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On Albert Problem and Irreducible Modules

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Abstract: Motivated by the relation between Albert's Problem and irreducible modules within the class of commutative power-associative algebras, in this paper, we show some equivalences to Albert's Problem. Furthermore, we study some properties of irreducible modules for the zero algebra of dimension n and we concluded that there are no irreducible modules of dimension four.

Keywords: Albert's problem; irreducible modules; power-associative algebra

MSC: 17A30; 17A05; 17A04

1. Introduction

Given a class of algebras, it is always interesting to have knowledge about the simple objects within that class. In the class of power-associative nilalgebras, there exists an intriguing unsolved problem: classifying the finite-dimensional simple nilalgebras. Unlike some well-known classes of algebras, for instance, associative, Jordan and other ones where there are no simple nilalgebras, in the class of power-associative nilalgebras, simple nilalgebras are an open problem. This problem has an immediate equivalence indeed—Problem 1 described in [1], commonly known as Albert's Problem [2]: "*Is every (commutative) finite-dimensional power-associative nilalgebra over a field of characteristic different from two solvable?*" This problem has been studied by many authors, and initially, it was proved that such an algebra is not necessarily nilpotent since D. Suttes [3] discovered a solvable but not nilpotent five-dimensional commutative power-associative nilalgebra over any field of characteristic different from two. In certain specific cases, this problem has an affirmative answer [4–12]. In summary, if the characteristic of the base field is zero or sufficiently large, the nilalgebra $\mathcal{A}$ is solvable if nilind $\mathcal{A} \geq n-3$, where $\dim \mathcal{A} = n$ and nilind denotes the nilindex of the algebra $\mathcal{A}$. Furthermore, the same holds for the zero characteristic and $\dim \mathcal{A} \leq 9$.

Throughout this paper, k represents an algebraically closed field, and its characteristic does not divide 30. Moreover, it should be noted that all the algebras under consideration here are commutative power-associative algebras over the field k. In particular, $\mathcal{A}_n$ denotes the vector space over k of dimension n with zero product, i.e., the zero algebra of dimension n.

I. P. Shestakov (see [13] Lemma 1) proposed another way to investigate Albert's Problem: with the study of irreducible bimodules over the class of commutative power-associative nilalgebras. Indeed:

Lemma 1. *Let A be an algebra with zero multiplication and M be a faithful irreducible A-bimodule in the variety of commutative power-associative algebras, both of a finite dimension. Assume that we can define a product on M with values in A, $(m, n) \mapsto m \cdot n \in A$, such that $M \cdot M = A$. If the vector space $Q = A \oplus M$ with the multiplication $(a+m)(b+n) = m \cdot n + (an + bm)$ is a commutative power-associative algebra, then Q is nil, simple and gives a counterexample to Albert's Problem.*

This approach is the first to study Albert's Problem via irreducible modules. Thus, the authors in [13] started the classification of the irreducible $\mathcal{A}_2$-modules. In their paper, they

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showed that, via isomorphism, the only nontrivial irreducible module has a dimension of three. Using this irreducible module, they constructed an irreducible module of dimension 3^l for any $l = 1, 2, \ldots, n-1$ over the zero algebra of dimension n. After that, in [14], the low commutative power-associative nilalgebras and their annihilator were studied. In [15], the author provided families of irreducible modules of dimension $3n$ for the zero algebra of dimension four, although a complete classification of finite-dimensional irreducible modules for this algebra was not achieved.

Thus, Lemma 1 provides the impetus for the classification of finite-dimensional irreducible modules over commutative power-associative algebras, even though no such constructions are currently known. Clearly, Lemma 1 is enough for the existence of simple nilalgebras. However, we adapt the thesis of this Lemma to find an equivalence to Albert's Problem.

The paper's structure unfolds as follows: Section 2 furnishes the fundamental insights into power-associative algebras. In Section 3, we use some well-known equivalences to Albert's Problem, and we prove a new equivalence to it. Finally, in Section 4, we study some general properties of the irreducible modules. They allow us to conclude that there are no irreducible modules of dimension four.

Let us recall that k represents a field with a characteristic distinct from two, three and five. Denote with $\mathcal{V}$ the class of commutative power-associative algebras. Consider $A \in \mathcal{V}$ and M an A-module such that there exists a bilinear map $m: M \times M \to A$. Define $Q_m = A \times M$ as the algebra with a product given by

$$(a, v) \cdot (b, w) = (ab + m(v, w), aw + bv).$$

Therefore, the main contribution of this paper is the next equivalence, which allows us to study Albert's Problem from another point of view:

1. Albert's Problem holds.
2. Given $A \in \mathcal{V}$ and M an irreducible A-module. If there exists m such that $Q_m \in \mathcal{V}$, then $\pi \circ m$ is not onto, where $\pi : A \to A/A^2$.

This new equivalence to Albert's Problem outspread some other equivalences already known (see Theorem 3).

2. Preliminaries

The concepts introduced in this section are the base for comprehending the subsequent ones. We delve into key concepts related to power-associative algebras and their associated modules.

Let $\mathcal{A}$ be an algebra and $x \in \mathcal{A}$. The (*right*) *powers* of x are defined inductively in the following way: $x^1 = x$, and for any $l \geq 2$, $x^l = x^{l-1}x$. If the algebra $\mathcal{A}$ satisfies the identities $x^i x^j = x^{i+j}$ for all positive integers i and j, then we say that $\mathcal{A}$ is a *power-associative* algebra. These algebras generalize various other algebraic structures, such as associative, alternative, Jordan and Lie algebras.

The properties and characteristics of commutative power-associative algebras were extensively studied by A. A. Albert in his seminal work [2]. He established that

Theorem 1. *A commutative algebra $\mathcal{A}$ is power-associative if and only if $x^2x^2 = x^4$ for any $x \in \mathcal{A}$.*

For the complete understanding of the objects we are studying, we define the following concept:

Definition 1. *A power-associative algebra $\mathcal{A}$ is referred to as nil or nilalgebra if, for every $x \in A$, there exists a positive integer n such that $x^n = 0$. If there exists a positive integer n such that $x^n = 0$ holds for all $x \in A$, then A is said to have a bounded nilindex. The smallest positive integer n for which $x^n = 0$ holds for all $x \in \mathcal{A}$ is known as the nilpotent index or nilindex of $\mathcal{A}$, denoted by* nilind $\mathcal{A}$.

Albert, in his article [2], posed a question regarding power-associative nilrings and stated "*One can then hardly expect to be able to prove that a nilring is nilpotent, but a limited result of this type is provable*". It is worth noting that there exist non-nilpotent nilalgebras. Hence, the modified problem posed by Albert is commonly referred to as:

Problem 1 (Albert's Problem)**.** *Every finite-dimensional commutative power-associative nilalgebra over a field of characteristics different from two is solvable.*

Two different approaches have been pursued to address this problem; specifically, this involves either constraining the dimension of the algebra or comparing the nilindex with the dimension of the algebra. In the former scenario, it has been established that the problem yields an affirmative solution for algebras with dimensions less than nine over the field k or for algebras with dimensions less than or equal to nine over a field of characteristic zero. In the latter case, for algebras of dimension n, the problem yields a positive solution if the nilindex is greater than or equal to $n-3$ for algebras over a field of characteristic of zero or that are sufficiently large.

The approach centered around bimodules, inspired by Lemma 1, necessitates a more profound comprehension of bimodules within the category of commutative power-associative algebras.

S. Eilenberg in [16] extended the theory of associative modules to encompass a broader class of algebras which are defined by multilinear identities:

Definition 2. *Let $\mathcal{V}$ be a class of algebras over a field k, and consider an algebra $\mathcal{A}$ belonging to $\mathcal{V}$. An $\mathcal{A}$-bimodule in the class $\mathcal{V}$, or simply a $\mathcal{V}$-bimodule, is a vector space M over the field k equipped with two bilinear maps $\mathcal{A} \times M \to M$ and $M \times \mathcal{A} \to M$, denoted by $(a,m) \mapsto am$ and $(m,a) \mapsto ma$, respectively. These maps satisfy the property that the algebra $E = \mathcal{A} \oplus M$, with the multiplication defined as $(a+m)(b+n) = ab + (an + mb)$ for all $a,b \in \mathcal{A}$ and $m,n \in M$, belongs to $\mathcal{V}$.*

It is worth noting that the notions of modules and bimodules over commutative algebras coincide.

From this point onward, we will use the notation $\mathcal{V}$ to refer to the class of commutative power-associative algebras, and $\mathcal{V}_4$ will denote the class of commutative power-associative nilalgebras with a nilindex less than or equal to four. There is interest in studying the irreducible $\mathcal{V}$-modules of the algebra $\mathcal{A}_n$.

In [13] we find the classification of the irreducible $\mathcal{A}_2$-modules:

Lemma 2. *Let $\mathcal{A}_2 = \mathrm{span}\{a,b\}$ be the two-dimensional algebra with zero multiplication. Then, every irreducible power-associative $\mathcal{A}_2$-module M has a dimension of one or three. If M has a dimension of one, then $AM = \{0\}$. If M has a dimension of three, there exists a suitable basis $\{u,v,w\}$ of M and a nonzero scalar $\lambda \in k$ such that*

$$au = v,\ av = w,\ aw = 0,\ bu = 0,\ bv = \lambda u,\ bw = -\lambda v.$$

So far, the complete classification of finite-dimensional $\mathcal{A}_n$-modules has only been achieved for the case of $n = 2$. For $n \geq 3$, examples of irreducible modules of dimensions $3, 9, \ldots, 3^{n-1}$ have been constructed by using the method described in ([13], Proposition 1). For $n = 4$, in [15], families of examples of dimension $3n$ for any $n \geq 2$ were constructed.

The class of commutative power-associative algebras $\mathcal{V}$ is characterized by a set of identities, as established by Albert in Theorem 1. These identities can be expressed as follows:

$$xy - yx = 0, \qquad x^2x^2 - x^4 = 0,$$

and $\mathcal{V}_4$ is defined by

$$xy - yx = 0, \qquad x^2x^2 = 0, \qquad x^4 = 0.$$

By employing the linearization technique of identities [17], we derive a set of useful identities for the variety $\mathcal{V}$:

$$\begin{aligned} x^3y + x(x^2y) + 2x(x(xy)) &= x^2(xy), \\ x(xy^2) + y(yx^2) + 2x(y(xy)) + 2y(x(xy)) &= 4(xy)^2 + 2x^2y^2. \end{aligned} \tag{1}$$

Analogously, some identities for the variety $\mathcal{V}_4$ are

$$x^3y + x(x^2y) + 2x(x(xy)) = 0, \quad x^2(xy) = 0, \tag{2}$$

$$x(xy^2) + y(yx^2) + 2x(y(xy)) + 2y(x(xy)) = 0, \quad 2(xy)^2 + x^2y^2 = 0, \tag{3}$$

$$2\big[x(x(yz)) + x(y(xz)) + x(z(xy)) + y(x(xz)) + z(x(xy))\big] + y(zx^2) + z(yx^2) = 0, \tag{4}$$

$$x^2(yz) + 2(xy)(xz) = 0, \tag{5}$$

$$s(x_1, x_2, x_3, x_4) = 0, \quad (xy)(zt) + (xz)(yt) + (xt)(yz) = 0, \tag{6}$$

where $s(x_1, x_2, x_3, x_4) := \frac{1}{2}\sum_{\sigma \in S_4} x_{\sigma(1)}(x_{\sigma(2)}(x_{\sigma(3)}x_{\sigma(4)}))$ and S_4 is the set of all permutations on the set $\{1, 2, 3, 4\}$.

Clearly, M is a $\mathcal{V}$-module for the algebra $\mathcal{A}$ if the following holds for any $x \in \mathcal{A}$:

$$L_{x^3} + L_x L_{x^2} + 2L_x^3 = L_{x^2} L_x, \tag{7}$$

where L_x denotes the left multiplication by x endomorphism of M.

By linearizing the identity $x^4 = 0$, it follows that M is an $\mathcal{A}_n$-module in the class $\mathcal{V}_4$ if and only if we have the following for any $x \in \mathcal{A}_n$:

$$L_x^3 = 0. \tag{8}$$

We shall denote with $\operatorname{span}\{X\}$ the vector subspace of A spanned by a subset X of A. Let M be an A-module and $v \in M$, whereby we use $\langle v \rangle$ to denote the submodule of M spanned by v.

In order to establish that irreducible modules of dimensions less than five over $\mathcal{A}_n$ are limited to those of $\mathcal{A}_2$, we use the following result in [18], where Fasoli classifies all maximal nilpotent linear subspaces of $\mathcal{M}(4, \mathbb{C})$. We denote with E_{ij} the 4×4 matrix with one in the (i, j) position and zeros everywhere else.

Theorem 2. *Every maximal nilpotent linear subspace of $\mathcal{M}(4, \mathbb{C})$ is conjugated to exactly one of the following six subspaces:*

$$\begin{aligned}
\mathcal{C}_1 &= \textit{all strictly upper triangular matrices}, \\
\mathcal{C}_2 &= \operatorname{span}_{\mathbb{C}}\{E_{12} + E_{23}, E_{21} - E_{32}, E_{41}, E_{42}, E_{43}\}, \\
\mathcal{C}_3 &= \operatorname{span}_{\mathbb{C}}\{E_{12} + E_{23}, E_{21} - E_{32}, E_{14}, E_{24}, E_{34}\}, \\
\mathcal{C}_4 &= \operatorname{span}_{\mathbb{C}}\{E_{12} + E_{34}, E_{31} - E_{42}, E_{23}\}, \\
\mathcal{C}_5 &= \operatorname{span}_{\mathbb{C}}\{E_{12} + E_{23} + E_{34}, E_{21} - E_{32}, E_{31} - E_{42}\}, \\
\mathcal{C}_6 &= \operatorname{span}_{\mathbb{C}}\{E_{12} + E_{24} + E_{34}, E_{12} + E_{23} + E_{34}, E_{21} + E_{31} - E_{32} - E_{42}\}.
\end{aligned}$$

3. The Equivalence

The investigation of irreducible modules over the class of commutative power-associative algebras extends beyond Lemma 1. Furthermore, we can utilize the irreducible modules of $\mathcal{A}_n$ to construct an irreducible module for an algebra $\mathcal{A}$ such that $\operatorname{codim} \mathcal{A}^2 = n$.

Lemma 3. *Let $\mathcal{A} \in \mathcal{V}$ such that* codim $\mathcal{A}^2 = n$ *and M be a $\mathcal{V}$-irreducible module of $\mathcal{A}_n$. Then, M could be considered a $\mathcal{V}$-module of $\mathcal{A}$.*

Proof. Let C be a basis of $\mathcal{A}^2$ and $B \cup C$ be a basis of A. Without a loss of generality, we can identify A_n with span$\{B\}$. It should be noted that any $x \in \mathcal{A}$ can be written uniquely as $x = x_B + a_2$, where $x_B \in A_n$ and $a_2 \in \mathcal{A}^2$. For any $v \in M$, we define $xv := x_B v$. It can be easily verified that Equation (7) holds since $\mathcal{A}^2 M = 0$ and $L^3_{x_B} = 0$. □

In the general case, we do not know if the annihilator of a module is an ideal of A. However, in the class $\mathcal{V}_4$, we have this weaker result:

Lemma 4. *Let M be a non trivial irreducible A-module over $\mathcal{V}_4$. Then,* ann *M is a subalgebra of A.*

Proof. For any $m \in M$, take $a \in A$ such that $am \neq 0$. Since M is irreducible, $m \in \langle am \rangle$. Thus, $m = \sum a_i v_i$ for some $a_i \in A$ and $v_i \in M$. Hence, without a loss of generality, suppose that $m = av$. If $b, c \in \text{ann}\, M$, then

$$0 = (bc)(av) + (ba)(cv) + (ca)(bv) = (bc)(av).$$

□

Now, observe that for any $\mathcal{V}$-module M of $\mathcal{A}_n$, the null split extension $A \oplus M$ belongs to $\mathcal{V}_4$.

Lemma 5. *Let M be a $\mathcal{V}$-module of the algebra A_n. Then, M is a $\mathcal{V}_4$-module.*

Proof. Note that for any $x \in \mathcal{A}$ and $v \in M$, we have $(a+v)^2 \in M$. Hence, $0 = (a+v)^2(a+v)^2 = (a+v)^4$. □

Consequently, the null split extension, in the notation introduced at the end of previously section Q_m, has the property that $Q_0 \in \mathcal{V}_4$.

Consider a commutative power-associative algebra A, M as an irreducible A-module, $m : M \times M \to A$ as a product and the algebra Q_m, where the product is given by

$$(a, v)(b, w) := (ab + m(v, w), aw + bv).$$

Consider the following statement:

Statement 1. *There exists A, M as an irreducible A-module and a product m such that $Q_m \in \mathcal{V}$ and $\pi \circ m$ is onto.*

In [14], we find a generalization of the next Lemma. However, we will use this weaker version:

Lemma 6. *If Statement 1 holds, then Q_m is nil, $Q_m^2 = Q_m$ and it gives a counterexample to Albert's Problem.*

Now, we establish several equivalences to Albert's Problem, some of which are already known, but we include the equivalence given for 1–2, which is the principal result of this paper.

Theorem 3. *The following are equivalent:*

1. *Albert's Problem holds.*
2. *Given $\mathcal{A} \in \mathcal{V}$ and M an irreducible A-module. If there exists m such that $Q_m \in \mathcal{V}$, then $\pi \circ m$ is not onto, where $\pi : A \to A/A^2$.*
3. *There are no simple commutative power-associative nilalgebras.*
4. *Given $\mathcal{A} \in \mathcal{V}$, there exists a nonzero symmetric associative bilinear form.*

Proof. Note that Lemma 6 gives that (1) implies (2). Now, suppose that there exists a simple nilalgebra $\mathcal{A}$. Since $V = \mathcal{A}$ is an irreducible $\mathcal{A}$-module, considering $m : V \times V \to \mathcal{A}$ as the product of $\mathcal{A}$, for any $a \in \mathcal{A}$ and $v \in V$,

$$\begin{aligned}
(a,v)^2 &= (a^2 + v^2, 2av), \\
(a,v)^2(a,v)^2 &= ((a^2+v^2)^2 + 4(av)^2, 4(av)(a^2+v^2)) \\
&= (a^4 + v^4 + 2a^2v^2 + 4(av)^2, 4a^2(av) + 4v^2(va)), \\
&= (a^4 + v^4 + 2a(v(av)) + 2v(a(av)) + v(v(a^2)) + a(a(v^2)), \\
&\quad a^3v + v(av^2) + 2v(v(av)) + a(v(a^2)) + v^3a + 2a(a(av))) \\
&= (a,v)^4.
\end{aligned}$$

Thus, $Q_m \in \mathcal{V}$, but $\pi \circ m$ is onto, which is contrary to (2). Now, suppose that $\mathcal{A}$ is not simple; then, there exists $x \in \mathcal{A} \setminus \mathcal{A}^2$. Define $f(x,x) = 1$ and $f(y,z) = 0$ otherwise. Note that $f(ab,c) = f(a,bc) = 0$; hence, f is a nonzero symmetric bilinear form. Since (4) implies (1) due to Theorem 1 [19], the equivalences are established. □

This equivalence enables us to approach Albert's Problem from various angles: concentrating solely on the study of nilalgebras, from the viewpoint of representation theory or even from the perspective of symmetric associative bilinear forms.

4. Irreducible Modules

The last theorem in the previous section emphasizes the importance of irreducible modules over the class $\mathcal{V}$. Thus, this section is dedicated to describing some properties that irreducible modules satisfy. We introduce the concept of the breadth of a module, and using Theorem 2, we are able to determine the irreducible modules of dimensions less than five over zero algebras.

Definition 3. *Consider V a subspace of $\mathcal{M}(n,k)$ and $\{A_1, A_1, \dots, A_l\}$ a basis of V and the $n \times nl$ matrix $N = [A_1|A_2|\cdots|A_l]$. We define the absolute breadth of V as*

$$\mathbf{B}(V) = \operatorname{rank} N.$$

Note that $\mathbf{B}(V)$ is independent of the choices of the basis of V.

If we consider $\mathcal{A} \in \mathcal{V}$ of dimension l and M an $\mathcal{A}$-module of dimension n, we transfer the absolute breadth to M in the following way:

Definition 4. *Let $\{a_1, \dots, a_l\}$ be a basis of $\mathcal{A}$ and $B = \{m_1, \dots, m_n\}$ be a basis of M. If A_i is the matrix representation of the operator $L_{a_i} : M \to M$ in the basis B, and $N = [A_1|A_2|\cdots|A_l]$, we define the absolute breadth of M as*

$$\mathbf{B}(M) = \operatorname{rank} N.$$

As well as above, the absolute breadth of a module is independent of the choice of both the basis of $\mathcal{A}$ as well as M.

The importance of the absolute breadth of a module is observed in the following Lemma:

Lemma 7. *Let M be an $\mathcal{A}$-module of dimension $n > 1$. If M is irreducible, then $\mathbf{B}(M) = n$*

Proof. Set $\{a_1, \dots, a_l\}$ as a basis of $\mathcal{A}$ and $B = \{m_1, \dots, m_n\}$ as a basis of M and denote with $[m]_B$ the coordinate vector of m for the basis B . There exist r_1, r_2 such that $v_1 = a_{r_1}m_1$, $v_2 = a_{r_2}v_1$ and $\{v_1, v_2\}$ is linearly independent. Since M is irreducible, there exists r_3, and $w_3 \in \operatorname{span}\{v_1, v_2\}$ such that $v_3 = a_{r_3}w_3$ is not belonging to $\operatorname{span}\{v_1, v_2\}$. Recurrently, for any $1 \leq j \leq n$, due to the irreducibility of M, there exists r_j and $w_j \in B_j = \text{span}\{v_1, \dots, v_{j-1}\}$ such that $v_j = a_{r_j} w_j \notin B_j$. Hence,

$$L = [[v_j]_B \,|\, 1 \leq j \leq n],$$

satisfies $\det L \neq 0$. Consequently, $\mathbf{B}(M) = n$. $\square$

Note that $\mathbf{B}(M) = n$ is just a necessary condition but it is not sufficient, as exhibited in the next example.

Example 1. *Consider M as the irreducible module of $\mathcal{A}_2$ given in Lemma 2 with $\lambda = 1$. Then,* $\mathbf{B}(M \oplus M) = 6$ *since*

$$M_1 = [L_a] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$M_2 = [L_b] = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where $\{a, b\}$ is a basis for $\mathcal{A}_2$. Thus, $N = [M_1|M_2]$ has rank 6; however, $M \oplus M$ is not irreducible.

Denoting with $\mathcal{M}_n$ the irreducible modules over the zero algebra of dimension n, we have the chain

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_n \subset \cdots$$

Definition 5. *Let M be an $\mathcal{A}_n$-irreducible module. We say that M is n-purely irreducible if $n = \min_l(\text{ann}_{\mathcal{A}_l}(M) = 0)$.*

Note that:

Lemma 8. *M is n-purely irreducible if M is not an $\mathcal{A}_{n-1}$-irreducible module.*

Proof. If M is not n-purely irreducible, then there exists $0 \neq a \in \text{ann}_{\mathcal{A}_n}(M)$. Thus, M is an $\mathcal{A}_n/ka$-irreducible module, which is a contradiction. $\square$

Thus, the classification of irreducible modules holds for any zero algebra of dimension $k \geq n$ if M is n-purely irreducible. Using Theorem 2 and Lemma 7, we will deduce that there are no irreducible modules of dimension 4, and consequently, the classification of irreducible modules of dimensions less than five is given for Lemma 2 for any zero algebra over an algebraically closed field of characteristic zero.

Lemma 9. *There are no irreducible $\mathcal{A}_n$-modules of dimension 4.*

Proof. Let M be an irreducible module of dimension 4. Since $L_x : M \to M$ is nilpotent for any $x \in \mathcal{A}_n$, then the matrix associated with L_x is nilpotent of index three and hence is contained in some $\mathcal{C}_i$ in Theorem 2. However, from Lemma 7, it is not possible to be in $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ due to $\mathbf{B}(\mathcal{C}_i) \leq 3$.

Now, if $L_x \in \mathcal{C}_i$ for $i = 4,5,6$, there exists $a_1, a_2, a_3 \in \mathcal{A}_n$ such that $L_x \in \operatorname{span}\{L_{a_1}, L_{a_2}, L_{a_3}\}$. Thus, Lemma 8 allows us to consider $\mathcal{A}_3$-modules. If $L_x \in \mathcal{C}_4$, then

$$[L_x] = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ c & 0 & 0 & a \\ 0 & -c & 0 & 0 \end{pmatrix}.$$

Since $L_x^3 = 0$, then $b = 0$ and $\mathbf{B}(M) = 3$.

If $L_x \in \mathcal{C}_5$, then there exists $a, b, c \in k$ such that

$$[L_x] = \begin{pmatrix} 0 & a & 0 & 0 \\ b & 0 & a & 0 \\ c & -b & 0 & a \\ 0 & -c & 0 & 0 \end{pmatrix}.$$

Once more, from $L_x^3 = 0$, implies $a = 0$ and $\mathbf{B}(M) = 3$.

If $L_x \in \mathcal{C}_6$, then there exists $a, b, c \in k$ such that

$$[L_x] = \begin{pmatrix} 0 & a+b & 0 & 0 \\ c & 0 & b & a \\ c & -c & 0 & a+b \\ 0 & -c & 0 & 0 \end{pmatrix},$$

and condition $L_x^3 = 0$ gives $ab = c = 0$, then $\mathbf{B}(M) \leq 3$.

Since any maximal nilpotent linear subspace with a nilindex less than four has $\mathbf{B}(M) \leq 3$, then there does not exist irreducible modules of dimension 4. $\square$

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