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One-Parameter Hyperbolic Spatial Locomotions and Invariants of the Axode

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Abstract: In this paper, based on the E. Study map, direct appearances were sophisticated for one-parameter hyperbolic dual spherical locomotions and invariants of the axodes. With the suggested technique, the Disteli formulae for the axodes were acquired and the correlations through kinematic geometry of a timelike line trajectory were provided. Then, a ruled analogy of the curvature circle of a curve in planar locomotions was expanded into generic spatial locomotions. Lastly, we present new hyperbolic proofs for the Euler–Savary and Disteli formulae.

Keywords: Serret–Frenet formulae; rectifying curve; helices

MSC: 53A04; 53A05; 53A17

1. Introduction

Line trajectory has a close association with spatial locomotions and has thus established achievements in robot kinematics and mechanism designs. In instantaneous locomotions, it is advantageous to examine the essential possesses of the line trajectory from the notions of the ruled surface in differential geometry [1–4]. In the context of spatial locomotions, the instantaneous screw axis ($\mathbb{ISA}$) of a mobile body tracing a couple of ruled surfaces, are named the mobile and immobile axodes, with $\mathbb{ISA}$ as their usual tracing in the movable space and in the steady space, respectively. Through the locomotion, the axodes glide and roll relative to each other in a certain procedure, such that the tangential alliance among the axodes is constantly maintained on the entire length of the two matting tracings (one being in each axode), which simultaneously set the $\mathbb{ISA}$ at any instant. Therefore, the investigation of a one-parameter spatial locomotion and axodes is of magnificent importance in several subjects such as mathematics, physics, and engineering. In view of kinematics, the axodes supply valuable details on the attitude of items as they proceed through space. One of the prime reasons for examining the axodes of one-parameter spatial locomotions is to understand the geometric and kinematic characteristics of mobile items. The axodes can be employed to characterize the trajectory, velocity, and acceleration of an item as it is mobile through space, providing perceptions into its physical behavior. Furthermore, they can be utilized to form mathematical samples of mobile frameworks, which can be utilized to resolve and optimize complex engineering frameworks [4–9]. Moreover, the axodes can be utilized to characterize the geometry of ruled surfaces, such as their curvature functions. This association among the axodes and ruled surfaces has significant implementations in subjects such as computer graphics, architecture mechanism designs, and robot kinematics [7–10].

Rather unexpectedly, dual numbers have been exercised to study the locomotion of a line space; their manifestation can even be the supplementary restful instruments for this end. In the theory of dual numbers, the E. Study map concludes that the set of the dual points on dual unit sphere in the dual 3-space $\mathbb{D}^3$ is in bijection with set of all directed

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lines in Euclidean 3-space $\mathbb{E}^3$. Via this map, a one-parameter set of points (a dual curve) on dual unit sphere matches to a one-parameter set of directed lines (ruled surface) in $\mathbb{E}^3$ [1–7]. In the Minkowski 3-space $\mathbb{E}_1^3$, Lorentzian metric can have one of three Lorentzian causal characters; it can be positive, negative, or zero, while the metric in the Euclidean 3-space $\mathbb{E}^3$ can only be positive definite. Therefore, the kinematic and geometric clarifications can be further meaningful in $\mathbb{E}_1^3$ [11–13].

This study is as follows. Section 2 shows how to design the E. Study map by the dual numbers and dual Lorentzian vectors manners. In Section 3, the hyperbolic dual spherical movements are explained in a short form. As a consequence, the invariants of the hyperbolic locomotions are gained to reveal the intrinsic properties of the axodes. Then, for the axodes, new proofs for Euler–Savary, and Disteli's formulae were acquired. Furthermore, the axodes are utilized to search the timelike line trajectory based on the Blaschke frame. In addition, the invariants of a timelike line trajectory and their monarchies are derived that of the axodes for a new proof of the Disteli formulae.

2. Basic Concepts

In this section, we give a short synopsis of the dual numbers theory, and dual Lorentzian vectors [11–15]. If a and a^* are real numbers, the term $\widehat{a} = a + \varepsilon a^*$ is named a dual number. Here, ε is a dual unit subject to $\varepsilon \neq 0,\ \varepsilon^2 = 0, \varepsilon \cdot 1 = 1 \cdot \varepsilon = \varepsilon$. Then, the set

$$\mathbb{D}^3 = \{\widehat{\mathbf{a}} := \mathbf{a} + \varepsilon \mathbf{a}^* = (\widehat{a}_1, \widehat{a}_2, \widehat{a}_3)\},$$

simultaneously with the Lorentzian inner product

$$\langle \widehat{\mathbf{a}}, \widehat{\mathbf{a}} \rangle = -\widehat{a}_1^2 + \widehat{a}_2^2 + \widehat{a}_3^2,$$

is titled dual Lorentzian 3-space $\mathbb{D}_1^3$. Thereby, a point $\widehat{\mathbf{a}} = (\widehat{a}_1, \widehat{a}_2, \widehat{a}_3)^t$ has dual coordinates $\widehat{a}_i = (a_i + \varepsilon a_i^*) \in \mathbb{D}$. If $\mathbf{a} \neq \mathbf{0}$, the norm of $\widehat{\mathbf{a}}$ is explained by

$$\|\widehat{\mathbf{a}}\| = \sqrt{|\langle \widehat{a}, \widehat{a} \rangle|} = \|\mathbf{a}\|(1 + \varepsilon \frac{\langle \mathbf{a}, \mathbf{a}^* \rangle}{\|\mathbf{a}\|^2}).$$

Thus, $\widehat{\mathbf{a}}$ is a spacelike (timelike) dual unit vector if $\|\widehat{\mathbf{a}}\|^2 = 1 (\|\widehat{\mathbf{a}}\|^2 = -1)$. It is evident that

$$\|\widehat{\mathbf{a}}\|^2 = \pm 1 \Longleftrightarrow \|\mathbf{a}\|^2 = \pm 1,\ \langle \mathbf{a}, \mathbf{a}^* \rangle = 0.$$

The hyperbolic and Lorentzian (de Sitter space) dual unit spheres, respectively, are:

$$\mathbb{H}_+^2 = \{\widehat{\mathbf{a}} \in \mathbb{D}_1^3 \mid \|\widehat{\mathbf{a}}\|^2 := -\widehat{a}_1^2 + \widehat{a}_2^2 + \widehat{a}_3^2 = -1\},$$

and

$$\mathbb{S}_1^2 = \{\widehat{\mathbf{a}} \in \mathbb{D}_1^3 \mid \|\widehat{\mathbf{a}}\|^2 := -\widehat{a}_1^2 + \widehat{a}_2^2 + \widehat{a}_3^2 = 1\}.$$

Then, the E. Study map concludes that the ring shaped hyperboloid is in bijection with the set of spacelike lines, the common asymptotic cone is in bijection with the set of null-lines, and the oval shaped hyperboloid is in bijection with the set of timelike lines (see Figure 1).

In view of the E. Study map, a smooth curve at the dual unit sphere $\mathbb{H}_+^2$ (or $\mathbb{S}_1^2$) can locally be spacelike, timelike or null (lightlike), if all of its tangent vectors are spacelike, timelike or null (lightlike), respectively. The smooth curve $t \in \mathbb{R} \mapsto \widehat{a}(t) \in \mathbb{H}_+^2$ (or $\mathbb{S}_1^2$) matches a timelike ruled surface $(\widehat{a})$ in Minkowski 3-space $\mathbb{E}_1^3$. Similarly, the smooth dual curve on $\mathbb{S}_1^2$ matches a spacelike or timelike ruled surface in $\mathbb{E}_1^3$. $\widehat{a}(t)$ are specified with the rulings of the surface and, from now on, we do not distinguish among ruled surface and its clarifying dual curve.

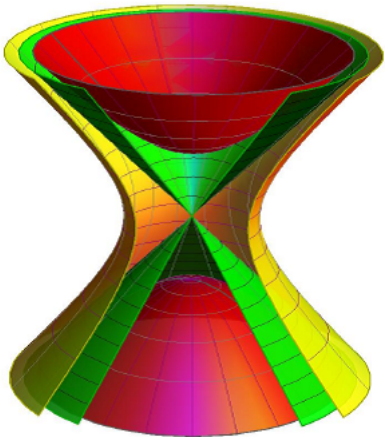


Figure 1. The dual hyperbolic and dual Lorentzian unit spheres.

Definition 1. *For any two (non-null) dual vectors* $\widehat{\mathbf{a}}$ *and* $\widehat{\mathbf{b}}$ *in* $\mathbb{D}_1^3$*, we have [11–15]:*

(i) If $\widehat{\mathbf{a}}$ *and* $\widehat{\mathbf{b}}$ *are two spacelike dual vectors, then*

- *If they specify a spacelike dual plane, there is a single dual number* $\widehat{\alpha} = \alpha + \varepsilon\alpha^*$; $0 \leq \alpha \leq \pi$, *and* $\alpha^* \in \mathbb{R}$ *such that* $\langle \widehat{\mathbf{a}}, \widehat{\mathbf{b}} \rangle = \|\widehat{\mathbf{a}}\| \left\|\widehat{\mathbf{b}}\right\| \cos\widehat{\alpha}$. *This number is the spacelike dual angle through* $\widehat{\mathbf{a}}$ *and* $\widehat{\mathbf{b}}$.
- *If they specify a timelike dual plane, there is a single dual number* $\widehat{\alpha} = \alpha + \varepsilon\alpha^* \geq 0$ *such that* $\langle \widehat{\mathbf{a}}, \widehat{\mathbf{b}} \rangle = \epsilon\|\widehat{\mathbf{a}}\| \left\|\widehat{\mathbf{b}}\right\| \cosh\widehat{\alpha}$, *where* $\epsilon = +1$ *or* $\epsilon = -1$ *via* $sign(\widehat{a}_2) = sign(\widehat{b}_2)$ *or* $sign(\widehat{a}_2) \neq sign(\widehat{b}_2)$, *respectively. This number is the central dual angle through* $\widehat{\mathbf{a}}$ *and* $\widehat{\mathbf{b}}$.

(ii) If $\widehat{\mathbf{a}}$ *and* $\widehat{\mathbf{b}}$ *are two timelike dual vectors, then there is a single dual number* $\widehat{\alpha} = \alpha + \varepsilon\alpha^* \geq 0$ *such that* $\langle \widehat{\mathbf{a}}, \widehat{\mathbf{b}} \rangle = \epsilon\|\widehat{\mathbf{a}}\| \left\|\widehat{\mathbf{b}}\right\| \cosh\widehat{\alpha}$, *where* $\epsilon = +1$ *or* $\epsilon = -1$ *via* $\widehat{\mathbf{a}}$ *and* $\widehat{\mathbf{b}}$ *have different time-direction or the same time-direction, respectively. This dual number is the Lorentzian timelike dual angle through* $\widehat{\mathbf{a}}$ *and* $\widehat{\mathbf{b}}$.

(iii) If $\widehat{\mathbf{a}}$ *is spacelike dual, and* $\widehat{\mathbf{b}}$ *is timelike dual, then there is a single dual number* $\widehat{\alpha} = \alpha + \varepsilon\alpha^* \geq 0$ *such that* $\langle \widehat{\mathbf{a}}, \widehat{\mathbf{b}} \rangle = \epsilon\|\widehat{\mathbf{a}}\| \left\|\widehat{\mathbf{b}}\right\| \sinh\widehat{\alpha}$, *where* $\epsilon = +1$ *or* $\epsilon = -1$ *via* $sign(\widehat{a}_2) = sign(\widehat{b}_1)$ *or* $sign(\widehat{a}_2) \neq sign(\widehat{b}_1)$. *This number is the Lorentzian timelike dual angle through* $\widehat{\mathbf{a}}$ *and* $\widehat{\mathbf{b}}$.

3. One-Parameter Hyperbolic Dual Spherical Locomotions

Let us consider that $\mathbb{H}^2_{+m}$ and $\mathbb{H}^2_{+f}$ are two hyperbolic dual unit spheres with $\widehat{\mathbf{0}}$ as a mutual center. Let $\{\widehat{\mathbf{e}}_m\}$=$\{\widehat{\mathbf{e}}_1$(timelike), $\widehat{\mathbf{e}}_2$, $\widehat{\mathbf{e}}_3\}$ and $\{\widehat{\mathbf{e}}_f\}$=$\{\widehat{\mathbf{f}}_1$(timelike), $\widehat{\mathbf{f}}_2$, $\widehat{\mathbf{f}}_3\}$ be two orthonormal dual coordinate frames strictly attached to $\mathbb{H}^2_{+m}$ and $\mathbb{H}^2_{+f}$, respectively. We assume that $\{\widehat{\mathbf{e}}_f\}$ is steady, whereas the elements of the set $\{\widehat{\mathbf{e}}_m\}$ are functions of a real parameter $t \in \mathbb{R}$ (say, time). Then, we say that $\mathbb{H}^2_{+m}$ are movables relative to the steady hyperbolic dual unit sphere $\mathbb{H}^2_{+f}$. The explanation of this is as follows: $\mathbb{H}^2_{+m}$ strictly attached with $\{\widehat{\mathbf{e}}_m\}$ movables over $\mathbb{H}^2_{+f}$ strictly attached with $\{\widehat{\mathbf{e}}_f\}$. This locomotion is named a one-parameter dual spherical locomotion, and will symbolized by $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$. If $\mathbb{H}^2_{+m}$ and $\mathbb{H}^2_{+f}$ correlate with the hyperbolic line spaces $\mathbb{L}_m$ and $\mathbb{L}_f$, respectively, then $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$ correlate with the one-parameter hyperbolic spatial locomotion $\mathbb{L}_m/\mathbb{L}_f$. Hence $\mathbb{L}_m$ is the movable space relative to the steady space $\mathbb{L}_f$. Since $\{\widehat{\mathbf{e}}_m\}$ and $\{\widehat{\mathbf{e}}_f\}$ has the same orientation, one frame is gained by using another when rotated about $\widehat{\mathbf{0}}$. By letting $< \widehat{\mathbf{r}}_i, \widehat{\mathbf{f}}_j >= \widehat{a}_{ij} = a_{ij} + \varepsilon a^*_{ij}$,

and assuming the dual matrix $\widehat{a}(t) = (\widehat{a}_{ij})$, it then follows that the signature matrix ϵ, explaining the inner product of dual Lorentzian 3-space $\mathbb{D}_1^3$, is specified by

$$\epsilon = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the dual matrix $\widehat{a}(t) = a_{ij}(t) + \varepsilon a_{ij}^*(t)$ has the characteristics $\widehat{a}^T = \epsilon \widehat{a}^{-1} \epsilon$, and $\widehat{a}^{-1} = \epsilon \widehat{a}^T \epsilon$. Therefore, we have

$$\widehat{a}\widehat{a}^{-1} = \widehat{a}\epsilon\widehat{a}^T\epsilon a = \widehat{a}^{-1}\widehat{a} = \epsilon\widehat{a}^T\epsilon\widehat{a} = I, \tag{1}$$

which shows it is an orthogonal matrix. This result signs that when a one-parameter hyperbolic spatial locomotion is described in $\mathbb{E}_1^3$, we can acquire a correlate hyperbolic dual orthogonal 3×3 matrix $\widehat{a}(t) = (\widehat{a}_{ij})$, where $(\widehat{a}_{ij})$ are dual functions of one variable $t \in \mathbb{R}$. As the set of real hyperbolic orthogonal matrices, the set of hyperbolic dual orthogonal 3×3 matrices, symbolized by $\mathbb{O}(\mathbb{D}_1^{3\times3})$, form a group with matrix multiplication as the group procedure (real hyperbolic orthogonal matrices are subgroup of hyperbolic dual orthogonal matrices). The identity element of $\mathbb{O}(\mathbb{D}_1^{3\times3})$ is the 3×3 unit matrix. Since the center of the hyperbolic dual unit sphere in $\mathbb{D}_1^3$ should rest constant, the transformation group in $\mathbb{D}_1^3$ (the image of hyperbolic locomotions in the Minkowski 3-space $\mathbb{E}_1^3$) does not contain any translations. Thus, in order to performs the Lorentzian locomotions in $\mathbb{D}_1^3$, we can state the next theorem:

Theorem 1. *The group of all one-parameter hyperbolic spatial locomotions in $\mathbb{E}_1^3$-space is in bijection with the group of hyperbolic dual orthogonal matrices $\mathbb{O}(\mathbb{D}_1^{3\times3})$ in $\mathbb{D}_1^3$-space.*

To have a component of the dual Lie algebra $L(\mathbb{O}_{\mathbb{D}_1^{3\times3}})$ of the dual group $\mathbb{O}(\mathbb{D}_1^{3\times3})$, we take a hyperbolic dual curve of such dual matrices $\widehat{a}(t)$ such that $\widehat{a}(0)$ is the identity. By making the derivative of Equation (1) with respect to t, we acquire:

$$\widehat{a}'\widehat{a}^{-1} + \widehat{a}\left(\widehat{a}^{-1}\right)' = 0;\ 0 \text{ is zero } 3 \times 3 \text{ matrix.}$$

If we set $\widehat{\omega}(t) = \widehat{a}'\widehat{a}^{-1}$, we see that $\widehat{\omega}^T + \epsilon\widehat{\omega}\epsilon = 0$, i.e., the matrix $\widehat{\omega}$ is a skew-adjoint matrix. Thus, via Theorem 1, the dual Lie algebra $L(\mathbb{O}_{\mathbb{D}_1^{3\times3}})$ of the dual Lorentzian group $\mathbb{O}(\mathbb{D}_1^{3\times3})$ is the dual algebra of skew-adjoint 3×3 dual matrices

$$\widehat{\omega}(t) = \begin{pmatrix} 0 & \widehat{\omega}_3 & \widehat{\omega}_2 \\ \widehat{\omega}_3 & 0 & \widehat{\omega}_1 \\ \widehat{\omega}_2 & -\widehat{\omega}_1 & 0 \end{pmatrix} \Leftrightarrow \widehat{\boldsymbol{\omega}}(t) = \begin{pmatrix} \widehat{\omega}_1 \\ \widehat{\omega}_2 \\ \widehat{\omega}_3 \end{pmatrix} \in L(O_{\mathbb{D}_1^{3\times3}}).$$

As a result, we may write the vectors from $\mathbb{D}_1^3$ in two forms as skew-adjoint matrix 3×3 dual matrices or as vectors. In what follows, we will employ both of these possibilities according to which of the two will be more advantageous in the specified case.

Let $\widehat{\mathbf{a}}_m$ and $\widehat{\mathbf{a}}_f$ symbolize a timelike dual unit vector $\widehat{\mathbf{a}}$ indicated in $\mathbb{H}_{+m}^2$ and $\mathbb{H}_{+f}^2$, respectively. Then,

$$\widehat{\mathbf{a}}_f = \widehat{a}\widehat{\mathbf{a}}_m. \tag{2}$$

The inverse transformation is

$$\widehat{\mathbf{a}}_m = \epsilon\widehat{a}^T\epsilon\widehat{\mathbf{a}}_f. \tag{3}$$

Taking the differentiation of Equation (2) with respect to t, we obtain

$$\widehat{\mathbf{a}}_f' = \widehat{a}'\widehat{\mathbf{a}}_m + \widehat{a}\widehat{\mathbf{a}}_m'.$$

So, for $\widehat{\mathbf{a}}_m$ fixed in $\mathbb{H}^2_{+m}$, we may realize

$$\widehat{\mathbf{a}}_f^{'} = \widehat{a}^{'}\widehat{\mathbf{a}}_m. \tag{4}$$

Expressing Equation (4) in $\mathbb{H}^2_{+f}$, and employing Equation (2), we have

$$\widehat{\mathbf{a}}_f^{'} = \widehat{\omega}_f(t)\widehat{\mathbf{a}}_f, \tag{5}$$

so that

$$\widehat{\omega}_f(t) = \widehat{a}^{'}\epsilon\widehat{a}^{T}\epsilon = \begin{pmatrix} 0 & \widehat{\omega}_{3f} & \widehat{\omega}_{2f} \\ \widehat{\omega}_{3f} & 0 & \widehat{\omega}_{1f} \\ \widehat{\omega}_{2f} & -\widehat{\omega}_{1f} & 0 \end{pmatrix}. \tag{6}$$

Once again, expressing Equation (4) in $\mathbb{H}^2_{+m}$ and transforming it to $\mathbb{H}^2_{+f}$ by Equation (2), the result can be equated once more with Equation (4) to yield

$$\widehat{a}\widehat{\omega}_m\widehat{\mathbf{a}}_m = \widehat{a}^{'}\widehat{\mathbf{a}}_m,$$

so that

$$\widehat{\omega}_m(t) = \epsilon\widehat{a}^{T}\epsilon\widehat{a}^{'} = \begin{pmatrix} 0 & \widehat{\omega}_{3m} & \widehat{\omega}_{2m} \\ \widehat{\omega}_{3m} & 0 & \widehat{\omega}_{1m} \\ \widehat{\omega}_{2m} & -\widehat{\omega}_{1m} & 0 \end{pmatrix}. \tag{7}$$

Then, from the above two equations, we have

$$\widehat{\omega}_f(t) = \widehat{a}(t)\widehat{\omega}_m(t), \text{ and } \|\widehat{\omega}_m(t)\| = \left\|\widehat{\omega}_f(t)\right\|. \tag{8}$$

3.1. Spatial Kinematics and Invariants of the Axodes

Through $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$, the spacelike dual curve (polode) $\widehat{\mathbf{r}}_m(t) = \widehat{\boldsymbol{\omega}}_m(t)\|\widehat{\boldsymbol{\omega}}_m(t)\|^{-1}$ is the position of the instantaneous screw axis ($\mathbb{ISA}$) on $\mathbb{H}^2_{+m}$. This position is timelike movable axode in $\mathbb{L}_m$-space. This axode is traced by the $\mathbb{ISA}$ as viewed from the movable space $\mathbb{L}_m$, and let us indicate this surface by π_m. Further, the $\mathbb{ISA}$ on $\mathbb{H}^2_{+f}$ is also a spacelike dual curve (polode) $\widehat{\mathbf{r}}_f(t) = \widehat{\boldsymbol{\omega}}_f(t)\left\|\widehat{\boldsymbol{\omega}}_f(t)\right\|^{-1}$. This curve also symbolizes the unmovable spacelike axode π_f. This axode is traced by those timelike lines in $\mathbb{L}_f$-space which at some instant matching with a timelike line in the movable space $\mathbb{L}_m$ having zero dual velocity. It is worth noting that $\left\|\widehat{\boldsymbol{\omega}}_f(t)\right\| = \|\widehat{\boldsymbol{\omega}}_m(t)\| = \omega(t) + \varepsilon\omega^*(t) = \widehat{\omega}(t)$ is the dual angular speed of the locomotion $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$. $\omega(t)$ and $\omega^*(t)$ match the rotation locomotions and the translation locomotions of $\mathbb{L}_m/\mathbb{L}_f$, respectively. Throughout the locomotion $\mathbb{L}_m/\mathbb{L}_f$, at any instant $t \in \mathbb{R}$, the pitch may be designated by

$$h(t) = \frac{\langle \boldsymbol{\omega}, \boldsymbol{\omega}^* \rangle}{\|\boldsymbol{\omega}\|^2} = \frac{-\omega_1\omega_1^* + \omega_1\omega_1^* + \omega_1\omega_1^*}{-\omega_1^2 + \omega_1^2 + \omega_1^2}.$$

Corollary 1. *For a one-parameter hyperbolic dual spherical locomotion $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$, the tangent vectors of the movable and unmovable spacelike polodes are linked by*

$$\pi_f : \widehat{\mathbf{r}}_f^{'} = \widehat{a}\widehat{\mathbf{r}}_m^{'}\widehat{a}^{-1}. \tag{9}$$

Proof. Without loss of generality, we set the variable $t \in \mathbb{R}$ as the canonical variable of $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$, i.e., $\left\|\widehat{\boldsymbol{\omega}}_f(t)\right\| = \|\widehat{\boldsymbol{\omega}}_m(t)\| = 1$. Then:

$$\pi_m : \widehat{\mathbf{r}}_m = \widehat{a}^{-1}\widehat{a}^{'}, \text{ and } \pi_f : \widehat{\mathbf{r}}_f = \widehat{a}^{'}\widehat{a}^{-1},$$

Further, we have:

$$\widehat{\mathbf{r}}_m' = \widehat{a}'^{-1}\widehat{a}' + \widehat{a}^{-1}\widehat{a}'',$$

and

$$\left.\begin{array}{l} \widehat{a}\widehat{\mathbf{r}}_m'\widehat{a}^{-1} = \widehat{a}(\widehat{a}'^{-1}\widehat{a}' + \widehat{a}^{-1}\widehat{a}'')\widehat{a}^{-1} \\ = \widehat{a}\widehat{a}'^{-1}\widehat{a}'\widehat{a}^{-1} + \widehat{a}\widehat{a}^{-1}\widehat{a}''\widehat{a}^{-1}. \end{array}\right\}$$

Furthermore, we acquire:

$$\widehat{\mathbf{r}}_f' = \widehat{a}''\widehat{a}^{-1} + \widehat{a}'\widehat{a}'^{-1}.$$

From the equation $\widehat{a}\widehat{a}^{-1} = I$, we gain $\widehat{a}'\widehat{a}^{-1} + \widehat{a}\widehat{a}'^{-1} = 0$. Substituting into the expression for $\widehat{a}\widehat{\mathbf{r}}_m'\widehat{a}^{-1}$, we attain:

$$\left.\begin{array}{l} \widehat{a}\widehat{\mathbf{r}}_m'\widehat{a}^{-1} = -\widehat{a}\widehat{a}'^{-1}\widehat{a}\widehat{a}'^{-1} + \widehat{a}''\widehat{a}^{-1} \\ = \widehat{a}'\widehat{a}^{-1}\widehat{a}\widehat{a}'^{-1} + \widehat{a}''\widehat{a}^{-1} \\ = \widehat{a}'\widehat{a}'^{-1} + \widehat{a}''\widehat{a}^{-1} = \widehat{\mathbf{r}}_f'. \end{array}\right\}$$

□

Equation (9) contains only first-order derivatives of $\pi_i (i = m, f)$; it is a first-order ownership of the timelike axodes, particularly is its dual unit speed. Furthermore, we have that:

$$\left\|\widehat{\mathbf{r}}_f'\right\| = \sqrt{\left|\langle \widehat{\mathbf{r}}_f', \widehat{\mathbf{r}}_f' \rangle\right|} = \sqrt{\left|\langle \widehat{a}\widehat{\mathbf{r}}_m'\widehat{a}^{-1}, \widehat{a}\widehat{\mathbf{r}}_m'\widehat{a}^{-1} \rangle\right|} = \left\|\widehat{\mathbf{r}}_m'\right\|,$$

which leads to $\left\|\widehat{\mathbf{r}}_f'\right\|$ and $\left\|\widehat{\mathbf{r}}_m'\right\|$ can be replaced by $\widehat{p}(t) = p(t) + \varepsilon p^*(t)$. Thus, the mutual dual arc-length parameter is $d\widehat{s} = ds + \varepsilon ds^* = \widehat{p}(t)dt$. Therefore, the mutual distribution parameter of the timelike axodes is

$$\mu(s) := \frac{p^*(t)}{p(t)} = \frac{ds^*}{ds}. \tag{10}$$

Hence, we have the following:

Corollary 2. *For a one-parameter hyperbolic dual locomotion $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$, the spacelike polode curves roll without slipping over each other. By using E. Study map, the timelike axodes contact each other along and rolls around the $\mathbb{ISA}$ in the first-order (Figure 2).*

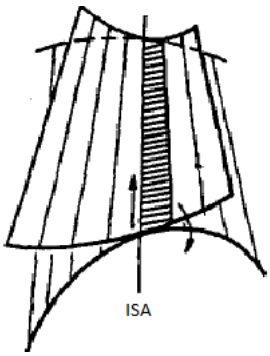


Figure 2. Typical portions of axodes.

3.2. Euler–Savary Formula for the Timelike Axodes

In the context of planar locomotions, at each point of a smooth curve, there occurs only one curvature circle of the curve. The center and radius of this circle can be specified by the Euler–Savary formula, if the place of the point is specified in the movable plane. In different types of geometry, the Euler–Savary formula had been generalized for a line trajectory, i.e., the construction of the Disteli formulae [4–7,11–13] . Therefore, we now shall look to the Euler–Savary and Disteli formulae for the timelike axodes by utilizing

the equipment just acquired above. Then, as in spherical geometry, we set a hyperbolic Blaschke frame of the polodes $\widehat{\mathbf{r}}_i(t) \in \mathbb{H}^2_{+i}(i = m, f)$ as:

$$\left.\begin{array}{l} \widehat{\mathbf{r}}_i{=}\widehat{\mathbf{r}}_i(t),\ \widehat{\mathbf{t}}_i(t) = \widehat{\mathbf{r}}_i^{'}\left\|\widehat{\mathbf{r}}_i^{'}\right\|^{-1},\ \widehat{\mathbf{g}}_i(t) = \widehat{\mathbf{r}}_i \times \widehat{\mathbf{t}}_i, \\ \widehat{\mathbf{r}}_i \times \widehat{\mathbf{t}}_i{=}\widehat{\mathbf{g}}_i, \widehat{\mathbf{r}}_i \times \widehat{\mathbf{g}}_i{=} -\widehat{\mathbf{t}}_i, \widehat{\mathbf{t}}_i \times \widehat{\mathbf{g}}_i{=}\widehat{\mathbf{r}}_i, \\ -\langle \widehat{\mathbf{r}}_i, \widehat{\mathbf{r}}_i \rangle{=}\langle \widehat{\mathbf{t}}_i, \widehat{\mathbf{t}}_i \rangle = \langle \widehat{\mathbf{g}}_i, \widehat{\mathbf{g}}_i \rangle = 1. \end{array}\right\} \tag{11}$$

With the dual unit vectors $\widehat{\mathbf{r}}_i$, $\widehat{\mathbf{t}}_i$, and $\widehat{\mathbf{g}}_i$ three alternately orthogonal lines appear, and their intersection is the mutual central point $\mathbf{c}(t)$ on the timelike axodes $\pi_i(i = m, f)$; $\widehat{\mathbf{t}}_i(t)$ and $\widehat{\mathbf{g}}_i(t)$, respectively, are the central normal and the central tangent. Then, the Blaschke formula is:

$$\begin{pmatrix} \widehat{\mathbf{r}}_i^{'} \\ \widehat{\mathbf{t}}_i^{'} \\ \widehat{\mathbf{g}}_i^{'} \end{pmatrix} = \begin{pmatrix} 0 & \widehat{p} & 0 \\ \widehat{p} & 0 & \widehat{q}_i \\ 0 & -\widehat{q}_i & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{r}}_i \\ \widehat{\mathbf{t}}_i \\ \widehat{\mathbf{g}}_i \end{pmatrix} = \widehat{\boldsymbol{\omega}}_i(\widehat{t}) \times \begin{pmatrix} \widehat{\mathbf{r}}_i \\ \widehat{\mathbf{t}}_i \\ \widehat{\mathbf{g}}_i \end{pmatrix}, \tag{12}$$

where $\widehat{\boldsymbol{\omega}_i}(\widehat{t}) = \widehat{q}_i\widehat{\mathbf{r}}_i - \widehat{p}\widehat{\mathbf{g}}_i$ is the Darboux vector. $\widehat{p}(t) = p(t) + \varepsilon p^*(t) = \left\|\widehat{\mathbf{r}}_1^{'}\right\|$, and $\widehat{q}_i(t) = q_i(t) + \varepsilon q_i^*(t) = \left\langle \widehat{\mathbf{r}}_i^{''}, \widehat{\mathbf{r}}_i^{'} \times \widehat{\mathbf{r}}_i \right\rangle \left\|\widehat{\mathbf{r}}_i^{'}\right\|^{-2}$ are Blaschke invariants of the timelike axode π_i. We shall disregard the pure translational locomotions, i.e., $\widehat{q}_m \neq \widehat{q}_f$. Furthermore, we disregard zero divisors, that is, $q_m^* \neq q_f^*$.

Under the condition $|\widehat{q}_i| > |p|$, the timelike Disteli-axis (striction axis or curvature axis) is pointed as:

$$\widehat{\mathbf{b}}_i(t) := \mathbf{b_i}{+}\varepsilon\mathbf{b_i^*} = \frac{\widehat{\boldsymbol{\omega}}_i}{\|\widehat{\boldsymbol{\omega}}_i\|} = \frac{\widehat{q}_i}{\sqrt{\widehat{q}_i^2 - \widehat{p}^2}}\widehat{\mathbf{r}}_i - \frac{\widehat{p}}{\sqrt{\widehat{q}_i^2 - \widehat{p}^2}}\widehat{\mathbf{g}}_i, \tag{13}$$

If $\widehat{\varphi}_i = \varphi_i + \varepsilon\varphi_i^*$ is the Lorentzian timelike dual angle (radius of curvature) through $\widehat{r}_i$, and $\widehat{\mathbf{b}}_i$, then

$$\widehat{\mathbf{b}}_i = \cosh\widehat{\varphi}_i\widehat{\mathbf{r}}_i - \sinh\widehat{\varphi}_i\widehat{\mathbf{g}}_i. \tag{14}$$

Here,

$$\widehat{\gamma}_i(t) = \gamma_i + \varepsilon\gamma_i^* = \frac{\widehat{q}_i}{\widehat{p}} = \coth\widehat{\varphi}_i, \text{ with } \widehat{p} \neq 0, \tag{15}$$

is the dual geodesic curvature of the timelike axode π_i. Then,

$$\widehat{\gamma}_f{-}\widehat{\gamma}_m = \coth\widehat{\varphi}_f{-}\coth\widehat{\varphi}_m = \frac{\widehat{\omega}}{\widehat{p}}, \text{ with } \widehat{\omega} = \widehat{q}_f - \widehat{q}_m. \tag{16}$$

Equation (16) is an unprecedented hyperbolic dual spherical Euler–Savary formula (compared with [1–5]). The two Disteli's axes $\widehat{\mathbf{b}}_f$ and $\widehat{\mathbf{b}}_m$ can be looked as envelopes of the spacelike polodes. From the real and dual parts of Equation (16), respectively, we obtain:

$$\coth\varphi_f - \coth\varphi_m = \frac{\omega}{p}, \tag{17}$$

and

$$\frac{\varphi_m^*}{\sinh^2\varphi_m} - \frac{\varphi_f^*}{\sinh^2\varphi_f} = \frac{\omega}{p}(\mu - h). \tag{18}$$

Equations (17) and (18) are new Disteli formulae of hyperbolic spatial locomotions for the timelike axodes. Note that ω, h and μ are invariants of the choice of the reference point.

Velocity and Acceleration for a Timelike Line Trajectory

Through the one-parameter hyperbolic spatial locomotion $\mathbb{L}_m/\mathbb{L}_f$, each constant timelike line of the $\mathbb{L}_m$-space, generally, creates a timelike ruled surface in the $\mathbb{L}_f$-space, symbolized by $(\widehat{x})$ and its creator by $\widehat{\mathbf{x}}$. In kinematics, this timelike ruled surface is named the timelike line trajectory. Then, we may write $\widehat{\mathbf{x}}$ as

$$\left.\begin{array}{l}\widehat{\mathbf{x}}(\widehat{s}) = \widehat{x}_1(\widehat{s})\widehat{\mathbf{r}}_i(\widehat{s}) + \widehat{x}_2(\widehat{s})\widehat{\mathbf{t}}_i(\widehat{s}) + \widehat{x}_3(\widehat{s})\widehat{\mathbf{g}}_i(\widehat{s}),\\ -\widehat{x}_1^2 + \widehat{x}_2^2 + \widehat{x}_3^2 = -1,\end{array}\right\} \tag{19}$$

where $\widehat{x}_1(\widehat{s})$, $\widehat{x}_2(\widehat{s})$, $\widehat{x}_3(\widehat{s})$ are its dual coordinates with respect to the axode π_i. Timelike line trajectories with specific values of velocity and acceleration have several characteristics in kinematics. Hence, the first derivative of $\widehat{\mathbf{x}}$ with respect to $\widehat{s}$ is

$$\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}\,|_i = (\frac{d\widehat{x}_1}{d\widehat{s}} + \widehat{x}_2)\widehat{\mathbf{r}}_i + (\frac{d\widehat{x}_2}{d\widehat{s}} + \widehat{x}_1 - \widehat{x}_3\widehat{\gamma}_i)\widehat{\mathbf{t}}_i + (\frac{d\widehat{x}_3}{d\widehat{s}} + \widehat{x}_2\widehat{\gamma}_i)\widehat{\mathbf{g}}_i. \tag{20}$$

Thus, if $\widehat{\mathbf{x}}$ is a constant in $\mathbb{L}_m$, we have $\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}\,|_m = 0$, i.e.,

$$\frac{d\widehat{x}_1}{d\widehat{s}} + \widehat{x}_2 = 0,\ \frac{d\widehat{x}_2}{d\widehat{s}} + \widehat{x}_1 - \widehat{x}_3\widehat{\gamma}_m = 0,\ \frac{d\widehat{x}_3}{d\widehat{s}} + \widehat{x}_2\widehat{\gamma}_m = 0. \tag{21}$$

Substituting Equation (21) into Equation (20) and facilitating it, we rewrite Equation (20) as

$$\frac{d\widehat{\mathbf{x}}}{d\widehat{s}} = \widehat{\gamma}(-\widehat{x}_3\widehat{\mathbf{t}}_f + \widehat{x}_2\widehat{\mathbf{g}}_f), \tag{22}$$

where $\widehat{\gamma}(\widehat{s}) = \gamma + \varepsilon\gamma^* = \widehat{\gamma}_f - \widehat{\gamma}_m$ is the relative dual geodesic curvature. For the kinematic significance of $\widehat{\gamma}$, one more expressions of $\widehat{x}'$ can be gained by Equation (12), which is

$$\frac{d\widehat{\mathbf{x}}}{d\widehat{s}} = \widehat{\mathbf{x}}'\frac{dt}{d\widehat{s}} = \widehat{\omega}(-\widehat{x}_3\widehat{\mathbf{t}}_f + \widehat{x}_2\widehat{\mathbf{g}}_f)\frac{dt}{d\widehat{s}}. \tag{23}$$

From Equations (22), and (23), it follows that

$$\widehat{\omega}dt = \widehat{\gamma}(1 + \varepsilon\mu)ds, \tag{24}$$

which implies the kinematic meaning of $\widehat{\gamma}$. By differentiating Equation (22) with respect to $\widehat{s}$ once more and simplifying it, we obtain

$$\frac{d^2\widehat{\mathbf{x}}}{d\widehat{s}^2} = \widehat{x}_3\widehat{\gamma}\widehat{\mathbf{r}}_f - (\widehat{x}_2\widehat{\gamma}^2 + \widehat{x}_3\frac{d\widehat{\gamma}}{d\widehat{s}})\widehat{\mathbf{t}}_f + (\widehat{x}_2\frac{d\widehat{\gamma}}{d\widehat{s}} - \widehat{x}_1\widehat{\gamma} - \widehat{x}_3\widehat{\gamma}^2)\widehat{\mathbf{g}}_f. \tag{25}$$

3.3. Euler–Savary and Disteli Formulae for $(\widehat{x})$

We now shall derive a ruled analogy of the concept of the curvature circle of a timelike line trajectory by using the completely obtained equipment. To obtain this, we set a spacelike circle on the hyperbolic dual unit sphere $\mathbb{H}^2_{+f}$ by the equation

$$\langle\widehat{\mathbf{x}}, \widehat{\mathbf{b}}\rangle = \cosh\widehat{\varphi}, \tag{26}$$

where $\widehat{\varphi} = \varphi + \varepsilon\varphi^* \geq 0$ is a specified Lorentzian timelike dual angle (dual spherical radius of curvature) and $\widehat{\mathbf{b}}$ is a constant timelike dual unit vector which locates the spacelike circle's center. Equation (26) describes a timelike linear line congruence (it is specified by two linear equations of the Plucker coordinates) [1–5]. Furthermore, the osculating spacelike circle should have touch of at least second-order with the curve $\widehat{\mathbf{x}}(\widehat{s}) \in \mathbb{H}^2_{+f}$ if the following equations are holds:

$$\langle\widehat{\mathbf{x}}, \widehat{\mathbf{b}}\rangle = \cosh\widehat{\varphi},\ \langle\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}, \widehat{\mathbf{b}}\rangle = 0, \text{ and } \langle\frac{d^2\widehat{\mathbf{x}}}{d\widehat{s}^2}, \widehat{\mathbf{b}}\rangle = 0. \tag{27}$$

Then, the first and the last two equations, respectively, describe the timelike linear line congruence of the timelike trajectory of $\widehat{\mathbf{x}}$ and its Disteli's axis $\widehat{\mathbf{b}}$. The noteworthiness of the Disteli's axis is that it is the axis of osculating helicoidal timelike surface of the trajectory of $(\widehat{x})$.

For researching the geometrical ownership of $\widehat{\mathbf{x}}$, the Blaschke frame is set up as follows:

$$\widehat{\mathbf{x}}=\widehat{\mathbf{x}}(\widehat{s}),\ \widehat{\mathbf{t}}(\widehat{s})=\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}\left\|\frac{d\widehat{\mathbf{x}}}{d\widehat{s}}\right\|^{-1},\ \widehat{\mathbf{g}}(\widehat{s})=\widehat{\mathbf{x}}\times\widehat{\mathbf{t}}, \tag{28}$$

where

$$\begin{gathered}-\langle\widehat{\mathbf{x}},\widehat{\mathbf{x}}\rangle=\langle\widehat{\mathbf{t}},\widehat{\mathbf{t}}\rangle=\langle\widehat{\mathbf{g}},\widehat{\mathbf{g}}\rangle=1,\\ \widehat{\mathbf{x}}\times\widehat{\mathbf{t}}=\widehat{\mathbf{g}},\widehat{\mathbf{x}}\times\widehat{\mathbf{g}}=-\widehat{\mathbf{t}},\widehat{\mathbf{t}}\times\widehat{\mathbf{g}}=\widehat{\mathbf{x}}.\end{gathered} \tag{29}$$

At any instant, it is clear from Equations (22) and (28) that

$$\langle\widehat{\mathbf{t}},\widehat{\mathbf{r}}_f\rangle=\langle\widehat{\mathbf{x}},\widehat{\mathbf{t}}\rangle=\langle\widehat{\mathbf{t}},\widehat{\mathbf{b}}\rangle=0. \tag{30}$$

For the Euler–Savary and Disteli formulae for the timelike line $\widehat{\mathbf{x}}$, we locate $\widehat{\mathbf{t}}$ regarding to $\{\widehat{\mathbf{r}}_f,\ \widehat{\mathbf{t}}_f,\ \widehat{\mathbf{g}}_f\}$ by its intercept distance ψ^*, metrical on the $\mathbb{ISA}$ and the angle ψ, metrical regarding to $\widehat{\mathbf{g}}_f$. We specify the dual angles $\widehat{\vartheta}=\vartheta+\varepsilon\vartheta^*$and $\widehat{\alpha}=\alpha+\varepsilon\alpha^*$, which assign the locations of $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{b}}$ on $\widehat{\mathbf{t}}$. These dual angles all result regarding the $\mathbb{ISA}$ (see Figure 3). The dependent agreement rules that the signs $(\vartheta,\ \vartheta^*)$ and $(\alpha,\ \alpha^*)$ are via the right-hand screw rule with the thumb pointing on $\widehat{\mathbf{t}}$; the sense of $\widehat{\mathbf{t}}$ is such that $0\leq\psi\leq\pi$. $(\psi,\ \psi^*)$ are specified with the thumb in the orientation of the $\mathbb{ISA}$. Since $\widehat{\mathbf{x}}$ is a timelike dual unit vector, we can write the following form:

$$\widehat{\mathbf{x}}=\cosh\widehat{\vartheta}\widehat{\mathbf{r}}_f+\sinh\widehat{\vartheta}\widehat{\mathbf{m}}_f;\ \ \widehat{\mathbf{m}}_f=\cos\widehat{\psi}\widehat{\mathbf{t}}_f+\sin\widehat{\psi}\widehat{\mathbf{g}}_f. \tag{31}$$

Similarly, the timelike Disteli's axis is

$$\widehat{\mathbf{b}}=\cosh\widehat{\alpha}\widehat{\mathbf{r}}_f+\sinh\widehat{\alpha}\widehat{\mathbf{m}}_f. \tag{32}$$

Substituting from Equations (25) and (32) into the third equation of Equation (27) produces

$$\widehat{\gamma}\widehat{x}_3\coth\widehat{\alpha}+(\widehat{x}_2^2\widehat{\gamma}+\widehat{x}_3\frac{d\widehat{\gamma}}{d\widehat{s}})\cos\widehat{\psi}-(-\widehat{x}_1\widehat{\gamma}+\widehat{x}_2\frac{d\widehat{\gamma}}{d\widehat{s}}-\widehat{x}_3^2\widehat{\gamma})\sin\widehat{\psi}=0. \tag{33}$$

Into Equation (33), we substitute from Equation (31) to acquire

$$\coth\widehat{\alpha}-\coth\widehat{\vartheta}=\frac{\widehat{\gamma}}{\sin\widehat{\psi}}. \tag{34}$$

This is precisely the dual extension of the Euler–Savary equation from ordinary spherical kinematics. Furthermore, by the real and the dual parts of (34), respectively, we obtain

$$\coth\alpha-\coth\vartheta=\frac{\gamma}{\sin\psi}, \tag{35}$$

and

$$\frac{\vartheta^*}{\sinh^2\vartheta}-\frac{\alpha^*}{\sinh^2\alpha}=\frac{1}{\sin\psi}(\gamma^*-\psi^*\cot\psi). \tag{36}$$

The hyperbolic spherical Euler–Savary equation in Equations (35) and (36) are Disteli formulae of hyperbolic spatial kinematics. As the angles ϑ and α are known, Equation (36) gives the correlation through ϑ^* and α^* in terms of ψ and ψ^* and the second-order invariant γ^*. Via Figure 3, the sign of α^* (+ or −) in Equation (36) designates that the locations of the Disteli's axis $\widehat{\mathbf{b}}$ are situated on the positive or negative orientation over the mutual central normal $\widehat{\mathbf{t}}_f$.

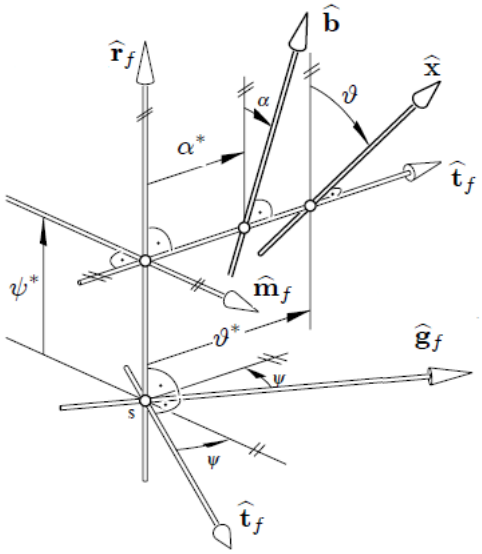


Figure 3. Illustrationsof $\hat{\mathbf{x}}$ and the Disteli-axis $\hat{\mathbf{b}}$.

On the other hand, we can obtain (34) as follows: combining Equations (28) and (31), with notice to Equation (22), we have

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} \cosh\hat{\vartheta} & \sinh\hat{\vartheta}\cos\hat{\psi} & \sinh\hat{\vartheta}\sin\hat{\psi} \\ 0 & -\sin\hat{\psi} & \cos\hat{\psi} \\ -\sinh\hat{\vartheta} & -\cosh\hat{\vartheta}\cos\hat{\psi} & -\cosh\hat{\vartheta}\sin\hat{\psi} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}}_f \\ \hat{\mathbf{t}}_f \\ \hat{\mathbf{g}}_f \end{pmatrix}. \tag{37}$$

Further, the Blaschke formula is:

$$\frac{d}{d\hat{s}} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{g}} \end{pmatrix} = \begin{pmatrix} 0 & \hat{p}_x & 0 \\ \hat{p}_x & 0 & \hat{q}_x \\ 0 & -\hat{q}_x & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{t}} \\ \hat{\mathbf{g}} \end{pmatrix}, \tag{38}$$

where

$$\left.\begin{aligned} \hat{p}_x &:= p_x + \varepsilon p_x^* = \hat{\gamma}\sinh\hat{\vartheta}, \\ \hat{q}_x &= q_x + \varepsilon q_x^* = \det\left(\hat{\mathbf{x}}, \tfrac{d\hat{\mathbf{x}}}{d\hat{s}}, \tfrac{d^2\hat{\mathbf{x}}}{d\hat{s}^2}\right)\left\|\tfrac{d\hat{\mathbf{x}}}{d\hat{s}}\right\|^{-3} = \hat{\gamma}\cosh\hat{\vartheta} + \tfrac{\cos\hat{\psi}}{\sinh\hat{\vartheta}}. \end{aligned}\right\} \tag{39}$$

By Equations (32) and (37), moreover, we have:

$$\left.\begin{aligned} \hat{\mathbf{r}}_f &= \cosh\hat{\vartheta}\hat{\mathbf{x}} + \sinh\hat{\vartheta}\hat{\mathbf{g}}, \\ \hat{\mathbf{b}} &= \cosh\hat{\varphi}\hat{\mathbf{x}} + \sinh\hat{\varphi}\hat{\mathbf{g}}. \end{aligned}\right\} \tag{40}$$

Note that $\hat{\varphi} = \hat{\vartheta} - \hat{\alpha}$=$\cosh^{-1}(\langle\hat{\mathbf{x}}, \hat{\mathbf{b}}\rangle)$ is the dual radius of curvature of $(\hat{x})$. It follows from the derivatives of Equation (40) that:

$$\left.\begin{aligned} \tfrac{d\hat{\mathbf{r}}_f}{d\hat{s}} &= (\sinh\hat{\vartheta}\hat{\mathbf{x}} + \cosh\hat{\vartheta}\hat{\mathbf{g}})\tfrac{d\hat{\vartheta}}{d\hat{s}} + (\hat{p}_x\cosh\hat{\vartheta} - \hat{q}_x\sinh\hat{\vartheta})\hat{\mathbf{t}}, \\ \tfrac{d\hat{\mathbf{b}}}{d\hat{s}} &= (\sinh\hat{\varphi}\hat{\mathbf{x}} + \cosh\hat{\varphi}\hat{\mathbf{g}})\tfrac{d\hat{\alpha}}{d\hat{s}} + (\hat{p}_x\cosh\hat{\varphi} - \hat{q}_x\sinh\hat{\varphi})\hat{\mathbf{t}}. \end{aligned}\right\} \tag{41}$$

The timelike dual unit vector $\hat{\mathbf{b}}$ is too named the evolute of $(\hat{x})$. Therefore, the dependent condition should be fulfilled

$$\langle\frac{d\hat{\mathbf{b}}}{d\hat{s}}, \hat{\mathbf{t}}\rangle = 0 \Leftrightarrow 0 = \hat{p}_x\cosh\hat{\varphi} - \hat{q}_x\sinh\hat{\varphi},$$

which leads to

$$\widehat{\gamma}_x = \gamma_x + \varepsilon\gamma_x^* = \frac{\widehat{q}_x}{\widehat{p}_x} = \tanh\widehat{\varphi}. \tag{42}$$

This equation assigns the correlation among the dual spherical curvature $\widehat{\gamma}_x$ and the dual radius of curvature $\widehat{\varphi}$. By Equation (41), we also attain

$$\langle \frac{d\widehat{\mathbf{r}}_f}{d\widehat{s}}, \widehat{\mathbf{t}} \rangle = \widehat{p}_x \cosh\widehat{\vartheta} - \widehat{q}_x \sinh\widehat{\vartheta}. \tag{43}$$

Furthermore, from Equations (40), (42) and (43), a simple calculation displays that:

$$\begin{aligned}
\langle \tfrac{d\widehat{\mathbf{r}}_f}{d\widehat{s}}, \widehat{\mathbf{t}} \rangle &= (\widehat{\gamma}\sinh\widehat{\vartheta}\cosh\widehat{\vartheta} - \widehat{\gamma}\sinh\widehat{\vartheta}\coth\widehat{\alpha}\sinh\widehat{\vartheta}) \\
&= \widehat{\gamma}\sinh\widehat{\vartheta}(\cosh\widehat{\vartheta} - \coth\widehat{\varphi}\sinh\widehat{\vartheta}) \\
&= \tfrac{\widehat{\gamma}\sinh\widehat{\vartheta}}{\sinh\widehat{\varphi}}(\sinh\widehat{\varphi}\cosh\widehat{\vartheta} - \cosh\widehat{\varphi}\sinh\widehat{\vartheta}) \\
&= \tfrac{\widehat{\gamma}\sinh\widehat{\vartheta}\sinh(\widehat{\varphi}-\widehat{\vartheta})}{\sinh\widehat{\varphi}} \\
&= \tfrac{\widehat{\gamma}\sinh\widehat{\vartheta}\sinh(\widehat{\varphi}-\widehat{\vartheta})}{\sinh(\widehat{\varphi}-\widehat{\vartheta})\cosh\widehat{\vartheta}+\cosh(\widehat{\varphi}-\widehat{\vartheta})\sinh\widehat{\vartheta}} \\
&= \tfrac{\widehat{\gamma}}{\coth\widehat{\vartheta}+\coth(\widehat{\varphi}-\widehat{\vartheta})} \\
&= \tfrac{\widehat{\gamma}}{\coth\widehat{\vartheta}-\coth\widehat{\alpha}}.
\end{aligned} \tag{44}$$

Moreover, we have

$$\begin{aligned}
\frac{d\widehat{\mathbf{r}}_f}{d\widehat{s}} &= \widehat{\boldsymbol{\omega}}_f \times \widehat{\mathbf{r}}_f, \\
&= \widehat{\boldsymbol{\omega}}_f \times (\cosh\widehat{\vartheta}\widehat{\mathbf{x}} + \sinh\widehat{\vartheta}\widehat{\mathbf{g}}), \\
&= \sinh\widehat{\vartheta}\cos\widehat{\psi}\widehat{\mathbf{x}} + \sin\widehat{\psi}\widehat{\mathbf{t}} + \cos\widehat{\psi}\widehat{\mathbf{g}}.
\end{aligned}$$

Thus, we obtain

$$\langle \frac{d\widehat{\mathbf{r}}_f}{d\widehat{s}}, \widehat{\mathbf{t}} \rangle = \sin\widehat{\psi}. \tag{45}$$

Upon substitution of Equation (45) into Equation (44), we have Equation (34), as claimed. However, we can derive a further version of the dual Euler–Savary formula in Equation (34) as follows: from Equations (39) and (42) one readily finds

$$\coth\widehat{\vartheta} - \tanh\widehat{\varphi} = \frac{\cos\widehat{\psi}}{\widehat{\gamma}\sinh^2\widehat{\vartheta}}.$$

This is a hyperbolic dual spherical Euler–Savary formula for the spacelike dual curve $\widehat{\mathbf{x}}(t) \in \mathbb{H}^2_+$, which matches a timelike ruled surface and its osculating spacelike circle in terms of the dual angle $\widehat{\psi}$ as well as the second-order invariant $\widehat{\gamma}$. By the real and the dual parts, respectively, we obtain

$$\coth\vartheta - \tanh\varphi = \frac{\cos\psi}{\gamma\sinh^2\vartheta},$$

and

$$\varphi^* = \frac{1}{\gamma}\frac{\sinh^2\varphi}{\sinh^2\vartheta}\{(\gamma - 2\coth\vartheta\cos\psi)\vartheta^* + \frac{\gamma^*}{\gamma}\cos\psi + \psi^*\sin\psi\}. \tag{46}$$

The above two equations are Disteli's formulae of a timelike line trajectory. Via Figure 3, the signal of φ^* ($+$ or $-$) in the last equation signalizes that the place of the timelike Disteli's axis $\widehat{\mathbf{b}}$ is on the $+$ or $-$ orientation of the central normal $\widehat{\mathbf{t}}$ at the central point $\mathbf{c}(\vartheta^*, \psi^*)$. Since the central points of $(\widehat{x})$ are on the normal plane, the Disteli's formulae can be displayed in the timelike plane $Sp\{\widehat{\mathbf{r}}_f, \widehat{\mathbf{t}}\}$ (or $Sp\{\widehat{\mathbf{r}}_m, \widehat{\mathbf{t}}\}$). Thus, any random point $\mathbf{c}(\vartheta^*, \psi^*)$ on the timelike plane $Sp\{\widehat{\mathbf{r}}_f, \widehat{\mathbf{t}}\}$ is displayed as central point of the timelike line trajectory whose

generators are the timelike oriented line $\widehat{\mathbf{x}}$ and the radius can be defined by Equation (46); its length is the line segment from $\mathbf{s}$ to the point $\mathbf{c}$ on the timelike plane $Sp\{\widehat{\mathbf{r}}_f, \widehat{\mathbf{t}}\}$. Further, the central point $\mathbf{c}$ is in the orientation of $\widehat{\mathbf{t}}$ if $\alpha^* > 0$ and in the opposite orientation of $\widehat{\mathbf{t}}$ if $\alpha^* < 0$. The central point $\mathbf{c}(\vartheta^*, \psi^*)$ can be on the $\mathbb{ISA}$ if $\vartheta^* = 0$ ($\varphi^* = -\alpha^*$) and on the timelike Disteli's axis $\widehat{\mathbf{b}}$ if $\alpha^* = 0 \Leftrightarrow \varphi^* = \vartheta^*$. In the latter case, the central point $\mathbf{c}(\vartheta^*, \psi^*)$ can be pointed by letting $\varphi^* = 0$ in Equation (46), which is simplified as a linear equation

$$L : \psi^* = (-\gamma + 2\coth\vartheta\cos\psi)\vartheta^* - \frac{\gamma^*}{\gamma}\cot\psi. \tag{47}$$

Equation (47) is linear in the position coordinates $\psi^*$and ϑ^* of the timelike oriented line $\widehat{\mathbf{x}}$. Therefore, for a one-parameter hyperbolic spatial locomotion $\mathbb{L}_m/\mathbb{L}_f$ the timelike lines in a steady orientation in $\mathbb{L}_m$-space lie on a timelike plane. The line L will convert its location if ϑ has various values but ψ is steady. However, a set of lines envelope a spacelike curve on the timelike plane $Sp\{\widehat{\mathbf{r}}_f, \widehat{\mathbf{t}}\}$. Further, the attitude of the timelike plane is various if the parameter ψ of a timelike line has various values but ϑ is steady. Therefore, the set of all timelike lines L specified by Equation (47) is a timelike line congruence for all values of (ϑ^*, ψ^*).

At the end of this section, we want to derive another hyperbolic dual Euler–Savary formula for the timelike axodes as follows: substituting $\widehat{\vartheta} = \widehat{\varphi}_m, \widehat{\alpha} = \widehat{\varphi}_f, \psi = \frac{\pi}{2}$ and $\psi^* = 0$ into Equation (34), we find

$$\coth\widehat{\varphi}_f - \coth\widehat{\varphi}_m = \widehat{\gamma}, \tag{48}$$

which is a dual Euler–Savary formula. By the real and the dual parts, respectively, we find

$$\coth\varphi_f - \coth\varphi_m = \gamma,$$

and

$$\frac{\varphi_m^*}{\sinh^2\varphi_m} - \frac{\varphi_f^*}{\sinh^2\varphi_f} = \gamma^*,$$

as claimed.

Corollary 3. *For a one-parameter hyperbolic dual locomotion $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$, the common correlation of timelike axodes is specified by dual Euler–Savary Formula (48).*

4. Conclusions

The main interest of this work is to address one-parameter hyperbolic spatial locomotions in view of the E. Study map. Then, new proofs for Euler–Savary and Disteli's formulae were deduced. In addition, a ruled identification of the curvature circle for a timelike line trajectory has been realized. The geometrical distinctive of the line trajectory are examined, which include the Euler–Savary analogue for the line trajectory and the osculating conditions of the curvature circle. Our calculations in this paper can contribute to the field of hyperbolic spatial kinematics and have practical applications in mechanical mathematics and engineering. In future work, we plan to study some implementations of kinematic-geometry of one-parameter hyperbolic spatial movement merge with singularity theory, submanifold theory, etc., in [5,9,16–19] to search additional outcomes and theorems concerning with symmetric properties of this topic.

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References

1. Bottema, O.; Roth, B. *Theoretical Kinematics*; North-Holland Press: New York, NY, USA, 1979.
2. Karger, A.; Novak, J. *Space Kinematics and Lie Groups*; Gordon and Breach Science Publishers: New York, NY, USA, 1985.
3. Stachel, H. *Instantaneous Spatial Kinematics and the Invariants of the Axodes*; Technical Report 34; Institute fur Geometrie, TU Wien: Wien, Austria, 1996.
4. Pottman, H.; Wallner, J. *Computational Line Geometry*; Springer: Berlin/Heidelberg, Germany, 2001.
5. Abdel-Baky, R.A.; Al-Solamy, F.R. A new geometrical approach to one-parameter spatial motion. *J. Eng. Math.* **2008**, *60*, 149–172. [CrossRef]
6. Tosun, M.; Gungor, M.A.; Okur, I. On the one-Parameter Lorentzian spherical motions and Euler–Savary formula. *Sep. J. Appl. Mech.* **2007**, *74*, 972–977. [CrossRef]
7. Al-Ghefari, R.A.; Abdel-Baky, R.A. Kinematic geometry of a line trajectory in spatial motion. *J. Mech. Sci. Technol.* **2015**, *29*, 3597–3608. [CrossRef]
8. Aslan, M.C.; Sekerci, G.A. Dual curves associated with the Bonnet ruled surfaces. *Int. J. Geom. Methods Mod. Phys.* **2020**, *17*, 2050204. [CrossRef]
9. Alluhaibi, N. Ruled surfaces with constant Disteli-axis. *AIMS Math.* **2020**, *5*, 7678–7694. [CrossRef]
10. Abdel-Baky, R.A. On the curvature theory of a line trajectory in spatial kinematics. *Commun. Korean Math. Soc.* **2019**, *34*, 333–349.
11. Gungor, M.A.; Ersoy, S.; Tosun, S. Dual Lorentzian spherical movements and dual Euler–Savary formula. *Eur. J. Mech. A/Solids* **2009**, *28*, 820–826. [CrossRef]
12. Alluhaibi, N.A.; Abdel-Baky, R.A. On the kinematic-geometry of one-parameter Lorentzian spatial movement. *Int. J. Adv. Manuf. Tech.* **2022**, *121*, 7721–7731. [CrossRef]
13. Li, Y.; Alluhaibi, N.; Abdel-Baky, R.A. One-parameter Lorentzian dual spherical movements and invariants of the axodes. *Symmetry* **2022**, *14*, 1930. [CrossRef]
14. Walfare, J. Curves and Surfaces in Minkowski Space. Ph.D. Thesis, K.U. Leuven, Faculty of Science, Leuven, Belgium, 1995.
15. O'Neil, B. *Semi-Riemannian Geometry Geometry, with Applications to Relativity*; Academic Press: New York, NY, USA, 1983.
16. Li, Y.; Aldossary, M.T.; Abdel-Baky, R.A. Spacelike circular surfaces in Minkowski 3-space. *Symmetry* **2023**, *15*, 173. [CrossRef]
17. Li, Y.; Chen, Z.; Nazra, S.H.; Abdel-Baky, R.A. Singularities for timelike developable surfaces in Minkowski 3-space. *Symmetry* **2023**, *15*, 277. [CrossRef]
18. Zhang, X.; Zhang, J.; Pang, B.; Zhao, X. An accurate prediction method of cutting forces in 5-axis flank milling of sculptured surface. *Int. J. Mach. Tools Manuf.* **2016**, *104*, 26–36.
19. Calleja, A.; Bo, P.; Gonzalez, H.; Barton, M.; Norberto, L.; de Lacalle, L. Highly accurate 5-axis flank CNC machining with conical tools. *Int. J. Adv. Manuf. Technol.* **2018**, *97*, 1605–1615.

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