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Article

Nonlinear Skew Lie-Type Derivations on ∗-Algebra

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Abstract: Let $\mathcal{A}$ be a unital $*$-algebra over the complex fields $\mathbb{C}$. For any $H_1, H_2 \in \mathcal{A}$, a product $[H_1, H_2]_\bullet = H_1H_2 - H_2H_1^*$ is called the skew Lie product. In this article, it is shown that if a map $\xi : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) satisfies $\xi(P_n(H_1, H_2, \ldots, H_n)) = \sum_{i=1}^{n} P_n(H_1, \ldots, H_{i-1}, \xi(H_i), H_{i+1}, \ldots, H_n)$ $(n \geq 3)$ for all $H_1, H_2, \ldots, H_n \in \mathcal{A}$, then ξ is additive. Moreover, if $\xi(i\frac{e}{2})$ is self-adjoint, then ξ is $*$-derivation. As applications, we apply our main result to some special classes of unital $*$-algebras such as prime $*$-algebra, standard operator algebra, factor von Neumann algebra, and von Neumann algebra with no central summands of type I_1.

Keywords: additive $*$-derivation; mixed bi-skew Jordan triple derivation; $*$-algebras; von Neumann algebra

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1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. A mapping $\xi : \mathcal{A} \to \mathcal{A}$ is called an additive derivation if $\xi(A + B) = \xi(A) + \xi(B)$ and $\xi(AB) = \xi(A)B + A\xi(B)$ hold for all $A, B \in \mathcal{A}$. Moreover, ξ is said to be an additive $*$-derivation if it is an additive derivation, and $\xi(A^*) = \xi(A)^*$ hold for all $A \in \mathcal{A}$. For $A, B \in \mathcal{A}$, define the Lie product and skew Lie product of A and B by $[A, B] = AB - BA$ and $[A, B]_\bullet = AB - BA^*$, respectively. A map $\xi : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is said to be a nonlinear Lie derivation (resectively, a nonlinear Lie triple derivation) if

$$\begin{aligned} \xi([A,B]) &= [\xi(A),B] + [A,\xi(B)] \\ (\textit{respectively},\ \ \xi([[A,B],C]) &= [[\xi(A),B],C] + [[A,\xi(B)],C] + [[A,B],\xi(C)] \end{aligned}$$

hold for all $A, B, C \in \mathcal{A}$. Analogously, a map $\mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a nonlinear skew Lie derivation (respectively, a nonlinear skew Lie triple derivation) if

$$\begin{aligned} \xi([A,B]_\bullet) &= [\xi(A),B]_\bullet + [A,\xi(B)]_\bullet \\ (\textit{resp.}\ \ \xi([[A,B]_\bullet,C]_\bullet) &= [[\xi(A),B]_\bullet,C]_\bullet + [[A,\xi(B)]_\bullet,C]_\bullet + [[A,B]_\bullet,\xi(C)]_\bullet \end{aligned}$$

hold for all $A, B, C \in \mathcal{A}$; many authors have studied the structure of nonlinear Lie derivation (respectively, nonlinear Lie triple derivation) and nonlinear skew Lie derivation (respectively, nonlinear skew Lie triple derivation) on various $*$-algebra (see [1–4]). In the last decade, many mathematicians have devoted themselves to the study of mappings involving new products on various kind of rings and algebras. These kind of new products are playing a more important role in some research topics, and their study has attracted many authors' attention (see [2,5–15]). Define the sequence of polynomials as $P_1(X_1) = X_1$, $P_2(X_1, X_2) = [X_1, X_2]_\bullet = X_1X_2 - X_2X_1^*$, $P_3(X_1, X_2, X_3) = [P_2(X_1, X_2), X_3]_\bullet$, $\ldots$, $P_n(X_1, X_2, \ldots, X_{n-1}, X_n) = [P_{n-1}(X_1, X_2, \ldots, X_{n-1}), X_n]_\bullet$, where

$P_n(X_1, X_2, \ldots, X_{n-1}, X_n) = [P_{n-1}(X_1, X_2, \ldots, X_{n-1}), X_n]_\bullet$ is called skew Lie n- product. A map $\xi : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is said to be nonlinear skew Lie n-derivation if

$$\begin{aligned}\xi(P_n(H_1, H_2, \ldots, H_n)) \quad &= \quad P_n(\xi(H_1), H_2, \ldots, H_n) + P_n(H_1, \xi(H_2), \ldots, H_n) \\ &\quad + \cdots + P_n(H_1, H_2, \ldots, \xi(H_n))\end{aligned}$$

holds for all $H_1, H_2, \ldots, H_n \in \mathcal{A}$.

A nonlinear skew Lie 2-derivation is called a nonlinear skew Lie derivation, and a nonlinear skew Lie 3-derivation is called a nonlinear skew Lie triple derivation. A nonlinear skew Lie 2-derivation, nonlinear skew Lie 3-derivation and nonlinear skew Lie n-derivation are collectively called nonlinear skew Lie-type derivations.

Remember the definition of $*$-algebra: first of all, define the involution $\star$ on ring R; then, define the involution $*$ on algebra $\mathcal{A}$. An additive map $\star$ on ring is called an involution if $(r_1 r_2)^\star = r_2^\star r_1^\star$ and $(r^\star)^\star = r$, for all $r_1, r_2, r \in R$. Defining the involution $*$ on algebra is an additive mapping satisfying $(ab)^* = b^* a^*$, $(a^*)^* = a$, and $(ra)^* = r^\star a^*$, for all $a, b \in \mathcal{A}$ and $r \in R$. An R-algebra with involution $*$ is called $*$-algebra. A set of complex numbers with conjugation as involution is an $*$-algebra. Let H be the complex Hilbert space and $B(H)$ be the algebra of bounded operator on H over the complex field $\mathbb{C}$, and define involution $*$ on $B(H)$ as the adjoint of x for all $x \in B(H)$. Therefore, $B(H)$ is an $*$-algebra. The class of $*$-algebras is very important and has many applications in many fields; the behavior of operators on Hilbert spaces is studied using $*$-algebras. The class of $*$- algebra is a more general class than prime $*$-algebra, standard operator algebra, factor von Neumann algebra, and von Neumann algebra with no central summands of type of I_1. Consequently, it would be crucial to describe a map on $*$-algebras. Lin [16] proved that every multiplicative skew Lie-type derivation on standard operator algebra is an additive $*$-derivations. In 2016, Zhang [12] studied nonlinear skew Jordan derivations on factor von Neumann algebras and proved that every nonlinear skew Jordan derivation on a factor von Neumann algebra is an addtive $*$-derivation. Later, this result has been extended to skew Jordan triple derivation and skew Jordan-type derivation on $*$-algebras in [13,15], respectively. Lin [17] proved that every multiplicative skew Lie-type derivation on von Neumann algebra is an additive $*$-derivation. In [15], Li et al. proved that every nonlinear $*$-Jordan type derivation on $*$-algebra is an additive $*$-derivation. Motivated by the above cited work, in this article, we define skew Lie-type mapping on a more general setting of arbitrary unital $*$-algebra. Correspondingly, a map $\xi : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a nonlinear skew Lie-type derivation if

$$\begin{aligned}\xi(P_n(H_1, H_2, \ldots, H_n)) \quad &= \quad P_n(\xi(H_1), H_2, \ldots, H_n) + P_n(H_1, \xi(H_2), \ldots, H_n) \\ &\quad + \cdots + P_n(H_1, H_2, \ldots, \xi(H_n))\end{aligned}$$

holds for all $H_1, H_2, \ldots, H_n \in \mathcal{A}$.

The aim of this article is to study the nonlinear skew Lie derivations on arbitrary $*$-algebras. More precisely, we show that under mild assumptions, every nonlinear skew Lie-type derivation on an unital $*$-algebra is an additive $*$-derivation. Finally, we apply our main result to some special classes of unital $*$-algebras such as prime $*$-algebra, standard operator algebra, factor von Neumann algebra, and von Neumann algebra with no central summands of type I_1.

2. The Main Results

The main results of this article are presented in this section.

Theorem 1. *Let $\mathcal{A}$ be a unital $*$-algebra with unit e containing a nontrivial projection P_1, and $P_2 = I - P_1$ satisfies*

$$X\mathcal{A}P_k = 0 \Longrightarrow X = 0 \ \ (k = 1, 2). \tag{1}$$

Then, if a map $\xi : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) satisfies

$$\xi(P_n(H_1, H_2, \ldots, H_n)) = \sum_{i=1}^{n} P_n((H_1), \ldots, H_{i-1}, \xi(H_i), H_{i+1}, \ldots, H_n) \ (n \geq 3) \quad (2)$$

for all $H_1, H_2, \ldots, H_n \in \mathcal{A}$, then ξ is additive. Moreover, if $\xi(i\frac{e}{2})$ is self-adjoint, then ξ is $*$-derivation.

Proof. Assume $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$. Then, by the Pierce decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Clearly, any $H \in \mathcal{A}$ can be written as $H = H_{11} + H_{12} + H_{21} + H_{22}$, where $H_{ij} \in \mathcal{A}_{ij}$ for $i, j = 1, 2$. □

We prove the above theorem using several lemmas. Putting $H_1 = H_2 = \cdots = H_n = 0$ into (2), we easily establish the following Lemma.

Lemma 1. $\xi(0) = 0$.

Lemma 2. *For any $H_{12} \in \mathcal{A}_{12}$ and $H_{21} \in \mathcal{A}_{21}$, we have*

$$\xi(H_{12} + H_{21}) = \xi(H_{12}) + \xi(H_{21}).$$

Proof. Assume that $T = \xi(H_{11} + H_{12}) - \xi(H_{11}) - \xi(H_{12})$. Our target is to show that $T = 0$. Invoking the fact that $P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), H_{12}) = P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), H_{21}) = 0$ and Lemma 1, we have

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), (H_{12} + H_{21})))\\
&= \xi(P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), H_{12})) + \xi(P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), H_{21})).\\
&= P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), H_{12}) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \frac{e}{2}, \ldots, (P_1 - P_2), H_{12})\\
&\quad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_1 - P_2), H_{12}) + P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), \xi(H_{12}))\\
&\quad + P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), H_{21}) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \frac{e}{2}, \ldots, (P_1 - P_2), H_{21})\\
&\quad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_1 - P_2), H_{21}) + P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), \xi(H_{21})).\\
&= P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), (H_{12} + H_{21})) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \frac{e}{2}, \ldots, (P_1 - P_2), (H_{12} + H_{21}))\\
&\quad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_1 - P_2), (H_{12} + H_{21})) + P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), \xi(H_{12}) + \xi(H_{21})).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), (H_{12} + H_{21})))\\
&= P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), (H_{12} + H_{21})) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \frac{e}{2}, \ldots, (P_1 - P_2), (H_{12} + H_{21}))\\
&\quad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_1 - P_2), (H_{12} + H_{21})) + P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), \xi(H_{12} + H_{21})).
\end{aligned}$$

Comparing the above two expressions for $\xi(P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), (H_{12} + H_{21})))$, we obtain $P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, (P_1 - P_2), T) = 0$. This leads to $T_{11} = 0$ and $T_{22} = 0$.

Invoking the fact that $P_n(i\frac{e}{2}, \frac{e}{2}, \frac{e}{2}, \ldots, H_{12}, P_1) = 0$ and Lemma 1, we find that

$$
\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,(H_{12}+H_{21}),P_1)) \\
&= \xi(P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,H_{12},P_1))+\xi(P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,H_{21},P_1)). \\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\frac{e}{2},\ldots,H_{12},P_1)+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\frac{e}{2},\ldots,H_{12},P_1) \\
&\quad +\cdots+P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,\xi(H_{12}),P_1)+P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,H_{12},\xi(P_1)) \\
&\quad +P_n(\xi(i\frac{e}{2}),\frac{e}{2},\frac{e}{2},\ldots,H_{21},P_1)+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\frac{e}{2},\ldots,H_{21},P_1) \\
&\quad +\cdots+P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,\xi(H_{21}),P_1),+P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,H_{21},\xi(P_1)). \\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\frac{e}{2},\ldots,(H_{12}+H_{21}),P_1)+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\frac{e}{2},\ldots,(H_{12}+H_{21}),P_1) \\
&\quad +\cdots+P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,\xi(H_{12})+\xi(H_{21}),P_1)+P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,(H_{12}+H_{12}),\xi(P_1)).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,(H_{12}+H_{21}),P_1)) \\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\frac{e}{2},\ldots,(H_{12}+H_{21}),P_1)+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\frac{e}{2},\ldots,(H_{12}+H_{21}),P_1) \\
&\quad +\cdots+P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,\xi(H_{12}+H_{21}),P_1)+P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,(H_{12}+H_{12}),\xi(P_1)).
\end{aligned}
$$

From the last two expressions for $\xi(P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,(H_{12}+H_{21}),P_1))$, we obtain $P_n(i\frac{e}{2},\frac{e}{2},\frac{e}{2},\ldots,T,P_1=0)$. On simplifying, we obtain $T_{21}=0$, and similarly, we can obtain $T_{12}=0$.

Hence, $T=0$; that is, $\xi(H_{12}+H_{21})=\xi(H_{12})+\xi(H_{21})$. □

Lemma 3. *For any $H_{11}\in\mathcal{A}_{11},H_{12}\in\mathcal{A}_{12},H_{21}\in\mathcal{A}_{21}$, and $H_{22}\in\mathcal{A}_{22}$, we have*

$$\xi(H_{11}+H_{12}+H_{21})=\xi(H_{11})+\xi(H_{12})+\xi(H_{21})$$

and

$$\xi(H_{12}+H_{21}+H_{22})=\xi(H_{12})+\xi(H_{21})+\xi(H_{22}).$$

Proof. Let $T=\xi(H_{11}+H_{12}+H_{22})-\xi(H_{11})-\xi(H_{12})-\xi(H_{22})$. We show that $T=0$. Using the fact that $P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),H_{12})=P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),H_{21})=0$ and Lemma 1, we have

$$
\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),(H_{11}+H_{12}+H_{21}))) \\
&= \xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),H_{11}))+\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),H_{12})) \\
&\quad +\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),H_{21})). \\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,(P_1-P_2),H_{11})+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,(P_1-P_2),H_{11}) \\
&\quad +\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1-P_2),H_{11})+P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),\xi(H_{11})) \\
&\quad +P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,(P_1-P_2),H_{12})+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,(P_1-P_2),H_{12}) \\
&\quad +\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1-P_2),H_{12})+P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),\xi(H_{12})) \\
&\quad +P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,(P_1-P_2),H_{21})+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,(P_1-P_2),H_{21}) \\
&\quad +\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1-P_2),H_{21})+P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),\xi(H_{21})). \\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,(P_1-P_2),(H_{11}+H_{12}+H_{21}))+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,(P_1-P_2),(H_{11}+H_{12}+H_{21})) \\
&\quad +\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1-P_2),(H_{11}+H_{12}+H_{21})) \\
&\quad +P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),\xi(H_{11})+\xi(H_{12})+\xi(H_{21})).
\end{aligned}
$$

On the other hand, we obtain

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),(H_{11}+H_{12}+H_{21})))\\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,(P_1-P_2),(H_{11}+H_{12}+H_{21}))+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,(P_1-P_2),(H_{11}+H_{12}+H_{21}))\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1-P_2),(H_{11}+H_{12}+H_{21}))\\
&\quad+P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),\xi(H_{11}+H_{12}+H_{21})).
\end{aligned}$$

Comparing the above two expressions for $\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),(H_{11}+H_{12}+H_{21})))$, we find that $P_n(i\frac{e}{2},\frac{e}{2},\ldots,(P_1-P_2),T)=0$, which leads us to $T_{11}=0$ and $T_{22}=0$.

Invoking the fact that $P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,H_{11})=0$ and using Lemmas 1 and 2, we find that

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,(H_{11}+H_{12}+H_{21})))\\
&= \xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,H_{11}))+\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,H_{12}))+\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,H_{21}))\\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_2,H_{11})+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_2,H_{11})\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_2),H_{11})+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,\xi(H_{11}))\\
&\quad+P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_2,H_{12})+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_2,H_{12})\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_2),H_{12})+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,\xi(H_{12}))\\
&\quad+P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_2,H_{21})+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_2,H_{21})\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_2),H_{21})+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,\xi(H_{21})).\\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_2,(H_{11}+H_{12}+H_{21}))+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_2,(H_{11}+H_{12}+H_{21}))\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_2),(H_{11}+H_{12}+H_{21}))\\
&\quad+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,\xi(H_{11})+\xi(H_{12})+\xi(H_{21})).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,(H_{11}+H_{12}+H_{21})))\\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_2,(H_{11}+H_{12}+H_{21}))+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_2,(H_{11}+H_{12}+H_{21}))\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_2),(H_{11}+H_{12}+H_{21}))+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,\xi(H_{11}+H_{12}+H_{21})).
\end{aligned}$$

Comparing the above two expressions for $\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,(H_{11}+H_{12}+H_{21})))$, we obtain that $P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_2,T)=0$, which further implies that $T_{12}=0$ and $T_{21}=0$. Hence $T=0$; that is,

$$\xi(H_{11}+H_{12}+H_{21})=\xi(H_{11})+\xi(H_{12})+\xi(H_{21}).$$

Similarly, we can show that $\xi(H_{12}+H_{21}+H_{22})=\xi(H_{12})+\xi(H_{21})+\xi(H_{22})$. □

Lemma 4. *For any $H_{11}\in\mathcal{A}_{11},H_{12}\in\mathcal{A}_{12},H_{21}\in\mathcal{A}_{21}$, and $H_{22}\in\mathcal{A}_{22}$, we have*

$$\xi(H_{11}+H_{12}+H_{21}+H_{22})=\xi(H_{11})+\xi(H_{12})+\xi(H_{21})+\xi(H_{22}).$$

Proof. Let $T=\xi(H_{11}+H_{12}+H_{21}+H_{22})-\xi(H_{11})-\xi(H_{12})-\xi(H_{21})-\xi(H_{22})$. We show that $T=0$. Using the fact that $P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,H_{22})=0$ and Lemmas 1 and 3, we find that

$$
\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(H_{11}+H_{12}+H_{21}+H_{22})))\\
&= \xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(H_{11}+H_{12}+H_{21})))+\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,H_{22})).\\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_1,(H_{11}+H_{12}+H_{21}))+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_1,(H_{11}+H_{12}+H_{21}))\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1),(H_{11}+H_{12}+H_{21}))+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,\xi(H_{11})+\xi(H_{12})+\xi(H_{21}))\\
&\quad+P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_1,H_{22})+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_1,H_{22})\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1),H_{22})+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,\xi(H_{22})).\\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_1,(H_{11}+H_{12}+H_{21}+H_{22}))+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_1,(H_{11}+H_{12}+H_{21}+H_{22}))\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1),(H_{11}+H_{12}+H_{21}+H_{22}))\\
&\quad+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(\xi(H_{11})+\xi(H_{12})+\xi(H_{21})+\xi(H_{22}))).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(H_{11}+H_{12}+H_{21}+H_{22})))\\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_1,(H_{11}+H_{12}+H_{21}+H_{22}))+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_1,(H_{11}+H_{12}+H_{21}+H_{22}))\\
&\quad+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1),(H_{11}+H_{12}+H_{21}+H_{22}))\\
&\quad+P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(\xi(H_{11}+H_{12}+H_{21}+H_{22}))).
\end{aligned}
$$

Comparing the above two expressions for $\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(H_{11}+H_{12}+H_{21}+H_{22})))$, we obtain that $P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,T)=0$, which further implies that $T_{11}=T_{12}=T_{21}=0$. Similarly, we can show that $T_{22}=0$. Thus $T=0$, that is,

$$\xi(H_{11}+H_{12}+H_{21}+H_{22})=\xi(H_{11})+\xi(H_{12})+\xi(H_{21})+\xi(H_{22}).$$

□

Lemma 5. *For any $H_{12},H'_{12}\in\mathcal{A}_{12}$ and $H_{21},H'_{21}\in\mathcal{A}_{21}$, we have*

$$\xi(H_{12}+H'_{12})=\xi(H_{12})+\xi(H'_{12})\ \text{ and }\ \xi(H_{21}+H'_{21})=\xi(H_{21})+\xi(H'_{21}).$$

Proof. Using the fact that $P_n(-i\frac{e}{2},\frac{e}{2},\ldots,(P_1+H_{12}),i(P_2+H'_{12}))=H_{12}+H'_{12}+H^*_{12}+H'_{12}H^*_{12}$ and Lemma 4, we have

$$
\begin{aligned}
&\xi(H_{12}+H'_{12})+\xi(H^*_{12})+\xi(H'_{12}H^*_{12})\\
&= \xi(P_n(-i\frac{e}{2},\frac{e}{2},\ldots,(P_1+H_{12}),i(P_2+H'_{12}))).\\
&= P_n(\xi(-i\frac{e}{2}),\frac{e}{2},\ldots,(P_1+H_{12}),i(P_2+H'_{12}))+P_n(-i\frac{e}{2},\xi(\frac{e}{2}),\ldots,(P_1+H_{12}),i(P_2+H'_{12}))\\
&\quad+\cdots+P_n(-i\frac{e}{2},\frac{e}{2},\ldots,(\xi(P_1)+\xi(H_{12})),i(P_2+H'_{12}))\\
&\quad+P_n(-i\frac{e}{2},\frac{e}{2},\ldots,(P_1+H_{12}),(\xi(iP_2)+\xi(iH'_{12}))).\\
&= \xi(P_n(-i\frac{e}{2},\frac{e}{2},\ldots,P_1,iP_2))+\xi(P_n(-i\frac{e}{2},\frac{e}{2},\ldots,H_{12},iH'_{12}))+\xi(P_n(-i\frac{e}{2},\frac{e}{2},\ldots,P_1,iH'_{12}))\\
&\quad+\xi(P_n(-i\frac{e}{2},\frac{e}{2},\ldots,H_{12},iP_2)).\\
&= \xi(H'_{12})+\xi(H_{12}+H^*_{12})+\xi(H'_{12}H^*_{12})\\
&= \xi(H'_{12})+\xi(H_{12})+\xi(H^*_{12})+\xi(H'_{12}H^*_{12})
\end{aligned}
$$

Hence, $\xi(H_{12} + H'_{12}) = \xi(H_{12}) + \xi(H'_{12})$ for any $H_{12} \in \mathcal{A}_{12}$ and $H'_{12} \in \mathcal{A}_{12}$. Similarly, we can prove other part. $\square$

Lemma 6. *For any $H_{ii}, H'_{ii} \in \mathcal{A}_{ii}$ for $(i = 1, 2)$, we have*

$$\xi(H_{11} + H'_{11}) = \xi(H_{11}) + \xi(H'_{11}) \text{ and } \xi(H_{22} + H'_{22}) = \xi(H_{22}) + \xi(H'_{22}).$$

Proof. Let $T = \xi(H_{11} + H'_{11}) - \xi(H_{11}) - \xi(H'_{11})$; we show that $T = 0$.

Using the fact that $P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, H_{11}) = P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, H'_{11}) = 0$ and Lemma 1, we obtain

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, (H_{11} + H'_{11}))) \\
&\quad= \xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, H_{11}) + \xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, H'_{11}). \\
&\quad= P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \ldots, P_2, H_{11}) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \ldots, P_2, H_{11}) \\
&\qquad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_2), H_{11}) + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, \xi(H_{11})) \\
&\qquad + P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \ldots, P_2, H'_{11}) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \ldots, P_2, H'_{11}) \\
&\qquad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_2), H'_{11}) + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, \xi(H'_{11})). \\
&\quad= P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \ldots, P_2, H_{11} + H'_{11}) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \ldots, P_2, H_{11} + H'_{11}) \\
&\qquad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_2), H_{11} + H'_{11}) + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, \xi(H_{11}) + \xi(H'_{11})).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, (H_{11} + H'_{11}))) \\
&\quad= P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \ldots, P_2, H_{11} + H'_{11}) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \ldots, P_2, H_{11} + H'_{11}) \\
&\qquad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_2), H_{11} + H'_{11}) + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, \xi(H_{11} + H'_{11})).
\end{aligned}$$

Comparing the above two expressions for $\xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, (H_{11} + H'_{11})))$, we find that $P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_2, T) = 0$, which in turn gives $T_{12} = T_{21} = T_{22} = 0$.

Next, we show that $T_{11} = 0$. Let $X_{12} \in \mathcal{A}_{12}$, and it is easy to observe that $P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, H_{11}, X_{12}), P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, H'_{11}, X_{12}) \in \mathcal{A}_{12}$. Thus, using Lemma 5, we find that

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, (H_{11} + H'_{11}), X_{12})) \\
&\quad= \xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, H_{11}, X_{12})) + \xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, H'_{11}, X_{12})) \\
&\quad= P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \ldots, P_1, H_{11}, X_{12}) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \ldots, P_1, H_{11}, X_{12}) \\
&\qquad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_1), H_{11}, X_{12}) + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, \xi(H_{11}), X_{12}) \\
&\qquad + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, H_{11}, \xi(X_{12})) + P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \ldots, P_1, H'_{11}, X_{12}) \\
&\qquad + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \ldots, P_1, H'_{11}, X_{12}) + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_1), H'_{11}, X_{12}) \\
&\qquad + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, \xi(H'_{11}), X_{12}) + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, H'_{11}, \xi(X_{12})). \\
&\quad= P_n(\xi(i\frac{e}{2}), \frac{e}{2}, \ldots, P_1, (H_{11} + H'_{11}), X_{12}) + P_n(i\frac{e}{2}, \xi(\frac{e}{2}), \ldots, P_1, (H_{11} + H'_{11}), X_{12}) \\
&\qquad + \cdots + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \xi(P_1), (H_{11} + H'_{11}), X_{12}) \\
&\qquad + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, (\xi(H_{11}) + \xi(H'_{11})), X_{12}) + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, P_1, (H_{11} + H'_{11}), \xi(X_{12})).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(H_{11}+H'_{11}),X_{12})) \\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,P_1,(H_{11}+H'_{11}),X_{12}) + P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,P_1,(H_{11}+H'_{11}),X_{12}) \\
&\quad + \cdots + P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(P_1),(H_{11}+H'_{11}),X_{12}) \\
&\quad + P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(\xi(H_{11}+H'_{11})),X_{12}) + P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(H_{11}+H'_{11}),\xi(X_{12})).
\end{aligned}$$

From the last two expressions for $\xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,(H_{11}+H'_{11}),X_{12}))$, we obtain $P_n(i\frac{e}{2},\frac{e}{2},\ldots,P_1,T,X_{12})=0$, which implies $T_{11}XP_2=0$. Application of condition (1) yields $T_{11}=0$. Hence, $T=0$; that is, $\xi(H_{11}+H'_{11})=\xi(H_{11})+\xi(H'_{11})$. Symmetrically, one can prove that $\xi(H_{22}+H'_{22})=\xi(H_{22})+\xi(H'_{22})$. □

Lemma 7. *ξ is additive on $\mathcal{A}$.*

Proof. For any $H,K\in\mathcal{A}$, we have $H=H_{11}+H_{12}+H_{21}+H_{22}$ and $K=K_{11}+K_{12}+K_{21}+K_{22}$. With the help of Lemmas 4–6, we obtain

$$\begin{aligned}
\xi(H+K) &= \xi(H_{11}+H_{12}+H_{21}+H_{22}+K_{11}+K_{12}+K_{21}+K_{22}) \\
&= \xi(H_{11}+K_{11})+\xi(H_{12}+K_{12})+\xi(H_{21}+K_{21})+\xi(H_{22}+K_{22}) \\
&= \xi(H_{11})+\xi(K_{11})+\xi(H_{12})+\xi(K_{12})+\xi(H_{21})+\xi(K_{21})+\xi(H_{22})+\xi(K_{22}) \\
&= \xi(H_{11}+H_{12}+H_{21}+H_{22})+\xi(K_{11}+K_{12}+K_{21}+K_{22}) \\
&= \xi(H)+\xi(K).
\end{aligned}$$

□

Lemma 8. $\xi(\frac{e}{2})^*=\xi(\frac{e}{2})$.

Proof. It follows from $P_n(\frac{e}{2},\frac{e}{2},\ldots,\frac{e}{2})=0$ and Lemma 1 that we have

$$\begin{aligned}
0 &= \xi(P_n(\frac{e}{2},\frac{e}{2},\ldots,\frac{e}{2})). \\
&= P_n(\xi(\frac{e}{2}),\frac{e}{2},\ldots,\frac{e}{2})+P_n(\frac{e}{2},\xi(\frac{e}{2}),\ldots,\frac{e}{2})+\cdots+P_n(\frac{e}{2},\frac{e}{2},\ldots,\xi(\frac{e}{2})).
\end{aligned}$$

On simplifying, we obtain $\xi(\frac{e}{2})^*=\xi(\frac{e}{2})$. □

Lemma 9. *If $\xi(i\frac{e}{2})^*=\xi(i\frac{e}{2})$, then $\xi(i\frac{e}{2})=\xi(\frac{e}{2})=0$.*

Proof. Using the fact that $P_n(i\frac{e}{2},\frac{e}{2},\ldots,\frac{e}{2})=i\frac{e}{2}$ and Lemma 8, we obtain

$$\begin{aligned}
\xi(i\frac{e}{2}) &= \xi(P_n(i\frac{e}{2},\frac{e}{2},\ldots,\frac{e}{2})). \\
&= P_n(\xi(i\frac{e}{2}),\frac{e}{2},\ldots,\frac{e}{2})+P_n(i\frac{e}{2},\xi(\frac{e}{2}),\ldots,\frac{e}{2})+\cdots+P_n(i\frac{e}{2},\frac{e}{2},\ldots,\xi(\frac{e}{2})). \\
&= (n-1)i\xi(\frac{e}{2}).
\end{aligned}$$

Taking the adjoint on both side of the above relation, we obtain

$$\xi(i\frac{e}{2})^* = -(n-1)i\xi(\frac{e}{2}),$$

since $\xi(i\frac{e}{2})$ is self-adjoint. On combining the last two relations, we obtain $\xi(i\frac{e}{2})=0$ and $\xi(\frac{e}{2})=0$. □

Lemma 10. *For any $H \in \mathcal{A}$, we have $\xi(H^*) = \xi(H)^*$.*

Proof. Observe that $P_n(-i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, H, i\frac{e}{2}) = \frac{H+H^*}{2}$ for any $H \in \mathcal{A}$. Using Lemmas 7 and 9, we find that

$$\begin{aligned} \xi\left(\frac{H+H^*}{2}\right) &= \xi(P_n(-i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, H, i\frac{e}{2})) \\ &= P_n(-i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, \xi(H), i\frac{e}{2}) \\ &= \frac{1}{2}(\xi(H) + \xi(H)^*), \end{aligned}$$

which implies

$$\xi(H^*) = \xi(H)^*.$$

□

Lemma 11. *$\xi(iH) = i\xi(H)$ for every $H \in \mathcal{A}$.*

Proof. Observe that $P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, H) = iH$ for every $H \in \mathcal{A}$, and using Lemma 9, we obtain

$$\begin{aligned} \xi(iH) &= \xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, H)) \\ &= P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, \xi(H)) \\ &= i\xi(H). \end{aligned}$$

□

Lemma 12. *$\xi(HK) = \xi(H)K + H\xi(K)$ for all $H, K \in \mathcal{A}$.*

Proof. Observe that $P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, H, K) = i(HK + KH^*)$ for any $H, K \in \mathcal{A}$, and using Lemmas 7 and 9–11, we obtain

$$\begin{aligned} i\xi(HK + KH^*) &= \xi(P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, H, K)) \\ &= P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, \xi(H), K) + P_n(i\frac{e}{2}, \frac{e}{2}, \ldots, \frac{e}{2}, H, \xi(K)) \\ &= i\xi(H)K + iH\xi(K) + i\xi(K)H^* + iK\xi(H)^*, \end{aligned}$$

which implies that

$$\xi(HK + KH^*) = \xi(H)K + H\xi(K) + \xi(K)H^* + K\xi(H)^*. \quad (3)$$

Equation (3) implies that

$$\begin{aligned} \xi(HK - KH^*) &= \xi((iH)(-iK) + (-iK)(iH)^*) \\ &= \xi(iH)(-iK) + (iH)\xi(-iK) + \xi(-iK)(iH)^* + (-iK)\xi((iH)^*) \\ &= \xi(H)K + H\xi(K) - \xi(K)H^* - K\xi(H)^*. \end{aligned}$$

Hence,

$$\xi(HK - KH^*) = \xi(H)K + H\xi(K) - \xi(K)H^* - K\xi(H)^*. \quad (4)$$

On combining (3) and (4), we obtain

$$\xi(HK) = \xi(H)K + H\xi(K).$$

□

By Lemmas 7, 10, and 12, ξ is an additive $*$-derivation. This completes the proof of Theorem 1.

3. Applications of Theorem 1

In this section, we apply Theorem 1 to certain special classes of $*$-algebras, namely prime $*$-algebras, standard operator algebras, factor von Neumann algebras, and von Neumann algebras with no central summands of type I_1.

Recall that an algebra $\mathcal{A}$ is prime if for any $A, B \in \mathcal{A}$, $A\mathcal{A}B = \{0\}$ implies that either $A = 0$ or $B = 0$. It is easy to verify that every prime $*$-algebra satisfies (1). Therefore, as a direct consequence of Theorem 1, we have the following result:

Corollary 1. *Let $\mathcal{A}$ be a unital prime $*$-algebra containing a nontrivial projection. Then, if a map $\xi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\xi(P_n(H_1, H_2, \ldots, H_n)) = \sum_{i=1}^{n} P_n((H_1), \ldots, H_{i-1}, \xi(H_i), H_{i+1}, \ldots, H_n) \ \ (n \geq 3) \tag{5}$$

for all $H_1, H_2, \ldots, H_n \in \mathcal{A}$, then ξ is additive. Moreover, if $\xi(i\frac{e}{2})$ is self-adjoint, then ξ is $*$-derivation.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ denote the subalgebra of all bounded finite rank operators. A subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a standard operator algebra if it contains $\mathcal{F}(\mathcal{H})$. Now, we have the following result:

Corollary 2. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. Then, if a map $\xi : \mathcal{A} \to \mathcal{B}(\mathcal{A})$ satisfies*

$$\xi(P_n(H_1, H_2, \ldots, H_n)) = \sum_{i=1}^{n} P_n((H_1), \ldots, H_{i-1}, \xi(H_i), H_{i+1}, \ldots, H_n) \ \ (n \geq 3) \tag{6}$$

for all $H_1, H_2, \ldots, H_n \in \mathcal{A}$, then ξ is additive. Moreover, if $\xi(i\frac{e}{2})$ is self-adjoint, then ξ is $*$-derivation. Moreover, there exists an operator $T \in \mathcal{B}(\mathcal{H})$ satisfying $T + T^* = 0$ such that $\xi(A) = AT - TA$ for all $A \in \mathcal{A}$; that is, ξ is inner.

Proof. Since $\mathcal{A}$ is a unital prime $*$-algebra containing nontrivial projections, then by Corollary 1, we see that ξ is an additive $*$-derivation. It follows from [18] that ξ is a linear inner derivation; that is, there exists an operator $S \in \mathcal{B}(\mathcal{H})$ such that $\xi(A) = AS - SA$ for all $A \in \mathcal{A}$. Using the fact that $\xi(A^*) = \xi(A)^*$, we have

$$A^*S - SA^* = \xi(A^*) = \xi(A)^* = S^*A^* - A^*S^*$$

for any $A \in \mathcal{A}$, This leads to $A^*(S + S^*) = (S + S^*)A^*$. Hence, $S + S^* = \lambda I$ for some $\lambda \in \mathbb{R}$. Letting $T = S - \frac{1}{2}\lambda I$, one can check that $T + T^* = 0$ and $\xi(A) = AT - TA$ for all $A \in \mathcal{A}$. □

A von Neumann algebra $\mathcal{A}$ is a weakly closed self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. A von Neumann algebra $\mathcal{A}$ is a factor von Neumann algebra if its center contains only the scalar operators. It is well known that a factor von Neumann algebra is prime; thus, it always satisfies (1). Hence, as an immediate consequence of Corollary 1, we obtain

Corollary 3. *Let $\mathcal{A}$ be a factor von Neumann algebra with dim$(\mathcal{A}) \geq 2$. Then, if a map $\xi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\xi(P_n(H_1, H_2, \ldots, H_n)) = \sum_{i=1}^{n} P_n((H_1), \ldots, H_{i-1}, \xi(H_i), H_{i+1}, \ldots, H_n) \ (n \geq 3) \quad (7)$$

for all $H_1, H_2, \ldots, H_n \in \mathcal{A}$, then ξ is additive. Moreover, if $\xi(i\frac{e}{2})$ is self-adjoint, then ξ is $*$-derivation.

Further, it is well known that every von Neumann algebra with no central summands of type I_1 satisfies (1) (see [8,19] for details). Therefore, applying Theorem 1, we have the following result:

Corollary 4. *Let $\mathcal{A}$ be a von Neumann algebra having no central summands of type I_1. Then, if a map $\xi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\xi(P_n(H_1, H_2, \ldots, H_n)) = \sum_{i=1}^{n} P_n((H_1), \ldots, H_{i-1}, \xi(H_i), H_{i+1}, \ldots, H_n) \ (n \geq 3) \quad (8)$$

for all $H_1, H_2, \ldots, H_n \in \mathcal{A}$, then ξ is additive. Moreover, if $\xi(i\frac{e}{2})$ is self-adjoint, then ξ is $*$-derivation.

4. Conclusions

In this article, we examine the pattern of nonlinear skew Lie-type derivation (ξ) on $*$-algebra $\mathcal{A}$. In fact, we proved that such a map is an additive derivation, preserving the $*$-structure of algebra $\mathcal{A}$, i.e., $\xi(H^*) = \xi(H)^*$ for all $H \in \mathcal{A}$. One can further investigate the structure of nonlinear skew Lie-type derivations on a variety of algebras such as incidence algebras, nest algebras, etc.

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