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An Optimal Control Problem Related to the RSS Model

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Abstract: In this paper, we consider a discrete-time optimal control problem related to the model of Robinson, Solow and Srinivasan. We analyze this optimal control problem without concavity assumptions on a non-concave utility function which represents the preferences of the planner and establish the existence of good programs and optimal programs which are Stiglitz production programs.

Keywords: choice of technique; good program; Stiglitz production program; weakly maximal program

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1. Introduction and Preliminaries

The analysis of the existence and the structure of approximate optimal solutions for variational problems, optimal control problems and dynamic games on unbounded domains has been a rapidly growing area of research [1–10] which has various applications in engineering [2,6], in models of economic growth [2,11–15], in model predictive control [16] and in the theory of thermodynamic equilibrium for materials [17,18]. Discrete-time optimal control problems were considered in [1,19–21], finite-dimensional continuous-time problems were analyzed in [2,6,22,23] and infinite-dimensional optimal control was studied in [2,24–28], while solutions of dynamic games were discussed in [29–32].

In this paper, we study the existence of good programs and optimal programs, which are the Stiglitz production programs, for optimal control problems over infinite horizons related to a model of an economy originally formulated by Robinson [33], Solow [34] and Srinivasan [35] (henceforth, the RSS model). This model was studied in the late nineteen-sixties and early nineteen-seventies in [33,36–40] and it was revisited by Khan and Mitra [41]. This seminal paper became a starting point for recent research on the RSS model. Many results of the RSS model are collected in [8].

It should be mentioned that Khan and Mitra [41] assumed that the function which represents the preferences of the planner is concave. This is a usual assumption in the theory of economic growth. In particular, Khan and Mitra [41] showed the existence of good and optimal programs, which are Stigliltz production programs. In the current paper, we will extend some of their results to problems without convexity assumptions.

We assume that R^1 (R^1_+) is the collection of all real (non-negative) numbers and that R^n is a finite-dimensional Euclidean space ordered by a non-negative orthant $R^n_+ = \{u \in R^n : \ u_j \geq 0, \ j = 1, \dots, n\}$. For every pair of vectors $u, v \in R^n$, let the inner product $uv = \sum_{j=1}^n u_j v_j$, and $u >> v$, $u > v$, $u \geq v$ have their usual meaning. Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and e be an element of R^n_+, all of whose coordinates are unity. For every point $u \in R^n$, let $\|u\|$ denote the Euclidean norm of u.

Let $a = (a_1, \dots, a_n) >> 0$, $b = (b_1, \dots, b_n) >> 0$ and let $d \in (0, 1]$.

In this paper, we study an economy which produces a finite number n of alternative types of machines. For every $i = 1, \dots, n$, one unit of machine of type i requires $a_i > 0$ units of labor to construct it, and together with one unit of labor, each unit of it can produce $b_i > 0$ units of a single consumption good. Therefore, the vectors a, b represent the production possibilities of the economy.

We assume that all machines depreciate at a rate of $d \in (0,1]$. For every integer $t \geq 0$, let $x(t) = (x_1(t), \dots, x_n(t)) \geq 0$ denote the amounts of the n types of machines which are available in time-period t, and let $z(t+1) = (z_1(t+1), \dots, z_n(t+1)) \geq 0$ be the gross investments in the n types of machines during period $t+1$. Thus, $z(t+1) = (x(t+1) - x(t)) + dx(t)$. Let $y(t) = (y_1(t), \dots, y_n(t))$ be the amounts of the n types of machines used for the production of the consumption good, $by(t)$, during period $t+1$. We assume that the total labor force of the economy is unity. Evidently, gross investment, $z(t+1)$, requires $az(t+1)$ units of labor in period t and $y(t)$ requires $ey(t)$ units of labor in period t. Therefore, the equation $az(t+1) + ey(t) \leq 1$ is true. For a more detailed discussion of the model, see [8,41]. We now give a formal description of this technological structure.

A sequence $\{x(t), y(t)\}_{t=0}^{\infty}$ is called a program if, for every non-negative integer t,

$$(x(t), y(t)) \in R^n_+ \times R^n_+, \ x(t+1) \geq (1-d)x(t),$$

$$0 \leq y(t) \leq x(t), \ a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1. \tag{1}$$

Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A pair of sequences

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$$

is called a program if $x(T_2) \in R^n_+$, and for every integer t which satisfies $T_1 \leq t < T_2$, Equation (1) is valid. Note that, here, $x(\cdot)$ is the state function, while $y(\cdot)$ is the control function.

Let $w : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function which represents the preferences of the planner.

For each $x_0 \in R^n_+$ and each integer $T > 0$, set

$$U(x_0, T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) :$$

$$(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1}) \text{ is a program such that } x(0) = x_0. \tag{2}$$

In the sequel, we assume that the supremum of the empty set is $-\infty$ and that the sum over the empty set is zero.

Let $x_0, \ \tilde{x}_0 \in R^n_+$ and let T be a natural number. Set

$$U(x_0, \tilde{x}_0, T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) :$$

$$(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1}) \text{ is a program such that } x(0) = x_0, \ x(T) \geq \tilde{x}_0 \tag{3}$$

The following result is easily deduced from the continuity of w.

Proposition 1. *For each $x_0 \in R^n_+$ and each natural number T there exists a program $(\{x(t)\}_{t=0}^{T}$, $\{y(t)\}_{t=0}^{T-1})$ such that $x(0) = x_0$ and $\sum_{t=0}^{T-1} w(by(t)) = U(x_0, T)$.*

Set

$$\Omega = \{(x, x') \in R^n_+ \times R^n_+ : \ x' \geq (1-d)x \text{ and } a(x' - (1-d)x) \leq 1\}. \tag{4}$$

Define a set-valued mapping $\Lambda : \Omega \to R^n_+$ by

$$\Lambda(x, x') = \{y \in R^n_+ : \ 0 \leq y \leq x \text{ and}$$

$$ey \leq 1 - a(x' - (1-d)x)\}, \ (x, x') \in \Omega. \tag{5}$$

Let $M_0 > 0$ and let T be a natural number. Set

$$\widehat{U}(M_0, T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) :$$

$$(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1}) \text{ is a program such that } x(0) \leq M_0 e \tag{6}$$

Evidently, $\widehat{U}(M_0, T)$ is finite. The following result is easily deduced from the continuity of w.

Proposition 2. *For each $M_0 > 0$ and each natural number T there exists a program $(\{x(t)\}_{t=0}^{T}$, $\{y(t)\}_{t=0}^{T-1})$ such that $x(0) \leq M_0 e$ and $\sum_{t=0}^{T-1} w(by(t)) = \widehat{U}(M_0, T)$.*

In this paper, we use the next simple Lemma (see Lemma 5.3 of [8]).

Lemma 1. *Let a number $M_0 > \max\{(a_i d)^{-1} : \ i = 1, \dots, n\}$, $(x, x') \in \Omega$ and let $x \leq M_0 e$. Then, $x' \leq M_0 e$.*

The study of the RSS model is a well-established area of research (see [8,9] and the references mentioned therein). Because of its simplicity, it allows us to study problems which cannot be solved for more complicated models. In particular, here, under certain assumptions, we obtain good programs on which investments are made only in the best of machines. Programs with such a property are called Stiglitz production programs. In [41], it was shown the existence of good and optimal programs are Stiglitz production programs in the case when the function w is concave. Here, we obtained analogous results without concavity assumptions.

Now, we present the main results of [42], which will be used in the sequel. They are extensions of some results [41] obtained when the function w was concave. It should be mentioned that the main goal in the study of models of economic growth is to show the existence of good and optimal programs. Usually, in the literature, their existence is shown when the function w representing the preferences of the planner is concave or even strictly concave. In this section, we present the results of our work [42], which show the existence of good and optimal programs without concavity assumptions on w.

We begin with the following result, which allows us to define the constant μ.

Theorem 1. *Let $M_1, M_2 > (da_j)^{-1}, \ j = 1, \dots, n$. Then, there exist finite limits*

$$\lim_{p \to \infty} \widehat{U}(M_i, p)/p, \ i = 1, 2$$

and

$$\lim_{p \to \infty} \widehat{U}(M_1, p)/p = \lim_{p \to \infty} \widehat{U}(M_2, p)/p.$$

Define

$$\mu = \lim_{p \to \infty} \widehat{U}(M, p)/p \tag{7}$$

where $M > \max\{(da_i)^{-1} : \ i = 1, \dots, n\}$. By Theorem 1, the constant μ is well-defined and it does not depend on M.

Theorem 2. *Assume that $M_0 > (da_j)^{-1} : \ j = 1, \dots, n$. Then, there exists a positive number M such that*

$$|\widehat{U}(M_0, p) - p\mu| \leq M \text{ for all integers } p \geq 1.$$

Corollary 1. *Let $M_0 > da_j^{-1}$: $j = 1, \dots, n$. Then, there exists a positive number M such that for every program $\{x(t), y(t)\}_{t=0}^{\infty}$ which satisfies $x(0) \le M_0 e$ and every natural number T, the inequality*

$$\sum_{t=0}^{T-1} [w(by(t)) - \mu] \le M$$

is valid.

Proposition 3. *Assume that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program. Then, either the sequence $\{\sum_{t=0}^{T-1}[w(by(t)) - \mu]\}_{T=1}^{\infty}$ is bounded or*

$$\lim_{T \to \infty} \sum_{t=0}^{T-1} [w(by(t)) - \mu] = -\infty.$$

In this paper, we use the following notion introduced by Gale [11].

A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is called good if there exists $M \in R^1$ such that

$$\sum_{t=0}^{T} (w(y(t)) - \mu) \ge M \text{ for all integers } T \ge 0.$$

A program is called bad if

$$\lim_{T \to \infty} \sum_{t=0}^{T} (w(y(t)) - \mu) = -\infty.$$

Proposition 3 implies that every program which is not good is bad.

Set

$$x(t) = (2nd \max\{a_i : \ i = 1, \dots, n\})^{-1} e,$$

$$y(t) = \min\{(2n)^{-1}, (2nd \max\{a_i : \ i = 1, \dots, n\})^{-1}\} e \text{ for all integers } t \ge 0.$$

It is clear that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program. Corollary 1 implies that

$$\mu \ge \lim_{T \to \infty} T^{-1} \sum_{t=0}^{T-1} w(by(t)) > w(0).$$

Thus,

$$\mu > w(0). \tag{8}$$

Theorem 3. *Let $M_0 > \max\{(da_i)^{-1} : \ i = 1, \dots, n\}$. Then, there exists $M > 0$ such that for each $x_0 \in R_+^n$ satisfying $x_0 \le M_0 e$, there exists a program $\{x(t), y(t)\}_{t=0}^{\infty}$ such that $x(0) = x_0$, for each integer $T_1 \ge 0$ and each integer $T_2 > T_1$*

$$\left| \sum_{t=T_1}^{T_2-1} w(by(t)) - \mu(T_2 - T_1) \right| \le M$$

and that for each integer $T > 0$

$$\sum_{t=0}^{T-1} w(by(t)) = U(x(0), x(T), T). \tag{9}$$

A program $\{x(t), y(t)\}_{t=0}^{\infty}$ is called weakly maximal if equality (9) holds for all integers $T > 0$.

Theorem 4. *Let* $\{x(t), y(t)\}_{t=0}^{\infty}$ *be a weakly maximal program such that* $\limsup_{t\to\infty} by(t) > 0$. *Then, the program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *is good.*

Many other results on optimal control problems related to models of economic growth are collected in [9,10].

2. The Main Results

Assume that there exists $\sigma \in \{1, \dots, n\}$ such that for each $i \in \{1, \dots, n\} \setminus \{\sigma\}$,

$$b_\sigma a_\sigma^{-1} \geq b_i a_i^{-1} \tag{10}$$

and

$$a_\sigma \geq a_i. \tag{11}$$

Under these assumptions, the machine σ is the most effective. It is natural to make investments only in the σ-type of machine. Programs with such a property are called Stiglitz production programs. In [41], it was shown the existence of good and optimal programs are Stiglitz production programs in cases when the function w is concave. Here, we obtained analogous results without concavity assumptions. Our results are of interest and importance since most results in the theory of economic growth are obtained under concavity assumptions on the function w.

It is clear that there exists a natural number $\tau \geq 4$ such that

$$\begin{aligned} &w(b(\min\{(2n)^{-1},\ (2nd\max\{a_j :\ j = 1, \dots, n\})^{-1}\})e) \\ &\geq \max\{b_j :\ j = 1, \dots, m\}(1-d)^{\tau}, \end{aligned} \tag{12}$$

$$\begin{aligned} &(\tau - 1)w(b(\min\{(2n)^{-1},\ (2nd\max\{a_j :\ j = 1, \dots, n\})^{-1}\})e) \\ &\geq \sum_{j=0}^{\tau} w(\max\{b_j :\ j = 1, \dots, m\}(1-d)^j). \end{aligned} \tag{13}$$

Our results will follow from the following Lemma, which is proven in the next section.

Lemma 2. *Assume that* $T_2 > T_1 \geq 0$ *are integers,* $(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$ *is a program, for each* $t \in \{T_1, \dots, T_2 - 1\}$,

$$z(t) = x(t+1) - (1-d)x(t), \tag{14}$$

$$y^{(i)}(t) \in R_+^n,\ i = 1, 2,$$

$$y^{(2)}(t) \leq (1-d)^{t-T_1} x(T_1), \tag{15}$$

$$y^{(1)}(t) \leq x(t) - (1-d)^{t-T_1} x(T_1), \tag{16}$$

$$y^{(1)}(t) + y^{(2)}(t) = y(t), \tag{17}$$

$$\tilde{x}(T_1) = x(T_1) \tag{18}$$

and that for each $t \in \{T_1, \dots, T_2 - 1\}$,

$$\tilde{x}_i(t+1) = (1-d)\tilde{x}_i(t),\ i \in \{1, \dots, n\} \setminus \{\sigma\}, \tag{19}$$

$$\tilde{x}_\sigma(t+1) = (1-d)\tilde{x}_\sigma(t) + a_\sigma^{-1}(a(x(t+1) - (1-d)x(t)), \tag{20}$$

$$\tilde{y}^{(2)}(t) = y^{(2)}(t),\ \tilde{y}^{(1)}(t) = a_\sigma^{-1} a e y^{(1)}(t) e(\sigma), \tag{21}$$

$$\tilde{y}^{(1)}(t) + \tilde{y}^{(2)}(t) = \tilde{y}(t). \tag{22}$$

Then, $(\{\tilde{x}(t)\}_{t=T_1}^{T_2}, \{\tilde{y}(t)\}_{t=T_1}^{T_2-1})$ *is a program and for each* $t \in \{T_1, \dots, T_2 - 1\}$*,*

$$\tilde{y}(t) - y(t) = a_\sigma^{-1} a y^{(1)}(t) e(\sigma) - y^{(1)}(t),$$

$$b\tilde{y}(t) - by(t) = \sum_{i=1}^{n} (b_\sigma a_\sigma^{-1} a_i - b_i) y_i^{(1)}(t) \geq 0. \tag{23}$$

Lemma 2 and Proposition 1 imply the following result.

Proposition 4. *For each* $x_0 \in R_+^n$ *and each natural number* T*, there exists a program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *such that* $x(0) = x_0$*,* $\sum_{t=0}^{T-1} w(by(t)) = U(x_0, T)$ *and that for each* $t \in \{T_1, \dots, T_2 - 1\}$*,*

$$\tilde{x}_i(t+1) = (1-d)\tilde{x}_i(t),\ i \in \{1, \dots, n\} \setminus \{\sigma\}.$$

Lemma 2 and Proposition 2 imply the following result.

Proposition 5. *For each* $M_0 > 0$ *and each natural number* T*, there exists a program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *such that* $x(0) \leq M_0 e$*,* $\sum_{t=0}^{T-1} w(by(t)) = \widehat{U}(M_0, T)$ *and that for each* $t \in \{T_1, \dots, T_2 - 1\}$*,*

$$\tilde{x}_i(t+1) = (1-d)\tilde{x}_i(t),\ i \in \{1, \dots, n\} \setminus \{\sigma\}.$$

Theorem 5. *Let* $M_0 > \max\{(da_i)^{-1} :\ i = 1, \dots, n\}$*. Then, there exists* $M > 0$ *such that for each* $x_0 \in R_+^n$ *satisfying* $x_0 \leq M_0 e$*, there exists a program* $\{x(t), y(t)\}_{t=0}^{\infty}$ *such that* $x(0) = x_0$*, for each integer* $T_1 \geq 0$ *and each integer* $T_2 > T_1$

$$|\sum_{t=T_1}^{T_2-1} w(by(t)) - \mu(T_2 - T_1)| \leq M,$$

for each integer $T > 0$

$$\sum_{t=0}^{T-1} w(by(t)) = U(x(0), x(T), T)$$

and that for each integer $t \geq 0$*,*

$$\tilde{x}_i(t+1) = (1-d)\tilde{x}_i(t),\ i \in \{1, \dots, n\} \setminus \{\sigma\}.$$

Proof. By Proposition 4, for each integer $k \geq 1$, there exists a program $(\{x^{(k)}(t)\}_{t=0}^{k}, \{y^{(k)}(t)\}_{t=0}^{k-1})$ such that

$$x^{(k)}(0) = x_0,$$

$$\sum_{t=0}^{k-1} w(by^{(k)}(t)) = U(x_0, k)$$

and that for each $t \in \{0, \dots, k-1\}$,

$$x_i^{(k)}(t+1) = (1-d)x_i^{(k)}(t),\ i \in \{1, \dots, n\} \setminus \{\sigma\}.$$

It was shown in the proof of Theorem 5.8 of [8] that there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ such that, for every non-negative integer t, there exists

$$\widehat{x}(t) = \lim_{j \to \infty} x^{(k_j)}(t),\ \widehat{y}(t) = \lim_{j \to \infty} y^{(k_j)}(t)$$

such that $\{\widehat{x}(t), \widehat{y}(t)\}_{t=0}^{\infty}$ is a program. For each integer $T_1 \geq 0$ and each integer $T_2 > T_1$,

$$|\sum_{t=T_1}^{T_2-1} w(b\widehat{y}(t)) - \mu(T_2 - T_1)| \leq M,$$

where M depends only on M_0 and that, for each integer $T > 0$,

$$\sum_{t=0}^{T-1} w(by(t)) = U(x(0), x(T), T).$$

It is clear that, for each integer $t \geq 0$,

$$\tilde{x}_i(t+1) = (1-d)\tilde{x}_i(t),\ i \in \{1, \dots, n\} \setminus \{\sigma\}.$$

Theorem 5 is proved. □

Lemma 2 implies the following result.

Proposition 6. *Assume that $\{x(t), y(t)\}_{t=0}^{\infty}$ is a program such that for each program $\{x'(t), y'(t)\}_{t=0}^{\infty}$ satisfying $x'(0) = x(0)$, the inequality*

$$\limsup_{T\to\infty}(\sum_{t=0}^{T-1} w(by'(t)) - \sum_{t=0}^{T-1} w(by(t)) \leq 0$$

$$(\limsup_{T\to\infty}(\sum_{t=0}^{T-1} w(by(t)) - \sum_{t=0}^{T-1} w(by'(t)) \geq 0 \ \textit{resp.})$$

holds. Then, there exists a program $\{\tilde{x}(t), \tilde{y}(t)\}_{t=0}^{\infty}$ satisfying $\tilde{x}(0) = x(0)$ such that for each program $\{x'(t), y'(t)\}_{t=0}^{\infty}$ satisfying $x'(0) = \tilde{x}(0)$ the inequality

$$\limsup_{T\to\infty}(\sum_{t=0}^{T-1} w(by'(t)) - \sum_{t=0}^{T-1} w(b\tilde{y}(t)) \leq 0$$

$$(\limsup_{T\to\infty}(\sum_{t=0}^{T-1} w(b\tilde{y}(t)) - \sum_{t=0}^{T-1} w(by'(t)) \geq 0 \ \textit{resp.})$$

and that for each integer $t \geq 0$,

$$\tilde{x}_i(t+1) = (1-d)\tilde{x}_i(t),\ i \in \{1, \dots, n\} \setminus \{\sigma\}.$$

3. Proof of Lemma 2

By (14), for each $t \in \{T_1, \dots, T_2 - 1\}$,

$$x(t+1) = (1-d)x(t) + z(t). \tag{24}$$

In view of (24), for each $s \in \{T_1 + 1, \dots, T_2\}$,

$$x(s) - (1-d)^{s-T_1} x(T_1)$$

$$= \sum_{t=0}^{s-1-T_1} ((1-d)^{s-T_1-t-1} x(T_1+t+1) - (1-d)^{S-T_1-t} x(T_1+t))$$

$$= \sum_{t=0}^{s-1-T_1} (1-d)^{s-T_1-t-1} z(T_1+t). \tag{25}$$

For each $t \in \{T_1, \dots, T_2 - 1\}$, set

$$\tilde{z}(t) = \tilde{x}(t+1) - (1-d)\tilde{x}(t). \tag{26}$$

Let $t \in \{T_1, \dots, T_2 - 1\}$. By (19)–(22) and (26),

$$\tilde{x}(t+1) \geq (1-d)\tilde{x}(t),\ \tilde{y}(t) \geq 0,\ \tilde{z}(t) \geq 0. \tag{27}$$

In view of (26), for each $s \in \{T_1 + 1, \dots, T_2\}$,

$$\tilde{x}(s) - (1-d)^{s-T_1}\tilde{x}(T_1)$$

$$= \sum_{t=0}^{s-1-T_1} ((1-d)^{s-T_1-t-1}\tilde{x}(T_1+t+1) - (1-d)^{S-T_1-t}\tilde{x}(T_1+t))$$

$$= \sum_{t=0}^{s-1-T_1} (1-d)^{s-T_1-t-1}\tilde{z}(T_1+t). \tag{28}$$

It follows from (19), (20) and (26) that, for each $t \in \{T_1, \dots, T_2 - 1\}$,

$$\tilde{z}(t) = \tilde{x}(t+1) - (1-d)\tilde{x}(t)$$

$$= a_\sigma^{-1} a(x(t+1) - (1-d)x(t))e(\sigma)$$

$$= a_\sigma^{-1} a z(t) e(\sigma). \tag{29}$$

Equations (18), (28) and (29) imply that, for each $s \in \{T_1, \dots, T_2\}$,

$$\tilde{x}(s) = (1-d)^{s-T_1}x(T_1)$$

$$+ a_\sigma^{-1} a \sum\{(1-d)^{s-T_1-i-1}z(T_1+i) : \ i \text{ is an integer},\ 0 \leq i \leq s-1-T_1\}e(\sigma). \tag{30}$$

We show that $(\{\tilde{x}(t)\}_{t=T_1}^{T_2}, \{\tilde{y}(t)\}_{t=T_1}^{T_2-1})$ is a program. Let $t \in \{T_1, \dots, T_2 - 1\}$. In view of (19) and (20),

$$a(\tilde{x}(t+1) - (1-d)\tilde{x}(t))$$

$$= a_\sigma(\tilde{x}_\sigma(t+1) - (1-d)\tilde{x}_\sigma(t))$$

$$= a(x(t+1) - (1-d)x(t)). \tag{31}$$

It follows from (15), (16), (21) and (22) that

$$\tilde{y}(t) = \tilde{y}^{(1)}(t) + \tilde{y}^{(2)}(t)$$

$$\leq (1-d)^{t-T_1}x(T_1) + a_\sigma^{-1}a((x(t) - (1-d)^{t-T_1}x(T_1))e_\sigma$$

$$= a_\sigma^{-1} a x(t) e(\sigma)$$

$$+ (1-d)^{t-T_1}x(T_1) - (1-d)^{t-T_1}a_\sigma^{-1} a x(T_1)e(\sigma). \tag{32}$$

By (15)–(18), (25), (30) and (32),

$$\tilde{y}(t) \leq (1-d)^{t-T_1}x(T_1)$$

$$+ a_\sigma^{-1} a \sum\{(1-d)^{t-T_1-i-1}z(T_1+i) :$$

$$i \text{ is an integer},\ 0 \leq i \leq t-1-T_1-1\}e(\sigma) = \widehat{x}(t). \tag{33}$$

It follows from (1), (17), (21), (22), (30) and (31) that

$$a(\tilde{x}(t+1) - (1-d)\tilde{x}(t)) + e\tilde{y}(t)$$

$$\leq a(x(t+1)-(1-d)x(t))+e\tilde{y}^{(1)}(t)+e\tilde{y}^{(2)}(t)$$
$$\leq a(x(t+1)-(1-d)x(t))+ey^{(2)}(t)+a_\sigma^{-1}ay^{(1)}(t)$$
$$\leq a(x(t+1)-(1-d)x(t))+ey^{(1)}(t)+ey^{(2)}(t)\leq 1. \quad (34)$$

By (27), (33) and (34), $(\{\tilde{x}(t)\}_{t=T_1}^{T_2}, \{\tilde{y}(t)\}_{t=T_1}^{T_2-1})$ is a program. By (17), (21) and (22), for each $t \in \{T_1, \dots, T_2-1\}$,

$$\tilde{y}(t)-y(t)=\tilde{y}^{(1)}(t)-y^{(1)}(t)=a_\sigma^{-1}ay^{(1)}(t)e(\sigma)-y^{(1)}(t),$$

$$b\tilde{y}(t)-by(t)=\sum_{i=1}^{n}(b_\sigma a_\sigma^{-1}a_i-b_i)y_i^{(1)}(t).$$

Lemma 2 is proved.

4. Optimal Programs

A program $\{x(t), y(t)\}_{t=0}^\infty$ is called optimal if, for each program $\{\tilde{x}(t), \tilde{y}(t)\}_{t=0}^\infty$ satisfying $\tilde{x}(0) = x(0)$, the inequality

$$\limsup_{T\to\infty}(\sum_{t=0}^{T-1} w(by(t)) - \sum_{t=0}^{T-1} w(b\tilde{y}(t)) \geq 0$$

holds.

Theorem 6. *Assume that*

$$d<1, \quad (35)$$

$$b_\sigma a_\sigma^{-1} > b_i a_i^{-1},\ i \in \{1,\dots,n\}\setminus\{\sigma\}, \quad (36)$$

$\{x(t), y(t)\}_{t=0}^\infty$ *is an optimal program and that*

$$z(t)=x(t+1)-(1-d)x(t),\ t=0,1,\dots.$$

Then, $z_i(t) = 0$ *for each integer* $t \geq 0$ *and each* $i \in \{1,\dots,n\}\setminus\{\sigma\}$.

Proof. For each integer $t \geq 0$ and each $i \in \{1,\dots,n\}$, set

$$y_i^{(1)}(t)=\max\{x_i(t)-(1-d)^t x_i(0),\ y_i(t)\}, \quad (37)$$

$$y_i^{(2)}(t)=y_i(t)-y_i^{(1)}(t).$$

Since our program is optimal, it is not difficult to see that for each integer $t \geq 0$ at least one of the following relations holds:

$$a(x(t+1)-(1-d)x(t))+ey(t)=1; \quad (38)$$

$$y(t)=x(t). \quad (39)$$

For each integer $t \geq 0$, set

$$\tilde{x}(0)=x(0),$$
$$\tilde{x}_i(t+1)=(1-d)\tilde{x}_i(t),\ i\in\{1,\dots,n\}\setminus\{\sigma\},$$
$$\tilde{x}_\sigma(t+1)=(1-d)\tilde{x}_\sigma(t)+a_\sigma^{-1}(a(x(t+1)-(1-d)x(t)),$$
$$\tilde{y}^{(2)}(t)=y^{(2)}(t),\ \tilde{y}^{(1)}(t)=a_\sigma^{-1}aey^{(1)}(t)e(\sigma),$$
$$\tilde{y}^{(1)}(t)+\tilde{y}^{(2)}(t)=\tilde{y}(t).$$

Lemma 2 and (23) imply that $\{\tilde{x}(t), \tilde{y}(t)\}_{t=0}^{\infty}$ is a program, for each integer $t \geq 0$,

$$0 \leq b\tilde{y}(t) - by(t) = \sum_{i=1}^{n}(b_{\sigma}a_{\sigma}^{-1}a_i - b_i)y_i^{(1)}(t)$$

and, by (36),

$$b\tilde{y}(t) = by(t)$$

if and only if

$$y_i(t) = 0,\ i \in \{1, \ldots, n\} \setminus \{\sigma\}.$$

Since the program $\{x(t), y(t)\}_{t=0}^{\infty}$ is optimal, this implies that, for each integer $t \geq 0$ and each $i \in \{1, \ldots, n\} \setminus \{\sigma\}$,

$$y_i^{(1)}(t) = 0. \tag{40}$$

We show that for each integer $p \geq 0$ and each $i \in \{1, \ldots, n\} \setminus \{\sigma\}$,

$$z_i(p) = 0.$$

Assume the contrary. Then, there exist integers $p \geq 0$ and $i \in \{1, \ldots, n\} \setminus \{\sigma\}$ such that

$$z_i(p) > 0. \tag{41}$$

We show that

$$x(p) = y(p).$$

By (35), (37), (40), (41) and the relation $d < 1$ for each integer $t \geq p + 1$,

$$x_i(t) - (1-d)^t x_i(0) > 0,\ y_i^{(1)}(t) = 0. \tag{42}$$

In view of (38), (39) and (42) for each integer $t \geq p + 1$,

$$y_i(t) = 0,\ a(x(t+1) - (1-d)x(t)) + ey(t) = 1. \tag{43}$$

Set

$$y^{(0)}(t) = y(t),\ t = 0, 1, \ldots,$$
$$x^{(0)}(t) = x(t),\ t = 0, 1, \ldots, p,$$
$$z^{(0)}(t) = z(t),\ t \in \{0, 1, \ldots, p\} \setminus \{p\},$$
$$z^{(0)}(p) = z(p) - 2^{-1}z_i(p)e(p),$$
$$x^{(0)}(p+1) = (1-d)x^{(0)}(p) + z^{(0)}(p),$$
$$x^{(0)}(t+1) = (1-d)x^{(0)}(t) + z^{(0)}(t)$$

for each integer $t \geq p$. By the equation above, (41) and (43), $\{x^{(0)}(t), y^{(0)}(t)\}_{t=0}^{\infty}$ is an optimal program such that

$$a(x^{(0)}(p+1) - (1-d)x^{(0)}(p)) + ey^{(0)}(p) \leq 1 - 2^{-1}z_i(p)a_i. \tag{44}$$

This implies that for each integer $s \geq 0$ at least one of the following relations holds:

$$a(x^{(0)}(s+1) - (1-d)x^{(0)}(s)) + ey^{(0)}(s) = 1;\ y^{(0)}(s) = x^{(0)}(s).$$

Together with (44), this implies that

$$x(p) = y(p).$$

We show that for each integer $s > p$,

$$z(s) = z_\sigma(s)e(\sigma).$$

Assume the contrary. Then, there exist integers $s > p$ and $j \in \{1, \dots, n\} \setminus \{\sigma\}$ such that

$$z_j(s) > 0. \tag{45}$$

By (41), (43) and (45),

$$y_j(t) = 0 \text{ for each integer } t > s. \tag{46}$$

In view of (42) and (45), choose a positive number

$$\delta < \min\{x_i(s),\ a_j z_j(s)\}. \tag{47}$$

Set

$$\widehat{y}(t) = y(t),\ \widehat{z}(t) = z(t), t \in \{0, 1, \dots\} \setminus \{s\}, \tag{48}$$

$$\widehat{z}(s) = z(s) - z_j(s)e(\sigma),\ \widehat{y}(s) = y(s) + \delta e(i), \tag{49}$$

$$\widehat{x}(t+1) = (1-d)\widehat{x}(t) + \widehat{z}(t),\ t = 0, 1, \dots. \tag{50}$$

Equations (47), (49) and (50) imply that

$$\widehat{y}_i(s) = \delta < x_i(s) = \widehat{x}_i(s),\ \widehat{y}(s) \le \widehat{x}(s). \tag{51}$$

It follows from (47) and (49) that

$$a\widehat{z}(s) + e\widehat{y}(s)$$

$$= az(s) - a_j z_j(s) + ey(s) + \delta$$

$$\le 1 - a_j z_j(s) + \delta < 1. \tag{52}$$

It follows from (45), (46) and (48)–(52) that $\{\widehat{x}(t), \widehat{y}(t)\}_{t=0}^{\infty}$ is a program. By (48) and (49), for each integer $T > S$,

$$\sum_{t=0}^{T} w(b\widehat{y}(t)) - \sum_{t=0}^{T} w(by(t))$$

$$= w(b\widehat{y}(s)) - w(by(s)) = w(by(s) + \delta b_i) - w(by(s)) > 0.$$

This contradicts the optimality of the program $\{x(t), y(t)\}_{t=0}^{\infty}$. The contradiction we have reached proves that

$$z(s) = z_\sigma(s)e(\sigma) \text{ for each integer } s > p. \tag{53}$$

Now, we show that $z(t) = 0$ for every integer $t > p$. Assume the contrary. Then, there exists an integer $s > p$ such that

$$z(s) > 0 \tag{54}$$

and

$$z(t) = 0 \text{ for each integer } t \text{ satisfying } p < t < s. \tag{55}$$

By (44) and (53),

$$z(s) = z_\sigma(s)e(\sigma) > 0. \tag{56}$$

Define

$$\bar{x}(t) = x(t),\ \bar{y}(t) = y(t),\ t = 0, \dots, p, \tag{57}$$

$$\bar{z}(t) = z(t),\ t \in \{0, \dots, p\} \setminus \{p\}, \tag{58}$$

$$\bar{z}(p) = z(p) + 2^{-1}a_\sigma^{-1}a_i z_i(p)e(\sigma) - z_i(p)e(i), \tag{59}$$

$$\bar{x}(p+1) = (1-d)\bar{x}(p) + \bar{z}(p). \tag{60}$$

By (57)–(60),

$$a\bar{z}(p) = az(p) - a_i z_i(p)/2, \tag{61}$$

$$a\bar{z}(p) + e\bar{y}(p) \leq 1 - a_i z_i(p)/2 \tag{62}$$

and that $(\{\bar{x}(t)\}_{t=0}^{p+1}, \{\bar{y}(t)\}_{t=0}^{p})$ is a program. For each integer t satisfying $p < t < s$, set

$$\bar{z}(t) = 0,\ \bar{y}(t) = y(t),\ \bar{x}(t+1) = (1-d)\bar{x}(t). \tag{63}$$

By (43), (49) and (61)–(63), $(\{\bar{x}(t)\}_{t=0}^{s}, \{\bar{y}(t)\}_{t=0}^{s-1})$ is a program. It follows from (55), (57)–(60) and (63) that

$$\begin{aligned}\bar{x}_\sigma(s) &= (1-d)^{s-p-1}\bar{x}_\sigma(p+1) \\ &= (1-d)^{s-p-1}(x_\sigma(p)(1-d) + \bar{z}_\sigma(p)) \\ &= (1-d)^{s-p-1}(x_\sigma(p)(1-d) + z_\sigma(p)e(\sigma) + 2^{-1}a_\sigma^{-1}a_i z_i(p)) \\ &= x_\sigma(s) + 2^{-1}(1-d)^{s-p-1}a_\sigma^{-1}a_i z_i(p).\end{aligned} \tag{64}$$

Choose a number $\delta \in (0,1)$ such that

$$\delta < 2^{-1}z_\sigma(s),\ \delta < 4^{-1}(1-d)^{s-p}\min\{a_\sigma^{-1}, a_\sigma^{-2}\}a_i z_i(p). \tag{65}$$

Set

$$\begin{aligned}\bar{z}(s) = z(s) - \delta e(\sigma),\ \bar{x}(s+1) = (1-d)\bar{x}(s) + \bar{z}(s), \\ \bar{y}(s) = y(s) + a_\sigma \delta e(\sigma).\end{aligned} \tag{66}$$

By (56) and (65),

$$\bar{z}(s) > 0.$$

In view of (64)–(66),

$$\bar{y}_\sigma(s) \leq x_\sigma(s) + a_\sigma\delta \leq \bar{x}_\sigma(s). \tag{67}$$

Equations (43), (49), (66) and (67) imply that

$$\bar{y}(s) \leq \bar{x}(s),$$

$$a\bar{z}(s) + e\bar{y}(s) = az(s) + ey(s) \leq 1.$$

It follows from the equation above that $(\{\bar{x}(t)\}_{t=0}^{s+1}, \{\bar{y}(t)\}_{t=0}^{s})$ is a program. By (64)–(66),

$$\begin{aligned}\bar{x}_\sigma(s+1) = (1-d)x_\sigma(s) + 2^{-1}(1-d)^{s-p}a_\sigma^{-1}a_i z_i(p) \\ + z_\sigma(s) - \delta \geq x_\sigma(s+1).\end{aligned}$$

In view of (57) and (66),

$$\begin{aligned}&\sum_{t=0}^{s} w(b\bar{y}(t)) - \sum_{t=0}^{s} w(by(t)) \\ &= w(by(s) + b_\sigma a_\sigma \delta) - w(b(s)) > 0.\end{aligned} \tag{68}$$

For every integer $t \geq s+1$, set

$$\bar{z}(t) = z(t),\ \bar{y}(t) = y(t),\ \bar{x}(t+1) = (1-d)\bar{x}(t) + \bar{z}(t).$$

It is not difficult to see that $\{\bar{x}(t), \bar{y}(t)\}_{t=0}^{\infty}$ is a program. By (68), for each integer $T > s$,

$$\sum_{t=0}^{T} w(b\bar{y}(t)) - \sum_{t=0}^{T} w(by(t))$$

$$= w(by(s) + b_{\sigma}a_{\sigma}\delta) - w(b(s)) > 0.$$

This contradicts the optimality of the program $\{x(t), y(t)\}_{t=0}^{\infty}$.
The contradiction we have reached implies that

$$z(t) = 0 \text{ for each integer } t > p.$$

This implies that

$$x(t) \to 0, \; w(by(t)) \to 0 \text{ as } t \to \infty.$$

By (8),

$$\lim_{T\to\infty} \sum_{t=0}^{T} w(by(t)) - \mu) = -\infty.$$

On the other hand, by Theorem 3, there exists a good program starting from the point $x(0)$. This contradicts the optimality of the program $\{x(t), y(t)\}_{t=0}^{\infty}$. The contradiction we have reached implies that

$$z_i(p) = 0$$

for each integer $p \geq 0$ and each $i \in \{1, \ldots, n\} \setminus \{\sigma\}$. Theorem 6 is proved. □

5. Conclusions

In our paper, we study a discrete-time optimal control problem which describes the model of Robinson, Solow and Srinivasan. We analyze this model with a non-concave utility function which represents the preferences of the planner and establish the existence of good programs and optimal programs which are Stiglitz production programs. Our results show that when we construct a good program, it is enough to make investments only in the best type of machine.

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