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Optimal Grouping of Dependent Components in Parallel-Series and Series-Parallel Systems with Independent Subsystems Equipped with Starting Devices

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Abstract: In this paper, we consider parallel-series and series-parallel systems comprising dependent components that are drawn from a heterogeneous population consisting of m different subpopulations, and each subsystem is equipped with a starter device. We also make the assumption that the components within each subpopulation are dependent, while the subsystems themselves are independent. The joint distribution of these subsystems is modeled using an Archimedean copula. Our research considers a general setting in which each subpopulation has a different Archimedean copula for its dependence. By adopting this general setup, we investigate the stochastic, hazard rate, and reversed hazard rate orders between these systems. Furthermore, we provide several numerical examples to demonstrate all the theoretical results established in this study. These results broaden the scope of the known results in the existing literature.

Keywords: archimedean copula; stochastic orders; parallel-series systems; series-parallel systems; starting devices

MSC: 62G30; 62N05; 60E15; 90B25

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Citation: Balakrishnan, N.; Saadat Kia, G.; Hosseinzadeh, A.; Sattari, M. Optimal Grouping of Dependent Components in Parallel-Series and Series-Parallel Systems with Independent Subsystems Equipped with Starting Devices. *Mathematics* **2023**, *11*, 3718. https://doi.org/10.3390/math11173718

Academic Editor: Stefano Bonnini

Received: 15 June 2023
Revised: 16 August 2023
Accepted: 28 August 2023
Published: 29 August 2023

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1. Introduction

Global competition has made the manufacturing process of technical systems quite demanding. A manufacturing process typically relies on parts or components coming from many different suppliers all over the world. Inevitably, this would result in differing quality and reliability measures for the components used in the systems. However, the supply and demand would warrant the producers to still order their parts from different suppliers and then use them suitably in their production process. Statistically, this poses great difficulties since the modelling of lifetimes of systems with such components becomes quite complex due to the dependence between the components within each subsystem and also the added dependence between the subsystems themselves. This, in turn, makes the comparative evaluation of different systems with regard to their reliability characteristics even more complex. One successful attempt in this direction is the recent work of Fang et al. [1] in which the authors considered a restricted set-up with the components within each subsystem being dependent (modelled by a copula), but the subsystems themselves functioning independently. Even though this work provided some key ideas and directions regarding the modelling of systems as well as their reliability comparisons, for the reasons stated above, in this work, we consider such a general set-up and then focus on series and parallel systems with dependent redundant components in both series and parallel structures and then establish various reliability comparative results between different systems.

It should be mentioned that though there are many coherent systems that could be formed and used, series, parallel, series-parallel, parallel-series and fail-safe systems are some of the most commonly used systems in practice. Series-parallel and parallel-series systems are commonly used in circuit systems. For this reason, we focus here on some forms of these systems under the very general setting described above, and then establish some reliability comparative results between different systems in terms of usual stochastic, hazard rate and reversed hazard rate orders. These comparisons are carried out for different cases: first, when the number of subsystems is fixed and the number of components allocated within subsystems vary, second, when the number of subsystems varies, and third when the selection probabilities or the distributions of subpopulations vary.

The optimal allocation of components on different series and parallel systems has been discussed extensively by many authors, including Di Crescenzo and Pellerey [2], Hazra et al. [3], Zhao et al. [4,5], and the references therein. Several authors have subsequently discussed this issue for parallel-series and series-parallel systems, including Coit and Smith [6], Ramirez-Marquez et al. [7], Sarhan et al. [8], Billionnet [9], Levitin and Amari [10] and Sun et al. [11]. Ling et al. [12] considered the optimal allocation strategy for series-parallel and parallel-series systems consisting of k subsystems with independent components drawn from a heterogeneous population. Ling and Wei [13] investigated the optimal allocation strategy of redundancies drawn from heterogeneous populations for a k-out-of-n system with n independent and heterogeneous components. Fang et al. [1] studied series-parallel and parallel-series systems consisting of n dependent components drawn from a heterogeneous population with m different subpopulations. Recently, Barmalzan et al. [14] extended the results of Fang et al. [1] to the case when the components within each subpopulation are dependent, and the subsystems themselves are also dependent, with their joint distribution being modelled by an Archimedean copula.

We shall now formulate the problem of interest statistically and then describe as to how the reliability systems are formed under this general set-up. Suppose a parallel-series system comprising n components is to be formed with k subsystems connected in parallel, with each subsystem being in series comprising $a_1, \cdots, a_k$ components, such that all $a_j > 0$ and $\sum_{j=1}^{k} a_j = n$. The components within each subsystem are assumed to come from the i-th subpopulation with probability p_i, $i = 1, 2, \cdots, m$, and also each subsystem is equipped with a starter device whose performance is modelled by a Bernoulli random variable, I_{q_j}, with $\mathbb{E}[I_{q_j}] = q_j$, for $j = 1, 2, \cdots, k$. Now, in a general set-up, we assume that the components within each subsystem are dependent and the subsystems themselves are independent. We also assume that all the subpopulations have different Archimedean copulas with generators ϕ_i, $i = 1, 2, \cdots, m$.

In the statistical literature, there are many ways to model dependence between random variables (see Kotz et al. [15], and the theory of copulas is one popular tool for this purpose; see, for example, Nelsen [16] for an elaborate discussion on the theory and applications of copulas. Though many copulas have been studied in the literature, Archimedean copulas have attracted considerable attention due to their flexibility and the fact that they include many well-known copulas, such as Clayton, Ali-Mikhail-Haq, Gumbel-Hougaard and Frank copulas, as special cases. In addition, they also include the independence copula as a special case and consequently comparison results established under Archimedean copula for the joint distribution of lifetimes of components in the system are quite general in form and structure, and so would naturally include the corresponding results developed in the recent work of Fang et al. [1]. Recently, Barmalzan et al. [14] extended the results of Fang et al. [1] for the case when the components within each subpopulation are dependent, and the subsystems themselves are also dependent, with their joint distribution being modeled by an Archimedean copula.

The rest of this paper proceeds as follows. In Section 2, we introduce some basic definitions and lemmas that are necessary for subsequent developments. In Section 3, we establish some comparative results for parallel-series systems by using the concepts of weak majorization and majorization orders. The usual stochastic and reversed hazard rate orders

between series-parallel systems are then established in Section 4. Several illustrative examples are presented based on Ali-Mikhail-Haq, Gumbel-Hougaard and Clayton copulas for the components and with starting performance probabilities for the subsystems. Finally, some concluding remarks are made in Section 5, including suggestions for possible future research.

2. Basic Definitions and Notions

In this section, we describe some well-known concepts about stochastic orders, majorization and associated orders, and copulas which are essential for subsequent developments. We assume all random variables under consideration to be non-negative, and use "increasing" to mean "nondecreasing" and similarly "decreasing" to mean "nonincreasing". We assume all involved expectations to exist, and for convenience in notation, we use $a \overset{sgn}{=} b$ to denote that both sides of an equality have the same sign. We also set

$$A = \Big\{ M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X}) : \ n,\ m,\ \boldsymbol{a} \in \mathbb{N}_+^k \ \textit{with} \ \sum_{j=1}^{k} a_j = n,\ \boldsymbol{q} \in [0,1]^k, \\ \boldsymbol{p} \in [0,1]^m \ \textit{such that} \sum_{i=1}^{m} p_i = 1 \Big\},$$

where $M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})$ denotes the lifetime of a system comprising k subsystems with each subsystem having been equipped with a starter device whose performance is modelled by a Bernoulli random variable, I_{q_j}, with $\mathbb{E}(I_{q_j}) = q_j$, for $j = 1, 2, \cdots, k$; the system has a total of n components coming from m suppliers in proportions $\boldsymbol{p} = (p_1, \cdots, p_m)$, and with $\boldsymbol{a} = (a_1, \cdots, a_k)$ denoting the number of components in the k subsystems and $\boldsymbol{X}$ denoting the component lifetimes.

2.1. Stochastic Orders

Let X and Y be two random variables with density functions f_X and f_Y, distribution functions F_X and F_Y, reliability functions $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$, hazard rate functions $h_X = f_X/\bar{F}_X$ and $h_Y = f_Y/\bar{F}_Y$, and reversed hazard rate functions $r_X = f_X/F_X$ and $r_Y = f_Y/F_Y$, respectively. Then, the following stochastic orders are well-known ones.

Definition 1. *X is said to be larger than Y in the*

(i) usual stochastic order (denoted by $X \geq_{\mathrm{st}} Y$) if $\bar{F}_X(t) \geq \bar{F}_Y(t)$, for all $t \in \mathbb{R}$, or equivalently, $\mathbb{E}[\phi(X)] \geq \mathbb{E}[\phi(Y)]$ for all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$;

(ii) hazard rate order (denoted by $X \geq_{\mathrm{hr}} Y$) if and only if $h_Y(t) \geq h_X(t)$, for all $t \in \mathbb{R}$, or equivalently, $\bar{F}_X(t)/\bar{F}_Y(t)$ is increasing in $t \in \mathbb{R}$;

(iii) reversed hazard rate order (denoted by $X \geq_{\mathrm{rh}} Y$) if and only if $r_X(t) \geq r_Y(t)$, for all $t \in \mathbb{R}$, or equivalently, $F_X(t)/F_Y(t)$ is increasing in $t \in \mathbb{R}$.

These orders are known to satisfy the following implications:

$$X \leq_{\mathrm{hr[rh]}} Y \Longrightarrow X \leq_{\mathrm{st}} Y.$$

Interested readers may be referred to the books by Müller and Stoyan [17] and Shaked and Shanthikumar [18] for exhaustive discussions on various stochastic orderings, and their inter-relationships and applications.

2.2. Majorization Orders

We now introduce some key majorization orders and associated results.

Definition 2. *Let $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ be vectors with increasing arrangements $a_{1:n} \leq \cdots \leq a_{n:n}$ and $b_{1:n} \leq \cdots \leq b_{n:n}$, respectively. Then:*

(i) $\mathbf{a}$ *is said to be majorized by vector* $\mathbf{b}$ *(denoted by* $\mathbf{a} \overset{m}{\preceq} \mathbf{b}$*) if* $\sum_{j=1}^{i} a_{j:n} \geq \sum_{j=1}^{i} b_{j:n}$*, for* $i = 1, \cdots, n-1$*, and* $\sum_{j=1}^{n} a_{j:n} = \sum_{j=1}^{n} b_{j:n}$*;*

(ii) $\mathbf{a}$ *is said to be weakly supermajorized by vector* $\mathbf{b}$ *(denoted by* $\mathbf{a} \overset{w}{\preceq} \mathbf{b}$*) if* $\sum_{j=1}^{i} a_{j:n} \geq \sum_{j=1}^{i} b_{j:n}$*, for* $i = 1, \cdots, n$*;*

(iii) $\mathbf{a}$ *is said to be weakly submajorized by vector* $\mathbf{b}$ *(denoted by* $\mathbf{a} \preceq_w \mathbf{b}$*) if* $\sum_{j=i}^{n} a_{j:n} \leq \sum_{j=i}^{n} b_{j:n}$*, for* $i = 1, \cdots, n$*.*

Definition 3. *A real-valued function* ϕ*, defined on a set* $\mathbb{A} \subseteq \mathbb{R}^n$*, is said to be Schur-convex (Schur-concave) on* $\mathbb{A}$*, if* $\mathbf{a} \overset{m}{\preceq} \mathbf{b}$ *implies* $\phi(\mathbf{a}) \leq (\geq) \phi(\mathbf{b})$ *for any* $\mathbf{a}, \mathbf{b} \in \mathbb{A}$*.*

For a detailed discussion on majorization and Schur functions, one may refer to Marshall et al. [19]. A result concerning the necessary and sufficient conditions for the characterization of Schur-convex and Schur-concave functions is as follows.

Lemma 1 (Marshall et al. [19], p. 84)**.** *Suppose* $J \subset \mathbb{R}$ *is an open interval and* $\phi : J^n \to \mathbb{R}$ *is a continuously differentiable function. Then, the necessary and sufficient conditions for* ϕ *to be Schur-convex (Schur-concave) on* J^n *are*

(i) ϕ *is symmetric on* J^n*;*

(ii) *for all* $i \neq j$ *and all* $\mathbf{z} \in J^n$*,*

$$(z_i - z_j)\left(\frac{\partial \phi(\mathbf{z})}{\partial z_i} - \frac{\partial \phi(\mathbf{z})}{\partial z_j}\right) \geq 0\,(\leq 0),$$

where $\partial \phi(\mathbf{z})/\partial z_i$ *denotes the partial derivative of* ϕ *with respect to its i-th argument.*

Lemma 2 (Marshall et al. [19], p. 87)**.** *Consider the real-valued function* ϕ*, defined on a set* $\mathbb{A} \subseteq \mathbb{R}^n$*. Then,*

(i) $\boldsymbol{u} \succeq_w \boldsymbol{v}$ *implies* $\phi(\boldsymbol{u}) \geq \phi(\boldsymbol{v})$ *if and only if* ϕ *is increasing and Schur-convex on* $\mathbb{A}$*;*

(ii) $\boldsymbol{u} \overset{w}{\succeq} \boldsymbol{v}$ *implies* $\phi(\boldsymbol{u}) \geq \phi(\boldsymbol{v})$ *if and only if* ϕ *is decreasing and Schur-convex on* $\mathbb{A}$*.*

2.3. Copulas

Many different stochastic comparisons of univariate random variables have been defined and discussed in the literature; see Müller and Stoyan [17] and Shaked and Shanthikumar [18] for pertinent details. They mostly involve comparisons of marginal distributions of the underlying variables, without taking dependence between the variables into account. Here, we discuss stochastic comparisons of series-parallel and parallel-series systems with dependent heterogeneous components, when the joint distribution is modelled by the flexible family of Archimedean copulas.

Archimedean copulas have been used extensively due to their ability to represent a wide range of dependence. For a decreasing and continuous function $\phi : [0, \infty) \longrightarrow [0, 1]$ such that $\phi(0) = 1$, $\phi(+\infty) = 0$, and $\psi = \phi^{-1}$ being the pseudo-inverse,

$$C_\phi(u_1, \cdots, u_n) = \phi(\psi(u_1) + \cdots + \psi(u_n)), \qquad \textit{for all } u_i \in [0,1],\ i = 1, \cdots, n,$$

is said to be an Archimedean copula with generator ϕ if $(-1)^k \phi^{[k]}(x) \geq 0$, for $k = 0, \cdots, n-2$, and $(-1)^{n-2}\phi^{[n-2]}(x)$ is decreasing and convex. Here, $\phi^{[k]}(x)$ denotes the k-th derivative of x. The Archimedean copula family includes many well-known copulas, such as independence (product), Clayton, Ali-Mikhail-Haq (AMH) and Gumbel-Hougaard copulas, all as special cases.

3. Results for Parallel-Series Systems

Here, we consider a parallel-series system that comprises k independent subsystems connected in parallel, with the j-th subsystem having a_j components connected in series, for $j = 1, 2, \cdots, k$. The components within each subsystem are assumed to come from the i-th subpopulation with probability p_i, $i = 1, 2, \cdots, m$, and also that each subsystem is equipped with a starter device whose performance is modelled by a Bernoulli random variable, I_{q_j}, with $\mathbb{E}[I_{q_j}] = q_j$, for $j = 1, 2, \cdots, k$. Next, we assume that the components within each subsystem are dependent, but the subsystems themselves are independent. We also assume that all the subpopulations have different Archimedean copulas with generators ϕ_i, $i = 1, 2, \cdots, m$. Then, with $M_n^m(k|\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{X})$ denoting the lifetime of such a general parallel-series system, its distribution function is given by

$$F_{H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) = \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i \, \phi_i\{a_j \, \psi_i(\bar{F}_i(t))\}\right], \qquad t \geq 0. \tag{1}$$

It is then of interest to determine how the n components from m different subpopulations could be distributed so that the resulting parallel-series system, with independent subsystems equipped with starting devices, will be optimal in some stochastic sense.

First, let us set

$$\mathcal{F}_k = \left\{(\boldsymbol{c}, \boldsymbol{d})' = \begin{pmatrix} c_1 \ldots c_k \\ d_1 \ldots d_k \end{pmatrix} : \, c_i, d_j > 0 \ \textit{and} \ (c_i - c_j)(d_i - d_j) \leq 0, \ i, j = 1, \cdots, k\right\}$$

and

$$\mathcal{E}_k = \left\{(\boldsymbol{u}, \boldsymbol{v})' = \begin{pmatrix} u_1 \ldots u_k \\ v_1 \ldots v_k \end{pmatrix} : \, u_i, v_j > 0 \ \textit{and} \ (u_i - u_j)(v_i - v_j) \geq 0, \ i, j = 1, \cdots, k\right\}.$$

To begin with, we first fix the number of independent subsystems within the parallel-series system to be k, with each subsystem being equipped with a starter device whose performance is modelled by a Bernoulli random variable, I_{q_j}, with $\mathbb{E}[I_{q_j}] = q_j$, for $j = 1, 2, \cdots, k$. We then show in the following theorem that if the allocation vector for any model with dependent components is weakly majorized by another allocation vector, then the parallel-series system corresponding to the first vector will be more reliable than the one corresponding to the second vector in the sense of usual stochastic order.

Theorem 1. *Let $H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ denote the lifetime of a parallel-series system equipped with starting devices, corresponding to $M_{n_1}^m(k|\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{X}) \in A$, with different Archimedean copula generators ϕ_i, for $i = 1, 2, \cdots, m$. Additionally, let $H_{M_{n_2}^m(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ denote the lifetime of a parallel-series system equipped with starting devices, corresponding to $M_{n_2}^m(k|\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{X}) \in A$, with the same set of different Archimedean copulas generators ϕ_i, for $i = 1, 2, \cdots, m$. Then, for $(\boldsymbol{a}, \boldsymbol{q})' \in \mathcal{F}_n$ and $(\boldsymbol{b}, \boldsymbol{q})' \in \mathcal{F}_n$, we have*

$$\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b} \Longrightarrow H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{st} H_{M_{n_2}^m(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}.$$

Proof. The distribution functions of parallel-series systems $H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ and $H_{M_{n_2}^m(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ are given by

$$F_{H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) = \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i \, \phi_i\{a_j \, \psi_i(\bar{F}_i(t))\}\right], \qquad t \geq 0,$$

$$F_{H_{M_{n_2}^m(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) = \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i\, \phi_i\{b_j\, \psi_i(\bar{F}_i(t))\}\right], \qquad t \geq 0,$$

respectively, where $a_1 + \cdots + a_k = n_1$, $b_1 + \cdots + b_k = n_2$, $\sum_{i=1}^{m} p_i = 1$ and $0 \leq q_j \leq 1$, for $j = 1, 2, \cdots, k$. Let us now define

$$H(\boldsymbol{a}) = F_{H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) = \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i\, \phi_i\{a_j\, \psi_i(\bar{F}_i(t))\}\right], \qquad t \geq 0.$$

Then, according to Lemma 2, we only need to show that $H(\boldsymbol{a})$ is increasing and Schur-concave in $(a_1, \cdots, a_k)$, for any fixed $t \geq 0$. Taking the derivative of $H(\boldsymbol{a})$ with respect to a_α, $\alpha \in \{1, 2, \cdots, k\}$, we obtain

$$\begin{aligned}
\frac{\partial H(\boldsymbol{a})}{\partial a_\alpha} &= -q_\alpha \left\{\sum_{i=1}^{m} p_i \psi_i(\bar{F}_i(t))\right\}\{\phi_i'(a_\alpha\, \psi_i(\bar{F}_i(t)))\} \\
&\times \prod_{j\neq\alpha=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i\, \phi_i\{a_j\, \psi_i(\bar{F}_i(t))\}\right] \\
&\geq 0.
\end{aligned}$$

Now, the assumption that $(\boldsymbol{a}, \boldsymbol{q})' \in \mathcal{F}_k$ implies that $(a_i - a_j)(q_i - q_j) \leq 0$, which means that $a_1 \leq a_2 \leq \cdots \leq a_k$ and $q_1 \geq q_2 \geq \cdots \geq q_k$, or $a_1 \geq a_2 \geq \cdots \geq a_k$ and $q_1 \leq q_2 \leq \cdots \leq q_k$. Here, we present the proof only for the case when $a_1 \leq a_2 \leq \cdots \leq a_k$ and $q_1 \geq q_2 \geq \cdots \geq q_k$, since the proof for the other case is quite similar.

Next, for $1 \leq \alpha < \beta \leq k$, we obtain

$$\begin{aligned}
J_1(\boldsymbol{a}) &:= (a_\alpha - a_\beta)\left(\frac{\partial H(\boldsymbol{a})}{\partial a_\alpha} - \frac{\partial H(\boldsymbol{a})}{\partial a_\beta}\right) \\
&= (a_\alpha - a_\beta)\, H(\boldsymbol{a}) \left\{ q_\beta \left\{\sum_{i=1}^{m} p_i \psi_i(\bar{F}_i(t))\right\}\{\phi_i'(a_\beta\, \psi_i(\bar{F}_i(t)))\} \right. \\
&\times \left[1 - q_\alpha \sum_{i=1}^{m} p_i\, \phi_i\{a_\alpha\, \psi_i(\bar{F}_i(t))\}\right] \\
&- \left. q_\alpha \left\{\sum_{i=1}^{m} p_i \psi_i(\bar{F}_i(t))\right\}\{\phi_i'(a_\alpha\, \psi_i(\bar{F}_i(t)))\} \left[1 - q_\beta \sum_{i=1}^{m} p_i\, \phi_i\{a_\beta\, \psi_i(\bar{F}_i(t))\}\right] \right\}.
\end{aligned}$$

For $i = 1, 2, \cdots, m$, according to the definition of Archimedean copula, we have

$$\phi_i'(x) \leq 0, \qquad \phi_i''(x) \geq 0.$$

So, for $a_\alpha \leq a_\beta$ and $q_\alpha \geq q_\beta$, we obtain

$$q_\beta \left\{\sum_{i=1}^{m} p_i \psi_i(\bar{F}_i(t))\right\}\{\phi_i'(a_\beta\, \psi_i(\bar{F}_i(t)))\} \geq q_\alpha \left\{\sum_{i=1}^{m} p_i \psi_i(\bar{F}_i(t))\right\}\{\phi_i'(a_\alpha\, \psi_i(\bar{F}_i(t)))\} \tag{2}$$

and

$$\left[1 - q_\alpha \sum_{i=1}^{m} p_i\, \phi_i\{a_\alpha\, \psi_i(\bar{F}_i(t))\}\right] \leq \left[1 - q_\beta \sum_{i=1}^{m} p_i\, \phi_i\{a_\beta\, \psi_i(\bar{F}_i(t))\}\right]. \tag{3}$$

From (2) and (3), we obtain

$$J_1(\boldsymbol{a}) = (a_\alpha - a_\beta)\left(\frac{\partial H(\boldsymbol{a})}{\partial a_\alpha} - \frac{\partial H(\boldsymbol{a})}{\partial a_\beta}\right) \leq 0,$$

which means that $H(\boldsymbol{a})$ is Schur-concave in $(a_1, \cdots, a_k)$, for any fixed $t \geq 0$. Hence, the theorem. □

Example 1. *Let us consider* $\bar{F}_i(t) = e^{-\lambda_i t}$, $k = 4$, $m = 6$. *Further, let* $\boldsymbol{a} = (8,6,4,3)$, $\boldsymbol{b} = (17,10,7,4)$, $\boldsymbol{q} = (0.15, 0.20, 0.25, 0.30)$, $\boldsymbol{p} = (0.1, 0.2, 0.1, 0.2, 0.1, 0.3)$, *and* $\boldsymbol{\lambda} = (1,2,3,4,5,6)$. *Now, let us consider the following situations:*

(i) *Suppose* $\phi_i(t) = \frac{1-\theta_i}{e^t - \theta_i}$, $\theta_i \in [0,1)$, $i = 1,2,\cdots,6$, *and* $\boldsymbol{\theta} = (1/2, 1/3, 1/5, 1/7, 1/9, 1/4)$. *Then, the expressions of* $F_{H_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}}(t)$ *and* $F_{H_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}}(t)$ *are given by*

$$F_{H_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{6} \frac{p_i(1-\theta_i)}{\left[(1-\theta_i)e^{\lambda_i t} + \theta_i\right]^{a_j} - \theta_i}\right),$$

$$F_{H_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{6} \frac{p_i(1-\theta_i)}{\left[(1-\theta_i)e^{\lambda_i t} + \theta_i\right]^{b_j} - \theta_i}\right);$$

(ii) *Suppose* $\phi_i(t) = e^{1-(1+t)^{\beta_i}}$, $\beta_i \in [1,\infty)$, $i = 1,2,\cdots,6$, *and* $\boldsymbol{\beta} = (1,2,3,4,5,6)$. *Then, the expressions of* $F_{H_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}}(t)$ *and* $F_{H_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}}(t)$ *are given by*

$$F_{H_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{6} p_i \exp\left[1 - \left\{1 + a_j\left[(1+\lambda_i t)^{1/\beta_i} - 1\right]\right\}^{\beta_i}\right]\right),$$

$$F_{H_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{6} p_i \exp\left[1 - \left\{1 + b_j\left[(1+\lambda_i t)^{1/\beta_i} - 1\right]\right\}^{\beta_i}\right]\right);$$

(iii) *Suppose* $\phi_i(t) = (\mu_i t + 1)^{-1/\mu_i}$, $\mu_i \in [0,\infty)$, $i = 1,2,\cdots,6$, *and* $\boldsymbol{\mu} = (1,2,3,4,5,6)$. *Then, the expressions of* $F_{H_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}}(t)$ *and* $F_{H_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}}(t)$ *are given by*

$$F_{H_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{6} p_i\left[1 + a_j(e^{\lambda_i \mu_i t} - 1)\right]^{-1/\mu_i}\right),$$

$$F_{H_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{6} p_i\left[1 + b_j(e^{\lambda_i \mu_i t} - 1)\right]^{-1/\mu_i}\right).$$

The plots of the distribution functions of parallel-series systems under Ali-Mikhail-Haq, Gumbel-Hougaard and Clayton copulas for the components and the starting device performance probabilities for the subsystems are presented in Figure 1. Because $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$, *with the choices of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *made, based on Theorem 1, we have* $H_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{st} H_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ *for all three cases.*

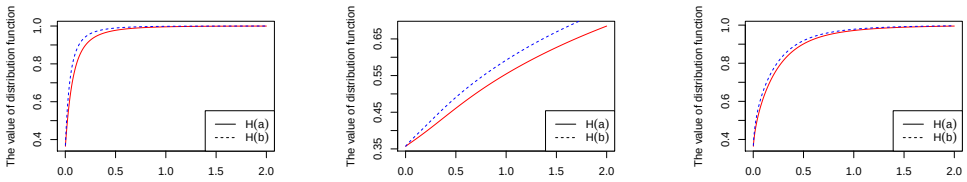

Figure 1. Plots of the distribution functions of parallel-series systems under Ali-Mikhail-Haq, Gumbel-Hougaard and Clayton copulas for the components and the starting device performance probabilities for the subsystems, from left to right.

From Figure 1, we observe that the distribution functions of the two parallel-series systems differ considerably under the Gumbel-Hougaard copula than under the other two copulas, with the Clayton copula having the least difference.

Remark 1. *Observe that Theorem 1 presents the usual stochastic order between parallel-series systems without any restriction on the generator functions of the copulas for the subsystems. Now, if we take* $\phi_i(t) = e^{-t}$*, corresponding to independence of the components, then the known results for this case are readily deduced.*

The following theorem shows that for a parallel-series system, under certain conditions, if the number of subsystems equipped with starting devices increases, then the system would become more reliable.

Theorem 2. *Let* $H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ *denote the lifetime of a parallel-series system equipped with starting devices, corresponding to* $M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X}) \in A$*, with different Archimedean copula generators* ϕ_i*, for* $i = 1, 2, \cdots, m$*. Additionally, let* $H_{M_{n_2}^m(k-1|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}_{-k},\boldsymbol{X})}$ *denote the lifetime of a parallel-series system equipped with starting devices, corresponding to* $M_{n_2}^m(k-1|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}_{-k},\boldsymbol{X}) \in A$*, with the same set of different Archimedean copula generators* ϕ_i*, for* $i = 1, 2, \cdots, m$*. Let* $\boldsymbol{a} = (a_1, \cdots, a_k)$*,* $\boldsymbol{b} = (b_1, \cdots, b_{k-1})$*,* $\boldsymbol{q} = (q_1, \cdots, q_k)$ *and* $\boldsymbol{q}_{-k} = (q_1, \cdots, q_{k-1})$*. Then, we have*

$$a_j \le b_j \text{ for all } j = 1, 2, \cdots, k-1 \Longrightarrow H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \ge_{st} H_{M_{n_2}^m(k-1|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}_{-k},\boldsymbol{X})}.$$

Proof. From (1), for all $t \ge 0$, we have

$$\begin{aligned} F_{H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) &= \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i \phi_i\{a_j \psi_i(\bar{F}_i(t))\}\right] \\ &\le \prod_{j=1}^{k-1}\left[1 - q_j \sum_{i=1}^{m} p_i \phi_i\{a_j \psi_i(\bar{F}_i(t))\}\right] \\ &\le \prod_{j=1}^{k-1}\left[1 - q_j \sum_{i=1}^{m} p_i \phi_i\{b_j \psi_i(\bar{F}_i(t))\}\right] \\ &= F_{H_{M_{n_2}^m(k-1|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}_{-k},\boldsymbol{X})}}(t), \end{aligned}$$

which implies that $H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \ge_{st} H_{M_{n_2}^m(k-1|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}_{-k},\boldsymbol{X})}$, as required. □

Let $M_n^m(1|n,\boldsymbol{p},q,\boldsymbol{X})$ denote a parallel-series system having one subsystem, with starting device performance probability q, with n components connected in series and $M_n^m(n|\boldsymbol{1},\boldsymbol{p},\boldsymbol{q}^+,\boldsymbol{X})$ denote a parallel-series system having n subsystems, with starting device performance probabilities $q_1, \cdots, q_n$, connected in parallel with each subsystem having only one component. Then, the following theorem shows that $M_n^m(n|\boldsymbol{1},\boldsymbol{p},\boldsymbol{q}^+,\boldsymbol{X})$ is an optimal system and $M_n^m(1|n,\boldsymbol{p},q,\boldsymbol{X})$ is the worst system among all possible such parallel-series systems, in the sense of reversed hazard rate order.

Theorem 3. *Let $H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ denote the lifetime of a parallel-series system equipped with starting devices, corresponding to $M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X}) \in A$ with different Archimedean copula generators ϕ_i, for $i = 1, 2, \cdots, m$. Let $\boldsymbol{q} = (q_1, \cdots, q_k)$ and $\boldsymbol{q}^+ = (q_1, \cdots, q_n)$ and $q \leq \min\{q_1, \cdots, q_k\}$. Then, we have*

$$H_{M_n^m(1|n,\boldsymbol{p},q,\boldsymbol{X})} \leq_{rh} H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \leq_{rh} H_{M_n^m\left(n|\mathbf{1},\boldsymbol{p},\boldsymbol{q}^+,\boldsymbol{X}\right)}.$$

Proof. The reversed hazard rate function of $H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ is given by

$$r_{H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) = \sum_{l=1}^{k} \frac{q_l \sum_{i=1}^{m} p_i a_l f_i(t) \psi_i'(\bar{F}_i(t)) \phi_i'(a_l \psi_i(\bar{F}_i(t)))}{1 - q_l \sum_{i=1}^{m} p_i \phi_i(a_l \psi_i(\bar{F}_i(t)))}.$$

Because $a_1 + a_2 + \cdots + a_k = n$, we have the following two special cases corresponding to $k = 1$ and $k = n$:

$$r_{H_{M_n^m(1|n,\boldsymbol{p},q,\boldsymbol{X})}}(t) = \sum_{l=1}^{k} \frac{q \sum_{i=1}^{m} p_i a_l f_i(t) \psi_i'(\bar{F}_i(t)) \phi_i'(n \psi_i(\bar{F}_i(t)))}{1 - q \sum_{i=1}^{m} p_i \phi_i(n \psi_i(\bar{F}_i(t)))},$$

$$r_{H_{M_n^m\left(n|\mathbf{1},\boldsymbol{p},\boldsymbol{q}^+,\boldsymbol{X}\right)}}(t) = \sum_{l=1}^{n} \frac{q_l \sum_{i=1}^{m} p_i a_l f_i(t) \psi_i'(\bar{F}_i(t)) \phi_i'(\psi_i(\bar{F}_i(t)))}{1 - q_l \sum_{i=1}^{m} p_i \phi_i(\psi_i(\bar{F}_i(t)))}.$$

Now, in order to prove that $H_{M_n^m(1|n,\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \leq_{rh} H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \leq_{rh} H_{M_n^m\left(n|\mathbf{1},\boldsymbol{p},\boldsymbol{q}^+,\boldsymbol{X}\right)}$, we need to show that, for all $l = 1, 2, \cdots, k$, that

$$\begin{aligned} 0 &\leq \sum_{l=1}^{k} \frac{q \sum_{i=1}^{m} p_i a_l f_i(t) \psi_i'(\bar{F}_i(t)) \phi_i'(n \psi_i(\bar{F}_i(t)))}{1 - q \sum_{i=1}^{m} p_i \phi_i(n \psi_i(\bar{F}_i(t)))} \\ &\leq \sum_{l=1}^{k} \frac{q_l \sum_{i=1}^{m} p_i a_l f_i(t) \psi_i'(\bar{F}_i(t)) \phi_i'(a_l \psi_i(\bar{F}_i(t)))}{1 - q_l \sum_{i=1}^{m} p_i \phi_i(a_l \psi_i(\bar{F}_i(t)))} \\ &\leq \sum_{l=1}^{n} \frac{q_l \sum_{i=1}^{m} p_i a_l f_i(t) \psi_i'(\bar{F}_i(t)) \phi_i'(\psi_i(\bar{F}_i(t)))}{1 - q_l \sum_{i=1}^{m} p_i \phi_i(\psi_i(\bar{F}_i(t)))}. \end{aligned} \tag{4}$$

For $x \geq 0$, let us consider

$$g(x, q_l) = \frac{q_l \sum_{i=1}^{m} p_i f_i(t) a_l \psi_i'(\bar{F}_i(t)) \phi_i'(x \psi_i(\bar{F}_i(t)))}{1 - q_l \sum_{i=1}^{m} p_i \phi_i(x \psi_i(\bar{F}_i(t)))}.$$

Then, the derivative of $g(x, q_l)$ with respect to x is given by

$$\begin{aligned} g'(x,q_l) &\overset{sgn}{=} q_l \sum_{i=1}^{m} p_i f_i(t) a_l \psi_i'(\bar{F}_i(t)) \phi_i''(x\psi_i(\bar{F}_i(t)))\, \psi_i(\bar{F}_i(t)) \\ &\times \left[1 - q_l \sum_{i=1}^{m} p_i \phi_i(x\psi_i(\bar{F}_i(t)))\right] \\ &+ q_l^2 \sum_{i=1}^{m} p_i f_i(t) a_l \phi_i'(x\psi_i(\bar{F}_i(t)))\psi_i(\bar{F}_i(t)) \\ &\times \left[\sum_{i=1}^{m} p_i f_i(t) a_l \psi_i'(\bar{F}_i(t)) \phi_i'(x\psi_i(\bar{F}_i(t)))\right] \\ &\leq 0, \end{aligned}$$

and so $g(x, q_l)$ is decreasing in $x \geq 0$. We can also show similarly that $g(x, q_l)$ is increasing in q_l, which means that (4) is true. Hence, the theorem. □

In the following theorem, we study the influence of selection probabilities and the distributions of subpopulations on the reliability of a parallel-series system equipped with starting devices for subsystems in terms of usual stochastic order.

Theorem 4. *Let $H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ denote the lifetime of a parallel-series system equipped with starting devices, corresponding to $M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X}) \in A$, with the same Archimedean copula generator ϕ. Additionally, let $H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q},\boldsymbol{X})}$ denote the random lifetime of a parallel-series system equipped with starting devices, corresponding to $M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q},\boldsymbol{X}) \in A$, with the same Archimedean copula generator ϕ. Then, for $(\boldsymbol{p},\bar{\boldsymbol{F}})' \in \mathcal{F}_m$ $((\boldsymbol{p},\bar{\boldsymbol{F}})' \in \mathcal{E}_m)$, we have*

$$\boldsymbol{p} \overset{m}{\succeq} \boldsymbol{p}^* \Longrightarrow H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{st} (\leq_{st}) H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q},\boldsymbol{X})}.$$

Proof. For all $t \geq 0$, we have

$$F_{H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) = \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i\, \phi\{a_j\, \psi(\bar{F}_i(t))\}\right],$$

$$F_{H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q},\boldsymbol{X})}}(t) = \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i^*\, \phi\{a_j\, \psi(\bar{F}_i(t))\}\right].$$

Now, let us define

$$H(\boldsymbol{p}) = F_{H_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) = \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i\, \phi\{a_j\, \psi(\bar{F}_i(t))\}\right].$$

Then, according to Lemma 2, we need to show that $H(\boldsymbol{p})$ is decreasing and Schur-convex (Schur-concave) in $(p_1, \cdots, p_m)$, for any fixed $t \geq 0$. Taking the derivative of $H(\boldsymbol{p})$ with respect to p_α, $\alpha \in \{1, 2, \cdots, k\}$, we obtain

$$\begin{aligned} \frac{\partial H(\boldsymbol{p})}{\partial p_\alpha} &= -\sum_{l=1}^{k} \frac{q_l \phi\{a_l\, \psi(\bar{F}_\alpha(t))\}}{1 - q_l \sum_{i=1}^{m} p_i\, \phi\{a_l\, \psi(\bar{F}_i(t))\}} \times H(\boldsymbol{p}) \\ &\leq 0. \end{aligned}$$

So,

$$\begin{aligned} J_2(\boldsymbol{p}) &:= (p_\alpha - p_\beta)\left(\frac{\partial H(\boldsymbol{p})}{\partial p_\alpha} - \frac{\partial H(\boldsymbol{p})}{\partial p_\beta}\right) \\ &= (p_\alpha - p_\beta)\, H(\boldsymbol{p}) \\ &\times \left(\sum_{l=1}^{k} \frac{q_l \phi\{a_l\, \psi(\bar{F}_\beta(t))\}}{1 - q_l \sum_{i=1}^{m} p_i\, \phi\{a_l\, \psi(\bar{F}_i(t))\}} - \sum_{l=1}^{k} \frac{q_l \phi\{a_l\, \psi(\bar{F}_\alpha(t))\}}{1 - q_l \sum_{i=1}^{m} p_i\, \phi\{a_l\, \psi(\bar{F}_i(t))\}}\right) \\ &= (p_\alpha - p_\beta)\, H(\boldsymbol{p}) \sum_{l=1}^{k} \frac{q_l(\phi\{a_l\, \psi(\bar{F}_\beta(t))\} - \phi\{a_l\, \psi(\bar{F}_\alpha(t))\})}{1 - q_l \sum_{i=1}^{m} p_i\, \phi\{a_l\, \psi(\bar{F}_i(t))\}}. \end{aligned}$$

The assumption that $(\boldsymbol{p}, \bar{\boldsymbol{F}})' \in \mathcal{F}_m$ ($(\boldsymbol{p}, \bar{\boldsymbol{F}})' \in \mathcal{E}_m$) implies that $\bar{F}_\alpha(t) \leq \bar{F}_\beta(t)$ and $p_\alpha \geq p_\beta$, or $\bar{F}_\alpha(t) \geq \bar{F}_\beta(t)$ and $p_\alpha \leq p_\beta$. Now, we present the proof only for the case when $\bar{F}_\alpha(t) \leq \bar{F}_\beta(t)$ and $p_\alpha \geq p_\beta$, since the proof for the other case is quite similar. It is easy to observe that

$$\phi\{a_l\, \psi(\bar{F}_\alpha(t))\} \leq (\geq) \phi\{a_l\, \psi(\bar{F}_\beta(t))\}$$

and then

$$\left(\frac{\partial H(\boldsymbol{p})}{\partial p_\alpha} - \frac{\partial H(\boldsymbol{p})}{\partial p_\beta}\right) \geq (\leq) 0,$$

which implies that $J_2(\boldsymbol{p}) \geq (\leq)\, 0$. Hence, the theorem. □

Example 2. *Let us consider* $\bar{F}_i(t) = e^{-\lambda_i t}$, $k = 4$, $m = 5$. *Further let* $\boldsymbol{a} = (9, 12, 6, 5)$, $\boldsymbol{p} = (0.4, 0.3, 0.1, 0.1, 0.1)$, $\boldsymbol{p}^* = (0.3, 0.2, 0.2, 0.2, 0.1)$, $\boldsymbol{q} = (0.25, 0.15, 0.4, 0.02)$, *and* $\boldsymbol{\lambda} = (10, 8, 6, 4, 2)$. *It is then easy to observe that* $\boldsymbol{p} \overset{m}{\succ} \boldsymbol{p}^*$. *Now, let us consider the following situations:*

(i) Suppose $\phi_i(t) = \frac{1-\theta_i}{e^t - \theta_i}$, $\theta_i \in [0, 1)$, $i = 1, 2, \cdots, 5$, *and* $\boldsymbol{\theta} = (1/2, 1/3, 1/5, 1/7, 1/9)$. *Then, the expressions of* $F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}(t)$ *and* $F_{H_{M_n^m}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}(t)$, *for* $t \geq 0$, *are given by*

$$F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{5} \frac{p_i(1-\theta_i)}{\left[(1-\theta_i)e^{\lambda_i t} + \theta_i\right]^{a_j} - \theta_i}\right),$$

$$F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{5} \frac{p_i^*(1-\theta_i)}{\left[(1-\theta_i)e^{\lambda_i t} + \theta_i\right]^{a_j} - \theta_i}\right);$$

(ii) Suppose $\phi_i(t) = e^{1-(1+t)^{\beta_i}}$, $\beta_i \in [1, \infty)$, $i = 1, 2, \cdots, 5$, *and* $\boldsymbol{\beta} = (1, 2, 3, 4, 5)$. *Then, the expressions of* $F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}(t)$ *and* $F_{H_{M_n^m}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}(t)$, *for* $t \geq 0$, *are given by*

$$F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{5} p_i \exp\left[1 - \left\{1 + a_j\left[(1+\lambda_i t)^{1/\beta_i} - 1\right]\right\}^{\beta_i}\right]\right),$$

$$F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{5} p_i^* \exp\left[1 - \left\{1 + a_j\left[(1+\lambda_i t)^{1/\beta_i} - 1\right]\right\}^{\beta_i}\right]\right);$$

(iii) Suppose $\phi_i(t) = (\mu_i t + 1)^{-1/\mu_i}$, $\mu_i \in [0, \infty)$, $i = 1, 2, \cdots, 5$, *and* $\boldsymbol{\mu} = (1, 2, 3, 4, 5)$. *Then, the expressions of* $F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}(t)$ *and* $F_{H_{M_n^m}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}(t)$, *for* $t \geq 0$, *are given by*

$$F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{5} p_i \left[1 + a_j(e^{\lambda_i \mu_i t} - 1)\right]^{-1/\mu_i}\right),$$

$$F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}(t) = \prod_{j=1}^{4}\left(1 - q_j \sum_{i=1}^{5} p_i \left[1 + a_j(e^{\lambda_i \mu_i t} - 1)\right]^{-1/\mu_i}\right).$$

The plots of the distribution functions of parallel-series systems under Ali-Mikhail-Haq, Gumbel-Hougaard and Clayton copulas for the components and the starting device performance probabilities for the subsystems are presented in Figure 2. Because $\boldsymbol{p} \overset{m}{\succeq} \boldsymbol{p}^*$, *based on Theorem 4, we have* $H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{st} H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q},\boldsymbol{X})}$ *for all three cases.*

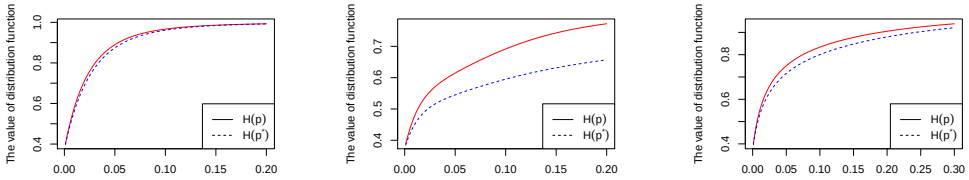

Figure 2. Plots of the distribution functions of parallel-series systems under Ali-Mikhail-Haq, Gumbel-Hougaard and Clayton copulas for the components and the starting device performance probabilities for the subsystems, from left to right.

In Figure 2, we observe once again that the distribution functions of the two parallel-series systems differ considerably under the Gumbel-Hougaard copula than under the other two copulas, with the Ali-Mikhail-Haq copula having the lease difference this time.

In the following theorem, we examine the influence of performance probabilities of starting devices for the subsystems on the system reliability of parallel-series systems in terms of usual stochastic order.

Theorem 5. *Let* $H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ *denote the lifetime of a parallel-series system equipped with starting devices, corresponding to* $M_{n_1}^m(k|\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{X}) \in A$, *with different Archimedean copula generators* ϕ_i, *for* $i = 1, 2, \cdots, m$. *Additionally, let* $H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{X})}$ *denote the lifetime of a parallel-series system equipped with starting devices, corresponding to* $M_n^m(k|\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}^*, \boldsymbol{X}) \in A$, *with the same set of different Archimedean copula generators* ϕ_i, *for* $i = 1, 2, \cdots, m$. *Then, for* $(\boldsymbol{a}, \boldsymbol{q})' \in \mathcal{F}_k$ *and* $(\boldsymbol{a}, \boldsymbol{q}^*)' \in \mathcal{F}_k$, *we have*

$$\boldsymbol{q} \succeq_w \boldsymbol{q}^* \Longrightarrow H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{st} H_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{X})}.$$

Proof. For all $t \geq 0$, we have

$$F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}(t) = \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i \, \phi_i\{a_j \, \psi_i(\bar{F}_i(t))\}\right],$$

$$F_{H_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{X})}(t) = \prod_{j=1}^{k}\left[1 - q_j^* \sum_{i=1}^{m} p_i \, \phi_i\{a_j \, \psi_i(\bar{F}_i(t))\}\right].$$

Now, let us define

$$H(\boldsymbol{q}) = F_{H_{M_{n_1}^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}(t) \quad = \quad \prod_{j=1}^{k}\left[1 - q_j \sum_{i=1}^{m} p_i\, \phi_i\{a_j\, \psi_i(\bar{F}_i(t))\}\right].$$

Then, according to Lemma 2, we need to show that $H(\boldsymbol{q})$ is decreasing and Schur-concave in $(q_1, \cdots, q_k)$, for any fixed $t \geq 0$. Taking the derivative of $H(\boldsymbol{q})$ with respect to q_α, $\alpha \in \{1, 2, \cdots, k\}$, we obtain

$$\begin{aligned} \frac{\partial H(\boldsymbol{q})}{\partial q_\alpha} &= -\frac{\sum_{i=1}^{m} p_i \phi_i\{a_\alpha\, \psi_i(\bar{F}_i(t))\}}{1 - q_\alpha \sum_{i=1}^{m} p_i\, \phi_i\{a_\alpha\, \psi_i(\bar{F}_i(t))\}} \times H(\boldsymbol{q}) \\ &\leq 0. \end{aligned}$$

So,

$$\begin{aligned} J_3(\boldsymbol{q}) &:= (q_\alpha - q_\beta)\left(\frac{\partial H(\boldsymbol{q})}{\partial q_\alpha} - \frac{\partial H(\boldsymbol{q})}{\partial q_\beta}\right) \\ &= (q_\alpha - q_\beta)\, H(\boldsymbol{q}) \\ &\times \left(\frac{\sum_{i=1}^{m} p_i \phi_i\{a_\beta\, \psi_i(\bar{F}_i(t))\}}{1 - q_\beta \sum_{i=1}^{m} p_i\, \phi_i\{a_\beta\, \psi_i(\bar{F}_i(t))\}} - \frac{\sum_{i=1}^{m} p_i \phi_i\{a_\alpha\, \psi_i(\bar{F}_i(t))\}}{1 - q_\alpha \sum_{i=1}^{m} p_i\, \phi_i\{a_\alpha\, \psi_i(\bar{F}_i(t))\}}\right). \end{aligned}$$

The assumption that $(\boldsymbol{a}, \boldsymbol{q})' \in \mathcal{F}_k$ implies $a_\alpha \leq a_\beta$ and $q_\alpha \geq q_\beta$, or $a_\alpha \geq a_\beta$ and $q_\alpha \leq q_\beta$. We present here the proof only for the case when $a_\alpha \leq a_\beta$ and $q_\alpha \geq q_\beta$, since the proof for the other case is quite similar. It is easy to observe that

$$\sum_{i=1}^{m} p_i \phi_i\{a_\alpha\, \psi_i(\bar{F}_i(t))\} \geq \sum_{i=1}^{m} p_i \phi_i\{a_\beta\, \psi_i(\bar{F}_i(t))\} \geq 0$$

and

$$\frac{1}{1 - q_\alpha \sum_{i=1}^{m} p_i\, \phi_i\{a_\alpha\, \psi_i(\bar{F}_i(t))\}} \geq \frac{1}{1 - q_\beta \sum_{i=1}^{m} p_i\, \phi_i\{a_\beta\, \psi_i(\bar{F}_i(t))\}} \geq 0,$$

which imply that

$$J_3(\boldsymbol{q}) \quad := \quad (q_\alpha - q_\beta)\left(\frac{\partial H(\boldsymbol{q})}{\partial q_\alpha} - \frac{\partial H(\boldsymbol{q})}{\partial q_\beta}\right) \leq 0.$$

Hence, the theorem. □

4. Results for Series-Parallel Systems

We now consider a series-parallel system that comprises k independent subsystems connected in series, with the j-th subsystem having a_j components connected in parallel, for $j = 1, 2, \cdots, k$. The components within each subsystem are assumed to come from the i-th subpopulation with probability p_i, $i = 1, 2, \cdots, m$, and also that each subsystem is equipped with a starter device whose performance is modelled by a Bernoulli random variable, I_{q_j}, with $\mathbb{E}[I_{q_j}] = q_j$, for $j = 1, 2, \cdots, k$. Next, we assume that the components within each subsystem are dependent, but the subsystems themselves are independent. We also assume that all the subpopulations have different Archimedean copulas with generators ϕ_i, $i = 1, 2, \cdots, m$. Then, with $M_n^m(k|\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{X})$ denoting the lifetime of such a series-parallel system, its reliability function is given by

$$\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}(t) = \prod_{j=1}^{k} q_j \left[1 - \sum_{i=1}^{m} p_i \phi_i\{a_j\, \psi_i(F_i(t))\}\right], \qquad t \geq 0. \tag{5}$$

It is then of interest to determine how the n components from m different subpopulations could be distributed so that the resulting series-parallel system, with independent subsystems equipped with starting devices, will be optimal in some stochastic sense. We first take the number of independent subsystems within the series-parallel system to be fixed as k. We then show in the following theorem that if the allocation vector for any model with dependent components is majorized by another allocation vector, then the series-parallel system corresponding to the second vector will be more reliable than the system corresponding to the first vector in the sense of usual stochastic order.

Theorem 6. *Let $S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ denote the lifetime of a series-parallel system equipped with starting devices, corresponding to $M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X}) \in A$, with different Archimedean copula generators ϕ_i, for $i = 1, 2, \cdots, m$. Additionally, let $S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{X})}$ denote the lifetime of a series-parallel system equipped with starting devices, corresponding to $M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{X}) \in A$, with the same set of different Archimedean copula generators ϕ_i, for $i = 1, 2, \cdots, m$. Further, let $\prod_{j=1}^k q_j^* \geq \prod_{j=1}^k q_j$. Then, we have*

$$\boldsymbol{a} \overset{w}{\succ} \boldsymbol{b} \Longrightarrow S^m_{n_1}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{X}) \geq_{st} S^m_{n_2}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X}).$$

Proof. The reliability functions of series-parallel systems $S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ and $S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ are given by

$$\begin{aligned}
\bar{F}_{S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) &= \prod_{j=1}^{k} q_j \left[1 - \sum_{i=1}^{m} p_i \phi_i\{a_j\, \psi_i(F_i(t))\}\right] \\
&= \Big(\prod_{j=1}^{k} q_j\Big) \prod_{j=1}^{k} \left[1 - \sum_{i=1}^{m} p_i \phi_i\{a_j\, \psi_i(F_i(t))\}\right], \qquad t \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
\bar{F}_{S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) &= \prod_{j=1}^{k} q_j^* \left[1 - \sum_{i=1}^{m} p_i \phi_i\{b_j\, \psi_i(F_i(t))\}\right] \\
&= \Big(\prod_{j=1}^{k} q_j^*\Big) \prod_{j=1}^{k} \left[1 - \sum_{i=1}^{m} p_i \phi_i\{b_j\, \psi_i(F_i(t))\}\right], \qquad t \geq 0,
\end{aligned}$$

respectively. Now, in view of Theorem 3.1 of Fang et al. [1], the required result follows immediately from the assumption that $\prod_{j=1}^k q_j^* \geq \prod_{j=1}^k q_j$. □

Example 3. *Let us consider $F_i(t) = 1 - e^{-\lambda_i t}$, $k = 3$, $m = 4$. Let us further choose $\boldsymbol{a} = (6, 8, 10)$, $\boldsymbol{b} = (10, 8, 17)$, $\boldsymbol{p} = (0.35, 0.25, 0.15, 0.25)$, $\boldsymbol{q} = (0.2, 0.6, 0.2)$, $\boldsymbol{q}^* = (0.6, 0.5, 0.1)$, and $\boldsymbol{\lambda} = (1, 2, 3, 4)$. Now, let us consider the following situations:*

(i) Suppose $\phi_i(t) = \frac{1-\theta_i}{e^t - \theta_i}$, $\theta_i \in [0, 1)$, $i = 1, 2, \cdots, 4$, and $\boldsymbol{\theta} = (1/2, 1/3, 1/5, 1/7)$. Then, the expressions of $\bar{F}_{S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}}(t)$ and $\bar{F}_{S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{\theta},\boldsymbol{X})}}(t)$ are given by

$$\bar{F}_{S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}}(t) = \prod_{j=1}^{3} q_j \left(1 - \sum_{i=1}^{4} \frac{p_i(1-\theta_i)}{\left[\frac{1-\theta_i e^{-\lambda_i t}}{1-e^{-\lambda_i t}}\right]^{a_j} - \theta_i}\right), \qquad t \geq 0,$$

$$\bar{F}_{S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{\theta},\boldsymbol{X})}}(t) = \prod_{j=1}^{3} q_j^* \left(1 - \sum_{i=1}^{4} \frac{p_i(1-\theta_i)}{\left[\frac{1-\theta_i e^{-\lambda_i t}}{1-e^{-\lambda_i t}}\right]^{b_j} - \theta_i}\right), \qquad t \geq 0;$$

(ii) *Suppose* $\phi_i(t) = e^{1-(1+t)^{\beta_i}}$, $\beta_i \in [1,\infty)$, $i = 1,2,\cdots,4$, *and* $\boldsymbol{\beta} = (1,2,3,4)$. *Then, the expressions of* $\bar{F}_{S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}}(t)$ *and* $\bar{F}_{S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{\beta},\boldsymbol{X})}}(t)$ *are given by*

$$\bar{F}_{S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}}(t) = \prod_{j=1}^{3} q_j \left(1 - \sum_{i=1}^{4} p_i \exp\left[1 - \left\{1 + a_j\left[\left\{1 + \log(1 - e^{-\lambda_i t})\right\}^{1/\beta_i} - 1\right]\right\}^{\beta_i}\right]\right), \quad t \geq 0,$$

$$\bar{F}_{S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{\beta},\boldsymbol{X})}}(t) = \prod_{j=1}^{3} q_j^* \left(1 - \sum_{i=1}^{4} p_i \exp\left[1 - \left\{1 + b_j\left[\left\{1 + \log(1 - e^{-\lambda_i t})\right\}^{1/\beta_i} - 1\right]\right\}^{\beta_i}\right]\right), \quad t \geq 0;$$

(iii) *Suppose* $\phi_i(t) = (\mu_i t + 1)^{-1/\mu_i}$, $\mu_i \in [0,\infty)$, $i = 1,2,\cdots,4$, *and* $\boldsymbol{\mu} = (1,2,3,4)$. *Then, the expressions of* $\bar{F}_{S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}}(t)$ *and* $\bar{F}_{S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{\mu},\boldsymbol{X})}}(t)$, *for* $t \geq 0$, *are given by*

$$\bar{F}_{S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}}(t) = \prod_{j=1}^{3} q_j \left(1 - \sum_{i=1}^{4} p_i \left[1 + a_j\left\{(1 - e^{-\lambda_i t})^{\mu_i} - 1\right\}\right]^{-1/\mu_i}\right),$$

$$\bar{F}_{S_{M^m_{n_2}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{\mu},\boldsymbol{X})}}(t) = \prod_{j=1}^{3} q_j^* \left(1 - \sum_{i=1}^{4} p_i \left[1 + b_j\left\{(1 - e^{-\lambda_i t})^{\mu_i} - 1\right\}\right]^{-1/\mu_i}\right).$$

The plots of the reliability functions of series-parallel systems under the Ali-Mikhail-Haq, Gumbel-Hougaard and Clayton copulas for the components and the starting device performance probabilities for the subsystems are presented in Figure 3. Because $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$, *with the choices of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *made, based on Theorem 6, we have* $S^m_{n_1}(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}^*,\boldsymbol{X}) \geq_{st} S^m_{n_2}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})$, *for all three cases.*

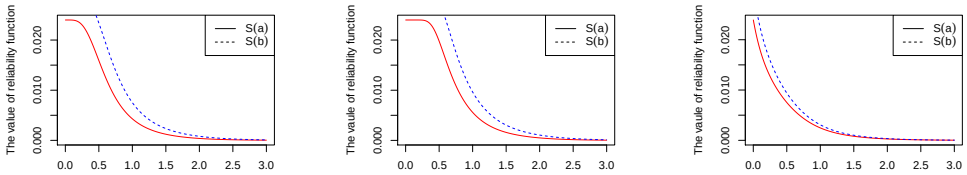

Figure 3. Plots of the reliability functions of series-parallel systems under the Ali-Mikhail-Haq, Gumbel-Hougaard and Clayton copulas for the components and the starting device performance probabilities for the subsystems, from left to right.

In Figure 3, we observe that the reliability functions of the two series-parallel systems differ considerably under the Gumbel-Hougaard and Ali-Mikhail-Haq copulas, and the Clayton copula has the least difference.

The following theorem shows that for a series-parallel system, under certain conditions, if the number of subsystems equipped with starting devices increases, then the system becomes more reliable.

Theorem 7. *Let* $S_{M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ *denote the lifetime of a series-parallel system equipped with starting devices, corresponding to* $M^m_{n_1}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X}) \in A$, *with different Archimedean copula generators* ϕ_i, *for* $i = 1,2,\cdots,m$. *Additionally, let* $S_{M^m_{n_2}(k-1|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ *denote the lifetime of a series-parallel system equipped with starting devices, corresponding to* $M^m_{n_2}(k-1|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X}) \in A$,

with the same set of different Archimedean copula generators ϕ_i, for $i = 1, 2, \cdots, m$. Let $\boldsymbol{a} = (a_1, \cdots, a_k)$, $\boldsymbol{b} = (b_1, \cdots, b_{k-1})$, $\boldsymbol{q} = (q_1, \cdots, q_k)$, and $\boldsymbol{q}_{-k} = (q_1, \cdots, q_{k-1})$. Then, we have

$$a_j \leq b_j \text{ for all } j = 1, 2, \cdots, k-1 \Longrightarrow S_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{st} S_{M_{n_2}^m(k-1|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}_{-k'},\boldsymbol{X})}.$$

Proof. From (5), for all $t \geq 0$, we have

$$\begin{aligned}
\bar{F}_{S_{M_{n_1}^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}}(t) &= \prod_{j=1}^{k} q_j \left[1 - \sum_{i=1}^{m} p_i \phi_i\{a_j\, \psi_i(F_i(t))\}\right] \\
&\leq \prod_{j=1}^{k-1} q_j \left[1 - \sum_{i=1}^{m} p_i \phi_i\{a_j\, \psi_i(F_i(t))\}\right] \\
&\leq \prod_{j=1}^{k-1} q_j \left[1 - \sum_{i=1}^{m} p_i \phi_i\{b_j\, \psi_i(F_i(t))\}\right] \\
&\leq \bar{F}_{S_{M_{n_2}^m(k|\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}_{-k'},\boldsymbol{X})}}(t),
\end{aligned}$$

as required. □

Let $M_n^m(1|n, \boldsymbol{p}, q, \boldsymbol{X})$ denote a series-parallel system having one subsystem, with starting device performance probability q, with n components connected in series and $M_n^m(n|\boldsymbol{1}, \boldsymbol{p}, \boldsymbol{q}^+, \boldsymbol{X})$ denoting a series-parallel system having n subsystems, with starting device performance probabilities $q_1, \cdots, q_n$, connected in series with each subsystem having only one component. Then, the following theorem shows that $M_n^m(n|\boldsymbol{1}, \boldsymbol{p}, \boldsymbol{q}^+, \boldsymbol{X})$ is an optimal system and $M_n^m(1|n, \boldsymbol{p}, q, \boldsymbol{X})$ is the worst system among all possible such series-parallel systems in the sense of hazard rate order.

Theorem 8. *Let $S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ denote the lifetime of a series-parallel system equipped with starting devices, corresponding to $M_n^m(k|\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{X}) \in A$, with different Archimedean copula generators ϕ_i, for $i = 1, 2, \cdots, m$. Then, we have*

$$S_{M_n^m(1|n,\boldsymbol{p},q,\boldsymbol{X})} \geq_{hr} S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{hr} S_{M_n^m\left(n|\boldsymbol{1},\boldsymbol{p},\boldsymbol{q}^+,\boldsymbol{X}\right)}.$$

Proof. It is easy to observe that the hazard rate function of $S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ does not depend on q_j's. Hence, the desired result is obtained readily from Theorem 3.3 of Fang et al. [1]. □

In the following theorem, we examine the influence of selection probabilities and the distributions of subpopulations on the system reliability of series-parallel systems equipped with starting devices for subsystems in terms of usual stochastic order.

Theorem 9. *Let $S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}$ denote the lifetime of a series-parallel system equipped with starting devices, corresponding to $M_n^m(k|\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{X}) \in A$, with the same Archimedean copula generator ϕ, for $i = 1, 2, \cdots, m$. Additionally, let $S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{X})}$ denote the lifetime of a series-parallel system equipped with starting devices, corresponding to $M_n^m(k|\boldsymbol{a}, \boldsymbol{p}^*, \boldsymbol{q}^*, \boldsymbol{X}) \in A$, with the same Archimedean copula generator ϕ, for $i = 1, 2, \cdots, m$. Further, suppose $(\boldsymbol{p}, \bar{\boldsymbol{F}})' \in \mathcal{F}_m$ $((\boldsymbol{p}, \bar{\boldsymbol{F}})' \in \mathcal{E}_m)$. Then, we have*

$$\prod_{j=1}^{k} q_j \geq (\leq) \prod_{j=1}^{k} q_j^* \quad \text{and} \quad \boldsymbol{p} \overset{m}{\succeq} \boldsymbol{p}^* \Longrightarrow S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{st} (\leq_{st}) S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{X})}.$$

Proof. The reliability functions of series-parallel systems $S_{M_{n_1}^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})$ and $S_{M_{n_2}^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{X})$ are given by

$$\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})}(t) = S(\boldsymbol{p}) = (\prod_{j=1}^{k} q_j) \prod_{j=1}^{k}\left[1-\sum_{i=1}^{m} p_i\phi_i\{a_j\,\psi_i(F_i(t))\}\right], \qquad t \geq 0,$$

and

$$\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{X})}(t) = S(\boldsymbol{p}^*) = (\prod_{j=1}^{k} q_j^*) \prod_{j=1}^{k}\left[1-\sum_{i=1}^{m} p_i^*\phi_i\{a_j\,\psi_i(F_i(t))\}\right], \qquad t \geq 0,$$

respectively. Now, in view of Theorem 3.4 of Fang et al. [1], the required result follows immediately from the assumption that $\prod_{j=1}^{k} q_j \geq (\leq) \prod_{j=1}^{k} q_j^*$. □

Example 4. *Let us consider* $F_i(t) = 1 - e^{-\lambda_i t}$, $k = 4$ *and* $m = 5$. *Further, let* $\boldsymbol{a} = (5, 10, 20, 8)$, $\boldsymbol{\lambda} = (1, 2, 3, 4, 5)$, $\boldsymbol{p} = (0.4, 0.3, 0.1, 0.1, 0.1)$, $\boldsymbol{p}^* = (0.3, 0.2, 0.2, 0.2, 0.1)$, $\boldsymbol{q} = (0.5, 0.1, 0.2, 0.4)$, *and* $\boldsymbol{q}^* = (0.4, 0.1, 0.2, 0.3)$. *Let us now consider the following situations:*

(i) Suppose $\phi_i(t) = \frac{1-\theta_i}{e^t-\theta_i}$, $\theta_i \in [0, 1)$, $i = 1, 2, \cdots, 5$, *and* $\boldsymbol{\theta} = (1/3, 1/2, 1/7, 1/9, 1/4)$. *Then, the expressions of* $\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}(t)$ *and* $\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{\theta},\boldsymbol{X})}(t)$ *are given by*

$$\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{X})}(t) = \prod_{j=1}^{4} q_j \left(1 - \sum_{i=1}^{5} \frac{p_i(1-\theta_i)}{\left[\frac{1-\theta_i e^{-\lambda_i t}}{1-e^{-\lambda_i t}}\right]^{a_j} - \theta_i}\right), \qquad t \geq 0,$$

$$\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{\theta},\boldsymbol{X})}(t) = \prod_{j=1}^{4} q_j^* \left(1 - \sum_{i=1}^{5} \frac{p_i^*(1-\theta_i)}{\left[\frac{1-\theta_i e^{-\lambda_i t}}{1-e^{-\lambda_i t}}\right]^{a_j} - \theta_i}\right), \qquad t \geq 0;$$

(ii) Suppose $\phi_i(t) = e^{1-(1+t)^{\beta_i}}$, $\beta_i \in [1, \infty)$, $i = 1, 2, \cdots, 5$, *and* $\boldsymbol{\beta} = (1, 2, 3, 4, 5)$. *Then, the expressions of* $\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}(t)$ *and* $\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{\beta},\boldsymbol{X})}(t)$ *are given by*

$$\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\beta},\boldsymbol{X})}(t) = \prod_{j=1}^{4} q_j \left(1 - \sum_{i=1}^{5} p_i \exp\left[1 - \left\{1 + a_j\left[\left\{1 + \log(1-e^{-\lambda_i t})\right\}^{1/\beta_i} - 1\right]\right\}^{\beta_i}\right]\right), \quad t \geq 0,$$

$$\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{\beta},\boldsymbol{X})}(t) = \prod_{j=1}^{4} q_j^* \left(1 - \sum_{i=1}^{5} p_i^* \exp\left[1 - \left\{1 + a_j\left[\left\{1 + \log(1-e^{-\lambda_i t})\right\}^{1/\beta_i} - 1\right]\right\}^{\beta_i}\right]\right), \quad t \geq 0;$$

(iii) Suppose $\phi_i(t) = (\mu_i t + 1)^{-1/\mu_i}$, $\mu_i \in [0, \infty)$, $i = 1, 2, \cdots, 5$, *and* $\boldsymbol{\mu} = (1, 2, 3, 4, 5)$. *Then, the expressions of* $\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}(t)$ *and* $\bar{F}_{S_{M_n^m}(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{\mu},\boldsymbol{X})}(t)$, *for* $t \geq 0$, *are given by*

$$\bar{F}_{S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{\mu},\boldsymbol{X})}}(t) = \prod_{j=1}^{4} q_j \left(1 - \sum_{i=1}^{5} p_i\left[1 + a_j\left\{(1-e^{-\lambda_i t})^{\mu_i} - 1\right\}\right]^{-1/\mu_i}\right),$$

$$\bar{F}_{S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{\mu},\boldsymbol{X})}}(t) = \prod_{j=1}^{4} q_j^* \left(1 - \sum_{i=1}^{5} p_i^* \left[1 + a_j \left\{(1 - e^{-\lambda_i t})^{\mu_i} - 1\right\}\right]^{-1/\mu_i}\right).$$

The plots of the reliability functions of these series-parallel systems are presented in Figure 4. Because $\boldsymbol{p} \overset{m}{\succeq} \boldsymbol{q}$*, based on Theorem 9, we have* $S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p},\boldsymbol{q},\boldsymbol{X})} \geq_{st} S_{M_n^m(k|\boldsymbol{a},\boldsymbol{p}^*,\boldsymbol{q}^*,\boldsymbol{X})}$ *for all three cases.*

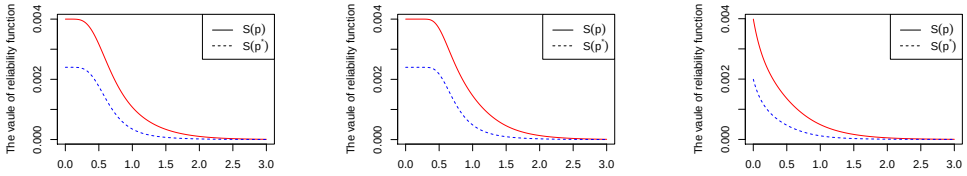

Figure 4. Plots of the reliability functions of series-parallel systems under the Ali-Mikhail-Haq, Gumbel-Hougaard and Clayton copulas for the components and the starting device performance probabilities for the subsystems, from left to right.

From Figure 4, we observe in this case that the reliability functions of the two series-parallel systems are considerably different under all three copulas.

5. Concluding Remarks

Many reliability systems are assembled with parts or components coming from many different suppliers from all over the world. The reliability properties of the components received would therefore vary; however, still acquiring then from different suppliers and with varying reliability characteristics becomes inevitable due to supply and demand situations.

In this paper, we have examined the reliability of parallel-series and series-parallel systems with starting devices. We have investigated three different scenarios: first, by fixing the number of subsystems and studying the relationships between allocation vectors; second, by varying the number of subsystems; and finally by changing the selection probabilities or the distributions of subpopulations. To compare the reliability characteristics of various systems that can be formed in these situations, we have utilized the theory of stochastic orders and majorization orders. Although our findings are applicable to the general family of Archimedean copulas, we have focused on specific examples such as the Ali-Mikhail-Haq, Gumbel-Hougaard, and Clayton copulas to illustrate our results.

It should be mentioned that this work is more theoretical. One of the shortcomings/limitations/restrictions of the present work is that there is no set of real data on parallel-series and series-parallel systems from reliability experiments in the literature to validate the results.

There are some natural generalizations of this work to consider as future work. We have assumed dependence between the components within the subsystems, but the subsystems themselves to be independent. It would be of interest to consider the more general situation in which the subsystems are also dependent, possibly being modelled by another Archimedean copula. Another possible extension is to consider general, e.g., *k*-out-of-*n* systems, instead of series or parallel structures, for the subsystems. We are currently working on these problems and hope report the findings in a future paper.

Author Contributions: Conceptualization, G.S.K., A.H. and M.S.; Methodology, N.B., G.S.K., A.H. and M.S.; Software, G.S.K.; Validation, G.S.K., A.H. and M.S.; Formal analysis, N.B., G.S.K., A.H. and M.S.; Investigation, G.S.K., A.H. and M.S.; Resources, G.S.K.; Data curation, Not applicable;Writing—Original draft preparation, G.S.K.; Review and editing, N.B.; Visualization, G.S.K., A.H. and M.S.; Supervision, N.B.; Project administration, N.B. and G.S.K.; Funding acquisition, N.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data sharing not applicable.

Acknowledgments: The authors express their sincere thanks to the Editor and the anonymous reviewers for their many useful suggestions on an earlier version of this manuscript which led to this improved version.

Conflicts of Interest: The authors declare no conflict of interest.

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