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Singular Surfaces of Osculating Circles in Three-Dimensional Euclidean Space

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Abstract: In this paper, we study the surfaces of osculating circles, which are the sets of all osculating circles at all points of regular curves. Since the surfaces of osculating circles may be singular, it is necessary to investigate the singular points of these surfaces. However, traditional methods and tools for analyzing singular properties have certain limitations. To solve this problem, we define the framed surfaces of osculating circles in the Euclidean 3-space. Then, we discuss the types of singular points using the theory of framed surfaces and show that generic singular points of the surfaces consist of cuspidal edges and cuspidal cross-caps.

Keywords: surfaces of osculating circles; framed surfaces; singularities

MSC: 53A05; 57R45

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1. Introduction

Surfaces in the Euclidean 3-space constructed or defined by curves are classical subjects in differential geometry [1]. Canal surfaces, ruled surfaces, surfaces of revolution and translation surfaces are all attractive objects applied extensively in geometric modeling and engineering. People tend to focus on the regular part and pay little attention to the singular points. However, singular points are essential in real life, such as the edges of some objects. In the cross-subject method, the singularity theory of surfaces has a wide range of applications such as physical optics, computer-aided geometric design and kinematics [2,3]. Therefore, it is necessary to explore the singular properties of surfaces.

The classification of singularities has a vital position in the singularity theory [4]. It has become an interesting area for many geometers. In 1985, Mond carried out a detailed study on the singularity classification of the mapping from the Euclidean plane to the Euclidean 3-space [5]. After Mond's classification, many geometers made great contributions to the singularity theory of surfaces in different spaces [6–8]. The decision theorem of singular points and geometric properties of fronts were investigated via flat surfaces in hyperbolic 3-space [9] and maximal surfaces in Lorentz-Minkowski 3-space [10]. For the classification of singular points of surfaces, it is common to use the unfolding theory of functions. Subsequently, Fukunaga and Takahashi explored another method to study singular surfaces [11]. They defined a smooth surface with a moving frame as the framed surface and gave criteria for singular points of the framed surface. Using this powerful tool, some researchers have studied the singular properties of different singular surfaces in recent years [12–15].

In this paper, we investigate a new class of surfaces called the surfaces of osculating circles, which are the sets of osculating circles at all points of regular curves. In [16], the authors studied the geometric properties of regular surfaces of osculating circles and gave a classification of these surfaces under some conditions on their curvature. However, surfaces of osculating circles may be singular. If singular points exist on these surfaces, we cannot define the normal vector fields at singular points. Therefore, we define the framed surfaces of osculating circles and obtain the geometric features of singular points. Then, we analyze the types of singular points that appear on the surfaces of osculating circles.

As far as we know, no one has ever considered the singular part of the surfaces of osculating circles. Based on this motivation, we analyze the singular properties of these surfaces to differentiate from previous research and provide a wider perspective for future work. The brief organization of the present paper is as follows. We review the notion of the surfaces of osculating circles in the Euclidean 3-space and discuss singular points of these surfaces in Section 3. In Section 4, we define the framed surfaces of osculating circles and calculate the basic invariants and curvature. We also present the main classification theorem of singular points of this paper (Theorem 1) in this section. Finally, we show several singular surfaces of osculating circles as examples in Section 5, which have cuspidal edges and cuspidal cross-caps.

All maps and manifolds we consider here are differentiable of class C^∞.

2. Preliminaries

Let $\mathbb{R}^3$ be the Euclidean 3-space with the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$, where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$. We define the unit sphere in $\mathbb{R}^3$ by $S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 \mid \|\boldsymbol{a}\| = 1\}$, where $\|\boldsymbol{a}\|$ is the norm of $\boldsymbol{a}$. We denote $\Delta = \{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 \mid \boldsymbol{a} \cdot \boldsymbol{b} = 0\}$. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a curve, and the arc–length is $s(t) = \int_{t_0}^{t} \|\boldsymbol{\gamma}'(v)\| dv$. The tangent vector with respect to s is $\boldsymbol{\gamma}'(s) = d\boldsymbol{\gamma}(s)/ds$ and $\|\boldsymbol{\gamma}'(s)\| = \|d\boldsymbol{\gamma}(s)/ds\| = 1$. We define three unit vectors as $\boldsymbol{T}(s) = \boldsymbol{\gamma}'(s)$, $\boldsymbol{N}(s) = \boldsymbol{\gamma}''(s)/\|\boldsymbol{\gamma}''(s)\|$ and $\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s)$, where "$\times$" denotes the vector product of two vectors. Then, the Frenet-Serret formula is as follows:

$$\{\boldsymbol{T}'(s), \boldsymbol{N}'(s), \boldsymbol{B}'(s)\} = \{\kappa(s)\boldsymbol{N}(s), -\kappa(s)\boldsymbol{T}(s) + \tau(s)\boldsymbol{B}(s), -\tau(s)\boldsymbol{N}(s)\},$$

where $\kappa(s)$ is the curvature function and $\tau(s)$ is the torsion function.

We review the theory of framed surfaces (cf. [11]). The framed surface is a great generalization of regular surfaces and frontals, at least locally.

Definition 1 ([11])**.** *We say $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$: $U \to \mathbb{R}^3 \times \Delta$ is a framed surface if $\boldsymbol{\psi}_s(s,u) \cdot \boldsymbol{\psi}_1(s,u) = 0$ and $\boldsymbol{\psi}_u(s,u) \cdot \boldsymbol{\psi}_1(s,u) = 0$ for all $(s,u) \in U$, where $\boldsymbol{\psi}_s(s,u) = (\partial\boldsymbol{\psi}/\partial s)(s,u)$ and $\boldsymbol{\psi}_u(s,u) = (\partial\boldsymbol{\psi}/\partial u)(s,u)$. We say $\boldsymbol{\psi} : U \to \mathbb{R}^3$ is a framed base surface if there exists $(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$: $U \to \Delta$ such that $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ is a framed surface.*

Definition 2 ([11])**.** *We define $(\boldsymbol{\psi}, \boldsymbol{\psi}_1)$: $U \to \mathbb{R}^3 \times S^2$ is a Legendre surface if $\boldsymbol{\psi}_s(s,u) \cdot \boldsymbol{\psi}_1(s,u) = 0$ and $\boldsymbol{\psi}_u(s,u) \cdot \boldsymbol{\psi}_1(s,u) = 0$ for all $(s,u) \in U$. A Legendre surface $(\boldsymbol{\psi}, \boldsymbol{\psi}_1)$ is called a Legendre immersion if $(\boldsymbol{\psi}, \boldsymbol{\psi}_1)$ is an immersion. We define $\boldsymbol{\psi} : U \to \mathbb{R}^3$ as a frontal (respectively, a front) if there exists $\boldsymbol{\psi}_1 : U \to S^2$ such that $(\boldsymbol{\psi}, \boldsymbol{\psi}_1)$ is a Legendre surface (respectively, a Legendre immersion).*

We denote $\boldsymbol{\psi}_3(s,u) = \boldsymbol{\psi}_1(s,u) \times \boldsymbol{\psi}_2(s,u)$; then, we can construct the moving frame along $\boldsymbol{\psi}(s,u)$ as $\{\boldsymbol{\psi}_1(s,u), \boldsymbol{\psi}_2(s,u), \boldsymbol{\psi}_3(s,u)\}$. Thus, we have:

$$\begin{pmatrix} \boldsymbol{\psi}_s \\ \boldsymbol{\psi}_u \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_2 \\ \boldsymbol{\psi}_3 \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} \boldsymbol{\psi}_{1s} \\ \boldsymbol{\psi}_{2s} \\ \boldsymbol{\psi}_{3s} \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\psi}_3 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\psi}_{1u} \\ \boldsymbol{\psi}_{2u} \\ \boldsymbol{\psi}_{3u} \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \boldsymbol{\psi}_3 \end{pmatrix}, \tag{2}$$

where a_i, b_i, e_i, f_i, g_i $(i = 1, 2)$ are called *basic invariants* of $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$. By the integrability conditions, we have $a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1$ [11]. We define the curvature $C_F = (J_F, K_F, H_F)$ of a framed surface $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ by

$$
\begin{aligned}
J_F &= \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \ K_F = \det\begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix}, \\
H_F &= -\frac{1}{2}\left(\det\begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} - \det\begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}\right).
\end{aligned}
$$

According to the above definition, p is a regular point of $\boldsymbol{\psi}$ if and only if $J^F(p) \neq 0$. If $\boldsymbol{\psi} : U \to \mathbb{R}^3$ is a regular surface, the first and second fundamental invariants are given by

$$
\begin{aligned}
&E = a_1^2 + b_1^2, \ F = a_1 b_1 + a_2 b_2, \ G = a_2^2 + b_2^2, \\
&L = -a_1 e_1 - b_1 f_1, \ M = -a_1 e_2 + b_1 f_2, \ N = -a_2 e_2 - b_2 f_2.
\end{aligned} \tag{3}
$$

The Gauss curvature and mean curvature of the framed surface $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ are expressed as

$$
K = \frac{K_F}{J_F}, H = \frac{H_F}{J_F}.
$$

We use the following definition and proposition in our paper.

Definition 3 ([6])**.** *A non-degenerate singular point p is the k-th kind if $\boldsymbol{\eta}^{(i-1)}\lambda(p) = 0$ for all $i \in \{1, \ldots, k\}$ and $\boldsymbol{\eta}^{(k)}\lambda(p) \neq 0$, where $\boldsymbol{\eta}^{(i)}$ denotes the i-th order directional derivative by $\boldsymbol{\eta}$.*

Proposition 1 ([11])**.** *Assume that $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface and $p \in U$. Then $(\boldsymbol{\psi}, \boldsymbol{\psi}_1)$ is a Legendre immersion around p if and only if $C^F(p) \neq 0$.*

For more details, see [7,11].

3. Singularities of the Surface of Osculating Circles

In this section, we retrospect the definition of the surface of osculating circles and give the sets of singular points of the surface [16]. Moreover, we show the characters of singular points of this surface.

Definition 4 ([16])**.** *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a curve parametrized by arc–length. Suppose that the radius of curvature is $r(s) = 1/\kappa(s)$, where $\kappa(s)$ is a non–zero function. The surface of osculating circles generated by $\boldsymbol{\gamma}$ is defined as the parametrized surface $\boldsymbol{\psi} : I \times \mathbb{R} \to \mathbb{R}^3$ given by*

$$
\boldsymbol{\psi}(s, u) = \boldsymbol{\gamma}(s) + r(s)(\sin u \, \boldsymbol{T}(s) + (1 - \cos u)\boldsymbol{N}(s)). \tag{4}
$$

This surface is denoted by $\mathcal{O}(\boldsymbol{\gamma})$.

In the present paper, we assume the curvature functions of all curves do not vanish in their domains.

The set of singular points of $\mathcal{O}(\boldsymbol{\gamma})$ is given in [16], where $u \in \mathbb{R}$. The following proposition indicates characteristics of the singular points, where $u \in [0, 2\pi)$.

Proposition 2 ([16])**.** *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3$ be a curve and $\boldsymbol{\psi} = \boldsymbol{\psi}(s, u)$ be the parametrization of $\mathcal{O}(\boldsymbol{\gamma})$. Then, the sets of singular points of $\mathcal{O}(\boldsymbol{\gamma})$ are given by*

$$
S_1 = \{(s, u) \in I \times U | \ u = 0\}, \ S_2 = \left\{(s, u) \in I \times U | \ r'(s) = 0, \tau(s) = 0, u \in (0, 2\pi)\right\}.
$$

According to the above proposition, we can easily know that the points of S_1 are located on the generating curve $\boldsymbol{\gamma}$. The characteristics of the singular points of $\mathcal{O}(\boldsymbol{\gamma})$ are as follows.

For a map germ $\boldsymbol{\psi} : (U, p) \to \mathbb{R}^3$, the point p is a cross-cap if $\boldsymbol{\psi}$ at p is $\mathcal{A}$-equivalent to the map germ $(s, u) \mapsto (s, su, u^3)$. For more details, see [17].

Proposition 3. *Let $\boldsymbol{\psi} : I \times \mathbb{R} \to \mathbb{R}^3$ be a surface of osculating circles generated by $\gamma(s)$. Then, cross-cap singular points do not appear on $\mathcal{O}(\gamma)$.*

Proof. Taking the partial derivative of $\boldsymbol{\psi}(s,u)$, we have

$$\begin{aligned}\frac{\partial \boldsymbol{\psi}(s,u)}{\partial s} &= (r'(s)\sin u + \cos u)\boldsymbol{T}(s) + (r'(s)(1-\cos u) + \sin u)\boldsymbol{N}(s)\\ &\quad + r(s)\tau(s)(1-\cos u)\boldsymbol{B}(s),\\ \frac{\partial \boldsymbol{\psi}(s,u)}{\partial u} &= r(s)(\cos u\boldsymbol{T}(s) + \sin u\boldsymbol{N}(s)).\end{aligned}$$

If $p = (s_0, u_0)$ is a singular point of $\boldsymbol{\psi}(s,u)$, then $\boldsymbol{\psi}_u(s_0,u_0) = r(s_0)\boldsymbol{\psi}_s(s_0,u_0) \neq 0$. In [18], Whitney proved if there exists a local coordinate system (s,u) centered at p such that

$$\frac{\partial \boldsymbol{\psi}}{\partial u}(s_0,u_0) = 0,\ \det\left(\frac{\partial \boldsymbol{\psi}}{\partial s}(s_0,u_0), \frac{\partial^2 \boldsymbol{\psi}}{\partial s \partial u}(s_0,u_0), \frac{\partial^2 \boldsymbol{\psi}}{\partial u^2}(s_0,u_0)\right) \neq 0,$$

then the type of singular point p is a cross-cap. Therefore, cross-cap singular points do not appear on $\mathcal{O}(\gamma)$. □

4. Surfaces of Osculating Circles as Framed Base Surfaces

In this section, we define a framed surface of $\mathcal{O}(\gamma)$. Then, we investigate the singular points of this surface using the criterion for a framed surface. For generic singular points of a frontal from $\mathbb{R}^2$ to $\mathbb{R}^3$, we show that $\mathcal{O}(\gamma)$ contains cuspidal edges and cuspidal cross-caps whose normal forms are (s,u^2,u^3) and (s,u^2,su^3). Cuspidal lips, cuspidal beaks, and Chen-Matsumoto-Mond singular points with normal forms $(3s^4+2s^2u^2, s^3+su^2, u)$, $(3s^4-2s^2u^2, s^3-su^2, u)$ and $(s,u^2,u^3(s^2 \pm u^2))$, respectively, do not appear on the surface of osculating circles.

Definition 5. *Let $\gamma(s)$ be a regular curve and $\boldsymbol{\psi}(s,u)$ be a surface of osculating circles generated by $\gamma(s)$. If there exists a smooth function $\iota : I \to \mathbb{R}^3$ such that $\langle \boldsymbol{\psi}_s(s,u), \boldsymbol{\psi}_1(s,u)\rangle = 0$ and $\langle \boldsymbol{\psi}_u(s,u), \boldsymbol{\psi}_1(s,u)\rangle = 0$, where $\boldsymbol{\psi}_1(s,u) = \cos\iota(s)(-\sin u\boldsymbol{T}(s) + \cos u\boldsymbol{N}(s)) + \sin\iota(s)\boldsymbol{B}(s)$ and $\boldsymbol{\psi}_2(s,u) = \cos u\boldsymbol{T}(s) + \sin u\boldsymbol{N}(s)$, then we have a framed surface $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : I \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$. We call $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ the framed surface of osculating circles and denote the framed base surface $\boldsymbol{\psi}$ as the OFB surface for short.*

Let $\boldsymbol{\psi}_3(s,u) = \boldsymbol{\psi}_1(s,u) \times \boldsymbol{\psi}_2(s,u)$. Therefore, we have

$$\boldsymbol{\psi}_3(s,u) = \sin\iota(s)(-\sin u\boldsymbol{T}(s) + \cos u\boldsymbol{N}(s)) - \cos\iota(s)\boldsymbol{B}(s).$$

From Equations (1) and (2), the basic invariants of $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ are given by

$$\begin{aligned} a_1 &= \mathrm{r}'(s)\sin u + 1, & a_2 &= r(s),\\ b_1 &= (\cos u - 1)(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s)), & b_2 &= 0,\\ e_1 &= -\kappa(s)\cos\iota(s) - \tau(s)\sin u\sin\iota(s), & e_2 &= -\cos\iota(s),\\ f_1 &= -(\iota_s(s) + \tau(s)\cos u), & f_2 &= 0,\\ g_1 &= \kappa(s)\sin\iota(s) - \tau(s)\sin u\cos\iota(s), & g_2 &= \sin\iota(s). \end{aligned}$$

We denote $W(s,u) = \sin u(r(s)\tau(s)\sin\iota(s) - r'(s)\cos\iota(s))$. Since $a_1e_2 + b_1f_2 = a_2e_1 + b_2f_1$, we have $W(s,u) \equiv 0$. The curvature $C_F = (J_F, K_F, H_F)$ of $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ is given by

$$
\begin{aligned}
J_F &= r(s)(1-\cos u)(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s)),\\
K_F &= -\cos\iota(s)(\iota_s(s) + \tau(s)\cos u),\\
H_F &= -\frac{1}{2}\Big(r(s)(\iota_s(s) + \tau(s)\cos u) + \cos\iota(s)(\cos u - 1)(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s))\Big).
\end{aligned}
$$

On the basis of the above results, we get the following propositions.

Proposition 4. *Let $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : I \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ be a framed surface of $\mathcal{O}(\gamma)$. We denote S_1 and S_2 as the sets of singular points of $\mathcal{O}(\gamma)$; then, we have*

(1) *If $p \in S_1 \cup S_2$,*
   (a) *since $J_F(p) = 0$, $\boldsymbol{\psi}(s,u)$ is not an immersion (a regular surface) around p;*
   (b) *$(\boldsymbol{\psi}, \boldsymbol{\psi}_1)$ is a Legendre immersion around p if and only if $K_F(p) \neq 0$ or $H_F(p) \neq 0$.*

(2) *If $p \notin S_1 \cup S_2$,*
   (c) *$\boldsymbol{\psi}(s,u)$ is an immersion (a regular surface) around p;*
   (d) *$(\boldsymbol{\psi}, \boldsymbol{\psi}_1)$ is a Legendre immersion around p.*

Proposition 5. *Let $\boldsymbol{\psi} : I \times \mathbb{R} \to \mathbb{R}^3$ be a regular surface of osculating circles. The fundamental invariants are expressed as*

$$
\begin{aligned}
E(s,u) =&\ (r'(s)\sin u + 1)^2 + (\cos u - 1)^2(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s))^2,\\
F(s,u) =&\ (r'(s)\sin u + 1)(\cos u - 1)(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s)),\\
G(s,u) =&\ r^2(s),\\
L(s,u) =&\ (r'(s)\sin u + 1)(\kappa(s)\cos\iota(s) + \tau(s)\sin u\sin\iota(s))\\
&+ (\iota_s(s) + \tau(s)\cos u)(\cos u - 1)(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s)),\\
M(s,u) =&\ (r'(s)\sin u + 1)\cos\iota(s),\\
N(s,u) =&\ r(s)\cos\iota(s).
\end{aligned}
$$

Proposition 6. *The Gauss curvature K and the mean curvature H of the regular surface $\boldsymbol{\psi}(s,u)$ are*

$$
K(s,u) = \frac{-\cos\iota(s)(\iota_s(s) + \tau(s)\cos u)}{r(s)(1-\cos u)(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s))},
$$

$$
H(s,u) = \frac{\cos\iota(s)}{2r(s)} - \frac{\iota_s(s) + \tau(s)\cos u}{2(1-\cos u)(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s))}.
$$

From Proposition 6, we can easily have the following results.

Corollary 1. *The regular surface $\boldsymbol{\psi} : I \times \mathbb{R} \to \mathbb{R}^3$ is developable if and only if $\cos\iota(s)(\iota_s(s) + \tau(s)\cos u) = 0$.*

Corollary 2. *The regular surface $\boldsymbol{\psi} : I \times \mathbb{R} \to \mathbb{R}^3$ is minimal if and only if $r(s)(\iota_s(s) + \tau(s)\cos u) - \cos\iota(s)(1-\cos u)(r'(s)\sin\iota(s) + r(s)\tau(s)\cos\iota(s)) = 0$.*

In order to analyze the singular properties of the OFB surface and simplify the proof process of Theorem 1, we give the following lemma to recognize whether or not the OFB surface is a front.

Lemma 1. *Let $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : I \times U \to \mathbb{R}^3 \times \Delta$ be a framed surface of $\mathcal{O}(\gamma)$. If p is a singular point of $\boldsymbol{\psi}(s,u)$, the OFB surface $\boldsymbol{\psi}(s,u)$ is a front near p if and only if $\iota_s(s_0) \neq 0$.*

Proof. According to Proposition 4, the OFB surface $\boldsymbol{\psi}(s,u)$ is a front around p if and only if $K_F(p) \neq 0$ or $H_F(p) \neq 0$. Since $K_F(p) = -\cos\iota(s_0)\iota_s(s_0)$ and $H_F(p) = -\dfrac{1}{2}r(s_0)\iota_s(s_0)$, then we have the result. □

For a map germ $f : (U \subseteq \mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, $p \in U$ is a singular point of f. We say p is of co-rank α if and only if $min(m,n) - rank(df_p) = \alpha$ [19]. For the purpose of discussing the types of singular points, it is necessary to calculate the co-rank of the singular points that appear on the OFB surfaces.

Lemma 2. *Let $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : I \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ be a framed surface of $\mathcal{O}(\gamma)$. If $p = (s_0, u_0)$ is a singular point of $\boldsymbol{\psi}(s,u)$, the OFB surface $\boldsymbol{\psi}(s,u)$ is parametrized by a co-rank one singular point at $\boldsymbol{\psi}(s_0,u_0)$.*

Proof. According to Proposition 2, p is a singular point of the OFB surface $\boldsymbol{\psi}(s,u)$ if and only if $r'(s_o) = \tau(s_0) = 0$, where $u \in [0, 2\pi)$. Next, we show that the type of the singular point p is of co-rank one. By a direct calculation, we have

$$\boldsymbol{\psi}_s(s_0,u_0) = \cos u_0 \boldsymbol{T}(s_0) + \sin u_0 \boldsymbol{N}(s_0),$$
$$\boldsymbol{\psi}_u(s_0,u_0) = r(s_0)(\cos u_0 \boldsymbol{T}(s_0) + \sin u_0 \boldsymbol{N}(s_0)).$$

Therefore, the rank of the differential of $\boldsymbol{\psi}(s,u)$ is equal to one, which means that the co-rank of the singular point is one. □

Now, we review the related definitions of non-degenerate singular points briefly. More details are in [8,9]. Let $\boldsymbol{\psi} : U \to \mathbb{R}^3$ be a frontal of Legendre surface $(\boldsymbol{\psi}, \boldsymbol{\psi}_1)$. Then, there exists a smooth function $\lambda(s,u)$ on U, which is called a *discriminant function*, such that $\lambda(s,u) = \det(\boldsymbol{\psi}_s, \boldsymbol{\psi}_u, \boldsymbol{\psi}_1)(s,u)$.

If the exterior derivative $d\lambda$ does not vanish at a singular point p, we call the singular point p non-degenerate. Parameterizing the singular point set, we have the singular curve $\boldsymbol{\delta}(s)$ satisfying $\boldsymbol{\delta}(s_0) = p$. Then, we can choose a null vector field $\boldsymbol{\eta}(s)$ along $\boldsymbol{\delta}(s)$ satisfying $d\boldsymbol{\psi}(\boldsymbol{\eta}(s)) = 0$, where $\boldsymbol{\eta}(s) = r(s)\frac{\partial}{\partial s} - \frac{\partial}{\partial u}$. Using the null vector field as a tool, we can identify the types of singular points that appear on $\mathcal{O}(\gamma)$.

Proposition 7. *Let $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : I \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ be a framed surface of $\mathcal{O}(\gamma)$. The singular point p of $\boldsymbol{\psi}(s,u)$ is non-degenerate if and only if $(1 - \cos u_0)(r(s_0)\tau'(s_0)\cos\iota(s_0) + r''(s_0)\sin\iota(s_0)) \neq 0$.*

Proof. Since p is a singular point of $\boldsymbol{\psi}(s,u)$, we define the density function as

$$\lambda(s,u) = r(s)(1 - \cos u)(r(s)\tau(s)\cos\iota(s) + r'(s)\sin\iota(s)).$$

Then $\lambda_s(p) = r(s_0)(1 - \cos u_0)(r(s_0)\tau'(s_0)\cos\iota(s_0) + r''(s_0)\sin\iota(s_0))$ and $\lambda_u(p) = 0$. Therefore, the singular point p of $\boldsymbol{\psi}(s,u)$ is non-degenerate if and only if $\lambda_s(p) \neq 0$, which completes the proof of this proposition. □

Corollary 3. *Let $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : I \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ be a framed surface of $\mathcal{O}(\gamma)$. The singular point p of $\boldsymbol{\psi}(s,u)$ is degenerate if and only if $r(s_0)\tau'(s_0)\cos\iota(s_0) + r''(s_0)\sin\iota(s_0) = 0$ or $u_0 = 0$.*

There are only the first kind singular points on the OFB surfaces, which are illustrated by the following proposition. Especially, the OFB surfaces have neither swallowtail singular points nor cuspidal butterfly singular points.

Proposition 8. *Let $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : I \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ be a framed surface of $\mathcal{O}(\gamma)$. Singular points of the OFB surface $\boldsymbol{\psi}(s,u)$ are all the first kind.*

Proof. Since

$$\boldsymbol{\eta}\lambda(p) = r^2(s_0)(1 - \cos(u_0))(r(s_0)\tau'(s_0)\cos\iota(s_0) + r''(s_0)\sin\iota(s_0)) \neq 0,$$

by Definition 3, the singular points of $\boldsymbol{\psi}(s,u)$ are all the first kind. □

The above propositions are all in preparation for the following theorem. According to the criteria of the types of singular points on the framed surfaces (Theorem 4 in [11], Proposition 1.3 in [9], and Theorem 1.4 in [10]), we have the following conclusions.

Theorem 1. *Let $\boldsymbol{\psi} : I \times \mathbb{R} \to \mathbb{R}^3$ be the surface of osculating circles generated by $\gamma(s)$. We assume that $(\boldsymbol{\psi}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2) : I \times U \to \mathbb{R}^3 \times \Delta$ is a framed surface of osculating circles and $(s_0, u_0) \in S_2$, where $S_2 = \{(s,u) \in I \times U |\ r'(s) = 0, \tau(s) = 0, u \in (0, 2\pi)\}$.*

(1) *If $(r(s_0) + s'(u_0))\iota_s(s_0) \neq 0$, then $\boldsymbol{\psi}(s,u)$ is a cuspidal edge at (s_0,u_0) if and only if $\sin\iota(s_0)r''(s_0) \neq 0$ or $\sin\iota(s_0) = 0, r''(s_0) \neq 0, \tau'(s_0) \neq 0$.*
(2) *If $\iota_s(s_0) = 0, s'(u_0)(r(s_0) + s'(u_0))(\tau'(s_0)\cos u_0 + \iota_{ss}(s_0)) \neq 0$, then $\boldsymbol{\psi}(s,u)$ is a cuspidal cross-cap at (s_0,u_0) if and only if $\sin\iota(s_0)r''(s_0) \neq 0$ or $\sin\iota(s_0) = 0, r''(s_0) \neq 0, \tau'(s_0) \neq 0$.*

Proof. Because of

$$d\boldsymbol{\psi}(p) = (\cos u_0 \boldsymbol{T}(s_0) + \sin u_0 \boldsymbol{N}(s_0))ds + (r(s_0)(\cos u_0 \boldsymbol{T}(s_0) + \sin u_0 \boldsymbol{N}(s_0)))du,$$

the null vector field $\boldsymbol{\eta}$ is defined as

$$\boldsymbol{\eta}(s) = r(s)\frac{\partial}{\partial s} - \frac{\partial}{\partial u}.$$

Suppose $(s_0,u_0) \in S_2$ is a non-degenerate singular point of $\boldsymbol{\psi}(s,u)$. By the definition of the OFB surface, we have

$$W(s,u) = \sin u(r(s)\tau(s)\sin\iota(s) - r'(s)\cos\iota(s)) = 0.$$

Since $W_s(s_0,u_0) = 0$, then we have (s_0,u_0) satisfy one of the following conditions:

(A) $r''(s_0) \neq 0$ and $\cos\iota(s_0) = \dfrac{r(s_0)\tau'(s_0)\sin\iota(s_0)}{r''(s_0)}$, or
(B) $\sin\iota(s_0) = 0$, $r''(s_0) = 0$ and $\tau'(s_0) \neq 0$.

For Case (A), since $\sin^2\iota(s) + \cos^2\iota(s) = 1$, we have $\sin\iota(s_0) \neq 0$; that is,

$$\boldsymbol{\eta}\lambda(p) = r^2(s_0)(1 - \cos u_0)(r(s_0)\tau'(s_0)\cos\iota(s_0) + r''(s_0)\sin\iota(s_0)) \neq 0.$$

We know $(s,u) \in U$ is a singular point of $\boldsymbol{\psi}(s,u)$ if and only if $\lambda(s,u) = 0$. According to the implicit function theorem, there exists a C^∞ function s with the condition $s(u_0) = s_0$ such that the singular curve is $\boldsymbol{\delta} = (s(u), u)$. By a straight calculation, we have

$$\begin{aligned}\Phi(u) &= \det((\boldsymbol{\psi} \circ \boldsymbol{\delta})', \boldsymbol{\psi}_1 \circ \boldsymbol{\delta}, d\boldsymbol{\psi}_1(\boldsymbol{\eta}))(u) \\ &= \Big(r\tau s'\sin\iota\sin u(1 - \cos u)(r\tau\cos\iota + r'\sin\iota) + r(\tau\cos u + \iota_s)((r + s') + r's'\sin u)\Big)(u)\end{aligned}$$

and

$$\Phi'(u_0) = \Big(rs'(r + s')(\tau'\cos u + \iota_{ss}) + r\iota_s(s'' + r''(s')^2\sin u)\Big)(u_0).$$

For Case (B), by the above conditions,

$$\boldsymbol{\eta}\lambda(p) = r^3(s_0)(1 - \cos u_0)\tau'(s_0)\cos\iota(s_0) \neq 0.$$

Then, we can also give the singular curve $\boldsymbol{\delta}$ as $\boldsymbol{\delta} = (s(u), u)$. Therefore, we have

$$\Phi(u_0) = r(s_0)(r(s_0) + s'(u_0))\iota_s(s_0)$$

and

$$\Phi'(u_0) = r(s_0)s'(u_0)(r(s_0) + s'(u_0))(\tau'(s_0)\cos u_0 + \iota_{ss}(s_0)).$$

Thus, we complete the proof of assertions (1) and (2). $\square$

If the singular point $(s_0, 0) \in S_1$, then $d\lambda(p) = 0$, which means the singular point p is degenerate. Then, the OFB surface $\boldsymbol{\psi}(s,u)$ at such a singular point $(s_0, 0)$ cannot be the cuspidal edge, swallowtail, or cuspidal cross-cap.

Next, we consider other singular points with co-rank one. Izumiya, Saji, and Takahashi first gave the criteria for cuspidal lips and cuspidal beaks in [20].

Proposition 9. *Let $\boldsymbol{\psi}(s,u)$ be the OFB surface. For cuspidal lips and cuspidal beaks, neither of them appears on $\boldsymbol{\psi}(s,u)$.*

Proof. Let $p = (s_0, u_0)$ be a degenerate singular point of $\boldsymbol{\psi} = \boldsymbol{\psi}(s,u)$. By Lemma 2, we get p is a co-rank one singular point. Hence,

$$\begin{aligned}
\lambda_{ss}(p) = {} & r(s_0)(1 - \cos u_0)(r(s_0)\tau''(s_0)\cos\iota(s_0) - 2r(s_0)\tau'(s_0)\iota_s(s_0)\sin\iota(s_0) \\
& + r^{(3)}(s_0)\sin\iota(s_0) + 2r''(s_0)\cos\iota(s_0)\iota_s(s_0)), \\
\lambda_{su}(p) = {} & \lambda_{uu}(p) = 0;
\end{aligned}$$

that means $\det(\mathrm{Hess}\lambda(p)) = 0$. According to Theorem 3.2 in [7], cuspidal lips and cuspidal beaks do not appear on $\boldsymbol{\psi}(s,u)$. $\square$

If p is a co-rank one singular point of $\boldsymbol{\psi} : (\mathbb{R}^2, p) \to (\mathbb{R}^3, p)$, then there exist two linearly independent vector fields $\boldsymbol{\xi}_0, \boldsymbol{\eta}_0 \in T_0\mathbb{R}^2$ near p such that $d\boldsymbol{\psi}_0(\boldsymbol{\eta}_0) = 0$. A function $\varphi : (\mathbb{R}^2, 0) \to \mathbb{R}$ is defined by $\varphi = \det(\boldsymbol{\xi}\boldsymbol{\psi}, \boldsymbol{\eta}\boldsymbol{\psi}, \boldsymbol{\eta}\boldsymbol{\eta}\boldsymbol{\psi})$.

Proposition 10. *Let $\boldsymbol{\psi}(s,u)$ be the OFB surface. Then, Chen-Matsumoto-Mond singular points do not appear on $\boldsymbol{\psi}(s,u)$.*

Proof. Let $p = (s_0, u_0)$ be a co-rank one singular point of $\boldsymbol{\psi} = \boldsymbol{\psi}(s,u)$. The function φ is defined as $\varphi(s,u) = \det(\boldsymbol{\xi}\boldsymbol{\psi}, \boldsymbol{\eta}\boldsymbol{\psi}, \boldsymbol{\eta}\boldsymbol{\eta}\boldsymbol{\psi})(s,u)$, where $\boldsymbol{\xi}(s) = r(s)\frac{\partial}{\partial u} + \frac{\partial}{\partial s}$. Then, we have

$$\begin{aligned}
\varphi(s,u) = {} & \det(\boldsymbol{\xi}\boldsymbol{\psi}, \boldsymbol{\eta}\boldsymbol{\psi}, \boldsymbol{\eta}\boldsymbol{\eta}\boldsymbol{\psi})(s,u) \\
= {} & r^3(s)(1 - \cos u)^2(r^2(s) + 1)\Big(r(s)(\tau(s)r''(s) + r(s)\tau^3(s)\cos u - r'(s)\tau'(s)) - 2(r'(s))^2\tau(s)\Big).
\end{aligned}$$

By a direct calculation, we have p is a critical point of φ. Additionally,

$$\begin{aligned}
&\varphi_{ss}(p) = r^4(s_0)(r^2(s_0) + 1)(1 - \cos u_0)^2(\tau'(s_0)r'''(s_0) - r''(s_0)\tau''(s_0)), \\
&\varphi_{su}(p) = \varphi_{uu}(p) = 0.
\end{aligned}$$

Thus, we get $\det(\mathrm{Hess}\varphi(p)) = 0$. According to Theorem 2.2 in [21], Chen-Matsumoto-Mond singular points do not appear on $\boldsymbol{\psi}(s,u)$. $\square$

5. Examples

For Theorem 1, we give specific examples of the OFB surfaces and analyze the types of singular points on the OFB surfaces. It can be seen that the surfaces of osculating circles may have cuspidal edges and cuspidal cross-caps.

Example 1 (Figure 1)**.** *Let $\boldsymbol{\beta} : (-\frac{\pi}{4}, \frac{3\pi}{4}) \to \mathbb{R}^3$ be*

$$\boldsymbol{\beta}(s) = \left(\frac{\sqrt{2}}{4}s - \frac{1}{4}\sin(2s + \frac{\pi}{4}),\ \frac{\sqrt{2}}{4}s - \frac{1}{4}\cos(2s + \frac{\pi}{4}),\ \sin(s + \frac{\pi}{4})\right).$$

By direct calculations, we have

$$\kappa(s) = \sqrt{2}\sin(s+\frac{\pi}{4}),\, \tau(s) = -\frac{1}{2}\cos(s+\frac{\pi}{4})$$

and

$$\begin{aligned}
\boldsymbol{T}(s) &= \Big(\sin(s+\frac{\pi}{4})\sin s,\ \sin(s+\frac{\pi}{4})\cos s,\ \cos(s+\frac{\pi}{4})\Big),\\
\boldsymbol{N}(s) &= \frac{1}{\sqrt{1+\sin^2(s+\frac{\pi}{4})}}\Big(\sin(2s+\frac{\pi}{4}),\ \cos(2s+\frac{\pi}{4}),\ -\sin(s+\frac{\pi}{4})\Big),\\
\boldsymbol{B}(s) &= \frac{1}{\sqrt{1+\sin^2(s+\frac{\pi}{4})}}\Big(a(s),\ b(s),\ -\sin^2(s+\frac{\pi}{4})\Big),
\end{aligned}$$

where

$$\begin{aligned}
a(s) &= \sin^2(s+\frac{\pi}{4})\cos s - \cos(s+\frac{\pi}{4})\cos(2s+\frac{\pi}{4}),\\
b(s) &= \sin^2(s+\frac{\pi}{4})\sin s + \cos(s+\frac{\pi}{4})\sin(2s+\frac{\pi}{4}).
\end{aligned}$$

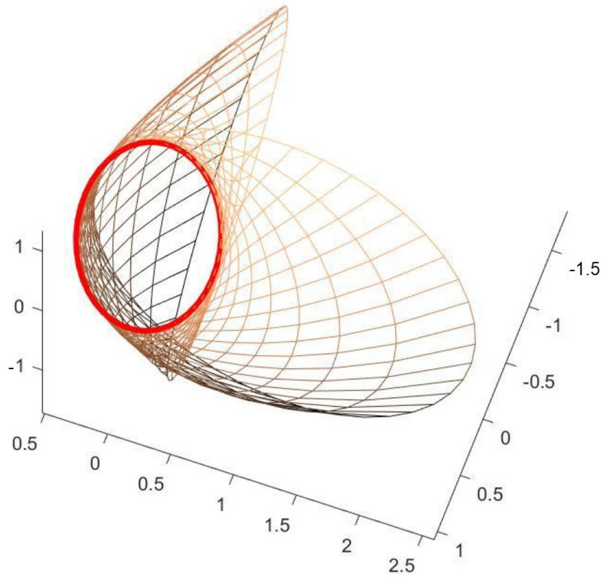

Figure 1. $\boldsymbol{\psi}(s,u)$ and cuspidal cross-caps (thick red circle).

The surface of osculating circles generated by the regular curve β is

$$\boldsymbol{\psi}(s,u) = \Big(f(s,u) + q(s,u)\sin(2s+\frac{\pi}{4}),\ g(s,u) + q(s,u)\cos(2s+\frac{\pi}{4}),\ h(s,u)\Big),$$

where

$$\begin{aligned}
f(s,u) =&\ \frac{\sqrt{2}}{4}s - \frac{1}{4}\sin(2s+\frac{\pi}{4}) + \frac{\sqrt{2}}{2}\sin u \sin s,\\
g(s,u) =&\ \frac{\sqrt{2}}{4}s - \frac{1}{4}\cos(2s+\frac{\pi}{4}) + \frac{\sqrt{2}}{2}\sin u \cos s,\\
h(s,u) =&\ \sin(s+\frac{\pi}{4}) + \frac{\sqrt{2}}{2}\sin u \cot(s+\frac{\pi}{4}) - q(s,u)\sin(s+\frac{\pi}{4}),\\
q(s,u) =&\ \frac{\sqrt{2}(1-\cos u)}{2\sin(s+\frac{\pi}{4})\sqrt{1+\sin^2(s+\frac{\pi}{4})}}.
\end{aligned}$$

Then, the set of singular points of $\boldsymbol{\psi}(s,u)$ can be given by

$$S = \left\{ \left(\frac{\pi}{4}, u \right) \middle| u \in [0, 2\pi) \right\}.$$

By Theorem 1, $\boldsymbol{\psi}(s,u)$ has cuspidal cross-cap singular points at $\left(\frac{\pi}{4}, u\right)$, where $u \neq \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Example 2 (Figure 2)**.** *Let $\gamma(s) : (-0.8, 0.8) \to \mathbb{R}^3$ be*

$$\gamma(s) = \left(\frac{1}{2}s^2, \frac{1}{4}s^4, \int_0^s \sqrt{1 - t^2 - t^6} dt \right).$$

Then, we have the Frenet–Serret formula as follows

$$\begin{aligned}
\boldsymbol{T}(s) &= \left(s, s^3, \sqrt{1 - s^2 - s^6} \right), \\
\boldsymbol{N}(s) &= \left(\frac{\sqrt{1 - s^2 - s^6}}{\sqrt{1 + 9s^4 - 4s^6}}, \frac{3s^2\sqrt{1 - s^2 - s^6}}{\sqrt{1 + 9s^4 - 4s^6}}, \frac{-s(1 + 3s^4)}{\sqrt{1 + 9s^4 - 4s^6}} \right), \\
\boldsymbol{B}(s) &= \left(\frac{s^2(2s^2 - 3)}{\sqrt{1 + 9s^4 - 4s^6}}, \frac{1 + 2s^6}{\sqrt{1 + 9s^4 - 4s^6}}, \frac{2s^3\sqrt{1 - s^2 - s^6}}{\sqrt{1 + 9s^4 - 4s^6}} \right).
\end{aligned}$$

We also have the curvature function and the torsion function

$$\begin{aligned}
\kappa(s) &= \frac{\sqrt{1 - s^2 - s^6}}{\sqrt{1 + 9s^4 - 4s^6}}, \\
\tau(s) &= \frac{2s(4s^8 - 12s^6 - 4s + 3)}{(1 + 9s^4 - 4s^6)\sqrt{1 - s^2 - s^6}}.
\end{aligned}$$

In this case, $r(s) = \dfrac{\sqrt{1 + 9s^4 - 4s^6}}{\sqrt{1 - s^2 - s^6}} \neq 0$ when $s \in (-0.8, 0.8)$. This curve is given in [22]; we consider the surface of osculating circles generated by this curve. In order to give a clear parameterized form of $\mathcal{O}(\gamma)$, let

$$\begin{aligned}
f(s,u) &= \frac{\sqrt{1 + 9s^4 - 4s^6}}{\sqrt{1 - s^2 - s^6}} \sin u, \\
g(s,u) &= \int_0^s \sqrt{1 - t^2 - t^6} dt + \sqrt{1 + 9s^4 - 4s^6} \sin u + \frac{(s + 3s^5)(\cos u - 1)}{\sqrt{1 - s^2 - s^6}}.
\end{aligned}$$

The surface of osculating circles generated by the regular curve γ is

$$\boldsymbol{\psi}(s,u) = \left(\frac{1}{2}s^2 + sf(s,u) + (1 - \cos u), \frac{1}{4}s^4 + s^3 f(s,u) + 3s^2(1 - \cos u), g(s,u) \right).$$

Then, the set of singular points of $\boldsymbol{\psi}(s,u)$ can be given by

$$S = \{(0,u) \,|u \in [0, 2\pi)\}.$$

By Theorem 1, $\boldsymbol{\psi}(s,u)$ has cuspidal edge singular points at $(0,u)$.

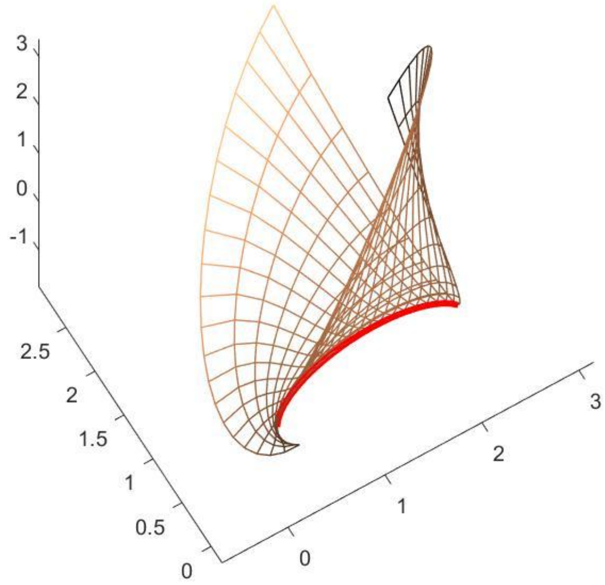

Figure 2. $\boldsymbol{\psi}(s,u)$ and cuspidal edges (thick red line).

6. Conclusions

In this paper, we investigated the singular properties of the surfaces of osculating circles. By the tool of framed surfaces, we analyzed the types of singular points that appear on surfaces and obtained the geometric conditions for these surfaces to have non-degenerate singular points, such as cuspidal edges and cuspidal cross-caps. For other higher-order degenerate singular points, we have not discussed their existence yet. Although this work may be difficult, we will precisely identify the types of singular points and generalize the same results in n-dimensional spaces in another paper.

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Abbreviations

The following abbreviations are used in this manuscript:

OFB surface	the framed base surface of osculating circles

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