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# New Criteria for Starlikness and Convexity of a Certain Family of Integral Operators

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**Abstract:** In this paper, we first modify one of the most famous theorems on the principle of differential subordination to hold true for normalized analytic functions with a fixed initial Taylor-Maclaurin coefficient. By using this modified form, we generalize and improve several results, which appeared recently in the literature on the geometric function theory of complex analysis. We also prove some simple conditions for starlikeness, convexity, and the strong starlikeness of several one-parameter families of integral operators, including (for example) a certain $\mu$-convex integral operator and the familiar Bernardi integral operator.

**Keywords:** analytic functions; univalent functions; principle of differential subordination; fixed initial Taylor-Maclarin coefficient; integral operators; starlike functions; convex functions; Janowski starlike function class; $\mu$-convex integral operator; Bernardi operator; Schwarz lemma

**MSC:** 33C45; 30C80

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## 1. Introduction and Motivation

As usual, we use the symbol $\mathcal{H}$ for denoting the set of analytic functions in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We set

$$\mathcal{H}[a, n] = \{f : f \in \mathcal{H} \quad \text{and} \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

where $a \in \mathbb{C}$ and $n \in \mathbb{N}$, and $\mathbb{C}$ and $\mathbb{N}$ are the sets of complex numbers and positive integers, respectively.

We also define the subclass $\mathcal{A}_n$ of $\mathcal{H}$ as follows:

$$\mathcal{A}_n = \{f : f \in \mathcal{H} \quad \text{and} \quad f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots\}.$$

In particular, we set $\mathcal{A}_1 = \mathcal{A}$. Furthermore, we let the subclass $\mathcal{S}$ of $\mathcal{A}$ be the class of all functions in $\mathcal{A}$ that are univalent in the open unit disk $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be in the class $f \in \mathcal{S}^*(\alpha)$ of normalized starlike functions of order $\alpha$ $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1).$$

Specifically, we put $\mathcal{S}^*(0) =: \mathcal{S}^*$. Every element in $\mathcal{S}^*$ is called a starlike function.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha$ $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1).$$

In particular, we put $\mathcal{K}(0) =: \mathcal{K}$. Every element in $\mathcal{K}$ is called a convex function.

Recently, by using different combinations of the representations of starlike and convex functions, many authors obtained simple conditions for the starlikeness and convexity of analytic functions. For example, by considering the quotient of the analytic representations of convex and starlike functions, Silverman [1] derived some new criteria for the starlikeness of analytic functions. Subsequently, Obradović and Tuneski [2] improved the work of Silverman [1].

Now, for analytic functions in $\mathbb{U}$ with a fixed initial coefficient, we define the class $\mathcal{H}_\beta[a,n]$ as follows:

$$\mathcal{H}_\beta[a,n] = \{f : f \in \mathcal{H} \quad \text{and} \quad f(z) = a + \beta z^n + a_{n+1}z^{n+1} + \cdots\},$$

where $n \in \mathbb{N}$, $a \in \mathbb{C}$, and $\beta \in \mathbb{C}$ are fixed complex numbers. Moreover, we assume that

$$\mathcal{A}_{n,b} = \{f : f \in \mathcal{H} \quad \text{and} \quad f(z) = z + bz^{n+1} + a_{n+2}z^{n+2} + \cdots\},$$

where $n \in \mathbb{N}$ and $b \in \mathbb{C}$ are fixed complex numbers. In addition, we set $\mathcal{A}_b := \mathcal{A}_{1,b}$.

For the functions $f$ and $g$ in $\mathcal{H}$, we say that the function $f$ is subordinate to $g$ in $\mathbb{U}$, which is written as $f \prec g$, if there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| \leqq |z| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

Moreover, if $g$ is an univalent function in $\mathbb{U}$, then we have the following equivalence:

$$f \prec g \quad (z \in \mathbb{U}) \iff f(0) = 0 \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

By considering the function $\frac{1+Az}{1+Bz}$, Janowski [3] generalized the class $\mathcal{S}^*$ of starlike functions as follows.

**Definition 1** (see [3])**.** *If $f \in \mathcal{A}$ and $-1 \leqq B < A \leqq 1$, then we say that the function $f$ is in the Janowski starlike function class $\mathcal{S}^*[A,B]$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U};\ -1 \leqq B < A \leqq 1). \tag{1}$$

It is easily observed that the Janowski function $\varphi(z)$, given by

$$\varphi(z) := \frac{1+Az}{1+Bz} \qquad (-1 < B < A < 1),$$

maps the open unit disk $\mathbb{U}$ onto the open disk with the center at $z = C$ and the radius $R$, where

$$C := \frac{1 - AB}{1 - B^2} \quad \text{and} \quad R := \frac{A - B}{1 - B^2}.$$

So, for all $f \in \mathcal{S}^*[A, B]$, the following two-sided inequality holds true:

$$\frac{1 - A}{1 - B} < \Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{1 + A}{1 + B}.$$

Hence, clearly, we have

$$\mathcal{S}^*[A, B] \subset \mathcal{S}^*\left(\frac{1 - A}{1 - B}\right).$$

Moreover, for several special values of the parameters $A$ and $B$, the Janowski starlike function class $\mathcal{S}^*[A, B]$ yields the following subclasses of $\mathcal{A}$:

$$\mathcal{S}^*[1, -1] =: \mathcal{S}^* \quad \text{and} \quad \mathcal{S}^*[1 - 2\alpha, -1] =: S^*(\alpha) \qquad (0 \leqq \alpha < 1).$$

We also have a special case of the Janowski starlike function class $\mathcal{S}^*[A, B]$ given by

$$\mathcal{S}^*[\alpha, 0] := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \left| \left(\frac{zf'(z)}{f(z)}\right) - 1 \right| < \alpha \quad (0 \leqq \alpha < 1) \right\}.$$

Under these and other conditions, the Janowski starlike function class $\mathcal{S}^*[A, B]$ has been investigated by several authors (see, for example, [4–7]).

It is important to note that the Taylor-Maclaurin coefficients of analytic functions play an important role in the geometric function theory of complex analysis. For example, the bound on the second coefficient of a univalent function leads to well-known results such as the growth, distortion, and covering theorems (see [8]). Recently, the subject of the second-order differential subordination for analytic functions with a fixed initial coefficient was considered by Ali et al. [9]. Furthermore, several authors (see, for example, [5,10,11]) discussed the various other properties of these functions. In addition, under some conditions of analytic functions $f$, it was concluded in [8] that a certain $\mu$-convex integral operator on $f$ can belong to the subclass $S^*[1, 0]$ of the Janowski starlike function class. Furthermore, Sharma et al. [12] made use of this same approach regarding analytic functions with a fixed initial coefficient. Motivated by the developments reported in [2,10,13–16], we propose first to extend some of the results of Sharma et al. [12]. In relation to analytic functions with a fixed initial Taylor-Maclaurin coefficient, we then determine some conditions by the means of which the $\mu$-convex integral operator belongs to the Janowski starlike function class $\mathcal{S}^*[A, B]$. Various other conditions for the starlikeness of analytic functions with a fixed initial coefficient are also discussed.

This article is organized as follows. In Section 2, we prove a main lemma that leads to the important result producing the functions in the class $\mathcal{S}^*[A, B]$, which will then be followed by the starlikeness of the $\mu$-convex integral operator on analytic functions with a fixed initial Taylor-Maclaurin coefficient. These results would extend some of the developments which were presented in [12]. Next, by assuming some conditions, we will show how the $\mu$-convex integral operator leads to the class of strongly starlike functions. In Section 3, we derive some sufficient conditions for the starlikeness of analytic functions with a fixed initial Taylor-Maclaurin coefficient. We also deduce some corollaries in Section 3. In addition, we establish the convexity of the Bernardi integral operator on the functions with a fixed initial coefficient that are not necessarily convex. Finally, in our concluding section (Section 4), we present a number of concluding remarks and observations which are based upon our investigation in this article.

In order to prove our main results, we require a definition and a basic lemma.

**Definition 2** (see [8]). *Let $Q$ denote the set of functions $q$ that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where*

$$E(q) := \left\{ \zeta : \zeta \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z \to \zeta} q(z) = \infty \right\},$$

*and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(q)$.*

**Lemma 1** (see [9]). *Let $q \in Q$ with $q(0) = a$ and $p \in \mathcal{H}_c[a,n]$ with $p(z) \not\equiv a$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$p(z_0) \in q(\partial\mathbb{U}) \quad \text{and} \quad p\big(\{z : |z| < |z_0|\}\big) \subset q(\mathbb{U}),$$

*then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$$

*and*

$$\Re\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geqq m\Re\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right),$$

*where*

$$q^{-1}\big(p(z_0)\big) = \zeta_0 = e^{i\theta_0}$$

*and*

$$m \geqq n + \frac{|q'(0)| - |c||z_0|^n}{|q'(0)| + |c||z_0|^n}.$$

## 2. A Set of Main Results

We begin this section by proving a main lemma that will provide an important tool in deriving the results of this article.

**Lemma 2.** *Let the function $q$ be univalent in $\mathbb{U}$ with $q(0) = a$. Suppose that the functions $\theta$ and $\phi$ are analytic in a domain $\mathbb{D} \subset \mathbb{C}$ containing $q(\mathbb{U})$ and that $\phi(z) \neq 0 \ \ (z \in \mathbb{U})$. Additionally, let $0 < \beta \leqq |q'(0)|$ and*

$$h(z) = \theta\big(q(z)\big) + \left(n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}\right) z q'(z) \phi\big(q(z)\big).$$

*Assume also that*
(i) *the function $h$ is convex or*
(ii) *the function $Q(z) = zq'(z)\phi\big(q(z)\big)$ is starlike,*
*and*
(iii) $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0.$
*If $p \in \mathcal{H}_\beta[a,n]$, $p(\mathbb{U}) \subset \mathbb{D}$, and*

$$\begin{aligned}\theta\big(p(z)\big) + zp'(z)\phi\big(p(z)\big) \prec \theta\big(q(z)\big) + \left(n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta}\right) z q'(z) \phi\big(q(z)\big) \\ = h(z),\end{aligned}$$

*then $p \prec q$.*

**Proof.** The proof of Lemma 2 is similar to that of a known result ([8], p. 132, Theorem 3.4h), so we choose to omit the details involved. Only for Case (ii), it is sufficient that we set

$$L(z,t) = \theta\big(q(z)\big) + \left(n + \frac{|q'(0)| - \beta}{|q'(0)| + \beta} + t\right) z q'(z) \phi\big(q(z)\big)$$

instead of ([8], p. 133, Equation (3.4–21)) and then proceed with the proof. □

**Definition 3.** *For $\mu > 0$, the $\mu$-convex integral operator $\mathfrak{A}_\mu$ is defined for $f \in \mathcal{A}$ by*

$$F(z) = \mathfrak{A}_\mu[f](z) := \left( \frac{1}{\mu} \int_0^z t^{-1} \, [f(t)]^{\frac{1}{\mu}} \, dt \right)^\mu \qquad (\mu > 0). \tag{2}$$

In our present investigation, we find it to be convenient to set

$$J(\mu, F; z) := (1-\mu)\frac{zF'(z)}{F(z)} + \mu\left(\frac{zF''(z)}{F'(z)} + 1\right),$$

It can be readily observed that (2) implies that

$$J(\mu, F; z) = \frac{zf'(z)}{f(z)}.$$

We now determine the conditions for functions in the class $\mathcal{A}_{n,b}$ and, by means of these conditions, the $\mu$-convex integral operator given by (2) will be placed in a special subclass of starlike functions. For this objective in view, we state and prove Lemma 3 below.

**Lemma 3.** *Let $n \in \mathbb{N}$, $\mu > 0$, and $-1 \leqq B < A \leqq 1$. Suppose also that $P \in \mathcal{H}_{-\beta(1+\mu n)}[1, n]$ with $B - A \leqq \beta < 0$ satisfies the following subordination condition:*

$$P(z) \prec \frac{1+Az}{1+Bz} + \left(n + \frac{A-B+\beta}{A-B-\beta}\right)\left(\frac{\mu(A-B)z}{(1+Az)(1+Bz)}\right) = h(z). \tag{3}$$

*If $p \in \mathcal{H}_\beta[1, n]$ and*

$$\mu z p'(z) + P(z)p(z) = 1, \tag{4}$$

*then*

$$p(z) \prec q(z) = \frac{1+Bz}{1+Az}.$$

**Proof.** Let us set

$$p_1(z) = \frac{1}{p(z)} \quad \text{and} \quad q_1(z) = \frac{1}{q(z)} = \frac{1+Az}{1+Bz}.$$

We then have $p_1 \in \mathcal{H}_{-\beta}[1, n]$ and the function $q_1$ is analytic and univalent in $\mathbb{U}$. Moreover, the Equations (3) and (4) yield

$$p_1(z) + \mu\frac{zp_1'(z)}{p_1(z)} \prec h(z),$$

where

$$h(z) = \theta[q_1(z)] + \left(n + \frac{|q_1'(0)| + \beta}{|q_1'(0)| - \beta}\right) zq_1'(z)\phi(q_1(z)),$$

$$\theta(z) = z \qquad \text{and} \qquad \phi(z) = \frac{\mu}{z}.$$

We now show that the conditions mentioned in Lemma 2 are satisfied. By setting

$$Q(z) = zq_1'(z)\phi(q_1(z)),$$

we have

$$Q(z) = \frac{\mu(A-B)z}{(1+Az)(1+Bz)}.$$

Consequently, after some computation, we obtain

$$\begin{aligned}\Re\left(\frac{zQ'(z)}{Q(z)}\right) &= 1 - \Re\left(\frac{Az}{1+Az}\right) - \Re\left(\frac{Bz}{1+Bz}\right) \\ &> \frac{1-|A||B|}{(1+|A|)(1+|B|)} \geqq 0,\end{aligned}$$

and so we obtain

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \frac{1}{\mu}\,\Re\left(\frac{1+Az}{1+Bz}\right) + \left(n + \frac{A-B+\beta}{A-B-\beta}\right)\Re\left(\frac{zQ'(z)}{Q(z)}\right) > 0.$$

Then, by applying Lemma 2, we deduce that $p_1(z) \prec q_1(z)$, which leads us to the following subordination: $p(z) \prec q(z) = \frac{1+Bz}{1+Az}$. This completes the proof of Lemma 3. □

**Remark 1.** *If we set $A = 1$ and $B = 0$, then Lemma 3 reduces to a result* ([12], Lemma 2.1). *Additionally, by putting $A = 1$, $B = 0$, and $\beta = -1$ into Lemma 3, it yields another known result* ([8], p. 253, Lemma 5.1a). *Furthermore, since*

$$1 + z + \frac{n\mu z}{1+z} \prec 1 + z + \left(n + \frac{1+\beta}{1-\beta}\right)\frac{\mu z}{1+z} \qquad (-1 \leqq \beta < 0),$$

*it is obvious that Lemma 3 would extend the aforementioned result* ([8], p. 253, Lemma 5.1a) *to hold true for functions in the class $\mathcal{H}_\beta[a,n]$.*

**Theorem 1.** *Let $n \in \mathbb{N}$, $\mu > 0$, and $-1 \leqq B < A \leqq 1$. Additionally, let $f \in \mathcal{A}_{n,b}$ and $F = \mathfrak{A}_\mu[f]$ with*

$$0 < \frac{bn}{n\mu+1} \leqq A - B,$$

*where $\mathfrak{A}_\mu$ is given by* (2). *If*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} + \left(n + \frac{(n\mu+1)(A-B)-bn}{(n\mu+1)(A-B)+bn}\right)\left(\frac{\mu(A-B)z}{(1+Az)(1+Bz)}\right),$$

*then*

$$\frac{zF'(z)}{F(z)} \prec \frac{1+Az}{1+Bz} \quad \textit{and} \quad \left|\frac{zF'(z)}{F(z)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}.$$

**Proof.** Let us define the function $p(z)$ as follows:

$$p(z) = \frac{1}{\mu[f(z)]^{\frac{1}{\mu}}} \int_0^z t^{-1}\,[f(t)]^{\frac{1}{\mu}}\,dt \qquad (\mu > 0).$$

Then, according to the known result ([8], p. 11, Lemma 1.2c), $p$ is well-defined, and

$$p \in \mathcal{H}_\beta[1,n] \qquad \left(\beta = -\frac{bn}{n\mu+1}\right).$$

By putting

$$P(z) = \frac{zf'(z)}{f(z)},$$

a simple computation shows that $p$ satisfies (4). Then, by applying Lemma 3, we deduce that

$$p(z) \prec \frac{1+Bz}{1+Az}.$$

We now define the function $F(z)$ as follows:

$$F(z) = f(z)[p(z)]^{\mu} \qquad (\mu > 0). \tag{5}$$

Since $p(z) \neq 0$, $F \in \mathcal{A}_{n,\frac{b}{n\mu+1}}$ is well-defined. Furthermore, it is easily seen that $F$ coincides with the function introduced in (2). Upon combining (4) and (5), we obtain

$$\frac{zF'(z)}{F(z)} = \frac{1}{p(z)} \prec \frac{1+Az}{1+Bz}.$$

Hence, clearly, we have

$$\left| \frac{zF'(z)}{F(z)} - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2},$$

as it is asserted by Theorem 1. □

**Remark 2.** *If we put $A = 1$ and $B = 0$, then Theorem 1 reduces to a known result* ([12], Theorem 2.3)*, Additionally, by setting $A = 1$, $B = 0$, and $b = \alpha + \frac{1}{n}$, Theorem 1 would yield the known result* ([8], p. 255, Theorem 5.1b)*. Since, for*

$$0 < \frac{nb}{nb+1} \leqq 1,$$

*we have*

$$1 + z + \frac{n\mu z}{1+z} \prec 1 + z + \left( n + \frac{\mu n + 1 - bn}{\mu n + 1 + bn} \right) \frac{\mu z}{1+z},$$

*it is fairly obvious that Theorem 1 extends the above-mentioned result* ([8], p. 255, Theorem 5.1b) *to hold true for functions $f \in \mathcal{A}_{n,b}$.*

**Theorem 2.** *Let $n \in \mathbb{N}$, $\mu > 0$, and $-1 \leqq B < A \leqq 1$. If $F \in \mathcal{A}_{n,c}$ with*

$$0 < c \leqq \frac{A-B}{n}$$

*and*

$$J(\mu, F; z) \prec \frac{1+Az}{1+Bz} + \left( n + \frac{A-B-nc}{A-B+nc} \right) \left( \frac{\mu(A-B)z}{(1+Az)(1+Bz)} \right),$$

*then*

$$\frac{zF'(z)}{F(z)} \prec \frac{1+Az}{1+Bz} \quad \text{and} \quad \left| \frac{zF'(z)}{F(z)} - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2}.$$

**Proof.** Let $f \in \mathcal{A}_{n,b}$. Then, because of the following equivalence:

$$J(\mu, F; z) = \frac{zf'(z)}{f(z)}$$

with (2), we can write

$$\frac{zf'(z)}{f(z)} = J(\mu, F; z) := (1-\mu)\frac{zF'(z)}{F(z)} + \mu\left( \frac{zF''(z)}{F'(z)} + 1 \right).$$

It can be readily seen that $b = c(1 + \mu n)$. Thus, by applying Theorem 1, we obtain the result asserted by Theorem 2. □

**Remark 3.** *If we let $A = 1$ and $B = 0$, then Theorem 2 reduces to a known result* ([12], Theorem 2.5). *Additionally, by putting $A = 1$, $B = 0$, and $c = \frac{1}{n}$, Theorem 2 reduces to another known result* ([8], p. 255, Theorem 5.1c). *Since, for*

$$0 < c \leqq \frac{1}{n},$$

*we have*

$$1 + z + \frac{n\mu z}{1+z} \prec 1 + z + \left(n + \frac{1-cn}{1+cn}\right)\frac{\mu z}{1+z},$$

*it is obvious that Theorem 2 extends the above-mentioned known result* ([8], p. 255, Theorem 5.1c) *to hold true for functions $f \in \mathcal{A}_{n,c}$.*

*Upon setting*

$$k(z) = z(1+Bz)^{\frac{A}{B}-1} \qquad (-1 \leqq B < A \leqq 1 \;\; B \neq 0)$$

*and*

$$k(z) = ze^{Az} \qquad (0 = B < A \leqq 1),$$

*if we consider $F \in \mathcal{A}_{n,c}$ with*

$$0 < c \leqq \frac{A-B}{n},$$

*then Theorem 2 can be shown to have the following symmetric form*:

$$J(\mu, F; z) \prec J\left(\mu\left[n + \frac{A-B-nc}{A-B+nc}\right], k; z\right) \qquad \Longrightarrow \qquad J(0, F; z) \prec J(0, k; z).$$

Next, we consider the class $\mathfrak{S}^*(\lambda)$ of *strongly starlike functions* of order $\lambda$ in $\mathbb{U}$, which was introduced by Brannan and Kirwan [11] as follows:

$$\mathfrak{S}^*(\lambda) = \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\,\lambda \qquad (z \in \mathbb{U};\; 0 < \lambda \leqq 1)\right\}.$$

We define the following subclass of the strongly starlike function class $\mathfrak{S}^*(\lambda)$:

$$\mathfrak{S}^*{}_{n,b}(\lambda) = \{ f : f \in \mathfrak{S}^*(\lambda) \quad \text{and} \quad f \in \mathcal{A}_{n,b}\}.$$

**Lemma 4.** *Let $n \in \mathbb{N}$, $\mu > 0$, $0 < \lambda \leqq 1$, and $-1 \leqq B < A \leqq 1$. Additionally, let $P \in \mathcal{H}_{-\beta(1+\mu n)}[1, n]$ with $(B - A)\lambda \leqq \beta < 0$ satisfy the following subordination condition*:

$$P(z) \prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda} + \left(n + \frac{\lambda(A-B)+\beta}{\lambda(A-B)-\beta}\right)\left(\frac{\lambda\mu(A-B)z}{(1+Az)(1+Bz)}\right) = h(z).$$

*If $p \in \mathcal{H}_\beta[1, n]$ and*

$$\mu z p'(z) + P(z)p(z) = 1,$$

*then*

$$p(z) \prec q(z) = \left(\frac{1+Bz}{1+Az}\right)^{\lambda}.$$

**Proof.** The proof of Lemma 4 is similar to that of Lemma 3. We, therefore, omit the analogous details of the proof. □

**Remark 4.** *If we set $B = 0$ and $A = \mu = \lambda = 1$, then Lemma 4 reduces to a known result* ([12], Lemma 2.11). *Additionally, by putting $B = 0$, $A = \mu = \lambda = 1$, and $\beta = -2$, Lemma 4 provides an extension of the known result* ([8], p. 46, Theorem 2.5b) *with $c = 1$.*

By analogously applying the arguments, which we used in our proof of Theorem 2, we can demonstrate each of the following theorems.

**Theorem 3.** *Let $n \in \mathbb{N}$, $\mu > 0$, $0 < \lambda \leqq 1$, and $-1 \leqq B < A \leqq 1$. Additionally, let $f \in \mathcal{A}_{n,b}$ and $F = \mathfrak{A}_\mu[f]$ with*

$$0 < \frac{bn}{n\mu + 1} \leqq (A - B)\lambda,$$

*where $\mathfrak{A}_\mu$ is given by (2). If*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1 + Az}{1 + Bz}\right)^\lambda + \left(n + \frac{\lambda(n\mu + 1)(A - B) - bn}{\lambda(n\mu + 1)(A - B) + bn}\right)\left(\frac{\mu\lambda(A - B)z}{(1 + Az)(1 + Bz)}\right) = h(z),$$

*then*

$$F \in \mathfrak{S}^*_{n,\frac{b}{n\mu+1}}(\lambda).$$

In its special case when $B = 0$ and $A = \lambda = 1$, Theorem 3 would yield the following corollary.

**Corollary 1.** *Let $\mu > 0$ and suppose that $f \in \mathcal{A}_{1,b}$ and $F = A_\mu[f]$ with*

$$0 < \frac{b}{\mu + 1} \leqq 2,$$

*where $\mu$ is given by (2). If*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1 + z}{1 - z}\right) + \frac{8\mu}{2 + b}\left(\frac{z}{1 - z^2}\right),$$

*then the function $F$ is starlike in $\mathbb{U}$.*

**Theorem 4.** *Let $n \in \mathbb{N}$, $\mu > 0$, $0 < \lambda \leqq 1$, and $-1 \leqq B < A \leqq 1$. If $F \in \mathcal{A}_{n,c}$ with*

$$0 < c \leqq \frac{(A - B)\lambda}{n}$$

*and*

$$J(\mu, F; z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^\lambda + \left(n + \frac{\lambda(A - B) - nc}{\lambda(A - B) + nc}\right)\left(\frac{\mu\lambda(A - B)z}{(1 + Az)(1 + Bz)}\right) = h(z),$$

*then $F \in \mathfrak{S}^*_{n,c}(\lambda)$.*

## 3. Starlikeness of Analytic Functions with Fixed Initial Taylor-Maclaurin Coefficient

Theorem 5 below provides our first set of criteria for the starlikeness of analytic functions with a fixed initial Taylor-Maclaurin coefficient.

**Theorem 5.** *Let $n \in \mathbb{N}$, $0 < \delta \leqq 1$,*

$$N = n + \frac{\delta_1 - 1}{\delta_1 + 1} \quad \text{and} \quad \mu_n = \frac{n + 2}{C_n}, \tag{6}$$

*where*

$$C_n = \frac{2n}{n + \frac{1-\delta}{1+\delta}}\left[\frac{n + \frac{1-\delta}{1+\delta}}{n} + \left(\frac{n + 2}{n}\right)\ln 2 - \int_0^1 \frac{t^{\frac{1}{N}}}{1 + t}\, dt\right] \tag{7}$$

*and*

$$\delta_1 = \frac{[n(1-\delta)+1] + \sqrt{[n(1-\delta)+1]^2 + 4\delta(n+1)(n+2)}}{2\delta(n+2)}.$$

*If $f \in \mathcal{A}_{n,b}$ with*

$$b = \frac{2\mu\delta}{(n+1)\left(n + \frac{1-\delta}{1+\delta}\right)} \qquad (0 < nb \leqq 2),$$

*where $0 \leqq \mu \leqq \mu_n$ and*

$$\Re\big(zf''(z)\big) > -\mu_n, \tag{8}$$

*then $f \in \mathcal{S}^*$.*

**Proof.** Let us define the functions $p(z)$ and $q(z)$ as follows:

$$p(z) = f'(z) \qquad \text{and} \qquad q(z) = 1 - \frac{2\mu}{n + \frac{1-\delta}{1+\delta}} \log{(1+z)}.$$

It is then readily seen that $p \in \mathcal{H}_{(n+1)b}[1, n]$ and that $q$ is a convex function. From (6) and (7), we find that the constants $\mu_n$ and $C_n$ are positive. Let

$$0 \leqq \mu \leqq \mu_n \quad \text{and} \quad \Re\big(zf''(z)\big) > -\mu.$$

We claim that $p(z) \prec q(z)$. Otherwise, if $p \not\prec q$, then (by Lemma 1), there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(q)$ such that

$$p(z_0) = q(\zeta_0) \qquad \text{and} \qquad z_0 p'(z_0) = m\zeta_0 q'(\zeta_0),$$

where

$$m \geqq n + \frac{1-\delta}{1+\delta}.$$

Thus, by taking $\zeta_0 = e^{it} \neq -1$, we deduce that

$$\begin{aligned}
\Re\big(z_0 p'(z_0)\big) = m\Re\big(\zeta_0 q'(\zeta_0)\big) &= -m\left(\frac{2\mu}{n + \frac{1-\delta}{1+\delta}}\right)\Re\left(\frac{e^{it}}{1+e^{it}}\right) \\
&= -m\left(\frac{\mu}{n + \frac{1-\delta}{1+\delta}}\right) \leqq -\mu,
\end{aligned}$$

which is a contradiction. Therefore, we conclude that $p \prec q$.

Now, since $q$ is convex and symmetric to the real axis, we have

$$\Re\big(f'(z)\big) > \beta = q(1) = 1 - \frac{2\mu}{n + \frac{1-\delta}{1+\delta}} \ln 2. \tag{9}$$

However, $\beta \geqq 0$ if

$$\mu \leqq \frac{n + \frac{1-\delta}{1+\delta}}{\ln 4}.$$

Consequently, we have

$$\Re\big(zf''(z)\big) > -\frac{n + \frac{1-\delta}{1+\delta}}{\ln 4}.$$

It follows that $\Re\big(f'(z)\big) > 0$.

On the other hand, by means of a simple computation, we have

$$\mu_n \leqq \frac{n + \frac{1-\delta}{1+\delta}}{\ln 4},$$

where $\mu_n$ is given by (6) and (7). Hence, according to (8), we conclude that $f$ is univalent. We will prove that $f$ is starlike in $\mathbb{U}$. For this purpose, if we let

$$P(z) = \frac{f(z)}{z} \qquad (z \in \mathbb{U}),$$

then we have $P \in \mathcal{H}_b[1, n]$ and

$$P(z) + zP'(z) = f'(z) \prec q(z) = 1 - \frac{2\mu}{n + \frac{1-\delta}{1+\delta}} \log(1+z).$$

We now consider the following differential equation (with the initial condition):

$$q_1(z) + Nzq_1'(z) = q(z) \qquad \big(q_1(0) = 1\big), \tag{10}$$

where $N$ is defined in the statement of Theorem 5. By solving the initial-value problem (10), we find the function $q_1(z)$ given by

$$q_1(z) = \frac{1}{Nz^{\frac{1}{N}}} \int_0^z q(t)\, t^{\frac{1}{N}-1}\, dt$$

as its solution. Since $q$ is convex, we can apply a known result ([8], p. 67, Theorem 2.6h) to conclude that the function $q_1$ is convex and, therefore, univalent in $\mathbb{U}$.

In order to apply Lemma 2, we need to investigate the conditions mentioned in it. For this purpose, it is sufficient to show that

$$\Re\left(\frac{zq'(z)}{Q(z)}\right) > 0,$$

where

$$Q(z) = zq_1'(z) \quad \text{and} \quad q(z) = 1 - \frac{2\mu}{n + \frac{1-\delta}{1+\delta}} \log(1+z).$$

However, in view of (10), we obtain

$$\Re\left(\frac{zq'(z)}{Q(z)}\right) = 1 + N\left(1 + \frac{zq_1''(z)}{q_1'(z)}\right) > 0.$$

Thus, by applying Lemma 2, we deduce that

$$\begin{aligned} P(z) \prec q_1(z) &= \frac{1}{Nz^{\frac{1}{N}}} \int_0^z q(t)\, t^{\frac{1}{N}-1}\, dt \\ &= 1 - \frac{2\mu}{N\left(n + \frac{1-\delta}{1+\delta}\right)z^{\frac{1}{N}}} \int_0^z t^{\frac{1}{N}-1} \log(1+t)\, dt. \end{aligned}$$

Since $q_1$ is convex and symmetric to the real axis, we have

$$\Re\big(P(z)\big) > \gamma = \gamma(\mu) = q_1(1) = 1 - \frac{2\mu}{n + \frac{1-\delta}{1+\delta}}\left(\ln 2 - \int_0^1 \frac{t^{\frac{1}{N}}}{1+t}\, dt\right).$$

Thus, if we put

$$p(z) = \frac{zf'(z)}{f(z)},$$

then $p \in \mathcal{H}_{nb}[1, n]$, and $f'(z) = P(z)p(z)$. Furthermore, it can be seen that

$$P(z)\big[zp'(z) + p^2(z)\big] = f'(z) + zf''(z).$$

Hence, from $\Re(zf''(z)) > -\mu$ and the Equation (9), we find that

$$\Re\big(P(z)\big[zp'(z) + p^2(z)\big]\big) > \beta - \mu. \tag{11}$$

We now show that

$$p(z) \prec q_3(z) = \frac{1+z}{1-z} \qquad (z \in \mathbb{U}).$$

Otherwise, if $p \not\prec q_3$, then (by Lemma 1) there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(q)$ such that

$$p(z_0) = q_3(\zeta_0) \qquad \text{and} \qquad z_0 p'(z_0) = m\zeta_0 q_3'(\zeta_0),$$

where

$$m \geqq n + \frac{2 - nb}{2 + nb}.$$

Thus, by taking

$$\zeta_0 = e^{it} \qquad (-\pi < t \leqq \pi),$$

we have

$$p(z_0) = ix \qquad \left(x = \cot\frac{t}{2} \in \mathbb{R}\right),$$

and so we get

$$\begin{aligned} \Re\big(P(z_0)\big[z_0 p'(z_0) + p^2(z_0)\big]\big) &= \Re\left(P(z_0)\left(-\frac{m(1+x^2)}{2} - x^2\right)\right) \\ &\leqq -\frac{m}{2}\Re\big(P(z_0)\big) \leqq -\frac{n}{2}\gamma. \end{aligned}$$

On the other hand, the Equations (6) and (7) imply that

$$-\frac{n}{2}\gamma \leqq \beta - \mu \qquad (0 \leqq \mu \leqq \mu_n).$$

This last inequality leads us to

$$\Re\big(P(z_0)\big[z_0 p'(z_0) + p^2(z_0)\big]\big) \leqq \beta - \mu,$$

which is in contradiction with (11). This completes the proof of Theorem 5. $\square$

Some corollaries and consequences of Theorem 5 are worth considering next.

**I.** By putting $n = 1$ and $\delta = \frac{1}{2}$ in the assumptions of Theorem 5, we obtain

$$\delta_1 = \frac{3 + \sqrt{57}}{6}, \quad C_1 = 4.59\cdots \quad \text{and} \quad \mu_1 = 0.65\cdots.$$

Thus, by applying Theorem 5, we have Corollary 2 below.

**Corollary 2.** *If $f \in \mathcal{A}_{1,b}$ with $b = \frac{3}{8}\mu$, where $0 \leqq \mu \leqq 0.65\cdots$ and*

$$\Re\big(zf''(z)\big) > -0.65\cdots,$$

*then $f \in \mathcal{S}^*$.*

**II.** By putting $n = 1$ and $\delta = \frac{3}{4}$ in the assumptions of Theorem 5, we find that

$$\delta_1 = \frac{5 + \sqrt{313}}{18}, \quad C_1 = 5.04\cdots \quad \text{and} \quad \mu_1 = 0.59\cdots.$$

Thus, by applying Theorem 5, we deduce Corollary 3 below.

**Corollary 3.** *If $f \in \mathcal{A}_{1,b}$ with $b = \frac{21}{16}\mu$, where*

$$0 \leqq \mu \leqq 0.59\cdots \qquad \text{and} \qquad \Re\big(zf''(z)\big) > -0.59\cdots,$$

*then $f \in \mathcal{S}^*$.*

**III.** By putting $n = 2$ and $\delta = \frac{3}{4}$ in the assumptions of Theorem 5, we obtain

$$\delta_1 = \frac{3+\sqrt{153}}{12}, \quad C_2 = 3.75\cdots \quad \text{and} \quad \mu_2 = 1.06\cdot\cdot$$

Thus, by appropriately applying Theorem 5, we have Corollary 4 below.

**Corollary 4.** *If $f \in \mathcal{A}_{2,b}$ with $b = \frac{21}{90}\mu$, where*

$$0 \leqq \mu \leqq 1.06\cdots \qquad \text{and} \qquad \Re\big(zf''(z)\big) > -1.06\cdots,$$

*then $f \in \mathcal{S}^*$.*

**Remark 5.** *If we compare Corollaries 2, 3, and 4 with the known result* ([8], p. 275, Theorem 5.2c), *we observe that, by choosing different values for n and b in Theorem 5, our results improve the known result* ([8], p. 275, Theorem 5.2c).

We turn now to the general Bernardi integral operator $L_\gamma \ (\gamma > -1)$, which is defined as follows (see, for details, [15]):

$$F(z) = L_\gamma[f](z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)\, t^{\gamma-1}\, dt \qquad \big(f \in \mathcal{A};\ \Re(\gamma) > -1\big). \tag{12}$$

It is recorded in ([8], p.67, Theorem 2.6h) that $L_\gamma[\mathcal{K}] \subset \mathcal{K}$ for $\Re(\gamma) \geqq 0$, with similar inclusion relations for the classes $\mathcal{S}^*$ and $\mathcal{C}$ of starlike and close-to-convex functions in $\mathbb{U}$. In our next result, we will present conditions for a function $f$ in the class $\mathcal{A}_{n,b}$ that are not necessarily convex, but the Bernardi operator $L_\gamma[f](z)$, given in (12), belongs to the class $\mathcal{K}$ (see also [15]).

**Theorem 6.** *Let $n \in \mathbb{N}$, $0 < \delta \leqq 1$, and*

$$N = n + \frac{\delta_1 - 1}{\delta_1 + 1}.$$

*Additionally, let*

$$0 < \gamma \leqq 1 \qquad \text{and} \qquad \alpha_n = \frac{n+2}{C_n(\gamma)}, \tag{13}$$

*where*

$$C_n(\gamma) = \frac{2n}{n + \frac{1-\delta}{1+\delta}}\left(\frac{(1+\gamma)\left(n+\frac{1-\delta}{1+\delta}\right)}{n} + \left(\frac{n+2}{n}\right)\ln 2 \right.$$
$$\left. - \frac{n+2\gamma^2}{n}\int_0^1 \frac{t^{\frac{1+\gamma}{N}}}{1+t}\, dt\right) \tag{14}$$

*and*

$$\delta_1 = \frac{1+(1-\delta)(n+\gamma)+\sqrt{[(\delta-1)(n+\gamma)-1]^2+4\delta(n+\gamma+1)(n+\gamma+2)}}{2\delta(n+\gamma+2)}.$$

*If $f \in \mathcal{A}_{n,b}$ with*

$$b = \frac{2\alpha\delta}{(n+1)\left(n+\frac{1-\delta}{1+\delta}\right)} \quad \text{and} \quad 0 < \frac{n(n+1)(1+\gamma)b}{n+\gamma+1} \leqq 2,$$

*where $0 \leqq \alpha \leqq \alpha_n$ and*

$$\Re\big(zf''(z)\big) > -\alpha_n,$$

*then $L_\gamma[f] \in \mathcal{K}$.*

**Proof.** Let us first define the functions $p(z)$ and $q(z)$ as follows:

$$p(z) = f'(z) \qquad \text{and} \qquad q(z) = 1 - \frac{2\alpha}{n+\frac{1-\delta}{1+\delta}} \log(1+z).$$

It can then be seen that $p \in \mathcal{H}_{(n+1)b}[1,n]$ and that $q$ is a convex function. Thus, from (13) and (14), we find that the constants $\alpha_n(\gamma)$ and $C_n(\gamma)$ are positive. Suppose that

$$0 \leqq \alpha \leqq \alpha_n \qquad \text{and} \qquad \Re\big(zf''(z)\big) > -\alpha.$$

By the same argument as in the demonstration of Theorem 5, we can infer that $p \prec q$, and so (9) holds true.

First of all, we show that $F$ is a univalent function. Indeed, upon differentiating both sides of (12) with respect to $z$, we find that

$$zF'(z) + \gamma F(z) = (\gamma+1)f(z),$$

which readily yields

$$zF''(z) + (\gamma+1)F'(z) = (\gamma+1)f'(z). \tag{15}$$

If we set $P(z) = F'(z)$, then $P \in \mathcal{H}_{\beta_1}[1,n]$, where

$$\beta_1 = \frac{(n+1)(\gamma+1)b}{n+\gamma+1},$$

and we find from (15) that

$$P(z) + \frac{zP'(z)}{1+\gamma} = f'(z) \prec q(z) = 1 - \left(\frac{2\alpha}{n+\frac{1-\delta}{1+\delta}}\right) \log(1+z)$$

We now consider the following differential equation (with the initial condition):

$$q_1(z) + \left(\frac{N}{1+\gamma}\right) zq_1'(z) = q(z) \qquad \big(q_1(0) = 1\big). \tag{16}$$

It can be seen that the function $q_1(z)$, given by

$$q_1(z) = \frac{\gamma+1}{Nz^{\frac{1+\gamma}{N}}} \int_0^z q(t)\, t^{\frac{\gamma+1}{N}-1}\, dt,$$

satisfies the initial-value problem (16). By analogously applying the argument used in the proof of Theorem 5, we can deduce that

$$\begin{aligned} P(z) \prec q_1(z) &= \frac{\gamma+1}{Nz^{\frac{1+\gamma}{N}}} \int_0^z q(t)\, t^{\frac{\gamma+1}{N}-1}\, dt \\ &= 1 - \frac{2\alpha(1+\gamma)}{N\left(n+\frac{1-\delta}{1+\delta}\right) z^{\frac{1+\gamma}{N}}} \int_0^z t^{\frac{\gamma+1}{N}-1} \log(1+t)\, dt. \end{aligned}$$

Since the function $q_1$ is convex and symmetric to the real axis, we can write

$$\Re\big(P(z)\big) > \gamma_1 = \gamma_1(\alpha) = q_1(1) = 1 - \frac{2\alpha}{n + \frac{1-\delta}{1+\delta}}\left(\ln 2 - \int_0^1 \frac{t^{\frac{1+\gamma}{N}}}{1+t}\,dt\right)$$

However, we note that

$$\gamma_1 \geqq 1 - \frac{2\alpha_n}{n + \frac{1-\delta}{1+\delta}}\ \ln 2$$

and

$$C_n(\gamma) \geqq 2\frac{\left(n + \frac{1-\delta}{1+\delta})(\gamma+1) - (n+2\gamma^2)\right)}{n + \frac{1-\delta}{1+\delta}} + \frac{(n+2)\ln 4}{n + \frac{1-\delta}{1+\delta}},$$

so we have

$$C_n(\gamma) \geqq \frac{(n+2)\ln 4}{n + \frac{1-\delta}{1+\delta}}.$$

Therefore, by combining the above relations, we obtain

$$\Re\big(F'(z)\big) > \gamma_1 \geqq 0. \tag{17}$$

If we let

$$p_1(z) = \frac{zF''(z)}{F'(z)} + 1 \qquad (z \in \mathbb{U}),$$

then we have $p_1 \in \mathcal{H}_{\beta_2}[1, n]$ with

$$\beta_2 = \frac{n(n+1)(1+\gamma)b}{n+\gamma+1}.$$

Moreover, we find from (15) that

$$P(z) \cdot p_1(z) = F'(z) + zF''(z) = (\gamma+1)f'(z) - \gamma F'(z). \tag{18}$$

Upon differentiating both sides of (18) with respect to $z$ and using the Equation (15), we obtain

$$P(z)[zp'(z) + p^2(z)] = \gamma^2 F'(z) + (1-\gamma^2)f'(z) + (1+\gamma)zf''(z).$$

Thus, by applying the Equations (9) and (17), in conjunction with the hypothesis of Theorem 6, we have

$$\Re\big(P(z)\big[zp'(z) + p^2(z)\big]\big) > \gamma^2\gamma_1 + (1-\gamma^2)\beta - (1+\gamma)\alpha.$$

Finally, just as in the case of Theorem 5 for proving the starlikeness of $f$, we can conclude that

$$p_1(z) \prec q_3(z) = \frac{1+z}{1-z}.$$

Hence, clearly, we have $L_\gamma[f] \in \mathcal{K}$. □

**IV.** By putting $n = 2$, $\gamma = 1$, and $\delta = \frac{3}{4}$ in Theorem 6, we find that

$$\delta_1 = \frac{7 + \sqrt{1009}}{30}, \quad C_2(1) = 5.38\cdots \quad \text{and} \quad \alpha_2 = 0.74\cdots.$$

Then, as a consequence of Theorem 6, we can deduce the following corollary.

**Corollary 5.** *If $f \in \mathcal{A}_{2,b}$ with $b = \frac{21}{90}\alpha$, where*

$$0 \leqq \alpha \leqq 0.74\cdots \qquad \text{and} \qquad \Re\big(zf''(z)\big) > -0.74\cdots,$$

*then $L[f] \in \mathcal{K}$.*

**Remark 6.** *If we compare Corollary 5 with the known result* ([8], p. 279, Theorem 5.2e), *we observe that, by choosing different values for $n$ and $b$ in Theorem 6, our result would improve this known result* ([8], p. 279, Theorem 5.2e).

We next state and prove the following result.

**Theorem 7.** *Let $n \in \mathbb{N}$, $\gamma > -1$ and $0 < \delta \leqq 1$. If $f \in \mathcal{A}_{n,b}$ with*

$$b = \frac{(n+\gamma+1)M\delta}{(1+n)\left(1+\gamma+n+\frac{1-\delta}{1+\delta}\right)}$$

*and*

$$|f'(z)-1| < M \qquad (z \in \mathbb{U}), \tag{19}$$

*where*

$$0 < M \leqq \frac{n+\frac{1-\delta}{1+\delta}+\gamma+1}{(1+\gamma)\sqrt{\left(n+\frac{1-\delta}{1+\delta}+\gamma+1\right)^2+(\gamma+1)^2}+|\gamma|}, \tag{20}$$

*then $L_\gamma[f] \in \mathcal{K}$.*

**Proof.** Let us introduce the functions $q(z)$ and $P(z)$ as follows:

$$P(z) = F'(z) \quad \text{and} \quad q(z) = 1 + \frac{(1+\gamma)Mz}{1+\gamma+n+\frac{1-\delta}{1+\delta}}.$$

It is then clear that $P \in \mathcal{H}_\beta[1,n]$ with

$$\beta = \frac{(1+\gamma)(1+n)b}{n+\gamma+1},$$

and that the function $q$ is convex.

Upon differentiating both sides of the Equation (12) with respect to $z$, we obtain

$$zF'(z) + \gamma F(z) = (\gamma+1)f(z)$$

and

$$zF''(z) + (\gamma+1)F'(z) = (\gamma+1)f'(z). \tag{21}$$

Consequently, the Equation (19) implies that

$$P(z) + \frac{zP'(z)}{1+\gamma} = f'(z) \prec 1 + Mz = h(z).$$

We now consider the following differential equation (with the initial condition):

$$q(z) + \left(n + \frac{1-\delta}{1+\delta}\right)\frac{zq'(z)}{1+\gamma} = 1 + Mz = h(z) \qquad \big(q(0)=1\big). \tag{22}$$

It can be easily seen that the function $q(z)$, given by

$$q(z) = 1 + \frac{(1+\gamma)Mz}{1+\gamma+n+\frac{1-\delta}{1+\delta}},$$

satisfies the initial-value problem (22). Thus, if we set

$$Q(z) = \frac{zq'(z)}{1+\gamma},$$

it is then obvious that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0.$$

Therefore, by applying Lemma 2, we can deduce that

$$P(z) = F'(z) \prec q(z) = 1 + \frac{(1+\gamma)Mz}{1+\gamma+n+\frac{1-\delta}{1+\delta}}. \tag{23}$$

If we put

$$R = \frac{(1+\gamma)M}{1+\gamma+n+\frac{1-\delta}{1+\delta}}, \tag{24}$$

then the Equation (23) yields

$$|P(z) - 1| < R. \tag{25}$$

In view of (24) and (20), we find that $R < 1$. Therefore, the equation (25) implies that $|F'(z) - 1| < 1$. Hence, clearly, $F$ is univalent. Thus, if we let

$$p(z) = \frac{zF''(z)}{F'(z)} + 1,$$

then $p \in \mathcal{H}_{\beta_1}[1, n]$ with

$$\beta_1 = \frac{n(n+1)(1+\gamma)b}{n+\gamma+1}.$$

Additionally, from the equation (21), we have

$$F'(z)[p(z) + \gamma] = (1+\gamma)f'(z). \tag{26}$$

Thus, if we first substitute (26) into (19) and then use (24), we obtain

$$|P(z)(p(z)+\gamma) - (\gamma+1)| < \left(1+\gamma+n+\frac{1-\delta}{1+\delta}\right)R. \tag{27}$$

We note that, since $\Re(F'(z)) > 0$, we have $P(z) \neq 0$. Thus, in order to prove that

$$\Re(p(z)) > 0 \ \ (z \in \mathbb{U}),$$

we suppose that there exists a point $z_0 \in \mathbb{U}$ such that $p(z_0) = i\rho \ \ (\rho \in \mathbb{R})$. We show that this leads to

$$|P(z_0)(i\rho+\gamma) - (\gamma+1)| \geqq \left(1+\gamma+n+\frac{1-\delta}{1+\delta}\right)R. \tag{28}$$

If we set

$$P(z_0) = u(z_0) + iv(z_0) = u + iv,$$

then we have

$$\begin{aligned} E &\equiv |P(z_0)[i\rho+\gamma]-(\gamma+1)|^2 \\ &=(u^2+v^2)\rho^2+2v(1+\gamma)\rho+(\gamma u-\gamma-1)^2+\gamma^2v^2 \\ &=(u^2+v^2)\rho^2+2v(1+\gamma)\rho+|\gamma P-(1+\gamma)|^2, \end{aligned}$$

which, in view of (25) and the triangle inequality, yields

$$|\gamma P(z)-(\gamma+1)| \geqq 1-|\gamma|R.$$

We thus find that

$$E \geqq (u^2+v^2)\rho^2+2v(1+\gamma)\rho+(1-|\gamma|R)^2.$$

Now, if

$$E-\left(\gamma+1+n+\frac{1-\delta}{1+\delta}\right)^2R^2 \geqq F(\rho) \geqq 0,$$

where

$$F(\rho) \equiv (u^2+v^2)\rho^2+2v(1+\gamma)\rho+(1-|\gamma|R)^2-\left(\gamma+1+n+\frac{1-\delta}{1+\delta}\right)^2R^2,$$

then the inequality (28) holds true. Moreover, since $u^2+v^2>0$, we have $F(\rho)\geqq 0$ if

$$(1+\gamma)^2v^2-(u^2+v^2)\left[(1-|\gamma|R)^2-\left(\gamma+1+n+\frac{1-\delta}{1+\delta}\right)^2R^2\right] \leqq 0,$$

that is,

$$\begin{aligned} &v^2\left[(1+\gamma)^2-(1-|\gamma|R)^2+\left(\gamma+1+n+\frac{1-\delta}{1+\delta}\right)^2R^2\right] \\ &\qquad \leqq u^2\left[(1-|\gamma|R)^2-\left(\gamma+1+n+\frac{1-\delta}{1+\delta}\right)^2R^2\right]. \end{aligned}$$

Upon some simple calculation and the use of (25), (24), and (20), we conclude that

$$\frac{v^2}{u^2} \leqq \frac{R^2}{1-R^2} \leqq \frac{1-|\gamma|R^2-\left(\gamma+1+n+\frac{1-\delta}{1+\delta}\right)^2R^2}{(1+\gamma)^2-(1-|\gamma|R)^2+\left(\gamma+1+n+\frac{1-\delta}{1+\delta}\right)^2R^2},$$

which completes the proof of Theorem 7. □

Lastly, in this section, we apply Theorem 7 in order to establish the following corollary.

**Corollary 6.** *Under the assumptions of Theorem 7, if $f \in \mathcal{A}_{n,b}$ and*

$$|f''(z)| < nM, \tag{29}$$

*then $L_\gamma[f] \in \mathcal{K}$, where $L_\gamma$ is given by (12) and*

$$0 < M \leqq \frac{n+\frac{1-\delta}{1+\delta}+\gamma+1}{(1+\gamma)\sqrt{\left(n+\frac{1-\delta}{1+\delta}+\gamma+1\right)^2+(\gamma+1)^2}+|\gamma|}.$$

**Proof.** By using the Schwarz lemma (see, for example, [17]), we find that

$$|f''(z)| \leqq nM|z|^{n-1},$$

which means that

$$|f'(z) - 1| = \left| \int_0^z f''(\zeta)\, d\zeta \right| = \left| z \int_0^1 f''(zt)\, dt \right| < \int_0^1 nMt^{n-1}\, dt = M. \tag{30}$$

Now, by applying Theorem 7, we complete the proof of Corollary 6. □

## 4. Concluding Remarks and Observations

In our present investigation, we have first modified one of the most famous theorems on the principle of differential subordination to hold true for normalized analytic functions with a fixed initial Taylor-Maclaurin coefficient. Then, by making use of this modified form, we have generalized and improved a number of results, which appeared, in recent years, in the literature on the geometric function theory of complex analysis. We have also proved some simple conditions for the starlikeness, convexity, and strong starlikeness of such one-parameter families of integral operators as (for example) the familiar Bernardi integral operator and a certain $\mu$-convex integral operator.

Here, in this article, we have established a total of seven main results (Theorems 1 to 7). By suitably specializing the parameters, which are involved in our main results, we have deduced several (known or new) corollaries and consequences thereof. Moreover, wherever possible, we have shown how some of our main results, as well as many of their corollaries and consequences, are related to various results, which are available in the current literature on the subject of our investigation. Remarkably, our Theorems 5 to 7, which involve a fixed second Taylor-Maclaurin coefficient $b$ of the functions in the normalized analytic function class $\mathcal{A}_{n,b}$, have been proven here for the first time and, to the best of our knowledge, even some of their corollaries would provide notable improvements of the available results in the literature.

The various results, which are proven in this article, together with their corollaries and consequences, are potentially useful in encouraging further researches on the subject.

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