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Article

On Miller–Ross-Type Poisson Distribution Series

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Abstract: The objective of the current paper is to find the necessary and sufficient conditions for Miller–Ross-type Poisson distribution series to be in the classes $\mathbb{S}^*_{\mathcal{T}}(\gamma,\beta)$ and $\mathbb{K}_{\mathcal{T}}(\gamma,\beta)$ of analytic functions with negative coefficients. Furthermore, we investigate several inclusion properties of the class $\mathcal{Y}^{\sigma}(V,W)$ associated of the operator $\mathbb{I}^{\varepsilon}_{\alpha,c}$ defined by this distribution. We also take into consideration an integral operator connected to series of Miller–Ross-type Poisson distributions. Special cases of the main results are also considered.

Keywords: analytic functions; starlike functions; convex functions; Hadamard product; Miller–Ross-type Poisson distribution series

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1. Definitions and Preliminaries

Special functions are very important in the study of geometric function theory, applied mathematics, physics, statistics and many other subjects. In [1], Kenneth S. Miller and Bertram Ross introduced the special function, which is called the Miller–Ross function defined by

$$\mathbb{E}_{\alpha,c}(\xi)=\xi^{\nu}\sum_{s=0}^{\infty}\frac{(c\xi)^s}{\Gamma(s+\alpha+1)},\quad (\alpha,c,\xi\in\mathbb{C}\ ). \tag{1}$$

Observe that the function $\mathbb{E}_{\alpha,c}$ contains many well-known functions as special cases, for example, $\mathbb{E}_{0,1}(\xi)=\xi e^{\xi}$, $\mathbb{E}_{1,1}(\xi)=e^{\xi}-1$, $\mathbb{E}_{2,1}(\xi)=\frac{2}{\xi}e^{\xi}-\frac{2}{\xi}-2$, $\mathbb{E}_{3,1}(\xi)=\frac{3(e^{\xi}-\xi^2-2\xi-2)}{\xi^2}$, $\mathbb{E}_{\frac{1}{2},\frac{1}{2}}(\xi)=e^{\frac{\xi}{2}}\sqrt{\frac{\pi}{2}}\sqrt{\xi}erf\sqrt{\frac{\xi}{2}}$, where erf is the error function defined by $erf(\xi)=\frac{2}{\sqrt{\pi}}\int_0^{\xi}e^{-t^2}dt$.

Let $E_{\varsigma,\mu}(\xi)$ be the two parameters Mittag–Leffler function [2] defined by

$$E_{\varsigma,\mu}(\xi)=\sum_{s=0}^{\infty}\frac{\xi^s}{\Gamma(\varsigma s+\mu)},\quad (\xi,\varsigma,\mu\in\mathbb{C},\mathrm{Re}(\varsigma)>0,\ \mathrm{Re}(\mu)>0\ ). \tag{2}$$

Several properties of Mittag–Leffler function and generalized Mittag–Leffler function can be found e.g., in ([3–11]).

If $\mu=1$, from (2) we get the Mittag–Leffler function of one parameter [12]

$$E_{\varsigma}(\xi)=\sum_{s=0}^{\infty}\frac{\xi^s}{\Gamma(\varsigma s+1)},\quad (\xi,\varsigma\in\mathbb{C},\mathrm{Re}(\varsigma)>0).$$

From (1) and (2), the Miller–Ross function can be expressed as

$$\mathbb{E}_{\alpha,c}(\xi)=\xi^{\alpha}E_{1,1+\alpha}(c\xi).$$

Let $\mathfrak{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ and $\mathcal{A}$ denote for the class of analytic functions given by the expansion

$$f(\xi) = \xi + \sum_{s=2}^{\infty} a_s \xi^s, \quad \xi \in \mathfrak{U}. \tag{3}$$

Further, let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(\xi) = \xi - \sum_{s=2}^{\infty} |a_s| \xi^s, \quad \xi \in \mathfrak{U}. \tag{4}$$

Given two functions $f, g \in \mathcal{A}$, where $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, their *Hadamard product* or *convolution* $f(z) * g(z)$ is defined by (see, [13,14])

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad (z \in \mathfrak{U}). \tag{5}$$

Let

$$\mathbb{S}^*(\gamma) = \left\{ f \in \mathcal{A} : \Re\left(\frac{\xi f'(\xi)}{f(\xi)} \right) > \gamma, \ \ \xi \in \mathfrak{U}, \right\}$$

and

$$\mathbb{K}(\gamma) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)} \right) > \gamma, \ \ \xi \in \mathfrak{U}, \right\}$$

denote the subclasses of $\mathcal{A}$ which are starlike and convex of order $\gamma (0 \leq \gamma < 1)$, respectively. Let $\mathbb{S}^*_{\mathcal{T}}(\gamma)$ and $\mathbb{K}_{\mathcal{T}}(\gamma)$ be the subfamilies of $\mathbb{S}^*(\gamma)$ and $\mathbb{K}(\gamma)$, respectively, whose functions are of the form (4).

The generalization of the classes $\mathbb{S}^*(\gamma)$ and $\mathbb{K}(\gamma)$ of functions $f \in \mathcal{A}$ given by the classes $\mathbb{S}^*(\gamma, \beta)$ and $\mathbb{K}(\gamma, \beta)$, which are satisfies the conditions:

$$\mathbb{S}^*(\gamma, \beta) = \left\{ f \in \mathcal{A} : \Re\left(\frac{\xi f'(\xi) + \beta \xi^2 f''(\xi)}{f(\xi)} \right) > \gamma, \xi \in \mathfrak{U}, 0 \leq \gamma < 1, 0 \leq \beta < 1 \right\}$$

and

$$\mathbb{K}(\gamma, \beta) = \left\{ f \in \mathcal{A} : \Re\left(\frac{\left(\xi f'(\xi) + \beta \xi^2 f''(\xi)\right)'}{f'(\xi)} \right) > \gamma, \xi \in \mathfrak{U}, 0 \leq \gamma < 1, 0 \leq \beta < 1 \right\},$$

respectively. Let

$$\mathbb{S}^*_{\mathcal{T}}(\gamma, \beta) = \mathbb{S}^*(\gamma, \beta) \cap \mathcal{T} \text{ and } \mathbb{K}_{\mathcal{T}}(\gamma, \beta) = \mathbb{K}(\gamma, \beta) \cap \mathcal{T}.$$

Clearly, we have $\mathbb{S}^*_{\mathcal{T}}(\gamma, 0) = \mathbb{S}^*_{\mathcal{T}}(\gamma)$ and $\mathbb{K}_{\mathcal{T}}(\gamma, 0) = \mathbb{K}_{\mathcal{T}}(\gamma)$.

For $\sigma \in \mathbb{C} \backslash \{0\}$ and $-1 \leq W < V \leq 1$, Dixit and Pal [15] introduced the class $\mathcal{Y}^{\sigma}(V, W)$ of all analytic functions in $\mathfrak{U}$, defined as:

$$\mathcal{Y}^{\sigma}(V, W) = \left\{ f : f \in \mathcal{A} \text{ and} \left| \frac{f'(\xi) - 1}{(V - W)\sigma - W[f'(\xi) - 1]} \right| < 1, \ \xi \in \mathfrak{U} \right\}.$$

In the recent years, there has been a tremendous lot of interest in the distributions of the random variables. In statistics and probability theory, their probability density functions in the real variable x and the complex variable ξ have been crucial. Distributions have so been the subject of much study. Numerous distribution types, including the Binomial distribution, negative binomial distribution, Poisson distribution and geometric distribution, emerged from real-world circumstances.

If the probability density function is given by:

$$P(X=k)=\frac{e^{-\varepsilon}}{k!}\varepsilon^k,\ k=0,1,2,\dots, \tag{6}$$

and $\varepsilon>0$ is the parameter of the distribution, then a random variable X follows a Poisson distribution.

Recently, with coefficients are Miller–Ross-type Poisson distribution Şeker et al. [16] (see also, [17]) defined the following power series

$$\mathbb{F}^{\varepsilon}_{\alpha,c}(\xi):=\xi+\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}\xi^s,\ \xi\in\mathfrak{U}, \tag{7}$$

where $\alpha>-1$, $c>0$.

We note that if we put $\alpha=0$ and $c=1$ in (7), we get the Poisson distribution series introduced by Porwal [18].

Furthermore, Şeker et al. [16] defined the series

$$\mathbb{k}^{\varepsilon}_{\alpha,c}(\xi):=2\xi-\mathbb{F}^{\varepsilon}_{\alpha,c}(\xi)=\xi-\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}\xi^s,\ \xi\in\mathfrak{U}. \tag{8}$$

Now by the convolution, we construct the linear operator $\mathbb{I}^{\varepsilon}_{\alpha,c}:\mathcal{A}\to\mathcal{A}$ to be

$$\mathbb{I}^{\varepsilon}_{\alpha,c}f(\xi):=\mathbb{F}^{\varepsilon}_{\alpha,c}(\xi)*f(\xi)=\xi+\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}a_s\xi^s,\quad(\xi\in\mathfrak{U},\ \alpha>-1,c>0). \tag{9}$$

In recent years, several researchers used this distribution series [19,20] and other distribution series such as Poisson distribution series [21–26], Pascal distribution series [27–30], hypergeometric distribution series [31–36], and the Mittag–Leffler-type Poisson distribution [37] to obtain some necessary and sufficient conditions for these distributions to belong to certain classes of analytic functions defined in $\mathfrak{U}$. In the present paper we obtain some necessary and sufficient conditions for the Miller–Ross-type Poisson distribution series $\mathbb{k}^{\varepsilon}_{\alpha,c}$ to be in our classes $\mathbb{S}^*_{\mathcal{T}}(\gamma,\beta)$ and $\mathbb{K}_{\mathcal{T}}(\gamma,\beta)$. Furthermore, we associate these subclasses with the class $\mathcal{Y}^{\sigma}(V,W)$, and finally, we give necessary and sufficient conditions for the function f such that the operator $\mathbb{G}^{\varepsilon}_{\alpha,c}f(\xi)=\int_0^{\xi}\frac{\mathbb{k}^{\varepsilon}_{\alpha,c}(t)}{t}dt$ belongs to class $\mathcal{Y}^{\sigma}(V,W)$.

2. Preliminary Lemmas

We require the following Lemmas in order to establish our main results.

Lemma 1 ([25]). *A function $f\in\mathcal{T}$ in the class $\mathbb{S}^*_{\mathcal{T}}(\gamma,\beta)$ if and only if*

$$\sum_{s=2}^{\infty}(s+\beta s(s-1)-\gamma)|a_s|\le 1-\gamma, \tag{10}$$

where $0\le\gamma<1, 0\le\beta<1$.

Lemma 2 ([25]). *A function $f\in\mathcal{T}$ in the class $\mathbb{K}_{\mathcal{T}}(\gamma,\beta)$ if and only if*

$$\sum_{s=2}^{\infty}s(s+\beta s(s-1)-\gamma)|a_s|\le 1-\gamma,$$

where $0\le\gamma<1, 0\le\beta<1$.

Lemma 3 ([15]). *If $f\in\mathcal{Y}^{\sigma}(V,W)$ is of the form (4), then*

$$|a_s|\le(V-W)\frac{|\sigma|}{s},\qquad s\in\mathbb{N}-\{1\}. \tag{11}$$

In this paper, we will assume that $0 \leq \gamma < 1, 0 \leq \beta < 1, \sigma \in \mathbb{C}\backslash\{0\}$, and $-1 \leq W < V \leq 1$ unless otherwise stated.

3. Necessary and Sufficient Conditions

The necessary and sufficient condition for $\Bbbk^{\varepsilon}_{\alpha,c}$ to be in the class $\mathbb{S}^*_{\mathcal{T}}(\gamma, \beta)$ is given by the following

Theorem 1. *Let* $\alpha > -1$ *and* $c > 0$, *then* $\Bbbk^{\varepsilon}_{\alpha,c} \in \mathbb{S}^*_{\mathcal{T}}(\gamma, \beta)$ *if and only if*

$$\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Big[\beta\varepsilon^2\mathbb{E}_{\alpha-1,c}(\varepsilon) + (2\beta(1-\alpha)+1)\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) + ((1-\alpha)(1-\beta\alpha)-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)\Big] \leq 1-\gamma. \tag{12}$$

Proof. Since $\Bbbk^{\varepsilon}_{\alpha,c}$ is defined by (8), in view of Lemma 1 it suffices to verify that

$$\sum_{s=2}^{\infty}(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)} \leq 1-\gamma. \tag{13}$$

Writing

$$s^2 = (\alpha+s-1)(\alpha+s-2) + (3-2\alpha)(\alpha+s-1) + (1-\alpha)^2$$

and

$$s = (\alpha+s-1) + (1-\alpha)$$

in (13), we have

$$\begin{aligned}
&\sum_{s=2}^{\infty}(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)}\\
&=\sum_{s=2}^{\infty}\Big(\beta s^2 + s(1-\beta)-\gamma\Big)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)}\\
&=\frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Bigg[\beta\sum_{s=2}^{\infty}(\alpha+s-1)(\alpha+s-2)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\\
&+(2\beta(1-\alpha)+1)\sum_{s=2}^{\infty}(\alpha+s-1)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\\
&+((1-\alpha)(1-\beta\alpha)-\gamma)\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\Bigg]\\
&=\frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Bigg[\beta\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha-2)} + (2\beta(1-\alpha)+1)\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha-1)}\\
&+((1-\alpha)(1-\beta\alpha)-\gamma)\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\Bigg]\\
&=\frac{c}{\mathbb{E}_{\nu,c}(\varepsilon)}\Bigg[\beta\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\nu}(c\varepsilon)^{s}}{\Gamma(s+\nu)} + (2\beta(1-\nu)+1)\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\nu}(c\varepsilon)^{s}}{\Gamma(s+\nu+1)}\\
&+((1-\alpha)(1-\beta\alpha)-\gamma)\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s}}{\Gamma(s+\alpha+2)}\Bigg]\\
&=\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Big[\beta\varepsilon^2\mathbb{E}_{\alpha-1,c}(\varepsilon) + (2\beta(1-\alpha)+1)\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) + ((1-\alpha)(1-\beta\alpha)-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)\Big],
\end{aligned}$$

which is bounded above by $1-\gamma$ if and only if

$$\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Big[\beta\varepsilon^2\mathbb{E}_{\alpha-1,c}(\varepsilon) + (2\beta(1-\alpha)+1)\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) + ((1-\alpha)(1-\beta\alpha)-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)\Big] \leq 1-\gamma.$$

□

Now, we obtain a necessary and sufficient condition for $\Bbbk^{\varepsilon}_{\alpha,c}$ to be in the class $\mathbb{K}_{\mathcal{T}}(\gamma,\beta)$.

Theorem 2. *Let $\alpha > -1$ and $c > 0$, then $\Bbbk^{\varepsilon}_{\alpha,c} \in \mathbb{K}_{\mathcal{T}}(\gamma,\beta)$ if and only if*

$$\begin{aligned}
&\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Big[\beta\varepsilon^3\mathbb{E}_{\alpha-2,c}(\varepsilon) + [\beta(6-3\alpha) + (1-\beta)]\varepsilon^2\mathbb{E}_{\alpha-1,c}(\varepsilon) \\
&+ \Big[\beta(3\alpha^2 - 9\alpha + 7) + (1-\beta)(3-2\alpha) - \gamma\Big]\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) \\
&+ \Big[\beta(1-\alpha)^3 + (1-\beta)(1-\alpha)^2 - \gamma(1-\alpha)\Big]\mathbb{E}_{\alpha+1,c}(\varepsilon)\Big] \\
&\leq 1-\gamma.
\end{aligned} \tag{14}$$

Proof. By Lemma 1 we show that

$$\sum_{s=2}^{\infty} s(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)} \leq 1-\gamma. \tag{15}$$

Writing

$$\begin{aligned}
s^3 = {} & (\alpha+s-1)(\alpha+s-2)(\alpha+s-3) + (6-3\alpha)(\alpha+s-1)(\alpha+s-2) \\
& (3\alpha^2 - 9\alpha + 7)(\alpha+s-1) + (1-\alpha)^3,
\end{aligned}$$

$$s^2 = (\alpha+s-1)(\alpha+s-2) + (3-2\alpha)(\alpha+s-1) + (1-\alpha)^2$$

and

$$s = (\alpha+s-1) + (1-\alpha)$$

in (15), we have

$$\begin{aligned}
&\sum_{s=2}^{\infty} s(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)} \\
&= \sum_{s=2}^{\infty}\Big(\beta s^3 + s^2(1-\beta) - \gamma s\Big)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)} \\
&= \frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Bigg[\beta\sum_{s=2}^{\infty}(\alpha+s-1)(\alpha+s-2)(\alpha+s-3)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)} \\
&+ [\beta(6-3\alpha) + (1-\beta)]\sum_{s=2}^{\infty}(\alpha+s-1)(\alpha+s-2)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)} \\
&+ \Big[\beta(3\alpha^2 - 9\alpha + 7) + (1-\beta)(3-2\alpha) - \gamma\Big]\sum_{s=2}^{\infty}(\alpha+s-1)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)} \\
&+ \Big[\beta(1-\alpha)^3 + (1-\beta)(1-\alpha)^2 - \gamma(1-\alpha)\Big]\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\Bigg] \\
&= \frac{1}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Bigg[\beta\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha-3)} \\
&+ [\beta(6-3\alpha) + (1-\beta)]\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha-2)} \\
&+ \Big[\beta(3\alpha^2 - 9\alpha + 7) + (1-\beta)(3-2\alpha) - \gamma\Big]\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha-1)} \\
&+ \Big[\beta(1-\alpha)^3 + (1-\beta)(1-\alpha)^2 - \gamma(1-\alpha)\Big]\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Bigg[\beta\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s}}{\Gamma(s+\alpha-1)} \\
&+ [\beta(6-3\alpha)+(1-\beta)]\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s}}{\Gamma(s+\alpha)} \\
&+ \Big[\beta(3\alpha^{2}-9\alpha+7)+(1-\beta)(3-2\alpha)-\gamma\Big]\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s}}{\Gamma(s+\alpha+1)} \\
&+\Big[\beta(1-\alpha)^{3}+(1-\beta)(1-\alpha)^{2}-\gamma(1-\alpha)\Big]\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s}}{\Gamma(s+\alpha+2)}\Bigg] \\
&= \frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Big[\beta\varepsilon^{3}\mathbb{E}_{\alpha-2,c}(\varepsilon)+[\beta(6-3\alpha)+(1-\beta)]\varepsilon^{2}\mathbb{E}_{\alpha-1,c}(\varepsilon) \\
&+ \Big[\beta(3\alpha^{2}-9\alpha+7)+(1-\beta)(3-2\alpha)-\gamma\Big]\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) \\
&+\Big[\beta(1-\alpha)^{3}+(1-\beta)(1-\alpha)^{2}-\gamma(1-\alpha)\Big]\mathbb{E}_{\alpha+1,c}(\varepsilon)\Big],
\end{aligned}
$$

but the final phrasing bounded above by $1-\gamma$ if and only if (14) holds. □

4. Inclusion Relations

The inclusion relations of the class $\mathcal{Y}^{\sigma}(V,W)$ associated of the operator $\mathbb{I}^{\varepsilon}_{\alpha,c}$ defined by (9) proved in this section.

Theorem 3. *Let $\alpha > -1$ and $c > 0$. If $f \in \mathcal{Y}^{\sigma}(V,W)$ and*

$$
\begin{aligned}
&\frac{(V-W)c|\sigma|}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Big[\beta\varepsilon^{2}\mathbb{E}_{\alpha-1,c}(\varepsilon)+(2\beta(1-\alpha)+1)\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) \\
&+((1-\alpha)(1-\beta\alpha)-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)] \\
&\leq 1-\gamma
\end{aligned} \tag{16}
$$

is satisfied then $\mathbb{I}^{\varepsilon}_{\alpha,c}f \in \mathbb{K}_{\mathcal{T}}(\gamma,\beta)$.

Proof. By Lemma 2 it is sufficient to show that

$$
\sum_{s=2}^{\infty} s(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}|a_s| \leq 1-\gamma. \tag{17}
$$

Since $f \in \mathcal{Y}^{\sigma}(V,W)$, then by Lemma 3, we have

$$
|a_s| \leq \frac{(V-W)|\sigma|}{s}.
$$

Therefore, it is enough to show that

$$
\begin{aligned}
&\sum_{s=2}^{\infty} s(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}|a_s| \\
&\leq (V-W)|\sigma|\left[\sum_{s=2}^{\infty}(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}\right] \leq 1-\gamma.
\end{aligned} \tag{18}
$$

Using the similar computations like in the proof of in Theorem 1 it follows that the inequality (18) is satisfied whenever (16) holds. □

Theorem 4. *Let $\alpha > -1$ and $c > 0$. If $f \in \mathcal{Y}^{\sigma}(V,W)$ and*

$$
\frac{(V-W)|\sigma|c}{\mathbb{E}_{\alpha,c}(\varepsilon)}[\beta\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon)+(1-\beta\alpha)\mathbb{E}_{\alpha+1,c}(\varepsilon)] \leq 1-\gamma \tag{19}
$$

is satisfied then $\mathbb{I}^{\varepsilon}_{\alpha,c} f \in \mathbb{S}^{*}_{\mathcal{T}}(\gamma,\beta)$.

Proof. By Lemma 1 it is sufficient to show that

$$\sum_{s=2}^{\infty}(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}|a_s| \leq 1-\gamma.$$

Since $f \in \mathcal{Y}^{\sigma}(V,W)$, using the inequality (11) of Lemma 3 , we have

$$\begin{aligned}
&\sum_{s=2}^{\infty}(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}|a_s| \\
&\leq (V-W)|\sigma|\left[\sum_{s=2}^{\infty}\left[(s\beta-\beta)+\left(1-\frac{\gamma}{s}\right)\right]\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}\right] \\
&\leq (V-W)|\sigma|\left[\sum_{s=2}^{\infty}[(s\beta-\beta)+1]\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}\right] \\
&= (V-W)|\sigma|\left[\sum_{s=2}^{\infty}[\beta(\alpha+s-1)+(1-\beta\alpha)]\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}\right] \\
&= \frac{(V-W)|\sigma|}{\mathbb{E}_{\alpha,c}(\varepsilon)}\left[\beta\sum_{s=2}^{\infty}(\alpha+s-1)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}+(1-\beta\alpha)\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\right] \\
&= \frac{(V-W)|\sigma|}{\mathbb{E}_{\alpha,c}(\varepsilon)}\left[\beta\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha-1)}+(1-\beta\alpha)\sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)}\right] \\
&= \frac{(V-W)|\sigma|c}{\mathbb{E}_{\alpha,c}(\varepsilon)}\left[\beta\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s}}{\Gamma(s+\alpha+1)}+(1-\beta\alpha)\varepsilon\sum_{s=0}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s}}{\Gamma(s+\alpha+2)}\right] \\
&= \frac{(V-W)|\sigma|c}{\mathbb{E}_{\alpha,c}(\varepsilon)}[\beta\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon)+(1-\beta\alpha)\mathbb{E}_{\alpha+1,c}(\varepsilon)],
\end{aligned}$$

this final phrasing is bounded above by $(1-\gamma)$ if and only if (19) holds. □

5. The Operator $\mathbb{G}^{\varepsilon}_{\alpha,c}(\xi)$

Theorem 5. *Let* $\alpha > -1$ *and* $c > 0$. *If the integral operator* $\mathbb{G}^{\varepsilon}_{\alpha,c}$ *is given by*

$$\mathbb{G}^{\varepsilon}_{\alpha,c}(\xi) := \int_0^{\xi}\frac{\mathbb{k}^{\varepsilon}_{\alpha,c}(t)}{t}dt,\ \xi \in \mathfrak{U}, \tag{20}$$

then $\mathbb{G}^{\varepsilon}_{\alpha,c} \in \mathbb{K}_{\mathcal{T}}(\gamma,\beta)$, *if and only if*

$$\begin{aligned}
&\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}\Big[\beta\varepsilon^2\mathbb{E}_{\alpha-1,c}(\varepsilon)+(2\beta(1-\alpha)+1)\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) \\
&+((1-\alpha)(1-\beta\alpha)-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)\Big] \leq 1-\gamma.
\end{aligned}$$

Proof. By (8) it follows that

$$\mathbb{G}^{\varepsilon}_{\alpha,c}(\xi) = \xi - \sum_{s=2}^{\infty}\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)}\frac{\xi^s}{s},\ \xi \in \mathfrak{U}.$$

Using Lemma 2, the integral operator $\mathbb{G}^{\varepsilon}_{\alpha,c}(\xi)$ belongs to $\mathbb{K}_{\mathcal{T}}(\gamma,\beta)$ if and only if

$$\sum_{s=2}^{\infty}(s+\beta s(s-1)-\gamma)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)} \leq 1-\gamma.$$

We omit the remaining part of the proof because the remaining proof of Theorem 5 is similar to that of Theorem 1. □

Theorem 6. *Let $\alpha > -1$ and $c > 0$. Then the integral operator $\mathbb{G}^{\varepsilon}_{\alpha,c}$ given by (20) is in the class $\mathbb{S}^*_{\mathcal{T}}(\gamma, \beta)$, if and only if*

$$\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}[\beta\mathbb{E}_{\alpha,c}(\varepsilon) + (1-\beta\alpha)\mathbb{E}_{\alpha+1,c}(\varepsilon)] \leq 1-\gamma.$$

Proof. Using Lemma 1, the integral operator $\mathbb{G}^{\varepsilon}_{\alpha,c}(\xi)$ belongs to $\mathbb{S}^*_{\mathcal{T}}(\gamma, \beta)$ if and only if

$$\sum_{s=2}^{\infty}(s\beta - \beta) + \left(1 - \frac{\gamma}{s}\right)\frac{\varepsilon^{\alpha}(c\varepsilon)^{s-1}}{\Gamma(s+\alpha)\mathbb{E}_{\alpha,c}(\varepsilon)} \leq 1-\gamma.$$

The complement is similar to proof of Theorem 4. □

6. Corollaries and Consequences

Putting $\beta = 0$ in the previous theorems, we get the following special cases.

Corollary 1. *Let $\alpha > -1$ and $c > 0$, then $\mathbb{k}^{\varepsilon}_{\alpha,c} \in \mathbb{S}^*_{\mathcal{T}}(\gamma)$ if and only if*

$$\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}[\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) + (1-\alpha-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)] \leq 1-\gamma.$$

Corollary 2. *Let $\alpha > -1$ and $c > 0$, then $\mathbb{k}^{\varepsilon}_{\alpha,c} \in \mathbb{K}_{\mathcal{T}}(\gamma)$ if and only if*

$$\begin{aligned}\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}&\Big[\varepsilon^2\mathbb{E}_{\alpha-1,c}(\varepsilon) + (3-2\alpha-\gamma)\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon)\\ &+(1-\alpha)(1-\alpha-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)\Big] \leq 1-\gamma.\end{aligned}$$

Corollary 3. *Let $\alpha > -1$ and $c > 0$. If $f \in \mathcal{Y}^{\sigma}(V, W)$ and*

$$\frac{(V-W)c|\sigma|}{\mathbb{E}_{\alpha,c}(\varepsilon)}[\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) + (1-\alpha-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)] \leq 1-\gamma$$

then $\mathbb{I}^{\varepsilon}_{\alpha,c}f \in \mathbb{K}_{\mathcal{T}}(\gamma)$.

Corollary 4. *Let $\alpha > -1$ and $c > 0$. If $f \in \mathcal{Y}^{\sigma}(V, W)$ and*

$$(V-W)|\sigma|c\frac{\mathbb{E}_{\alpha+1,c}(\varepsilon)}{\mathbb{E}_{\alpha,c}(\varepsilon)} \leq 1-\gamma$$

then $\mathbb{I}^{\varepsilon}_{\alpha,c}f \in \mathbb{S}^*_{\mathcal{T}}(\gamma)$.

Corollary 5. *Let $\alpha > -1$ and $c > 0$. Then the integral operator $\mathbb{G}^{\varepsilon}_{\alpha,c}$ given by (20) is in the class $\mathbb{K}_{\mathcal{T}}(\gamma)$, if and only if*

$$\frac{c}{\mathbb{E}_{\alpha,c}(\varepsilon)}[\varepsilon\mathbb{E}_{\alpha,c}(\varepsilon) + (1-\alpha-\gamma)\mathbb{E}_{\alpha+1,c}(\varepsilon)] \leq 1-\gamma.$$

Corollary 6. *Let $\alpha > -1$ and $c > 0$. Then the integral operator $\mathbb{G}^{\varepsilon}_{\alpha,c}$ given by (20) is in the class $\mathbb{S}^*_{\mathcal{T}}(\gamma)$, if and only if*

$$\frac{c\mathbb{E}_{\alpha+1,c}(\varepsilon)}{\mathbb{E}_{\alpha,c}(\varepsilon)} \leq 1-\gamma.$$

7. Conclusions

Several researchers have used certain distribution series such as Poisson distribution series, Pascal distribution series, hypergeometric distribution series, and the Mittag–Leffler-type Poisson distribution to obtain some necessary and sufficient conditions for these distributions to belong to certain classes of analytic functions defined in the open disk $\mathfrak{U}$. In our study, necessary and sufficient conditions for Miller–Ross-type Poisson distribution series to be in the classes $\mathbb{S}^*_{\mathcal{T}}(\gamma, \beta)$ and $\mathbb{K}_{\mathcal{T}}(\gamma, \beta)$ of analytic functions with negative coefficients is obtained. We also investigate several inclusion properties of the class $\mathcal{Y}^{\sigma}(V, W)$ associated of the operator $\mathbb{I}^{\varepsilon}_{\alpha,c}$ defined by this distribution. This study could inspire researchers to introduce new sufficient conditions for Miller–Ross-type Poisson distribution series to be in different classes of analytic functions with negative coefficients defined in $\mathfrak{U}$.

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