{p, \Gamma_{b}}-P_{p, \Gamma_{b}}=0
$$

and, thus, $Z_{r_{b}, \Gamma_{b}}=-\ell$. Because there are $-\ell$ zeros in the sense-reversing region and the sum of the orders of the zeros in $\mathbb{C}$ is $m-\ell$, there are $m$ zeros in the sense-preserving region. Then there are $m+\ell$ total zeros for $b<b_{0}$.

We now consider $b$ large. We apply Rouché's theorem to $p(z)=s z^{m}+\frac{a s}{\bar{z}^{k}}$ and $q(z)=(1-s) z^{n}+\frac{(1-s) b}{\bar{z}^{\ell}}-1$ on the critical circle, $\Gamma_{b}$. If $b>1$, then on this circle,

$$
\begin{aligned}
|q(z)| & =\left|(1-s) z^{n}+\frac{(1-s) b}{\bar{z}^{\ell}}-1\right| \\
& \leq(1-s)|z|^{n}+\frac{(1-s) b}{|z|^{\ell}}+1 \\
& =(1-s)\left(\frac{b \ell}{n}\right)^{\frac{n}{n+\ell}}+(1-s) b\left(\frac{b \ell}{n}\right)^{-\frac{\ell}{n+\ell}}+1 \\
& <(1-s)\left(\frac{\ell}{n}\right)^{\frac{n}{n+\ell}} b^{\frac{n}{n+\ell}}+(1-s)\left(\frac{\ell}{n}\right)^{-\frac{\ell}{n+\ell}} b^{\frac{n}{n+\ell}}+b^{\frac{n}{n+\ell}} \\
& =\left[(1-s)\left(\frac{\ell}{n}\right)^{\frac{n}{n+\ell}}+(1-s)\left(\frac{\ell}{n}\right)^{-\frac{\ell}{n+\ell}}+1\right] \frac{n}{b^{\frac{n}{n+\ell}}}
\end{aligned}
$$

Note that the expression in square brackets is positive.
On the other hand, on the critical curve,

$$
\begin{aligned}
|p(z)| & =\left|s z^{m}+\frac{a s}{\bar{z}^{k}}\right| \\
& \geq \frac{a s}{|z|^{k}}-s|z|^{m} \\
& =s\left(\frac{b \ell m}{n k}\right)\left(\frac{b \ell}{n}\right)^{-\frac{k}{n+\ell}}-s\left(\frac{b \ell}{n}\right)^{\frac{m}{n+\ell}} \\
& =s\left(\frac{\ell}{n}\right)^{\frac{m}{n+\ell}}\left(\frac{m-k}{k}\right) b^{\frac{m}{n+\ell}} .
\end{aligned}
$$

Because $m>k$, this expression is positive. Furthermore, $m>n$ implies that $\frac{m}{n+\ell}>\frac{n}{n+\ell}$. Together, these facts imply that there is some $b_{1}$, such that for all $b>b_{1}$,

$$
\left[(1-s)\left(\frac{\ell}{n}\right)^{\frac{n}{n+\ell}}+(1-s)\left(\frac{\ell}{n}\right)^{-\frac{\ell}{n+\ell}}+1\right] b^{\frac{n}{n+\ell}}<s\left(\frac{\ell}{n}\right)^{\frac{m}{n+\ell}}\left(\frac{m-k}{k}\right) b^{\frac{m}{n+\ell}}
$$

Then for all $b>b_{1}$, for $z$ on $\Gamma_{b}$,

$$
|q(z)|=\left|(1-s) z^{n}+\frac{(1-s) b}{\bar{z}^{\ell}}+1\right|<\left|s z^{m}+\frac{a s}{\bar{z}^{k}}\right|=|p(z)| .
$$

By Rouché's theorem, then, $Z_{r_{b}, \Gamma_{b}}-P_{r_{b}, \Gamma_{b}}=Z_{p, \Gamma_{b}}-P_{p, \Gamma_{b}}$.
One readily computes that $\Gamma_{b}$ is also the critical curve for $p$. A straight-forward calculation in the polar coordinates shows that all the zeros of $p$ lie on a circle whose radius exceeds that of the critical circle; thus, $p$ has no zeros inside $\Gamma_{b}$. Because it has a pole of order $-k$ at the origin,

$$
Z_{p, \Gamma_{b}}-P_{p, \Gamma_{b}}=0-(-k)=k
$$

Then $Z_{r_{b}, \Gamma_{b}}-P_{r_{b}, \Gamma_{b}}=k$, or, equivalently, $Z_{r_{b}, \Gamma_{b}}=-(\ell-k)$. Because the sum of the orders of the zeros of $r_{b}$ in $\mathbb{C}$ is $m-\ell$, there are $m-k$ zeros in the sense-preserving region, giving a total of $m+\ell-2 k$ zeros for $b>b_{1}$.

## 5. Future Directions

This paper does not provide a detailed analysis of families (4) and (5) but only considers subfamilies for which the complex dilatation has the very special form $\omega(z)=C z^{D}$. Many interesting questions remain for future work:

1. There are actually several inequivalent sets of conditions on the exponents $m, k, n$, $\ell$, and the parameters $a, b$, and $s$, for which the complex dilatation simplifies to $\omega(z)=C z^{D}$, each leading to a different subfamily. Do these other subfamilies exhibit the same behaviors observed in our main theorems?
2. As we saw in Section 2, a set of weaker assumptions on the exponents and parameters leads to a complex dilatation of the form $\omega(z)=C z^{D}\left(\frac{z^{B}+A}{1+A z^{B}}\right)$. (See, for example, (9)). Our subfamily arose from further requiring $A=1$. Are there other values of $A$ that give new or more general results?
3. Are there interesting results if we consider the convex combination

$$
f_{a, b, s}(z)=s\left(z^{m}+a \bar{z}^{k}-1\right)+(1-s)\left(z^{n}+\frac{b}{\bar{z}^{\ell}}-1\right) ?
$$

4. Does anything interesting happen when we consider a convex combination of three terms, such as

$$
f(z)=s_{1}\left(z^{m_{1}}+a_{1} \bar{z}^{k_{1}}-1\right)+s_{2}\left(z^{m_{2}}+a_{2} \bar{z}^{k_{2}}-1\right)+s_{3}\left(z^{m_{3}}+a_{3} \bar{z}^{k_{3}}-1\right)
$$

where $0 \leq s_{j} \leq 1$ for $j=1,2,3$ and $s_{1}+s_{2}+s_{3}=1$ ?

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