mathematics

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Credibilistic Multi-Period Mean-Entropy Rolling Portfolio Optimization Problem Based on Multi-Stage Scenario Tree

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Abstract: This study considers a time-consistent multi-period rolling portfolio optimization issue in the context of a fuzzy situation. Rolling optimization with a risk aversion component attempts to separate the time periods and psychological effects of one's investment in a mathematical model. Furthermore, a resilient portfolio selection may be attained by taking into account fuzzy scenarios. Credibilistic entropy of fuzzy returns is used to measure portfolio risk because entropy, as a measure of risk, is not dependent on any certain sort of symmetric membership function of stock returns and may be estimated using nonmetric data. Mathematical modeling is performed to compare the Rolling Model (RM) and the Unified Model (UM). Two empirical studies from the Tehran stock market (10 stocks from April 2017 to April 2019) and the global stock market (20 stocks from April 2021 to April 2023) are utilized to illustrate the applicability of the suggested strategy. The findings reveal that RM can limit the risk of the portfolio at each time, but the portfolio's return is smaller than that of UM. Furthermore, the suggested models outperform the standard deterministic model.

Keywords: portfolio optimization; fuzzy entropy; rolling optimization; credibility theory; scenario tree; multi-period portfolio model

MSC: 90-10; 90B90; 90C05; 90C29; 90C70; 90C90; 91-05; 91-10

Citation: Peykani, P.; Nouri, M.; Pishvaee, M.S.; Oprean-Stan, C.; Mohammadi, E. Credibilistic Multi-Period Mean-Entropy Rolling Portfolio Optimization Problem Based on Multi-Stage Scenario Tree. *Mathematics* **2023**, *11*, 3889. https://doi.org/10.3390/math11183889

Academic Editor: Andrea Scozzari

Received: 19 July 2023
Revised: 9 September 2023
Accepted: 10 September 2023
Published: 12 September 2023

1. Introduction

Investing in financial markets aims to achieve both the goals of maximizing expected return and minimizing the risk of return. Investment portfolio construction is one of the traditional approaches for achieving these objectives [1–5]. Markowitz [6] proposed a mean-variance model for portfolio selection, which serves as the foundation for contemporary portfolio theory. The basic goals of Markowitz's mean-variance model are to maximize anticipated return and reduce expected risk. In the actual world, portfolio strategies are frequently multi-period, allowing investors to reassess their investment strategy, despite the fact that various extensions have been developed based on the notion of Markowitz's mean-variance model in a single period horizon [7].

Despite the fact that [8,9] explored the multi-period stochastic programming model for portfolio selection problem, they provided more credibility to this topic by focusing on the subject of scenario tree usage and investment in their investigations. Mulvey et al. [10] presented a non-linear model that uses asset/liability management to control risk over long time periods. Dupaová [11] proposed chosen techniques for analyzing results obtained by solving stochastic programs, focusing on the moment problem and parametric optimization outcomes. Mulvey and Shetty [12] proposed a framework for modeling basic financial planning issues that was based on multi-stage optimization under scenarios as a technique for dealing with uncertainty and used interior-point methods to solve it. Hibiki [13] presented and evaluated a hybrid model based on multi-period stochastic optimization to the scenario tree model.

Edirisinghe and Patterson [14] examined a stochastic multi-stage programming model and advocated for the use of block separable recourse structures as well as ways for constructing such structures inside a layered L-shaped decomposition. Gülpinar and Rustem [15] developed a multi-period mean-variance optimization approach for the worst-case design of the stochastic features of the scenario tree. To deal with market uncertainty, Pinar [16] developed multi-stage portfolio selection models with a linear composed objective and a simulated market model. Şakar and Köksalan [17] examined a stochastic programming solution for a multi-criterion, multi-period portfolio optimization problem. They employed a single index model to predict stock returns from a market-representative index and a random walk model to build scenarios based on the probable values of the index return. Najafi and Mushakhian [18] proposed a multi-stage stochastic mean semi-variance Conditional Value at Risk (CVaR) model using scenario trees as a technique for dealing with uncertainty. To tackle this, they devised a hybrid metaheuristic based on the Genetic Algorithm (GA) and Particle Swarm Optimization (PSO) methods.

The scenario tree approach was used by Chen et al. [19] to transform the multi-period portfolio selection problem with terminal distortion risk measure into a deterministic convex programming problem. To cope with market uncertainty, Liu and Chen [20] established two multi-period robust risk measures inside a regime switching framework and used a scenario tree. Nesaz et al. [21] created a multi-period optimization issue with their contribution of a unique Lower Partial Moment (LPM) model computation and solved it with the meta-heuristic method Non-dominated Sorting Genetic method II (NSGA II). They also employed quantitative performance metrics to demonstrate the efficacy of the suggested strategy.

There are several research studies on multi-period portfolio selection issues in the literature, the majority of which propose a single target regardless of other investment goals. However, while the scenario tree is well known in finance as one of the most important tools for managing uncertainty, it cannot capture numerous non-probabilistic aspects in actual financial markets, such as social, economic, political, and psychological factors. Credibility theory can be used to define the uncertainty environment in the financial market. Mohebbi and Najafi [22] developed a bi-objective mean-VaR portfolio selection model that incorporates fuzzy credibility theory and a scenario tree. Their suggested model was non-linear, and their recommended solution algorithm was influenced by it. Furthermore, their model was not created using nodes.

Peykani et al. [23] proposed a novel Fuzzy Multi-Period Multi-Objective Portfolio Optimization (FMPMOPO) model that may be used in the face of ambiguous data and practical constraints such as budget, cardinality, and bound limits. It should be noted that the proposed FMPMOPO model considers three objectives, including terminal wealth, risk, and liquidity, as well as real-world constraints. The alpha-cut method is also used to deal with fuzzy data. Lam et al. [24] developed a mean-absolute deviation-entropy model for portfolio optimization which includes entropy maximization as a multi-objective optimization technique. Furthermore, the proposed model makes use of a goal-programming method to account for the optimal value of each objective function. The key aims of the proposed model are to optimize portfolio entropy, decrease absolute deviation, and maximize mean return. Peykani et al. [25] proposed a novel uncertain portfolio optimization method that may be applied when there are fuzzy data and linguistic elements present. It should be noted that the recommended Fuzzy Portfolio Optimization (FPO) model takes into consideration investment constraints, mean (return), absolute deviation (non-systematic risk measure), and beta (systematic risk measure). To deal with the uncertainty of financial data, the Possibilistic Programming (PP) and Chance-Constrained Programming (CCP) methodologies are used.

Novais et al. [26] calculated risk using entropy and mutual information rather than variance and covariance, and they compared the performance of the original Markowitz model both inside and outside of the sample to that of the recommended model, as well as other cutting-edge shrinkage approaches. Nasini et al. [27] proposed a novel optimization framework for multi-market portfolio management in which a centralized headquarter delegated market-wise portfolio selection to specialised affiliates. Because the headquarter is risk-averse, it calculates the maximum expected loss (in the form of conditional value at risk) for the affiliates endogenously. As a result, the affiliates develop portfolios and retain a portion of the expected investment returns as management fees.

Nouri et al. [28] suggested a multi-stage stochastic portfolio selection technique. To cope with market uncertainty, a three-pronged aim and a portfolio selection model based on a scenario tree called Wealth Mean Absolute Semi-Deviation Liquidity (WMAL) were designed. Because of their duration, continuity of the horizon, and unpredictability, the scenario tree is an ideal tool for modeling multi-period portfolio problems. It is critical to evaluate wealth, risk, the asset investment cap, transaction costs, and liquidity when considering the situation at hand. In this study, rebalancing and mean absolute semi-deviation are utilized as indices of portfolio risk. For long-term investment horizons, effective investment plans are constructed utilizing the Node-Based Modeling (NBM) approach. When the goal programming approach is used, the investigated multi-objective model becomes a single-objective model. Table 1 presents a comparative literature evaluation.

Table 1. Overview of uncertain multi-period portfolio optimization studies in the literature.

Year	Research	MC	NO	NBM	TS	NS	CF	BC	MR	TC	LQ	RAF	RM	Method(s)
1989	Mulvey & Vladimirou [8]	NLP	SO	✓	✓		✓	✓		✓			VAR	PDP
1997	Mulvey et al. [10]	NLP	SO	✓			✓	✓		✓			VAR	TBS
1999	Dupačová [11]	NLP	SO	✓	✓		✓	✓		✓			MAD	WCA
2004	Mulvey & Shetty [12]	NLP	SO	✓			✓	✓					VAR	IPM
2007	Edirisinghe & Patterson [14]	NLP	SO	✓	✓		✓	✓	✓	✓			VAR	NM
2007	Gülpınar & Rustem [15]	NLP	SO	✓	✓		✓	✓		✓			VAR	WCD/MMO
2007	Pinar [16]	NLP	SO	✓				✓		✓			SV	-
2013	Sakar & Köksalan [17]	NLP	MO	✓		✓	✓	✓	✓		✓		CVaR	SIM
2015	Najafi & Mushakhian [18]	NLP	SO				✓	✓	✓				SV/CVaR	HMA/TED
2016	Chen et al. [19]	LP	SO	✓		✓	✓	✓		✓			LPM	DCP
2018	Liu & Chen [20]	NLP	SO		✓			✓					CVaR	DP
2018	Mohebbi & Najafi [22]	NLP	MO		✓			✓			✓		VaR	DP
2020	Nasaz et al. [21]	NLP	MO			✓		✓		✓			LPM	NSGAII/TED
2021	Peykani et al. [23]	NLP	MO		✓	✓	✓	✓		✓	✓		MAD	GP
2021	Lam et al. [24]	LP	MO					✓	✓				AD/EP	GP
2021	Peykani et al. [25]	NLP	SO		✓	✓		✓					AD/Beta	PP/CCP
2022	Novais et al. [26]	NLP	SO					✓	✓			✓	EP	SQSLP/CVXOPT
2022	Nasini et al. [27]	MIP	SO		✓	✓							CVaR	BCA
2023	Nouri et al. [28]	LP	MO	✓	✓	✓	✓	✓	✓	✓	✓		SD	GP
2023	This work	LP	MO	✓	✓	✓	✓	✓	✓	✓	✓	✓	FEP	GP/RO/CRM

MC: Modelling Class, LP: Linear Programming, NLP: Non-Linear Programming, MIP: Mixed-Integer Programming, NO: Number of Objectives, SO: Single Objective, MO: Multi-Objective, NBM: Node Based Modelling, TS: Threshold, NS: No Short Selling, CFC: Cash Flow Constraint, BC: Budget Constraint, MR: Minimum Return Constraint, TC: Transaction Costs, LQ: Liquidity, RAF: Risk Aversion Factor, RM: Risk Measure, VAR; Variance, MAD: Mean Absolute Deviation, SV: Semi-Variance, CVaR: Conditional Value at Risk, VaR: Value at Risk, LPM: Low Partial Moments, AD: Absolute Deviation, EP: Entropy, SD: Semi-Deviation, FEP: Fuzzy Entropy, PDP: Proposed Decomposition Procedure, TBS: Tabu Search, WCA: Worst-Case Analysis, IPM: Interior-point methods, NM: New Method, WCD: Worst-Case Design, MMO: Min–Max Optimization, SIM: Single Index Model, HMA: Hybrid Metaheuristic Algorithm, TED: Taguchi Experimental Design, DCP: Deterministic Convex Programming, DP: Dynamic Programming, NSGAII: Non-Dominated Sorting Genetic Algorithm II, GP: Goal Programming, PP: Possibilistic Programming, CCP: Chance-Constraint Programming, RO: Rolling Optimization, CRM: Credibility Measure, BCA: Branch-and-Cut Algorithm.

In this study, for the first time, the portfolio selection issue is formulated using nodes in a scenario tree that are incorporated into fuzzy credibility theory. Two bi-objective mean-entropy portfolio selection models with static risk factor aversion and transaction costs are proposed for the two multi-period portfolio problem models. Both proposed models are totally linear and are constructed using nodes. The periods are continuous in the first model, known as the Unified Model (UM), but separated in the second model, known as the Rolling Model (RM). RM is defined as a sequence of rolling linear programming problems that can be solved using simplex approaches.

This paper is organized as follows: The definitions of the credibilistic mean, the credibilistic entropy of the fuzzy returns of assets, and other features are introduced in Section 2. Section 3 discusses the mean-entropy multi-period portfolio unified model under fuzzy situations within the context of credibility theory, where expected value and entropy are used to calculate the return and risk of each period's investment, respectively. The mean-entropy multi-period portfolio rolling optimization models are then constructed. The models are transformed into deterministic linear programming in this stage, which can be readily solved using simplex methods. Section 4 includes two empirical studies based on real data to demonstrate the efficacy of the modeling methodologies, and simulation tests are used to assess the suggested models. Section 5 summarizes the findings and future potential research areas.

2. Preliminaries

2.1. Credibility Measure

This section presents several concepts from credibility theory that will be useful in the following sections. Though both necessity and possibility measures satisfy the properties of normality, non-negativity, and monotonicity, neither fulfills the self-duality that establishes a powerful connection between the measures of an event and the measures of the opposed event (each determining the other) [29–35]. In both theory and practice, self-duality has been acknowledged as intuitive and necessary for laying the theoretical underpinnings of an efficient measurement theory [36]. A credibility measure is described by Liu and Liu [37] as the average value of the possibility measure and the necessity measure, which as a set function, attempt to meet the axioms of normalcy, monotonicity, self-duality, and maximality. We assume ξ is a fuzzy variable with the membership function $\mu(x)$ and r is a real number. The credibility of $Cr\{\xi \geq r\}$ is, therefore, defined as

$$Cr\{\xi \geq r\} = \frac{1}{2}(Pos(\xi \geq r) + Nec(\xi \geq r)) = \frac{1}{2}\left(\underset{x\geq r}{Sup}\mu(x) + 1 - \underset{x<r}{Sup}\mu(x)\right) \tag{1}$$

where $Cr\{\xi \geq r\} + Cr\{\xi < r\} = 1$, i.e., the credibility measure exhibits self-duality. A fuzzy event will almost surely occur if its credibility value is more than one, and it will fail if its credibility value is equal to zero.

Definition 1. *Suppose that ξ is a fuzzy variable, then its expected value is defined as*

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq r\}dx - \int_{-\infty}^{0} Cr\{\xi \leq r\}dx \tag{2}$$

The credibility distribution $\phi : \Re \to [0,1]$ of a fuzzy variable ξ is defined by Liu & Liu [37] as

$$\Phi(x) = Cr\{\theta \in \Theta | \, \xi(\theta) \leq x\} \tag{3}$$

Furthermore, according to Definition 1, Liu & Liu [37] proposed that we have $E[\lambda_1\xi + \lambda_2] = \lambda_1 E[\xi] + \lambda_2$ for any real numbers such as λ_1 and λ_2. Also, for ξ and η to be two independent fuzzy variables with finite anticipated values, and for any real integers like λ_1 and λ_2, we have $E[\lambda_1\xi + \lambda_2\eta] = \lambda_1 E[\xi] + \lambda_2 E[\eta]$.

Triangular and trapezoidal fuzzy variables, two well-known and useful fuzzy variables, are used to determine the believability of a fuzzy event.

Example 1. *Let $\xi = (a, b, c)$ be a triangular fuzzy variable $(a < b < c)$ with a member-ship function provided by Equation (4) (see Figure 1), and let $\{\xi \leq r\}$ be a fuzzy event with a credibility determined by Equation (5):*

$$\mu(r) = \begin{cases} \frac{r-a}{b-a}, & if \quad a \leq r \leq b \\ \frac{c-r}{c-b}, & if \quad b \leq r \leq c \\ 0, & otherwise. \end{cases} \tag{4}$$

$$\{\xi \leq r\} = \begin{cases} 0, & if \quad r \leq a, \\ \frac{r-a}{2(b-a)}, & if \quad a \leq r \leq b, \\ \frac{c-2b+r}{2(c-b)}, & if \quad b \leq r \leq c, \\ 1, & if \quad c \leq r. \end{cases} \tag{5}$$

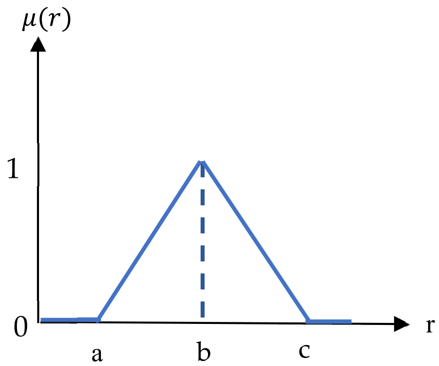

Figure 1. Triangular membership function.

The graphical demonstration of the credibility of fuzzy event $\{\xi \leq r\}$ is depicted in Figure 2.

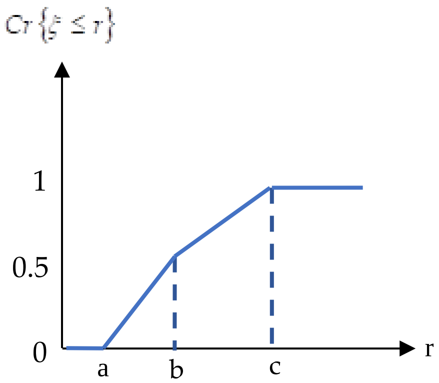

Figure 2. Credibility of triangular fuzzy variable.

If at least one of the two integrals is finite in Equation (3), the predicted value of the triangular fuzzy variable $\xi = (a, b, c)$ is given by

$$E[\xi] = (a + 2b + c)/4 \tag{6}$$

Example 2. *Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable $(a < b < c < d)$ with a membership function provided by Equation (7) (see Figure 3), and let $\{\xi \le r\}$ be a fuzzy event with a credibility determined by Equation (8):*

$$\mu(r) = \begin{cases} \frac{r-a}{b-a}, & if \quad a \le r \le b \\ 1, & if \quad b \le r \le c \\ \frac{c-r}{c-b}, & if \quad c \le r \le d \\ 0, & if \quad otherwise. \end{cases} \tag{7}$$

$$\{\xi \le r\} = \begin{cases} 0, & if \quad r \le a, \\ \frac{r-a}{2(b-a)}, & if \quad a \le r \le b, \\ \frac{1}{2} & if \quad b \le r \le c, \\ \frac{c-2b+r}{2(c-b)}, & if \quad c \le r \le d, \\ 1, & if \quad otherwise. \end{cases} \tag{8}$$

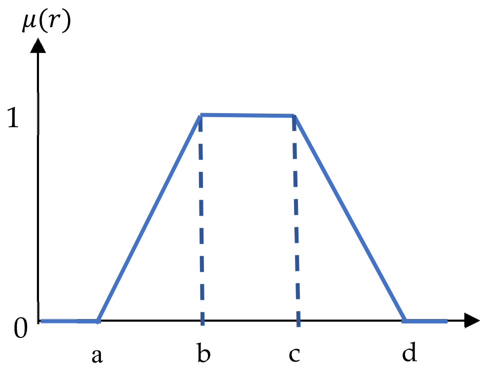

Figure 3. Trapezoidal membership function.

The graphical demonstration of the credibility of fuzzy event $\{\xi \le r\}$ is depicted in Figure 4.

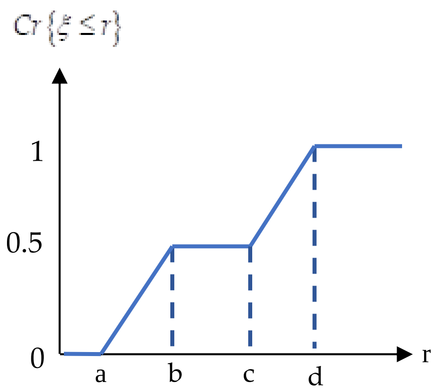

Figure 4. Credibility of trapezoidal fuzzy variable.

If at least one of the two integrals is finite in Equation (3), the predicted value of the trapezoidal fuzzy variable $\xi = (a, b, c, d)$ is given by

$$E[\xi] = (a + b + c + d)/4 \tag{9}$$

2.2. Fuzzy Entropy

Let us briefly review the notion of credibility theory because we will need it in the following section. In real financial markets, often investors have no or little accurate information on the stock returns and the fuzzy portfolio return cannot decrease the lack of information. Therefore, it is possible for an investor that his/her preferences are not met by the fuzzy portfolio returns. In this case, the portfolio entropy will attain its maximum value since the uncertainty of portfolio return and, consequently, the risk is minimized. In fact, entropy is the distribution of validity, which is a measure of uncertainties related to fuzzy variables. The entropy of credibility distribution can be partly regarded as an alternative measure for incompatibility [38]. Shannon [39] established entropy as an uncertainty measure in 1948, and Philippatos and Wilson [40] used it to choose portfolios as a new risk measure to replace variance in the Markowitz mean-variance model. According to research, entropy is more dynamic and deeper than variance and will improve portfolio optimization [41]. Recently, Li and Liu [42] proposed that when utilizing credibility distribution as an alternative risk measure, fuzzy entropy is useful in tackling fuzzy portfolio optimization issues. Huang [43] further demonstrates the efficiency of entropy in tackling fuzzy portfolio optimization issues.

Definition 2. *Assume ξ is a continuous fuzzy variable with a credibility function v on the credibility space $(\Theta, \mathcal{P}, Cr)$. The entropy is thus defined as follows [42]:*

$$H[\xi] = -\int_{-\infty}^{+\infty} S(v(x))dx \tag{10}$$

where $S(t)$ is a continuous and differentiable function, as illustrated in Figure 5, and is defined as follows:

$$S(t) = -t\ln t - (1-t)\ln(1-t) \tag{11}$$

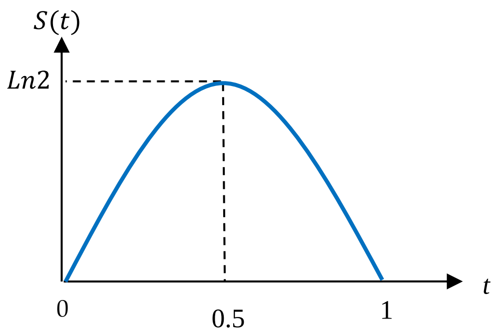

Figure 5. Function $S(t)$.

Theorem 1. *Assume ξ be a continuous fuzzy variable. Then for any real numbers of a and b, we have [42]:*

$$H[a\xi + b] = |a|H[\ \xi]. \tag{12}$$

Example 3. *The expected value of triangular fuzzy variable $\xi = (a, b, c)$ is determined by Equations (10) and (11):*

$$H[\xi] = (c-a)/2 \tag{13}$$

And the expected value of trapezoidal fuzzy variable $\xi = (a, b, c, d)$ is given by

$$H[\xi] = (b+d-a-c)/2 + (c-b)\ln 2. \tag{14}$$

2.3. Scenario Tree

Scenarios are commonly used to simulate random parameters in multi-period stochastic programming models. A tree structure is used to build scenarios. The model is based on an extension of the decision space that takes into account the scenario tree's conditional character. Each node makes conditional judgments based on the modeling restrictions (see Figure 6). The number of choice factors and restrictions in the scenario tree may expand exponentially to guarantee that the generated representative set of scenarios adequately covers the set of possibilities [13]. A scenario tree model is what this model is termed.

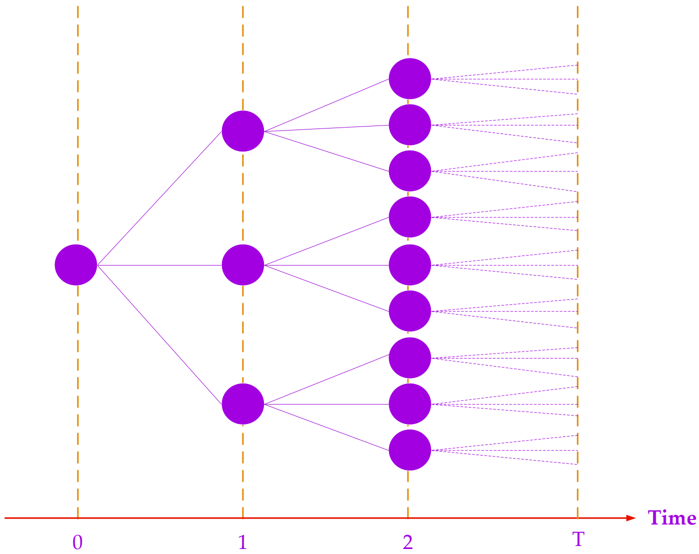

Figure 6. A scenario tree.

3. Model Formulation

In this section, UM is calculated without a risk aversion factor, assuming that the factor is equal to one. Furthermore, RM is not used in UM. A time-consistent multi-period rolling portfolio optimization problem is also developed.

3.1. The Unified Model: Before Rolling Model

A multi-period portfolio optimization issue with I hazardous assets in a financial market is studied in this section, and the optimal portfolio at each node is chosen using a psychological risk aversion factor. At the start of the first period, the investor simply intends to divide his or her whole available wealth (including prior period returns) among I risky assets. We use anticipated value as a return metric and entropy as a risk measure within the context of credibility theory. $\mathrm{W}^{\mathrm{s}}_{\mathrm{t+1}}$ is likewise a fuzzy variable, according to the Extension Principle [43]. We assume the investor has W available at the start of the first investing term. The following notations will be used from now on:

Index	
i	*risky asset* $i = 1, 2, 3, \ldots, I$
t	*investment period* $t = 1, 2, 3, \ldots, T$
s	*scenario* $s = 1, 2, 3, \ldots, s_t$
Parameter	
φ	*node in each scenario,*
$\xi^s_{t,i}$	*return of risky asset i at period t under scenario s, as a fuzzy variable*
$u^\varphi_{t,i}$	*upper bound of* $x^\varphi_{t,i}$
$l^\varphi_{t,i}$	*lower bound of* $x^\varphi_{t,i}$
$c_{t,i}$	*unit transaction cost of risky asset i at period t*
p^s_t	*the occurrence probability of scenario s at period t*
S_t	*the number of scenarios at period t that branch from each node*
ϕ_t	*the number of nodes at the beginning of period t*
W^s_t	*available wealth at the beginning of period t*
B^φ_t	*budget of investment in node* φ *at the beginning of period t*
C^φ_t	*total transaction cost of portfolio in node* φ *at the beginning of period t*
ω^φ_t	*risk aversion factor at the tth investment for node* φ *at the beginning of period t*

Variable	
$x_{t,i}^{\varphi}$	*invest of risky asset i at period t for node φ, as a decision variable*
X_t^{φ}	*portfolio at period t for node φ, i.e., $X_t^{\varphi} = \left(x_{t,1}^{\varphi}, x_{t,2}^{\varphi}, \ldots x_{t,I}^{\varphi}\right)$*

Patel and Subrahmanyam [44] showed that disregarding transaction costs during portfolio trading frequently results in suboptimal portfolios. The transaction costs are specified as a V-shape function of the difference between the tth and $t-1$th period portfolios [45], and the transaction cost of a hazardous asset at period t for a node is $c_{t,i}^{\varphi}\left|x_{t,i}^{\varphi} - x_{t-1,i}^{\acute{\varphi}}\right|$ with $\varphi\epsilon\phi_t$ and $\acute{\varphi}\epsilon\phi_{t-1}$. Therefore, the total transaction costs of the portfolio during period t for a node are given as $C_t^{\varphi} = \sum\limits_{i=1}^{I} c_{t,i}^{\varphi}\left|x_{t,i}^{\varphi} - x_{t-1,i}^{\acute{\varphi}}\right|$. The model is built using node-based scenario modeling with two objective functions. Objective 15 is the first objective function that minimizes the predicted entropy of the portfolio in late-stage situations. As the second function, Objective 16 computes the expected value of wealth at the end of the term. It is assumed that $x_{0,1}^{\varphi} = 0$ for $i\epsilon I$.

$$Min \sum_{s=1}^{S_T}\sum_{i=1}^{I} p_T^s.H\left[\xi_{T,i}^s\right].x_{T,i}^{\varphi} \tag{15}$$

$$Max \sum_{s=1}^{S_T} p_T^s.W_T^s \tag{16}$$

$$S.t. \quad \sum\nolimits_{i=1}^{I} x_{1,i}^{\varphi} + \sum\nolimits_{i=1}^{I} c_{1,i}^{\varphi} x_{1,i}^{\varphi} = W, \quad \varphi\epsilon\phi_1 \tag{17}$$

$$\sum\nolimits_{i=1}^{I} x_{t,i}^{\varphi} + \sum\nolimits_{i=1}^{I} c_{t,i}^{\varphi}\left|x_{t,i}^{\varphi} - x_{t-1,i}^{\acute{\varphi}}\right| = \sum\nolimits_{i=1}^{I} x_{t,i}^{\varphi}\left(1 + E\left[\xi_{t,i}^s\right]\right), \quad t = 2, \ldots, T, \quad \acute{\varphi}\epsilon\phi_{t-1}, \quad \varphi\epsilon\phi_t \tag{18}$$

$$W_T^s = \sum\nolimits_{i=1}^{I} x_{T,i}^{\varphi}\left(1 + E\left[\xi_{T,i}^s\right]\right) \quad s\epsilon S_T, \quad \varphi\epsilon\phi_T \tag{19}$$

$$l_{t,i} \le x_{t,i}^{\varphi} \le u_{t,i}, \quad i\epsilon I, \quad t \in T \tag{20}$$

$$x_{t,i}^{\varphi} \ge 0, \quad i\epsilon I, \quad t \in T \tag{21}$$

The first period's capital budget restriction is thus Constraint (17). The total of investment and transaction expenses equals the budget set in Constraint (18). For each node positioned beyond the first period, the capital budget limitation is taken into account. Constraint (19) is used to compute the final wealth at time T. Constraint (20) expresses the upper and lower limits to restrict the lowest and maximum proportion of money that may be invested in a single asset. Short selling of assets in any specific time period is also prohibited, as stipulated by Constraint (21).

3.2. The Rolling Modeling for Portfolio Optimization

Markowitz's mean-variance investing concept [46] states that an investor should strike a balance between maximizing mean and limiting variance. It is possible to combine these two opposing goals into a convex combination of $Var(W_t) - \omega E(W_t)$ by inserting a predetermined risk aversion factor ω.

The proposed model is developed on the premise that the best portfolio for each investment period is optimized in RM, which implies that the available wealth in each node at the start of each period is already chosen by the preceding node's optimal portfolio. As a result, the investor adapts his or her risk aversion to the next investment node period and distributes all the available capital among hazardous assets. By including time-consistent risk aversion in each node of investment periods in this RM, the investor can establish a more realistic multi-period portfolio investment plan.

Securities returns are believed to be independent of various investment periods. The calculated risk aversion factor is used to explain the investor's shifting risk attitude toward the next investment node period. The time consistent multi-period rolling portfolio optimization model comprises ϕ_t single period portfolio selection procedures with varied risk aversions at period t. Figure 7 depicts an overview of the Rolling Method.

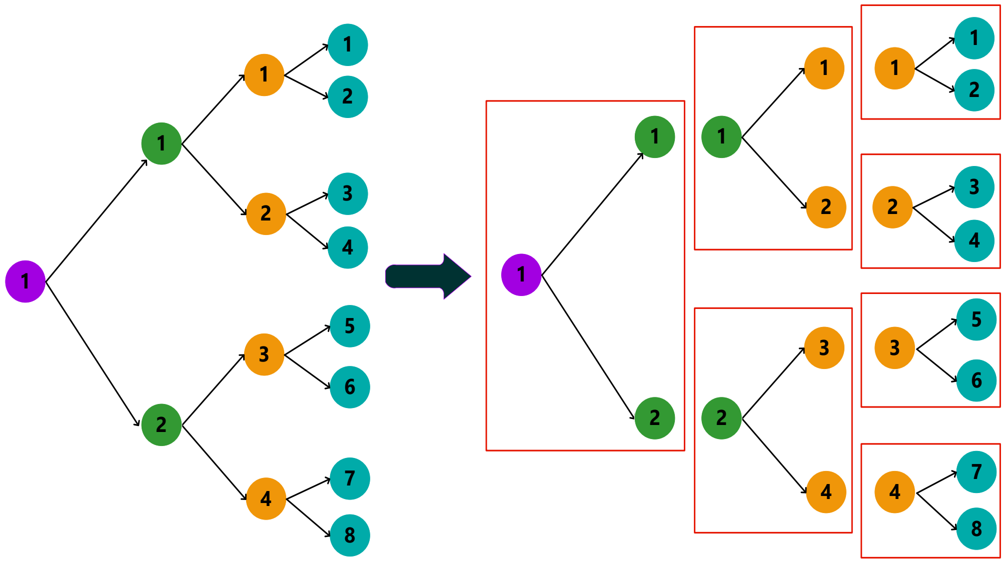

Figure 7. Overview of the scenario model analysis based on the Rolling Method.

At the beginning of the first period, there is just one node, $\phi_t = 1$, and the investor's risk aversion is considered to be ω_1^1. The capital budget is B_1^1 at the first period. Therefore, $\sum_{i=1}^{I} x_{1,i}^1 = B_1^1$ is the budget constraint. Since the number of nodes represents the number of models in each period, there is no transaction cost in the first node. In the first period, there is only one portfolio optimization model, $X_1^1 = \left(x_{1,1}^1, x_{1,2}^1, \ldots, x_{1,I}^1\right)$, which can be obtained by solving the following linear programming model M_1^1:

$$Min \left(\sum_{s=1}^{S_1} \sum_{i=1}^{I} p_1^s.H\left[\xi_{1,i}^s\right].x_{1,i}^1\right) - \omega_1^1 \left(\sum_{s=1}^{S_1} p_1^s.W_1^s\right) \tag{22}$$

$$S.t. \sum_{i=1}^{I} x_{1,i}^1 = B_1^1 \tag{23}$$

$$W_1^s = \sum_{i=1}^{I} x_{1,i}^1\left(1 + E\left[\xi_{1,i}^s\right]\right),\ s\epsilon S_1 \tag{24}$$

$$l_{1,i} \leq x_{1,i}^1 \leq u_{1,i},\ i\epsilon I \tag{25}$$

$$x_{1,i}^1 \geq 0,\ i\epsilon I \tag{26}$$

After solving the model $M_1 = \langle M_1^1 \rangle$ and obtaining the portfolio X_1^1, the budget value B_2^{φ} is calculated for each model in the second period. We know that $\phi_2 = S_1$, so we have:

$$B_2^{\varphi} = \sum_{i=1}^{I} x_{1,i}^1\left(1 + E\left[\xi_{1,i}^s\right]\right),\ (\varphi = s)\epsilon S_1 \tag{27}$$

In the second investment period, the set of linear models $M_2 = \left\langle M_2^1, M_2^2, \ldots, M_2^{\phi_2} \right\rangle$ is solved from the second to the last period and the transaction costs are also included. Model M_2^{φ} for $\varphi \epsilon \phi_2$ will be as follows:

$$Min\left(\sum_{s=1}^{S_2}\sum_{i=1}^{I} p_2^s.H\left[\xi_{2,i}^s\right].x_{2,i}^{\varphi}\right) - \omega_2^{\varphi}\left(\sum_{s=1}^{S_2} p_2^s.W_2^s\right) \tag{28}$$

$$S.t. \quad \sum\nolimits_{i=1}^{I} x_{2,i}^{\varphi} + \sum\nolimits_{i=1}^{I} c_{2,i}^{\varphi}\left|x_{2,i}^{\varphi} - x_{1,i}^{1}\right| = B_2^{\varphi}, \quad \acute{\varphi}\epsilon\phi_1 \tag{29}$$

$$W_2^s = \sum\nolimits_{i=1}^{I} x_{2,i}^{\varphi}\left(1 + E\left[\xi_{2,i}^s\right]\right),\ s\epsilon S_2 \tag{30}$$

$$l_{2,i} \le x_{2,i}^{\varphi} \le u_{2,i},\ i\epsilon I \tag{31}$$

$$x_{2,i}^{\varphi} \ge 0,\ i\epsilon I \tag{32}$$

After solving the model $M_2 = \left\langle M_2^1, M_2^2, \ldots, M_2^{\phi_2} \right\rangle$ and obtaining the portfolio X_2^{φ}, the budget value B_3^{φ} is calculated for each model in the second period. We know that $\phi_3 = S_1 * S_2$, so we have:

$$B_3^{\varphi} = \sum\nolimits_{i=1}^{I} x_{1,i}^{\acute{\varphi}}\left(1 + E\left[\xi_{1,i}^s\right]\right),\ \varphi\epsilon\phi_3, \quad \acute{\varphi}\epsilon\phi_2, \quad s\epsilon S_2 \tag{33}$$

Likewise, until the last period, we solve the set of linear models $M_T = \left\langle M_T^1, M_T^2, \ldots, M_T^{\phi_T} \right\rangle$. Consider that B_T^{φ} will be calculated for each model in the period $T-1$. We know that $\phi_T = S_{T-1} \times \ldots \times S_2 \times S_1$. Model M_T^{φ} for $\varphi\epsilon\phi_T$ is as follows:

$$Min\left(\sum_{s=1}^{S_T}\sum_{i=1}^{I} p_T^s.H\left[\xi_{T,i}^s\right].x_{T,i}^{\varphi}\right) - \omega_T^{\varphi}\left(\sum_{s=1}^{S_T} p_T^s.W_T^s\right) \tag{34}$$

$$S.t. \quad \sum\nolimits_{i=1}^{I} x_{T,i}^{\varphi} + \sum\nolimits_{i=1}^{I} c_{T,i}^{\varphi}\left|x_{T,i}^{\varphi} - x_{T-1,i}^{\acute{\varphi}}\right| = B_T^{\varphi},\ \acute{\varphi}\epsilon\phi_{T-1} \tag{35}$$

$$W_T^s = \sum\nolimits_{i=1}^{I} x_{T,i}^{\varphi}\left(1 + E\left[\xi_{T,i}^s\right]\right),\ s\epsilon S_T \tag{36}$$

$$l_{T,i} \le x_{T,i}^{\varphi} \le u_{T,i},\ i\epsilon I \tag{37}$$

$$x_{T,i}^{\varphi} \ge 0,\ i\epsilon I \tag{38}$$

4. Implementation and Evaluation

Two empirical investigations are used to test the proposed model for multi-period portfolio selection and the effectiveness of the rolling approach for solving it. The first empirical research explores a case study in the Tehran stock exchange (TSE) market, while the second empirical study presents a case study in an international stock exchange. In the following sections, these two cases are described.

4.1. First Empirical Case Study: Tehran Stock Market

The historical data of ten stocks were collected from April 2017 to April 2019, and each week was designated as an investing period to manage the data. We assume an investor wishes to invest in a three-period portfolio (i.e., T = 3) and needs ten particular equities from the Tehran stock exchange. Also, the number of scenarios at each period is two, i.e., $S_t = 2$ for $t \in T$. The creation of scenarios in each period is exponential. Thus, there is one scenario at beginning of the first period, two scenarios at beginning of the second period, four scenarios at the beginning of third period, and eight scenarios at the end of the third period (i.e., 1, 2, 4, and 8). At the start of each period, the investor invests all available capital in a few equities, assuming time-consistent risk aversion. Triangular fuzzy variables characterize the returns of the ten stocks across each time.

The beginning wealth of the investor is $W_1^1 = 1$, and the transaction cost at each period is 0.0001. The maximum bound of the stock investment proportion for each period is fixed at 0.3, while the lower bound is set at 0. The proposed model is solved in two ways: At first, it is solved in the UM and then by RM, with both models considering the risk aversion factor. The proposed model is then run in GAMS 24.1.2.

As stated earlier, the objective function is defined as $Var(W_t) - \omega E(W_t)$. The values of variables $x_{t,i}^{\varphi}$ obtained after solving UM are listed in Table 2. Then the return and entropy are obtained for each period. Table 2 presents these results. The portfolio is formed at the beginning of each period, but the return and entropy are calculated at the end of each period.

Table 2. Optimal portfolio from UM.

Asset	Period 1	Period 2		Period 3			
	Node 1	**Node 1**	**Node 2**	**Node 1**	**Node 2**	**Node 3**	**Node 4**
SK01	0	0	0	0	0	0	0
SK02	0	0	0	0.3	0.3	0.3	0.3
SK03	0.3	0.3	0	0	0	0	0
SK04	0	0	0.153	0.055	0.281	0	0
SK05	0	0	0	0.3	0	0.3	0.3
SK06	0	0	0.3	0.3	0	0	0
SK07	0.099	0.3	0.3	0	0	0	0
SK08	0.3	0	0.3	0	0	0.251	0.3
SK09	0.3	0.258	0	0	0.3	0.3	0
SK10	0	0.3	0	0.3	0.3	0	0.267

When RM is implemented in the problem for this multi-period portfolio selection model, the model is transformed into seven single-period models for portfolio selection in each period. Furthermore, seven risk aversion factors are required for this model. In this empirical example, it is assumed that the investor is risk-averse and is looking for low-risk investments. In RM, the risk aversion factors are assumed to be a fixed number in each node ($\omega = 1$). The optimal values of variables $x_{t,i}^{\varphi}$ are presented in Table 3.

Table 3. Optimal portfolio from RM at initial of each period.

Asset	Period 1	Period 2		Period 3			
	Node 1	**Node 1**	**Node 2**	**Node 1**	**Node 2**	**Node 3**	**Node 4**
SK01	0	0	0	0	0	0	0.3
SK02	0	0	0	0.3	0.3	0.3	0
SK03	0	0.3	0	0	0	0	0
SK04	0.099	0	0.145	0.008	0.246	0	0
SK05	0.3	0	0	0.3	0	0.3	0.3
SK06	0	0	0.3	0.3	0	0	0
SK07	0	0.3	0.3	0	0	0	0
SK08	0.3	0	0.3	0	0	0.241	0.3
SK09	0	0.222	0	0	0.3	0.3	0
SK10	0.3	0.3	0	0.3	0.3	0	0.258

Then the wealth and entropy are obtained for each period (see Table 4). The optimal portfolio is formed at the beginning of each period, but the wealth and entropy are calculated at the end.

Table 4. Wealth and entropy are obtained from the UM and RM at the end of each period.

Period	Node	UM		RM	
		Return	Entropy	Return	Entropy
Period 1	1	5.40%	0.344	4.00%	0.182
	2	15.90%	0.418	2.30%	0.207
Period 2	1	16.80%	0.406	16.00%	0.404
	2	15.30%	0.237	14.30%	0.235
	3	18.20%	0.155	14.80%	0.153
	4	25.70%	0.339	22.10%	0.337
Period 3	1	22.20%	0.180	22.20%	0.100
	2	18.90%	0.117	22.10%	0.077
	3	27.90%	0.267	26.90%	0.264
	4	18.70%	0.264	17.80%	0.263
	5	23.00%	0.215	19.00%	0.204
	6	17.10%	0.141	13.60%	0.136
	7	25.30%	0.215	20.50%	0.208
	8	23.40%	0.140	19.30%	0.128

In RM, the investor seeks low-risk investments, so he/she makes risk-averse decisions. At each period, he/she is trying to reduce his/her risk by buying low-risk stocks, which leads to a safer profit with low risk. Compared with UM, the risk (entropy) of the investment by RM is decreased about 19%. Usually, the assets that make the most returns are riskier than the stocks with the least returns. In RM, although we see a decrease in entropy, this decrease in risk results in a decrease in profit. This is a 17% decrease in profit compared with UM, as seen in Figures 8 and 9.

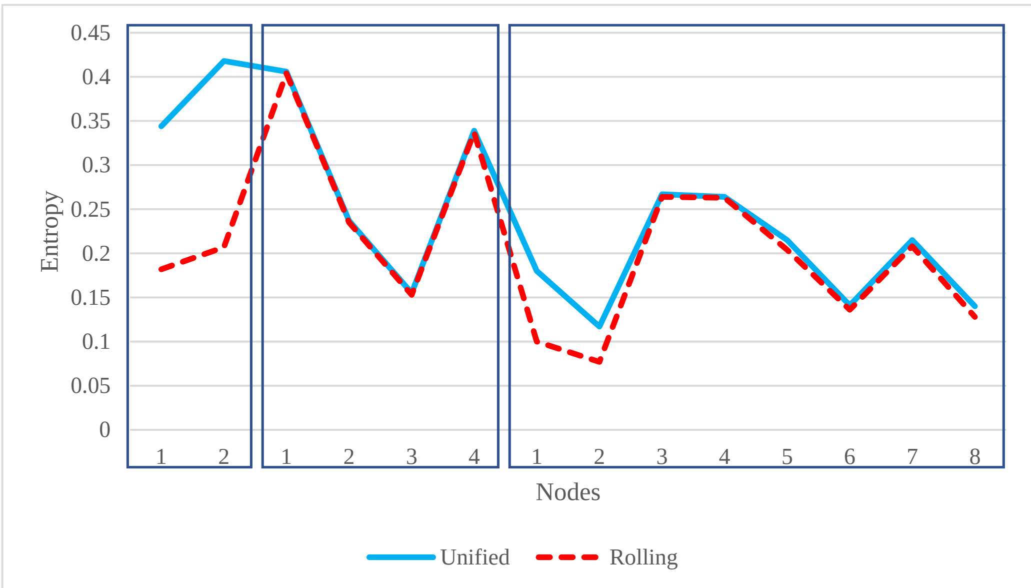

Figure 8. Entropy of RM and UM at end of each period.

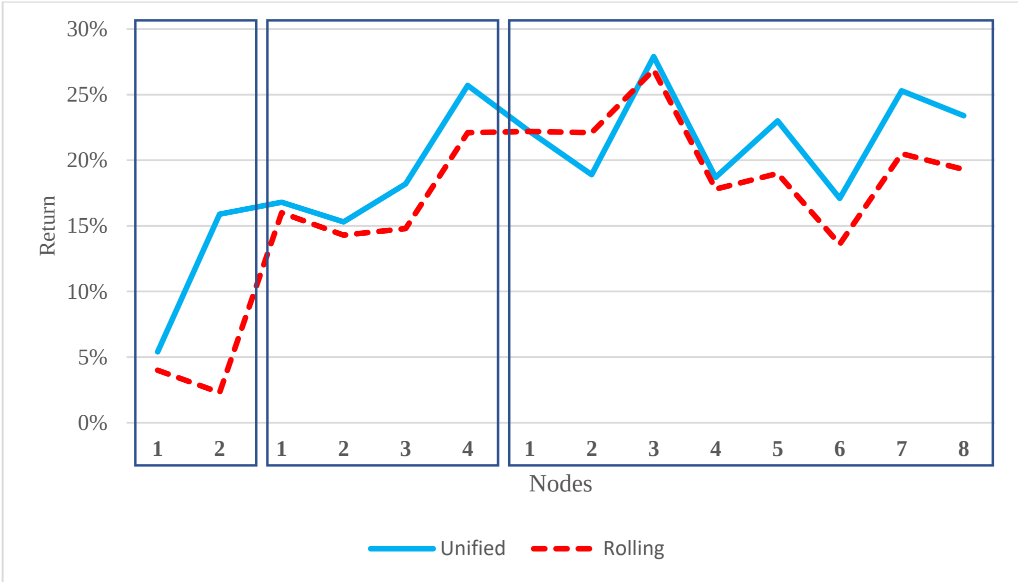

Figure 9. Return of RM and UM at end of each period.

In addition to investing at a lower risk at each period and achieving a robust profit, the investor can exit the investment at each period. This cut-off investment in RM is associated with the least loss because it carries the least risk. Generally, in RM, the average network profit is reduced by 17% and the entropy is reduced by 19%. That is, although profits have declined, entropy also declines at a higher rate and the investment is robust.

4.2. Second Empirical Case Study: Global Stock Market

The historical data of 20 stocks from April 2021 to April 2023 were collected and, to handle the data, every week was determined to be an investment period. We assume an investor wants to invest in a 10-period portfolio (i.e., T = 10) and requires some stocks from the international stock exchange market. Also, the number of scenarios at each period is two, i.e., $S_t = 2$ for $t \in T$. The creation of scenarios in each period is exponential. Thus, there is one scenario at beginning of first period, two scenarios at the beginning of the second period, and four scenarios at the beginning of third period. This continues until there are 512 scenarios at the beginning of the 10th period and 1024 scenarios at the end of the same period. (i.e., 1, 2, 4, . . ., 512 and 1024). At the start of each period, the investor invests all the available capital in some equities, assuming time-consistent risk aversion. Trapezoidal fuzzy variables characterize the returns of the 10 stocks across each time.

The beginning wealth of the investor is $W_1^1 = 1$, and the transaction cost at each period is 0.0001. The maximum bound of the stock investment proportion for each period is fixed at 0.3, while the lower bound is set at 0. The proposed model is solved in two ways: First, it is solved in a UM and then by RM, with both models considering the risk aversion factor. The proposed model is then run in GAMS 24.1.2.

As mentioned above, the creation of scenarios in each period is exponential. There are many scenarios in the later periods. For instance, these are 512 scenarios at the beginning of the 10th period. Thus, it is difficult to report the results in detail and it is only possible to compare the two models (RM and UM) in the main indices, efficiency and entropy, as a whole. Therefore, four main statistical indices are used to evaluate the UM and RM results in this case study: average, standard deviation, maximum, and minimum. The indices are calculated by the node's data such as return and entropy.

Figures 10 and 11 show the average, standard deviation, maximum and minimum of entropy in each period. Figure 10 shows that the average UM entropy is greater than the average RM entropy in each period. In period one, the biggest difference between RM and UM is observed (13.5%). The smallest difference between these two models is observed in the period 10 (1.3%). In the other periods, the differences show a sinusoidal pattern. In general, the average entropy in RM is not greater in any period, but only relying on this index cannot guarantee the dominance of RM over UM. The standard deviation is another index that describes the distribution of a scenario's entropy in each period. The trend in the standard deviation is similar until the 7th period. Up to period 4, the standard deviation of UM is less than that of RM. In the periods between 5 and 10, the standard deviation of RM less than that of UM. After the 7th period, the standard deviation of UM moves away from RM exponentially and reaches its peak at 19.4%. A large standard deviation for the risk index increases its intensity. If we examine the average and standard deviation indicators together, we find that RM appears to be superior to UM. The maximum and minimum values of entropy in each period are also reported to illustrate the distribution of the scenarios.

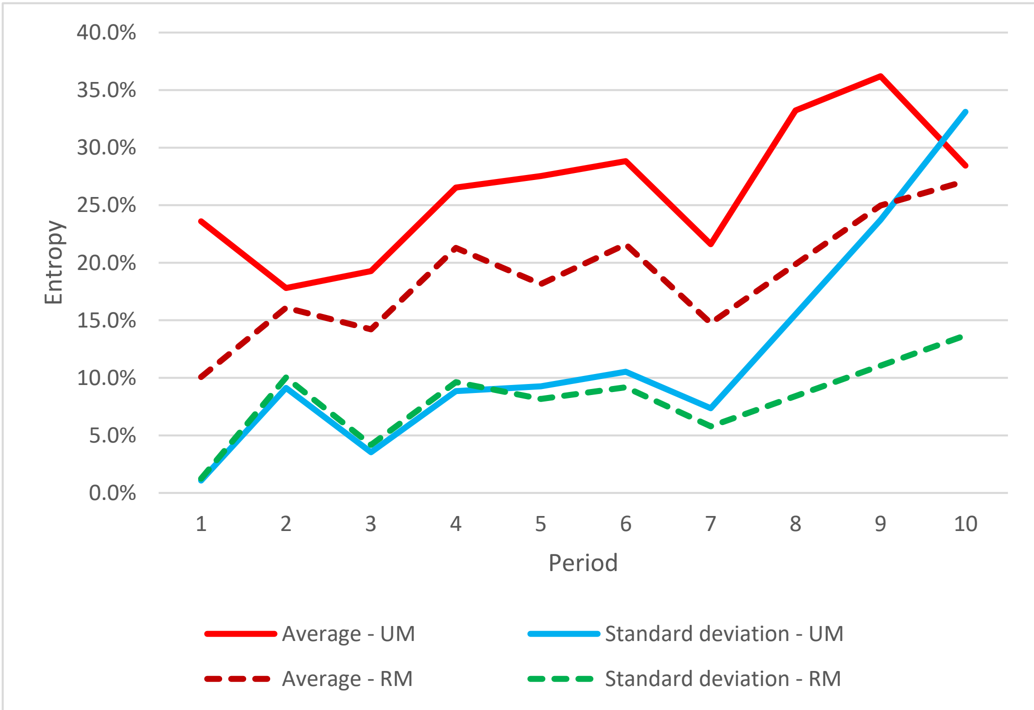

Figure 10. Average and standard deviation of entropy of RM and UM.

Figures 12 and 13 show the average, standard deviation, maximum and minimum of return in each period. Figure 12 shows that both the average and standard deviation of return of UM are greater than those of RM in each period. The growth of UM in higher periods compared with RM gradually increases and reaches its maximum value in the 10th period, which is 7.8%. In each period, the standard deviation of returns in the two models is in the range between −0.58 and 0.83, which is very narrow. The maximum and minimum values of return in each period are also reported to illustrate the distribution of the scenarios.

Thus, UM functions more effectively than RM when considering the average of return. Conversely, RM functions more effectively than UM in terms of the standard deviation of return.

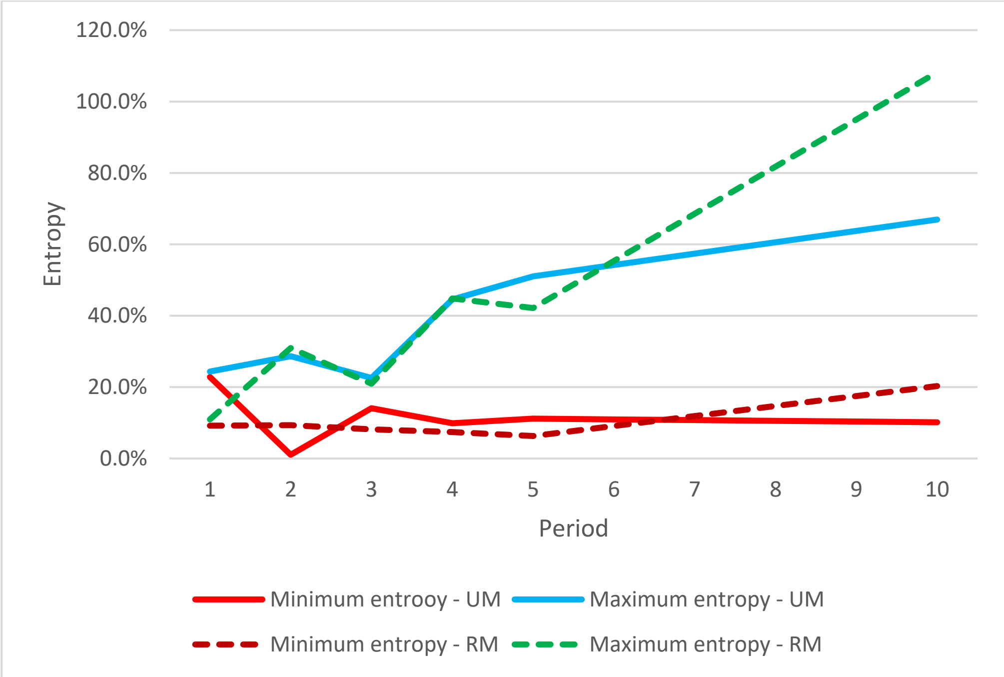

Figure 11. Maximum and minimum of entropy of RM and UM.

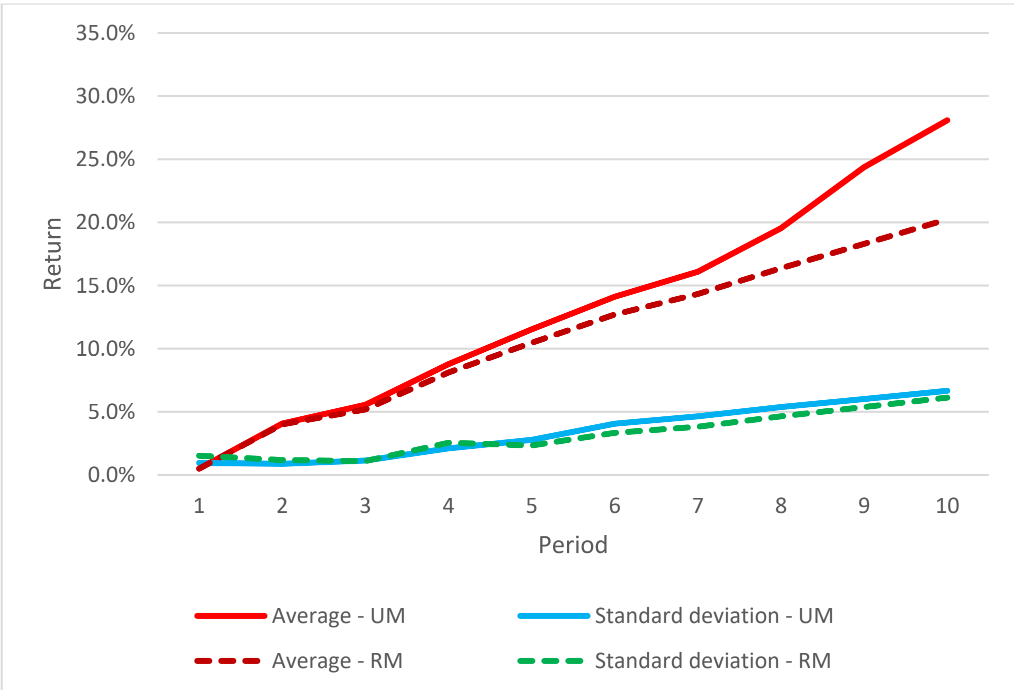

Figure 12. Average and standard deviation of return of RM and UM.

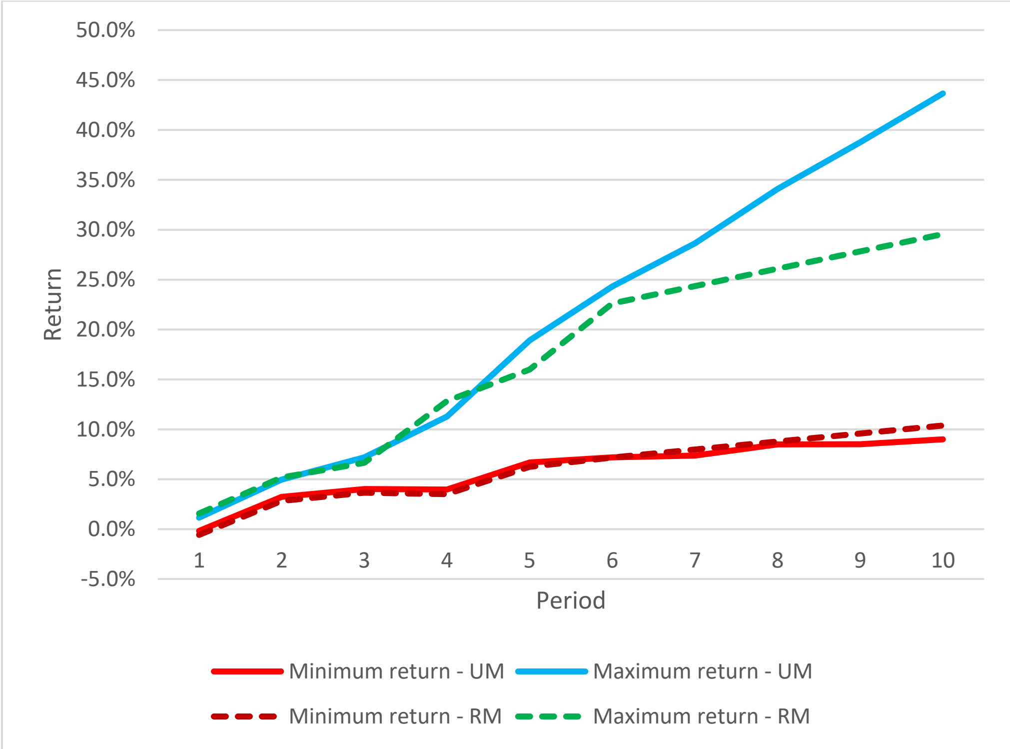

Figure 13. Maximum and minimum of entropy of RM and UM.

4.3. Evaluation

To perform an evaluation, it is necessary to construct a deterministic model to enable a comparison with the proposed model. To achieve this, the uncertainty parameters in UM are changed to crisp parameters and entropy as a measure risk is automatically deleted. Therefore, the deterministic model (DM) is introduced as:

$$Max \sum_{s=1}^{S_T} p_T.W_T \tag{39}$$

$$S.t. \quad \sum\nolimits_{i=1}^{I} x_{1,i} + \sum\nolimits_{i=1}^{I} c_{1,i}.x_{1,i} = W \tag{40}$$

$$\sum\nolimits_{i=1}^{I} x_{t,i} + \sum\nolimits_{i=1}^{I} c_{1,i}|x_{t,i} - x_{t-1,i}| = \sum\nolimits_{i=1}^{I} x_{t,i}\ (1+\xi_{t,i}),\ t = 2,\ldots,T \tag{41}$$

$$W_T = \sum\nolimits_{i=1}^{I} x_{T,i}\ \ (1+\xi_{T,i}),\ s\epsilon S_T \tag{42}$$

$$l_{t,i} \leq x_{t,i} \leq u_{t,i},\ i\epsilon I, \quad t \in T \tag{43}$$

$$x_{t,i} \geq 0,\ i\epsilon I, \quad t \in T \tag{44}$$

In order to evaluate UM, RM and DM by simulation tests, 100 realistic samples referring to historical data are used. The simulation test results are shown in Figures 14–16. The UM portfolio simulation test results show a 22.39% average return, with a standard deviation of 22.02%. In comparison with UM, RM has a lower average return (13.92%) and standard deviation (15.41%). The DM has a lower average of return compared with the two proposed models i.e., 11.19%, with a standard deviation of 20.88%. These results show that UM and RM have more return on average and that RM has the lowest standard deviation.

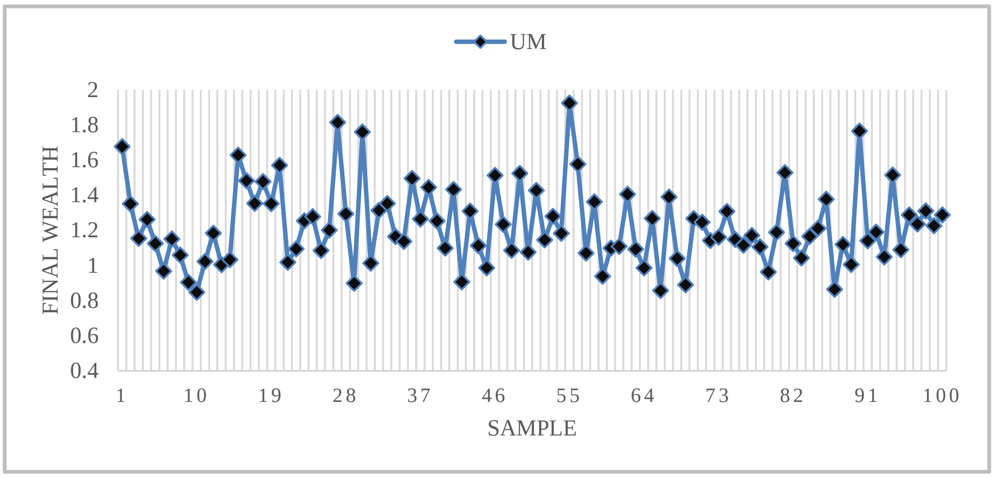

Figure 14. Results of UM in simulation with historical data.

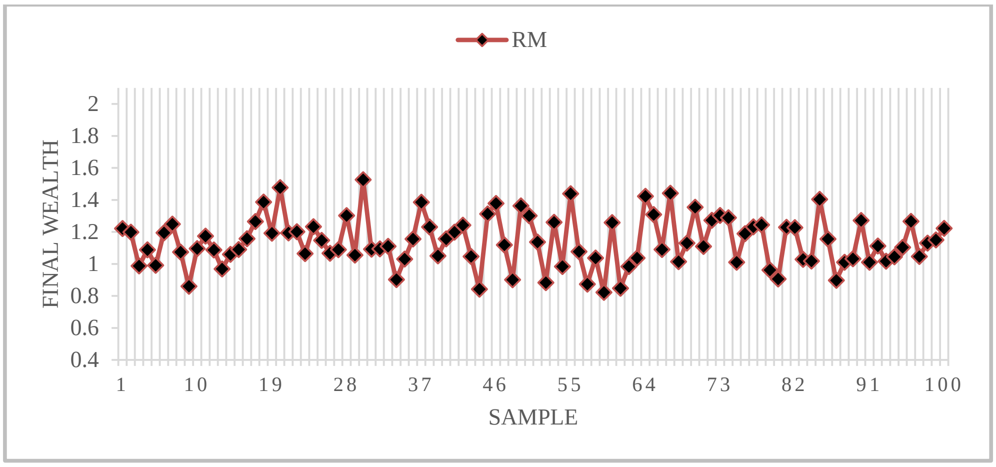

Figure 15. Results of RM in simulation with historical data.

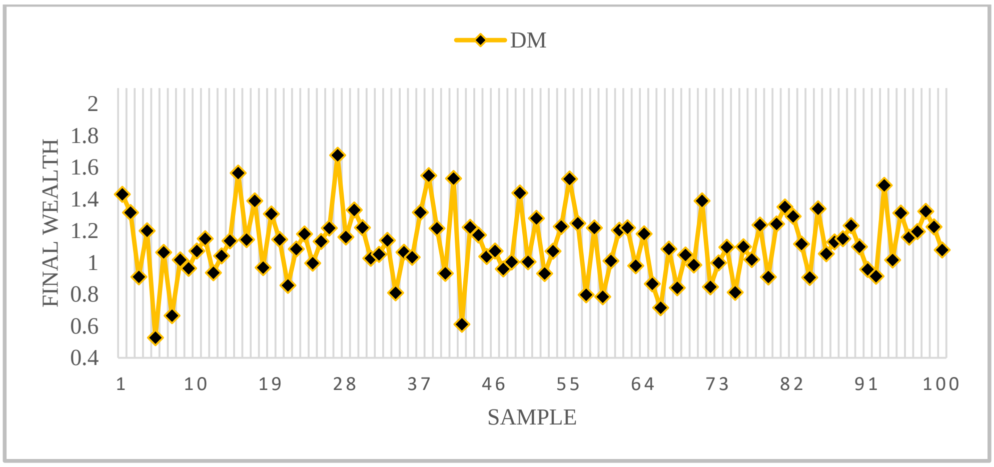

Figure 16. Results of DM in simulation with historical data.

Table 5 displays the summarized results of the simulation. In this summary, the average and standard deviation of the simulated returns in each step of the sensitivity analysis are presented.

Table 5. Average (AVG) and standard deviation (SD) of simulated return in each method.

Step	DM		UM		RM	
	AVG	**SD**	**AVG**	**SD**	**AVG**	**SD**
0.2	1.8681	0.3869	1.9202	0.2680	1.9096	0.2274
0.4	1.9032	0.4308	1.9210	0.2657	1.9102	0.2251
0.6	1.8816	0.4202	1.9218	0.2637	1.9102	0.2251
0.8	1.8574	0.4466	1.9225	0.2620	1.9053	0.2278
1	1.8120	0.4708	1.9227	0.2602	1.9047	0.2276

Furthermore, in order to prove the robustness of the UM and RM solutions, simulation tests are utilized for sensitivity analysis. In the implementation of sensitivity analysis, the fuzzy numbers domain becomes larger and the proposed models are solved again and again. To compare these numbers, it is necessary to compare the summarized information.

The sensitivity analysis results show that UM and RM are more robust than DM. Table 6 shows the percentage change in average of return and standard deviation for UM compared with DM by formulation (I) and RM compared with DM by formulation (II). UM is also compared with RM by formulation (III) under sensitivity analysis after simulation tests. Both UM and RM have an average of return greater than that of DM and both have a lower standard deviation than DM. Comparing UM and RM, UM has a higher average of return than RM, but as a negative point, UM has a higher standard deviation than RM.

$$\text{Formulation (I)}: \quad \Delta[\lambda]_{DM}^{UM} = \left[\frac{\lambda^{UM} - \lambda^{DM}}{\lambda^{DM}}\right] * 100 \quad \forall \lambda = AVG, SD.$$

$$\text{Formulation (II)}: \quad \Delta[\lambda]_{DM}^{RM} = \left[\frac{\lambda^{RM} - \lambda^{DM}}{\lambda^{DM}}\right] * 100 \quad \forall \lambda = AVG, SD.$$

$$\text{Formulation (III)}: \quad \Delta[\lambda]_{UM}^{RM} = \left[\frac{\lambda^{RM} - \lambda^{UM}}{\lambda^{UM}}\right] * 100 \quad \forall \lambda = AVG, SD.$$

Table 6. Percentage changes in average (AVG) and standard deviation (SD) of simulated returns by each method.

Step	$\Delta[\lambda]_{DM}^{UM}$		$\Delta[\lambda]_{DM}^{RM}$		$\Delta[\lambda]_{RM}^{UM}$	
	AVG	**SD**	**AVG**	**SD**	**AVG**	**SD**
0.2	2.79%	−30.72%	2.22%	−41.21%	0.56%	17.85%
0.4	0.94%	−38.33%	0.37%	−47.75%	0.57%	18.02%
0.6	2.14%	−37.25%	1.52%	−46.43%	0.61%	17.14%
0.8	3.50%	−41.34%	2.58%	−48.99%	0.90%	15.01%
1	6.11%	−44.72%	5.12%	−51.65%	0.94%	14.33%

5. Conclusions and Research Prospects

This study presents a comprehensive multi-period portfolio optimization model that addresses the challenges of risk aversion, fuzzy scenarios, and transaction costs. By incorporating time-consistent risk aversion in the rolling method, the RM allows investors to dynamically adapt their risk attitudes at each investment period, resulting in more realistic and adaptable investment strategies. The sensitivity analysis provides valuable insights into the robustness of the model, showcasing its stability and reliability under different scenarios. One of the notable strengths of the study is its use of triangular and trapezoidal fuzzy variables to represent uncertain asset returns, offering flexibility in handling various degrees of uncertainty. Applying UM and RM to the Tehran stock market (historical data of 10 stocks) and global stock market (historical data of 20 stocks) demonstrates the relevance of RM and its potential for practical investment decision-making. The proposed model has the ability to manage portfolio risk across all periods. Risk aversion factors and the psychological aspects of the investor were considered, which improves the consistency of investment risk and the investor's utility. The RM results show that the average net profit is reduced by 17% and its entropy is reduced by 19%. On the other hand, the entropy decreases at a higher rate, although the profit has decreased, and the investment is stable. To evaluate UM, RM and DM by simulation tests, 100 samples are generated. The simulation tests results are shown in Figure 6. In the simulation, the UM portfolio has 22.39% average return and a 22.02% standard deviation. In comparison with UM, RM has a lower average return i.e., 13.92%, with 15.41% standard deviation. In addition, DM has the lowest average of return of the two proposed models i.e., 11.19%, with a 20.88% standard deviation. The results show that UM and RM have more return on average and that RM has the lowest standard deviation. This study raises new problems and suggests new research avenues in the field of multi-period portfolio selection. As a result, we propose to expand this study in the following areas:

Dynamic Correlations between Stocks: The proposed model assumes that the returns of securities are independent across different investment periods. However, this might not always hold true in real-world financial markets, where correlations between assets may change over time [47–49].

Dynamic Risk-Aversion Factor: By assuming fixed risk aversion factors in each node, the model may overlook the dynamic and time-varying nature of investor risk preferences, which can vary due to changing market conditions and economic factors. This oversimplification could lead to suboptimal portfolio allocations and may not fully capture the complexities of real-world investment decisions. In this work, a static risk aversion factor is considered. Using a risk aversion factor function can improve the efficiency of the model results [50–55].

Liquidity: The liquidity of stocks in financial market could be used as an efficient objective or an important constraint in modeling to select appropriate stocks [56–61].

Cardinality: In this work, in order to avoid complicated mathematics, the cardinality constraint was not utilized, but it is suggested to use cardinality to extend the degree of the investor's control of the portfolio [62–67].

Portfolio Selection: Portfolio selection could be used as a stage before portfolio optimization. The stage helps to select appropriate stocks for investing. Data envelopment analysis is one of the common methods to attain this goal [68–75].

Other Hybrid Uncertainty: The proposed model combined fuzzy parameters and the scenario tree concept, but other hybrid uncertainties such as combinations of interval variables and scenario trees are suggested to extend this model [76–82].

Author Contributions: Conceptualization, P.P., M.N., M.S.P. and E.M.; Methodology, P.P., M.N., M.S.P. and E.M.; Software, P.P. and M.N.; Validation, P.P., M.N. and C.O.-S.; Formal Analysis, M.N., M.S.P. and C.O.-S.; Investigation, P.P., M.N., C.O.-S. and E.M.; Resources, P.P., M.N. and C.O.-S.; Data Curation, M.N., M.S.P. and E.M.; Writing—Original Draft Preparation, P.P. and M.N.; Writing—Review and Editing, P.P., M.N., M.S.P., C.O.-S. and E.M.; Visualization, P.P. and M.N.; Supervision, P.P., M.S.P. and E.M.; Project Administration, P.P., M.S.P. and C.O.-S.; All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by a *Hasso Plattner Excellence Research Grant* (LBUS-HPI-ERG-2020-04), financed by the Knowledge Transfer Centre of the Lucian Blaga University of Sibiu, Romania.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data used in the study are available from the authors and can be shared upon reasonable request.

Acknowledgments: The authors would like to thank the anonymous reviewers and the editor-in-chief for their constructive comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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