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Article

Certain Properties of Harmonic Functions Defined by a Second-Order Differential Inequality

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Abstract: The Theory of Complex Functions has been studied by many scientists and its application area has become a very wide subject. Harmonic functions play a crucial role in various fields of mathematics, physics, engineering, and other scientific disciplines. Of course, the main reason for maintaining this popularity is that it has an interdisciplinary field of application. This makes this subject important not only for those who work in pure mathematics, but also in fields with a deep-rooted history, such as engineering, physics, and software development. In this study, we will examine a subclass of Harmonic functions in the Theory of Geometric Functions. We will give some definitions necessary for this. Then, we will define a new subclass of complex-valued harmonic functions, and their coefficient relations, growth estimates, radius of univalency, radius of starlikeness and radius of convexity of this class are investigated. In addition, it is shown that this class is closed under convolution of its members.

Keywords: harmonic; univalent; starlikeness; convexity; convolution

MSC: 30C45; 30C80

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1. Introduction

In this section, some definitions and necessary information that we will use in this paper will be given. After these definitions, we will give some important properties about harmonic functions and introduce the representations of a few subclasses.

When $|z| < 1$ for $z \in \mathbb{C}$, notation $\mathbb{D}$ is called open unit disk. Here, $\mathbb{C}$ is complex number set and h and g are analytic in $\mathbb{D}$. Let $\mathbb{H}$ be the class of complex-valued harmonic functions $f = h + \overline{g}$ in the open unit disk $\mathbb{D}$, normalized so that $h(0) = 0 = g(0)$, $h'(0) = 1$. From this point of view, we can give the definition of $\mathbb{H}^0 = \{f = h + \overline{g} \in \mathbb{H} : g'(0) = 0\}$ for the $\mathbb{H}^0$ class. Every function $f \in \mathbb{H}^0$ has the canonical representation of a harmonic function $f = h + \overline{g}$ in the open unit disk $\mathbb{D}$ as the sum of an analytic function h and the conjugate of an analytic function g. The power series expansions of h and g functions be defined as follows:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

In this case, the functions in Relation (1) are analytic on the open unit disk. $f = h + \overline{g}$ is locally univalent and sense-preserving in the open unit disk $\mathbb{D}$ if and only if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Denote by $\mathbb{SH}^0$ the subclass of $\mathbb{H}^0$ that is univalent and sense-preserving in the open unit disk $\mathbb{D}$ (see [1,2]). Note that, with $g(z) = 0$, the classical family $\mathbb{S}$ of analytic univalent and normalized functions in the open unit disk $\mathbb{D}$ is a subclass of

$\mathbb{SH}^0$, just as the class $\mathbb{A}$ of analytic and normalized functions in the open unit disk $\mathbb{D}$ is a subclass of $\mathbb{H}^0$.

Let $\mathbb{K}, \mathbb{S}^*$ and $\mathbb{CK}$ be the subclasses of $\mathbb{S}$ mapping $\mathbb{D}$ onto convex, starlike and close-to-convex domains, respectively, just as $\mathbb{KH}^0$, $\mathbb{SH}^{*,0}$ and $\mathbb{CKH}^0$ are the subclasses of $\mathbb{SH}^0$ mapping the open unit disk to their respective domains.

The classes introduced above have been studied and developed by many researchers. One of these researchers, Ponussamy et al. [3], introduced the following class in 2013:

$$\mathbb{PH}^0 = \left\{ f \in \mathbb{H}^0 : \mathrm{Re}\left[h'(z)\right] > \left|g'(z)\right| \text{ for } z \in \mathbb{D} \right\}$$

and they proved that functions in $\mathbb{PH}^0$ are close-to-convex. After this study, the following subclass definition has been made using this class and some important features such as coefficient bounds, growth estimates, etc., are examined by Ghosh and Vasudevarao [4]:

$$\mathbb{WH}^0(\beta) = \left\{ f \in \mathbb{H}^0 : \mathrm{Re}\left[h'(z) + \beta z h''(z)\right] > \left|g'(z) + \beta z g''(z)\right| \text{ for } z \in \mathbb{D},\ \beta \geq 0 \right\}.$$

Nagpal and Ravichandran [5] studied a special version of subclass $\mathbb{WH}^0$ of functions $f \in \mathbb{H}^0$ which satisfy the inequality $\mathrm{Re}\left[h'(z) + z h''(z)\right] > \left|g'(z) + z g''(z)\right|$ for $z \in \mathbb{D}$ which is a harmonic analogue of the class $\mathbb{W}$ defined by Chichra [6] consisting of functions $h \in \mathbb{A}$, satisfying the condition $\mathrm{Re}\left[h'(z) + z h''(z)\right] > 0$ for $z \in \mathbb{D}$.

In 1977, Chichra [6] studied the class $\mathbb{G}(\alpha)$ for some $\alpha \geq 0$, where $\mathbb{G}(\alpha)$ consists of analytic function $h(z)$ such that

$$\mathrm{Re}\left[(1-\alpha)\frac{h(z)}{z} + \alpha h'(z)\right] > 0$$

for $|z| < 1$.

Recently, Liu and Yang [7] defined a class

$$\mathbb{G}^0_{\mathbb{H}}(\alpha; r) = \left\{ f \in \mathbb{H}^0 : \mathrm{Re}\left[(1-\alpha)\frac{h(z)}{z} + \alpha g'(z)\right] \right.$$
$$\left. > \left|(1-\alpha)\frac{h(z)}{z} + \alpha g'(z)\right| \text{ for } |z| < r \right\}$$

where $\alpha \geq 0, 0 < r \leq 1$.

Also, Rajbala and Prajapat [8] studied such properties of the class $\mathbb{WH}^0(\delta, \lambda)$ of functions $f \in \mathbb{H}^0$ satisfies the following inequality:

$$\mathrm{Re}\left[h'(z) + \delta z h''(z) - \lambda\right] > \left|g'(z) + \delta z g''(z)\right| \text{for } z \in \mathbb{D}, \delta \geq 0, 0 \leq \lambda < 1.$$

Apart from all these past studies, there are many ongoing studies today. For important studies that we can use as references in this article, References [5,9–12] can be consulted.

Denote by $\mathbb{WH}^0(\alpha, \beta)$ the class of functions $f = h + \overline{g} \in \mathbb{H}^0$, and satisfy the following inequality:

$$\begin{aligned} &\mathrm{Re}\left[(2\beta+1-\alpha)\frac{h(z)}{z} + (\alpha-2\beta)h'(z) + \beta z h''(z)\right] \\ &> \left|(2\beta+1-\alpha)\frac{g(z)}{z} + (\alpha-2\beta)g'(z) + \beta z g''(z)\right| \end{aligned} \tag{2}$$

where $\alpha \geq \beta \geq 0$. When $\beta = 0$ and $\alpha \geq 0$ are specially selected, previously studied $\mathbb{G}^0_{\mathbb{H}}(\alpha; 1)$ [7] and $\mathcal{G}^0_H(1-\alpha, \alpha, 0)$ [10] classes are obtained.

Let us define class $\mathbb{W}_{\gamma}(\alpha,\beta)$ as the class formed by the functions taken from class $f \in \mathbb{A}$ that satisfy the following inequality:

$$\operatorname{Re}\left[(2\beta+1-\alpha)\frac{f(z)}{z}+(\alpha-2\beta)f'(z)+\beta z f''(z)\right]>\gamma \ (\alpha \geq \beta \geq 0 \text{ and } \gamma<1).$$

See References [6,13–19] for important major articles studied in this class.

Now, we will give some basic examples for $f \in \mathbb{WH}^0(\alpha,\beta)$.

Example 1. *Let* $f = z + \dfrac{\overline{z}^3}{1+2(\alpha+\beta)} \in \mathbb{WH}^0(\alpha,\beta)$. *And let* $\alpha = 1/2$, $\beta = 1/2$. *Then,* $f = z + \dfrac{\overline{z}^3}{3} \in \mathbb{WH}^0(\dfrac{1}{2},\dfrac{1}{2})$. *Let us see where the function* $f(z)$ *defined here will map the unit disk. See Figure 1.*

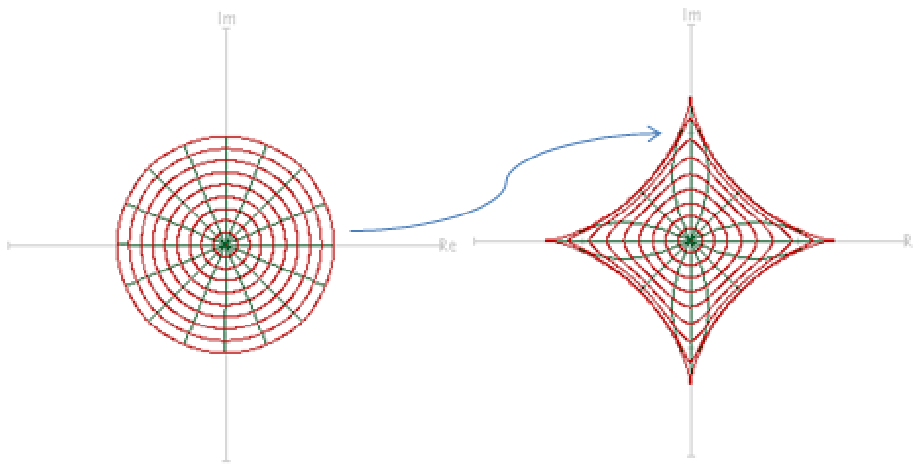

Figure 1. Image of the unit disk under the f function.

Example 2. *Let* $f = z + \dfrac{\overline{z}^2}{1+\alpha} \in \mathbb{WH}^0(\alpha,\beta)$. *And let* $\alpha = 1/2$ *and* $0 \leq \beta \leq 1/2$. *Then,* $f = z + \dfrac{2\overline{z}^2}{3} \in \mathbb{WH}^0(\dfrac{1}{2},\beta)$. *Let us see where the function* $f(z)$ *defined here will map the unit disk. See Figure 2.*

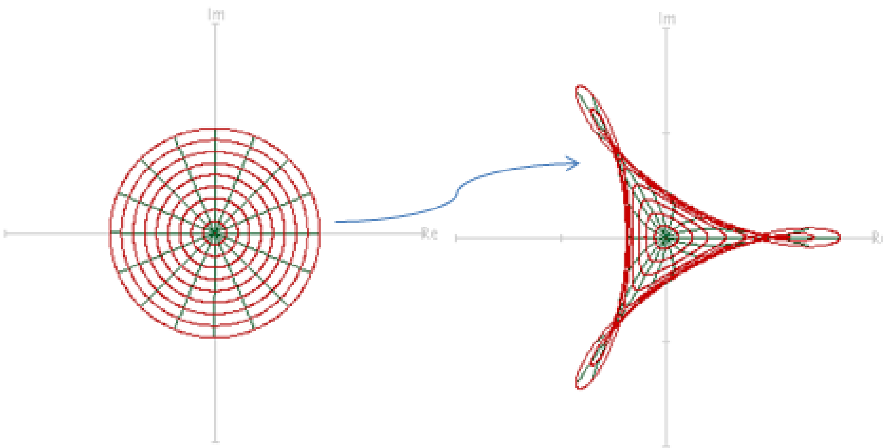

Figure 2. Image of the unit disk under the f function.

2. The Sharp Coefficient Estimates and Growth Theorems of $\mathbb{WH}^0(\alpha,\beta)$

In this section, we will examine the $\mathbb{WH}^0(\alpha,\beta)$ class.

The first theorem is about the conditions under which a given function will belong to the $\mathbb{WH}^0(\alpha,\beta)$ class, and what properties a function in the $\mathbb{WH}^0(\alpha,\beta)$ class has.

Theorem 1. *The function* $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ *defined in (1) is in class* $\mathbb{WH}^0(\alpha,\beta)$ *if and only if* $F_\sigma = \mathfrak{h} + \sigma\mathfrak{g} \in \mathbb{W}_0(\alpha,\beta)$ *for each* $\sigma(|\sigma| = 1)$.

Proof. Suppose $xtup\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}} \in \mathbb{WH}^0(\alpha,\beta)$. Then, by using (2) for $z \in \mathbb{D}$, and also for each complex number σ with $|\sigma| = 1$, we obtain

$$\begin{aligned}
&\mathrm{Re}\left[(2\beta+1-\alpha)\frac{\mathfrak{h}(z)+\sigma\mathfrak{g}(z)}{z} + (\alpha-2\beta)\big(\mathfrak{h}'(z)+\sigma\mathfrak{g}'(z)\big) + \beta z\big(\mathfrak{h}''(z)+\sigma\mathfrak{g}''(z)\big)\right] \\
&= \mathrm{Re}\left[(2\beta+1-\alpha)\frac{\mathfrak{h}(z)}{z} + (\alpha-2\beta)\mathfrak{h}'(z) + \beta z\mathfrak{h}''(z)\right] \\
&+ \mathrm{Re}\left[\sigma\left((2\beta+1-\alpha)\frac{\mathfrak{g}(z)}{z} + (\alpha-2\beta)\mathfrak{g}'(z) + \beta z\mathfrak{g}''(z)\right)\right] \\
&> \mathrm{Re}\left[(2\beta+1-\alpha)\frac{\mathfrak{h}(z)}{z} + (\alpha-2\beta)\mathfrak{h}'(z) + \beta z\mathfrak{h}''(z)\right] \\
&- \left|(2\beta+1-\alpha)\frac{\mathfrak{g}(z)}{z} + (\alpha-2\beta)\mathfrak{g}'(z) + \beta z\mathfrak{g}''(z)\right| \\
&> 0,\ z \in \mathbb{D}.
\end{aligned}$$

Thus, $F_\sigma \in \mathbb{W}_0(\alpha,\beta)$ for each $\sigma(|\sigma| = 1)$. Conversely, let $F_\sigma = \mathfrak{h} + \sigma\mathfrak{g} \in \mathbb{W}_0(\alpha,\beta)$. This implies that

$$\begin{aligned}
&\mathrm{Re}\left[(2\beta+1-\alpha)\tfrac{\mathfrak{h}(z)}{z} + (\alpha-2\beta)\mathfrak{h}'(z) + \beta z\mathfrak{h}''(z)\right] \\
&> -\mathrm{Re}\left[\sigma\left((2\beta+1-\alpha)\tfrac{\mathfrak{g}(z)}{z} + (\alpha-2\beta)\mathfrak{g}'(z) + \beta z\mathfrak{g}''(z)\right)\right].
\end{aligned} \tag{3}$$

Setting $\mathcal{A} := (2\beta+1-\alpha)\frac{\mathfrak{g}(z)}{z} + (\alpha-2\beta)\mathfrak{g}'(z) + \beta z\mathfrak{g}''(z)$, $\varphi_0 = arg[\mathcal{A}]$. Therefore, $\mathcal{A} = |\mathcal{A}|e^{i\varphi_0}$. For each fixed $z \in \mathbb{D}_r, r \in (0,1)$ and arbitrarily chosen complex number σ with $|\sigma| = 1$, i.e., $\sigma = e^{i(\pi-\varphi_0)}$, (3) becomes

$$\begin{aligned}
&\mathrm{Re}\left[(2\beta+1-\alpha)\frac{\mathfrak{h}(z)}{z} + (\alpha-2\beta)\mathfrak{h}'(z) + \beta z\mathfrak{h}''(z)\right] \\
&> -\mathrm{Re}\left[e^{i(\pi-\varphi_0)}\mathcal{A}\right] = -\mathrm{Re}\left[e^{i\pi-i\varphi_0}|\mathcal{A}|e^{i\varphi_0}\right] = -\mathrm{Re}(e^{i\pi})|\mathcal{A}| = |\mathcal{A}| \quad (z \in \mathbb{D}_r).
\end{aligned}$$

This shows that $\mathfrak{f} \in \mathbb{WH}^0(\alpha,\beta)$. □

Now, let us examine the coefficient relation of the co-analytical part of a function $\mathfrak{f}$ of class $\mathbb{WH}^0(\alpha,\beta)$.

Theorem 2. *Let* $\mathfrak{f}$ *be a function of type* $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ *in* $\mathbb{WH}^0(\alpha,\beta)$ *class. Then, for* $n \geq 2$,

$$|b_n| \leq \frac{1}{1+(-1+n)[\alpha+\beta(-2+n)]}. \tag{4}$$

The result is sharp and equality applies to the function $\mathfrak{f}(z) = z + \frac{1}{1+(-1+n)[\alpha+\beta(-2+n)]}\bar{z}^n$.

Proof. Let us assume that the function $\mathfrak{f}$ defined in type (1) belongs to class $\mathbb{WH}^0(\alpha,\beta)$. Using the series representation of $\mathfrak{g}(\rho e^{i\varphi})$, $0 \leq \rho < 1$ and $\varphi \in \mathbb{R}$, we derive

$$\begin{aligned}
&\rho^{n-1}[2\beta+1-\alpha+(\alpha-2\beta)n+\beta n(-1+n)]|b_n| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\left|(2\beta+1-\alpha)\frac{\mathfrak{g}(\rho e^{i\varphi})}{\rho e^{i\varphi}} + (\alpha-2\beta)\mathfrak{g}'(\rho e^{i\varphi}) + \beta\rho e^{i\varphi}\mathfrak{g}''(\rho e^{i\varphi})\right|d\varphi
\end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[(2\beta+1-\alpha)\frac{\hbar(\rho e^{i\varphi})}{\rho e^{i\varphi}} + (\alpha-2\beta)\hbar'(\rho e^{i\varphi}) + \beta\rho e^{i\varphi}\hbar''(\rho e^{i\varphi})\right] d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left[1+\sum_{n=2}^{\infty}(1+(-1+n)[\alpha+\beta(-2+n)])a_n\rho^{-1+n}e^{i(-1+n)\varphi}\right] d\varphi \\
&= 1.
\end{aligned}
$$

Allowing $\rho \to 1^-$, we prove the inequality (4). Moreover, the equality is achieved for $f(z) = z + \frac{1}{1+(-1+n)[\alpha+\beta(-2+n)]}\bar{z}^n$. □

The following theorem, which allows us to understand the relationship between the coefficients of the function f, also allows us to solve the problem of finding an upper bound for the coefficients of the functions in the $\mathbb{WH}^0(\alpha,\beta)$ class.

Theorem 3. *Let f be a function of type $f = \hbar + \overline{g}$ in $\mathbb{WH}^0(\alpha,\beta)$ class. Then, for $n \geq 2$, we have*

$$(i)\ |a_n| + |b_n| \leq \frac{2}{1+(-1+n)[\alpha+\beta(-2+n)]}$$

$$(ii)\ ||a_n| - |b_n|| \leq \frac{2}{1+(-1+n)[\alpha+\beta(-2+n)]}$$

$$(iii)\ |a_n| \leq \frac{2}{1+(-1+n)[\alpha+\beta(-2+n)]}.$$

All boundaries are sharp here. Conditions of equality for all boundaries are satisfied if $f(z) = z + \sum_{n=2}^{\infty} \frac{2}{1+(-1+n)[\alpha+\beta(-2+n)]} z^n$.

Proof. (i) Let us assume that the function f defined in type (1) belongs to class $\mathbb{WH}^0(\alpha,\beta)$. Then, from Theorem 1, $F_\sigma = \hbar + \sigma g \in \mathbb{W}_0(\alpha,\beta)$ for each σ ($|\sigma| = 1$). Thus, for each $|\sigma| = 1$, we have

$$\mathrm{Re}\left[(2\beta+1-\alpha)\frac{\hbar(z)+\sigma g(z)}{z} + (\alpha-2\beta)(\hbar(z)+\sigma g(z))' + \beta z(\hbar(z)+\sigma g(z))''\right] > 0$$

for $z \in \mathbb{D}$. From here, we see that there exists an analytical function $\mathfrak{p}$ of the type $\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} \mathfrak{p}_n z^n$, whose real part is positive, which satisfies Equation (5) in the open unit disk $\mathbb{D}$, such that

$$(2\beta+1-\alpha)\frac{\hbar(z)+\sigma g(z)}{z} + (\alpha-2\beta)(\hbar(z)+\sigma g(z))' + \beta z(\hbar(z)+\sigma g(z))'' = \mathfrak{p}(z) \quad (5)$$

If we equate the coefficients in Equation (5), we obtain the following relation

$$(1+(-1+n)[\alpha+\beta(-2+n)])(a_n+\sigma b_n) = \mathfrak{p}_{-1+n} \text{ for } n \geq 2.$$

According to Caratheodory (for detailed information, see [20]), since $|\mathfrak{p}_n| \leq 2$ for $n \geq 1$, and σ ($|\sigma| = 1$) is arbitrary, the proof of the first inequality is thus completed. The proof can be completed by using the method used in the first proof in other parts of the theorem. In all cases, the state of equality is provided by the function $f(z) = z + \sum_{n=2}^{\infty} \frac{2}{1+(-1+n)[\alpha+\beta(-2+n)]} z^n$. □

Theorem 4. *Let $\mathfrak{f}$ be a function of type $\mathfrak{f} = \hbar + \overline{g}$ in $\mathbb{H}^0$ class with*

$$\sum_{n=2}^{\infty} (1 + (-1+n)[\alpha + \beta(-2+n)])(|a_n| + |b_n|) \leq 1, \tag{6}$$

then $\mathfrak{f} \in \mathbb{WH}^0(\alpha, \beta)$.

Proof. Let us assume that the function $\mathfrak{f}$ defined in type (1) belongs to class $\mathbb{H}^0$. Then, using (6),

$$\begin{aligned}
& \mathrm{Re}\left[(2\beta+1-\alpha)\frac{\hbar(z)}{z} + (\alpha-2\beta)\hbar'(z) + \beta z \hbar''(z)\right] \\
= & \mathrm{Re}\left[1 + \sum_{n=2}^{\infty}(1+(-1+n)[\alpha+\beta(-2+n)])a_n z^{-1+n}\right] \\
> & 1 - \sum_{n=2}^{\infty}(1+(-1+n)[\alpha+\beta(-2+n)])|a_n| \\
\geq & \sum_{n=2}^{\infty}(1+(-1+n)[\alpha+\beta(-2+n)])|b_n| \\
> & \left|\sum_{n=2}^{\infty}(1+(-1+n)[\alpha+\beta(-2+n)])b_n z^{-1+n}\right| \\
= & \left|(2\beta+1-\alpha)\frac{g(z)}{z} + (\alpha-2\beta)g'(z) + \beta z g''(z)\right|.
\end{aligned}$$

Hence, $\mathfrak{f} \in \mathbb{WH}^0(\alpha, \beta)$. □

The following theorem determines the lower and upper bounds for the modulus of the function $\mathfrak{f}$.

Theorem 5. *Let $\mathfrak{f}$ be a function of type $\mathfrak{f} = \hbar + \overline{g}$ in $\mathbb{WH}^0(\alpha, \beta)$ class for $\alpha \geq \beta \geq 0$. Then,*

$$|z| + 2\sum_{n=2}^{\infty} \frac{(-1)^{-1+n}|z|^n}{1+(-1+n)[\alpha+\beta(-2+n)]} \leq |\mathfrak{f}(z)| \leq |z| + 2\sum_{n=2}^{\infty} \frac{|z|^n}{1+(-1+n)[\alpha+\beta(-2+n)]}$$

The result is sharp and equalities apply to the function $\mathfrak{f}(z) = z + \sum_{n=2}^{\infty} \frac{2}{1+(-1+n)[\alpha+\beta(-2+n)]} z^n$.

Proof. Let $\mathfrak{f}$ be a function of type $\mathfrak{f} = \hbar + \overline{g}$ in class $\mathbb{WH}^0(\alpha, \beta)$. Then, using Theorem 1, $F_\sigma = \hbar + \sigma g \in \mathbb{W}_0(\alpha, \beta)$, and for each $|\sigma| = 1$, we have $\mathrm{Re}\{\mathfrak{p}(z)\} > 0$, where

$$\mathfrak{p}(z) = (2\beta+1-\alpha)\frac{F_\sigma(z)}{z} + (\alpha-2\beta)F_\sigma'(z) + \beta z F_\sigma''(z).$$

If we then apply the method used by Rosihan et al. (Theorem 2.1 [16]), we get the following result

$$\mathfrak{p}(z) = uv z^{1-\frac{1}{u}} \frac{d}{dz}\left[z^{\frac{1}{u}-\frac{1}{v}+1}\frac{d}{dz}\left(z^{\frac{1}{v}-1}F_\sigma(z)\right)\right]$$

where u and v be two nonnegative real constants satisfying

$$u + v = \alpha - \beta \text{ and } uv = \beta.$$

Thus,

$$z^{\frac{1}{u}-\frac{1}{v}+1}\frac{d}{dz}\left(z^{\frac{1}{v}-1}F_\sigma(z)\right) = \frac{1}{uv}\int_0^z \omega^{\frac{1}{u}-1}\mathfrak{p}(\omega)d\omega.$$

$z s^u$ is written instead of ω and, after a few operations,

$$z^{\frac{1}{u}-\frac{1}{v}+1}\frac{d}{dz}\left(z^{\frac{1}{v}-1}F_\sigma(z)\right)=\frac{1}{v}\int_0^1 z^{\frac{1}{u}}\mathfrak{p}(zs^u)ds$$

and

$$\frac{d}{dz}\left(z^{\frac{1}{v}-1}F_\sigma(z)\right)=\frac{1}{v}\int_0^1 z^{\frac{1}{v}-1}\mathfrak{p}(zs^u)ds$$

and

$$\frac{F_\sigma(z)}{z}=\int_0^1\int_0^1 \mathfrak{p}(zt^v s^u)dsdt \tag{7}$$

is obtained. We say that an analytic function f is subordinate to an analytic function g, and write $f \prec g$, if there exists a complex valued function ϖ which maps $\mathbb{D}$ into oneself with $\varpi(0)=0$, such that $f(z)=g(\varpi(z))$ $(z\in\mathbb{D})$. Where $\prec$ shows subordination symbol, on the other hand, since $\mathrm{Re}\{\mathfrak{p}(z)\}>0$, then $\mathfrak{p}(z)\prec\frac{1+z}{1-z}$ [2]. Let $\phi(z)=1+\sum_{n=1}^{\infty}\frac{z^n}{(1+un)(1+vn)}$ and $p(z)=\frac{1+z}{1-z}=1+\sum_{n=1}^{\infty}2z^n$. Using Equality (7), we obtain

$$\begin{aligned}\frac{F_\sigma(z)}{z} &\prec (\phi * p)(z)\\ &= \left(1+\sum_{n=1}^{\infty}\frac{z^n}{(1+un)(1+vn)}\right)*\left(1+\sum_{n=1}^{\infty}2z^n\right)\\ &= \left(\int_0^1\int_0^1\frac{dtds}{1-zt^v s^u}\right)*\left(1+\sum_{n=1}^{\infty}2z^n\right)\\ &= 1+\sum_{n=1}^{\infty}\frac{2}{1+(-1+n)[\alpha+\beta(-2+n)]}z^n.\end{aligned}$$

Since

$$\begin{aligned}\left|\frac{F_\sigma(z)}{z}\right| &= \left|\frac{h(z)+\sigma g(z)}{z}\right|\\ &\le 1+2\sum_{n=1}^{\infty}\frac{|z|^n}{1+n[\alpha+\beta(-1+n)]}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{F_\sigma(z)}{z}\right| &= \left|\frac{h(z)+\sigma g(z)}{z}\right|\\ &\ge 1+2\sum_{n=1}^{\infty}\frac{(-1)^n|z|^n}{1+n[\alpha+\beta(-1+n)]}\end{aligned}$$

especially, we obtain

$$\left|\frac{h(z)}{z}\right|+\left|\frac{g(z)}{z}\right|\le 1+2\sum_{n=1}^{\infty}\frac{|z|^n}{1+n[\alpha+\beta(-1+n)]}$$

and

$$\left|\frac{h(z)}{z}\right|-\left|\frac{g(z)}{z}\right|\ge 1+2\sum_{n=1}^{\infty}\frac{(-1)^n|z|^n}{1+n[\alpha+\beta(-1+n)]}.$$

Then,

$$\begin{aligned}|\mathfrak{f}(z)| &\leq |h(z)|+|g(z)| \\ &\leq |z|+2\sum_{n=1}^{\infty}\frac{|z|^{n+1}}{1+n[\alpha+\beta(-1+n)]} \\ &= |z|+2\sum_{n=2}^{\infty}\frac{|z|^{n}}{1+(-1+n)[\alpha+\beta(-2+n)]}\end{aligned}$$

and

$$\begin{aligned}|\mathfrak{f}(z)| &\geq |h(z)|-|g(z)| \\ &\geq |z|+2\sum_{n=1}^{\infty}\frac{(-1)^{n}|z|^{1+n}}{1+n[\alpha+\beta(-1+n)]} \\ &= |z|+2\sum_{n=2}^{\infty}\frac{(-1)^{-1+n}|z|^{n}}{1+(-1+n)[\alpha+\beta(-2+n)]}.\end{aligned}$$

Since σ $(|\sigma|=1)$ is arbitrary, we have

$$|z|+2\sum_{n=2}^{\infty}\frac{(-1)^{-1+n}|z|^{n}}{1+(-1+n)[\alpha+\beta(-2+n)]} \leq |\mathfrak{f}(z)| \leq |z|+2\sum_{n=2}^{\infty}\frac{|z|^{n}}{1+(-1+n)[\alpha+\beta(-2+n)]} \quad, z \in \mathbb{D}.$$

□

3. Geometric Properties of Harmonic Mappings in $\mathbb{WH}^0(\alpha,\beta)$

In this section, we will examine the geometric properties of the functions in the $\mathbb{WH}^0(\alpha,\beta)$ class. We shall provide the radius of univalency, starlikeness and convexity for functions belonging to the class $\mathbb{WH}^0(\alpha,\beta)$. Let us consider and remember the three lemmas that will guide us in the theorems given in this section and shed light on the proofs.

Lemma 1 (Corollary 2.2 [21])**.** *Let $\mathfrak{f}=h+\overline{g}$ be a sense-preserving harmonic mapping in the open unit disk. If for all σ $(|\sigma|=1)$, the analytic functions $F_\sigma=h+\sigma g$ are univalent in $\mathbb{D}$, then $\mathfrak{f}$ is univalent in $\mathbb{D}$.*

Lemma 2 ([16])**.** *Let $F\in\mathbb{W}_0(\alpha,\beta)$, with $\alpha\geq\beta\geq 0$. Then, F is univalent in $|z|<r_0$, where r_0 is the smallest positive root of the equation $\mathfrak{v}(r)=0$ with*

$$\mathfrak{v}(r)=\begin{cases}\dfrac{1}{uv}\displaystyle\int_0^1\int_0^1 t^{\frac{1}{v}-1}s^{\frac{1}{u}-1}\left(\frac{2}{(1+str)^2}-1\right)dsdt, & \alpha>0,\ \beta>0, \\ \displaystyle\int_0^1\left(\frac{2}{(1+t^{\alpha}r)^2}-1\right)dt, & \alpha\geq 0,\ \beta=0\end{cases}. \tag{8}$$

This result is sharp.

Lemma 3 ([22,23])**.** *Let $\mathfrak{f}=h+\overline{g}$ be a harmonic mapping, where h and g have the form (1). If $\sum_{n=2}^{\infty}n(|a_n|+|b_n|)\leq 1$, then $\mathfrak{f}$ is starlike in $\mathbb{D}$; if $\sum_{n=2}^{\infty}n^2(|a_n|+|b_n|)\leq 1$, then $\mathfrak{f}$ is convex in $\mathbb{D}$.*

Theorem 6. *Let $\mathfrak{f}=h+\overline{g}\in\mathbb{WH}^0(\alpha,\beta)$ be a sense-preserving harmonic mapping in $\mathbb{D}$, then $\mathfrak{f}(z)$ is univalent in $|z|<r_0$, where $\mathfrak{v}(r)$, as given in (8), r_0 is the smallest positive root of the equation $\mathfrak{v}(r)=0$. This result is sharp.*

Proof. Let $\mathfrak{f}=h+\overline{g}\in\mathbb{WH}^0(\alpha,\beta)$. Then, using Theorem 1, $F_\sigma=h+\sigma g\in\mathbb{W}_0(\alpha,\beta)$ for each σ $(|\sigma|=1)$. Referring to Lemma 2, we derive that functions $F_\sigma=h+\sigma g\in\mathbb{W}_0(\alpha,\beta)$ are univalent in $|z|<r_0$ for all σ $(|\sigma|=1)$. Because of Lemma 1, we see that functions in $\mathbb{WH}^0(\alpha,\beta)$ are univalent in $|z|<r_0$. □

Theorem 7. *Let $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}} \in \mathbb{WH}^0(\alpha, \beta)$ be a sense-preserving harmonic mapping in $\mathbb{D}$ with $3\beta \leq \alpha \leq 1 + 2\beta$ and $0 \leq \beta \leq 1$, where $\mathfrak{h}$ and $\mathfrak{g}$ are the type in (1). Then, f is starlike in $|z| < r_1$, where r_1 is the smallest positive root in $(0, 1)$ of the equation*

$$r \int_0^1 \frac{v(u-1)\eta^{\frac{1}{u}} - u(v-1)\eta^{\frac{1}{v}}}{1 - r\eta} d\eta = \frac{u - v}{2},$$

where $u + v = \alpha - \beta$, $uv = \beta$ and $(u - v)^2 = (u + v)^2 - 4uv$.

Proof. Let $0 < r < 1$ and

$$\mathfrak{f}_r(z) = r^{-1}\mathfrak{f}(rz) = r^{-1}\mathfrak{h}(rz) + r^{-1}\overline{\mathfrak{g}(rz)}$$

so that

$$\mathfrak{f}_r(z) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}, \ z \in \mathbb{D}.$$

For convenience, we let

$$\Xi = \sum_{n=2}^{\infty} n(|a_n| + |b_n|) r^{n-1}.$$

According to Lemma 3, it suffices to show that $\Xi \leq 1$ for $r < r_1$. Using Theorem 3(i), Ξ gives that

$$\begin{aligned}
\Xi &\leq 2 \sum_{n=2}^{\infty} \frac{n}{1 + (-1+n)[\alpha + \beta(-2+n)]} r^{n-1} \\
&= 2 \sum_{n=2}^{\infty} \frac{n}{\left(-1 + n + \frac{1}{u}\right)\left(-1 + n + \frac{1}{v}\right)} r^{-1+n} \\
&= \frac{2v(u-1)}{u-v} \sum_{n=2}^{\infty} \frac{r^{-1+n}}{-1 + n + \frac{1}{u}} - \frac{2u(v-1)}{u-v} \sum_{n=2}^{\infty} \frac{r^{-1+n}}{-1 + n + \frac{1}{v}} \\
&= \frac{2v(u-1)}{u-v} r^{-\frac{1}{u}} \sum_{n=2}^{\infty} \int_0^r \xi^{-2+n+\frac{1}{u}} d\xi - \frac{2u(v-1)}{u-v} r^{-\frac{1}{v}} \sum_{n=2}^{\infty} \int_0^r \xi^{-2+n+\frac{1}{v}} d\xi \\
&= \frac{2v(u-1)}{u-v} r^{-\frac{1}{u}} \int_0^r \frac{\xi^{\frac{1}{u}}}{1-\xi} d\xi - \frac{2u(v-1)}{u-v} r^{-\frac{1}{v}} \int_0^r \frac{\xi^{\frac{1}{v}}}{1-\xi} d\xi \\
&= \frac{2v(u-1)}{u-v} r \int_0^1 \frac{\eta^{\frac{1}{u}}}{1-r\eta} d\eta - \frac{2u(v-1)}{u-v} r \int_0^1 \frac{\eta^{\frac{1}{v}}}{1-r\eta} d\eta \\
&= \frac{2r}{u-v} \int_0^1 \frac{v(u-1)\eta^{\frac{1}{u}} - u(v-1)\eta^{\frac{1}{v}}}{1-r\eta} d\eta.
\end{aligned}$$

It is easily seen from the last two inequalities that $\Xi \leq 1$ if $r < r_1$. □

Theorem 8. *Let $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}} \in \mathbb{WH}^0(\alpha, \beta)$ with $3\beta \leq \alpha \leq 1 + 2\beta$ and $0 \leq \beta \leq 1$, where $\mathfrak{h}$ and $\mathfrak{g}$ are the type in (1). Then, f is convex in $|z| < r_2$, where r_2 is the smallest positive root in $(0, 1)$ of the equation*

$$\frac{r}{1-r} + \frac{r}{uv(u-v)} \int_0^1 \frac{(uv-v)^2\eta^{\frac{1}{u}} - (uv-u)^2\eta^{\frac{1}{v}}}{1-r\eta} d\eta = \frac{1}{2}.$$

where $u + v = \alpha - \beta$, $uv = \beta$ and $(u - v)^2 = (u + v)^2 - 4uv$.

Proof. Let $0 < r < 1$ and

$$\mathfrak{f}_r(z) = r^{-1}\mathfrak{f}(rz) = r^{-1}h(rz) + r^{-1}\overline{g(rz)}$$

so that

$$\mathfrak{f}_r(z) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}.\ z \in \mathbb{D}.$$

For convenience, we let

$$\Upsilon = \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|)r^{n-1}.$$

According to Lemma 3, it suffices to show that $\Upsilon \leq 1$ for $r < r_2$. Using Theorem 3(i), Υ gives that

$$\begin{aligned}
\Upsilon &\leq 2\sum_{n=2}^{\infty} \frac{n^2}{1+(-1+n)[\alpha+\beta(-2+n)]} r^{-1+n} \\
&= \frac{2v(u-1)}{u-v} \sum_{n=2}^{\infty} \frac{n r^{-1+n}}{-1+n+\frac{1}{u}} - \frac{2u(v-1)}{u-v} \sum_{n=2}^{\infty} \frac{n r^{-1+n}}{-1+n+\frac{1}{v}} \\
&= \left(\frac{2v(u-1)}{u-v} \sum_{n=2}^{\infty} \frac{r^n}{-1+n+\frac{1}{u}} - \frac{2u(v-1)}{u-v} \sum_{n=2}^{\infty} \frac{r^n}{-1+n+\frac{1}{v}} \right)' \\
&= \left(\frac{2v(u-1)}{u-v} r^{1-\frac{1}{u}} \sum_{n=2}^{\infty} \int_0^r \xi^{-2+n+\frac{1}{u}} d\xi - \frac{2u(v-1)}{u-v} r^{1-\frac{1}{v}} \sum_{n=2}^{\infty} \int_0^r \xi^{-2+n+\frac{1}{v}} d\xi \right)' \\
&= \left(\frac{2v(u-1)}{u-v} r^{1-\frac{1}{u}} \int_0^r \frac{\xi^{\frac{1}{u}}}{1-\xi} d\xi - \frac{2u(v-1)}{u-v} r^{1-\frac{1}{v}} \int_0^r \frac{\xi^{\frac{1}{v}}}{1-\xi} d\xi \right)' \\
&= \frac{2r}{1-r} + \frac{2v(u-1)^2}{u(u-v)} r^{-\frac{1}{u}} \int_0^r \frac{\xi^{\frac{1}{u}}}{1-\xi} d\xi - \frac{2u(v-1)^2}{v(u-v)} r^{-\frac{1}{v}} \int_0^r \frac{\xi^{\frac{1}{v}}}{1-\xi} d\xi \\
&= \frac{2r}{1-r} + \frac{2v(u-1)^2}{u(u-v)} r \int_0^1 \frac{\eta^{\frac{1}{u}}}{1-r\eta} d\eta - \frac{2u(v-1)^2}{v(u-v)} r \int_0^1 \frac{\eta^{\frac{1}{v}}}{1-r\eta} d\eta \\
&= \frac{2r}{1-r} + \frac{2r}{uv(u-v)} \int_0^1 \frac{(uv-v)^2\eta^{\frac{1}{u}} - (uv-u)^2\eta^{\frac{1}{v}}}{1-r\eta} d\eta.
\end{aligned}$$

It is easily seen from the last two inequalities that $\Upsilon \leq 1$ if $r < r_2$. □

4. Convex Combinations and Convolutions

In this section, we investigate that the class $\mathbb{WH}^0(\alpha, \beta)$ is convolutions and closed under convex combinations of its members.

Theorem 9. *The class $\mathbb{WH}^0(\alpha, \beta)$ is closed under convex combinations.*

Proof. Suppose $\mathfrak{f}_i = h_i + \overline{g_i} \in \mathbb{WH}^0(\alpha, \beta)$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} c_i = 1$ $(0 \leq c_i \leq 1)$. The convex combination of functions $\mathfrak{f}_i$ $(i = 1, 2, \ldots, n)$ may be written as

$$\mathfrak{f}(z) = \sum_{i=1}^{n} c_i \mathfrak{f}_i(z) = h(z) + \overline{g(z)},$$

where

$$h(z) = \sum_{i=1}^{n} c_i h_i(z) \text{ and } g(z) = \sum_{i=1}^{n} c_i g_i(z).$$

Then, both h and g are analytic in the open unit disk $\mathbb{D}$ with $h(0) = g(0) = h'(0) - 1 = g'(0) = 0$ and

$$\begin{aligned} & \operatorname{Re}\left[(2\beta + 1 - \alpha)\frac{h(z)}{z} + (\alpha - 2\beta)h'(z) + \beta z h''(z)\right] \\ = & \operatorname{Re}\left[\sum_{i=1}^{n} c_i \left((2\beta + 1 - \alpha)\frac{h_i(z)}{z} + (\alpha - 2\beta)h_i'(z) + \beta z h_i''(z)\right)\right] \\ > & \sum_{i=1}^{n} c_i \left|(2\beta + 1 - \alpha)\frac{g_i(z)}{z} + (\alpha - 2\beta)g_i'(z) + \beta z g_i''(z)\right| \\ = & \left|(2\beta + 1 - \alpha)\frac{g(z)}{z} + (\alpha - 2\beta)g'(z) + \beta z g''(z)\right| \end{aligned}$$

showing that $f \in \mathbb{WH}^0(\alpha, \beta)$. □

A sequence $\{\lambda_n\}_{n=0}^{\infty}$ of non-negative real numbers is said to be a convex null sequence, if $\lambda_n \to 0$ as $n \to \infty$, and $\lambda_0 - \lambda_1 \geq \lambda_1 - \lambda_2 \geq \lambda_2 - \lambda_3 \geq \cdots \geq \lambda_{n-1} - \lambda_n \geq \cdots \geq 0$. The following lemmas are needed to complete the proof.

Lemma 4 ([24]). *If $\{\lambda_n\}_{n=0}^{\infty}$ is a convex null sequence, then function*

$$q(z) = \frac{\lambda_0}{2} + \sum_{n=1}^{\infty} \lambda_n z^n$$

is analytic and $Re[q(z)] > 0$ in $\mathbb{D}$.

Lemma 5 ([25]). *Let the function p be analytic in the open unit disk $\mathbb{D}$ with $p(0) = 1$ and $Re[p(z)] > 1/2$ in the open unit disk $\mathbb{D}$. Then, for any analytic function F in $\mathbb{D}$, the function $p * F$ takes values in the convex hull of the image of $\mathbb{D}$ under F.*

Lemma 6. *Let $F \in \mathbb{W}_0(\alpha, \beta)$, then $Re\left[\frac{F(z)}{z}\right] > \frac{1}{2}$.*

Proof. Suppose $F \in \mathbb{W}_0(\alpha, \beta)$ is given by $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, then

$$\operatorname{Re}\left[1 + \sum_{n=2}^{\infty} (1 + (-1 + n)[\alpha + \beta(-2 + n)])A_n z^{-1+n}\right] > 0 \quad (z \in \mathbb{D}).$$

This expression is equivalent to $\operatorname{Re}[p(z)] > \frac{1}{2}$ in $\mathbb{D}$, where

$$p(z) = 1 + \frac{1}{2} \sum_{n=2}^{\infty} (1 + (-1 + n)[\alpha + \beta(-2 + n)])A_n z^{-1+n}.$$

Now, consider a sequence $\{\lambda_n\}_{n=0}^{\infty}$ defined by

$$\lambda_0 = 2 \text{ and } \lambda_{-1+n} = \frac{2}{1 + (-1 + n)[\alpha + \beta(-2 + n)]} \quad \text{for } n \geq 2.$$

It can be easily seen that the sequence $\{\lambda_n\}_{n=0}^{\infty}$ is a convex null sequence. Using Lemma 4, this implies that the function

$$\mathfrak{q}(z) = 1 + \sum_{n=2}^{\infty} \frac{2}{1 + (-1+n)[\alpha + \beta(-2+n)]} z^{-1+n}$$

is analytic and $\mathrm{Re}[\mathfrak{q}(z)] > 0$ in $\mathbb{D}$. Writing

$$\frac{F(z)}{z} = \mathfrak{p}(z) * \left(1 + \sum_{n=2}^{\infty} \frac{2}{1 + (-1+n)[\alpha + \beta(-2+n)]} z^{-1+n}\right),$$

and making use of Lemma 5 gives that $\mathrm{Re}\left\{\frac{F(z)}{z}\right\} > \frac{1}{2}$ for $z \in \mathbb{D}$. □

Lemma 7. *Let $F_i \in \mathbb{W}_0(\alpha, \beta)$ for $i = 1, 2$. Then, $F_1 * F_2 \in \mathbb{W}_0(\alpha, \beta)$.*

Proof. Suppose $F_1(z) = z + \sum_{n=2}^{\infty} A_n z^n$ and $F_2(z) = z + \sum_{n=2}^{\infty} B_n z^n$. Then, the convolution of $F_1(z)$ and $F_2(z)$ is defined by

$$F(z) = (F_1 * F_2)(z) = z + \sum_{n=2}^{\infty} A_n B_n z^n.$$

Since $F'(z) = F_1'(z) * \frac{F_2(z)}{z}$, we have

$$\begin{aligned} &(2\beta + 1 - \alpha)\tfrac{F(z)}{z} + (\alpha - 2\beta)F'(z) + \beta z F''(z) \\ &= \left[(2\beta + 1 - \alpha)\tfrac{F_1(z)}{z} + (\alpha - 2\beta)F_1'(z) + \beta z F_1''(z)\right] * \frac{F_2(z)}{z}. \end{aligned} \tag{9}$$

Since $F_1 \in \mathbb{W}_0(\alpha, \beta)$,

$$\mathrm{Re}\left[(2\beta + 1 - \alpha)\frac{F_1(z)}{z} + (\alpha - 2\beta)F_1'(z) + \beta z F_1''(z)\right] > 0 \; (z \in \mathbb{D})$$

and, using Lemma 6, $\mathrm{Re}\left[\frac{F_2(z)}{z}\right] > \frac{1}{2}$ in $\mathbb{D}$. Now, applying Lemma 5 to (9) yields $\mathrm{Re}\left[(2\beta + 1 - \alpha)\frac{F(z)}{z} + (\alpha - 2\beta)F'(z) + \beta z F''(z)\right] > 0$ in $\mathbb{D}$. Thus, $F = F_1 * F_2 \in \mathbb{W}_0(\alpha, \beta)$. □

Now, using Lemma 7, we give the following theorem.

Theorem 10. *Let $f_i \in \mathbb{WH}^0(\alpha, \beta)$ for $i = 1, 2$. Then, $f_1 * f_2 \in \mathbb{WH}^0(\alpha, \beta)$.*

Proof. Suppose $f_i = h_i + \overline{g_i} \in \mathbb{WH}^0(\alpha, \beta)$ $(i = 1, 2)$. Then, the convolution of f_1 and f_2 is defined as $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$. In order to prove that $f_1 * f_2 \in \mathbb{WH}^0(\alpha, \beta)$, we need to prove that $F_\sigma = h_1 * h_2 + \sigma(g_1 * g_2) \in \mathbb{W}_0(\alpha, \beta)$ for each σ $(|\sigma| = 1)$. By Lemma 7, the class $\mathbb{W}_0(\alpha, \beta)$ is closed under convolutions for each σ $(|\sigma| = 1)$, $h_i + \sigma g_i \in \mathbb{W}_0(\alpha, \beta)$ for $i = 1, 2$. Then, both F_1 and F_2 given by

$$F_1 = (h_1 - g_1) * (h_2 - \sigma g_2) \text{ and } F_2 = (h_1 + g_1) * (h_2 + \sigma g_2),$$

belong to $\mathbb{W}_0(\alpha, \beta)$. Since $\mathbb{W}_0(\alpha, \beta)$ is closed under convex combinations, then the function

$$F_\sigma = \frac{1}{2}(F_1 + F_2) = h_1 * h_2 + \sigma(g_1 * g_2)$$

belongs to $\mathbb{W}_0(\alpha, \beta)$. Hence, $\mathbb{WH}^0(\alpha, \beta)$ is closed under convolution. □

Let us remember the following Hadamard product explained by Goodloe [26]:

$$\mathfrak{f} \widetilde{*} \phi = \mathfrak{h} * \phi + \overline{\mathfrak{g} * \phi},$$

where $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}}$ is harmonic function and ϕ is an analytic function in $\mathbb{D}$.

Theorem 11. *Let $\mathfrak{f} \in \mathbb{WH}^0(\alpha, \beta)$ and $\phi \in \mathbb{A}$ be such that $\mathrm{Re}\left[\frac{\phi(z)}{z}\right] > \frac{1}{2}$ for $z \in \mathbb{D}$, then $\mathfrak{f} \widetilde{*} \phi \in \mathbb{WH}^0(\alpha, \beta)$.*

Proof. Suppose that $\mathfrak{f} = \mathfrak{h} + \overline{\mathfrak{g}} \in \mathbb{WH}^0(\alpha, \beta)$, then $F_\sigma = \mathfrak{h} + \sigma\mathfrak{g} \in \mathbb{W}_0(\alpha, \beta)$ for each σ $(|\sigma| = 1)$. By Theorem 1, to show that $\mathfrak{f} \widetilde{*} \phi \in \mathbb{WH}^0(\alpha, \beta)$, we need to show that $\mathcal{T} = \mathfrak{h} * \phi + \sigma(\mathfrak{g} * \phi) \in \mathbb{W}_0(\alpha, \beta)$ for each σ $(|\sigma| = 1)$. Write $\mathcal{T}$ as $\mathcal{T} = F_\sigma * \phi$, and

$$\begin{aligned} &(2\beta + 1 - \alpha)\frac{\mathcal{T}(z)}{z} + (\alpha - 2\beta)\mathcal{T}'(z) + \beta z\mathcal{T}''(z) \\ = &\left((2\beta + 1 - \alpha)\frac{F_\sigma(z)}{z} + (\alpha - 2\beta)F_\sigma'(z) + \beta z F_\sigma''(z)\right) * \frac{\phi(z)}{z}. \end{aligned}$$

Since $\mathrm{Re}\left[\frac{\phi(z)}{z}\right] > \frac{1}{2}$ and $\mathrm{Re}\left[(2\beta + 1 - \alpha)\frac{F_\sigma(z)}{z} + (\alpha - 2\beta)F_\sigma'(z) + \beta z F_\sigma''(z)\right] > 0$ in $\mathbb{D}$, Lemma 5 proves that $\mathcal{T} \in \mathbb{W}_0(\alpha, \beta)$. □

Corollary 1. *Let $\mathfrak{f} \in \mathbb{WH}^0(\alpha, \beta)$ and $\phi \in \mathbb{K}$, then $\mathfrak{f} \widetilde{*} \phi \in \mathbb{WH}^0(\alpha, \beta)$.*

Proof. Suppose $\phi \in \mathbb{K}$, then $\mathrm{Re}\left[\frac{\phi(z)}{z}\right] > \frac{1}{2}$ for $z \in \mathbb{D}$. As a corollary of Theorem 11, $\mathfrak{f} \widetilde{*} \phi \in \mathbb{WH}^0(\alpha, \beta)$. □

5. Discussion

In this research, we examine some specific properties for harmonic functions defined by a second-order differential inequality. First, we gave the necessary definitions and preliminary information. Then, we define and prove the coefficient relations and growth theorems for the $\mathbb{WH}^0(\alpha, \beta)$ class. Then, we examined the geometric properties of the harmonic mappings belonging to the $\mathbb{WH}^0(\alpha, \beta)$ class. Finally, we proved the theorems about convex combinations and convolutions. Today, it is known that harmonic functions have a very wide field of study. Moreover, it is known that application areas are used by different disciplines. With this study, we aim to shed light on studies in other disciplines. We think that the results of this study, which will be used by many researchers in the future, will connect with different disciplines. In addition to all these, this study will act as a bridge between the articles written in the past and the articles to be written in the future.

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