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Ascoli-Type Theorem for Baire 1 Functions

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Abstract: Let X, Y be metric spaces and $\mathcal{B}_1(X,Y)$ be the space of Baire 1 functions from X to Y. The main purpose of this paper is to study compact subsets of $\mathcal{B}_1(X,Y)$ equipped with the topology τ_{UC} of uniform convergence on compacta and prove Ascoli-type theorem for locally bounded Baire 1 functions. The key notion in our paper is the notion of equi-Lebesgue family of functions from X to Y.

Keywords: Baire 1 function; topology of uniform convergence on compacta; equi-Lebesgue family; compactness

MSC: 26A21; 54D30; 54H05

1. Introduction

The Arzelà–Ascoli theorem is an important result of mathematical analysis. It deals with compactness in the space of real-valued continuous functions on a compact space. The notion of equicontinuity is a key notion in this theorem. It was introduced by the Italian mathematicians Cesare Arzelà [1,2] and Giulio Ascoli [3].

If X is a locally compact space and (Y,d) is a metric space, a subset $\mathcal{A}$ of $C(X,Y)$ of continuous functions from X to Y is compact in $(C(X,Y),\tau_{UC})$ if and only if $\mathcal{A}$ is closed in $(C(X,Y),\tau_{UC})$, $\mathcal{A}$ is equicontinuous and the closure of the set $\{f(x) : f \in \mathcal{A}\}$ is compact in Y for every $x \in X$ [4]. The set $\mathcal{A}$ is equicontinuous if for each $x \in X$ and every $\epsilon > 0$ there is a neighborhood G of x with $d(f(x),f(z)) < \epsilon$ for every $z \in G$ and every $f \in \mathcal{A}$.

We say that a function f between metric spaces X and Y is Baire 1 if the preimage of any open set is F_σ (cf. [5] [Def. 24.1]). Such functions are also called in the literature F_σ-measurable functions. The notion of the Baire 1 function is a classical one. In 1905, Lebesgue [6] characterized Baire 1 functions with values in $\mathbb{R}$. A detailed study of classical results concerning Baire 1 functions from a metric space into a separable metric space can be found in [7]. In the literature, we can find several characterizations of Baire 1 functions; see [5] [Thm 24.15]. Recently, a new characterization of Baire 1 functions was proved in [8] and [9]. It is known that if $Y = \mathbb{R}$, the Baire 1 function is a limit of a pointwise convergent sequence of continuous functions. There is rich literature concerning Baire 1 functions; see [8–14] and others.

In our paper, we prove Ascoli-type theorem for locally bounded Baire 1 functions using a special form of equi-Lebesgue family. The notion of equi-Lebesgue family of functions from X to Y was introduced in [14], and it also appeared in [10] and in [11]. If X is a separable metric space, then the notion of equi-Lebesgue family coincides with the notion of equi-Baire 1 family introduced by Lecomte in his paper [10]. Lecomte studied versions of Ascoli's theorem for Baire 1 functions. He also proved the following result. Let X be a locally compact separable metric space, Y be a separable metric space, $\mathcal{B}_1(X,Y)$ be the space of Baire 1 functions from X to Y, and τ_{uc} be the topology of uniform convergence on compacta. If $\mathcal{A} \subset \mathcal{B}_1(X,Y)$, equipped with τ_{uc}, is relatively compact in Y^X, then $\mathcal{A}$ is equi-Baire 1. Lecomte did not prove in [10] a characterization of compact subsets of $\mathcal{B}_1(X,Y)$ equipped with the topology of uniform convergence on compacta.

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In [15], Ascoli-type theorem for quasicontinuous locally bounded functions was studied, and in [16], Ascoli-type theorems for quasicontinuous subcontinuous functions were proved. In [17], the authors studied compact subsets of quasicontinuous functions, and in [18], they studied compact subsets of minimal usco and minimal cusco maps equipped with the topology of uniform convergence on compact sets. Ascoli-type theorems for so-called densely continuous forms and locally bounded densely continuous forms were proved in [19,20]. Notice that minimal usco/cusco maps and densely continuous forms are set-valued mappings, and the class of quasicontinuous mappings is entirely different from the class of Baire 1 functions. There are easy examples of Baire 1 functions that are not quasicontinuous, and also, there are quasicontinuous functions from $[0,1]$ to $[0,1]$ that are not Lebesgue-measurable [21]. The mentioned papers cannot be used for characterization of compact subsets in $(\mathcal{B}_1(X,Y),\tau_{UC})$. In [22], the authors studied Ascoli-type theorem for locally bounded functions. Some results concerning the compactness of locally bounded Baire 1 functions can be found in [22]; however, they are only for X, which is compact, and Y, which is boundedly compact.

The paper is organized as follows. In Section 2, we recall a characterization of Baire 1 functions between metric spaces X and Y, which will be important for our study. In Section 3, we study metrizability and complete metrizability of $(\mathcal{B}_1(X,Y),\tau_{UC})$. In Section 4, necessary conditions for compact subsets of $(\mathcal{B}_1(X,Y),\tau_{uc})$ are given. In Section 5, we prove Ascoli-type theorem for locally bounded Baire 1 functions with values in metric spaces in which every bounded set is totally bounded. In Section 6, we present our opinion on the importance of our paper.

2. Preliminaries

In our paper, we denote $\mathbb{R}$ the space of real numbers with the usual Euclidean metric and $\mathbb{N}$ the set of positive integers.

Let (Y,d) be a metric space. Denote by $B(y,\varepsilon)$ the open d-ball with center $y\in Y$ and radius $\varepsilon>0$ and by $B(A,\varepsilon)$ the ε-parallel body $\bigcup_{a\in A}B(a,\varepsilon)$ for a subset A. The closure of $A\subset Y$ will be denoted by $\overline{A}$.

Let X be a topological space, $\mathfrak{K}(X)$ be the family of all compact subsets of X, and (Y,d) be a metric space. The topology τ_{uc} of uniform convergence on compacta on Y^X is induced by the uniformity $\mathfrak{U}_{uc}$ of uniform convergence on compacta, which has a base consisting of sets of the form

$$S(A,\varepsilon)=\{(h,l):\ \forall\, x\in A\ \ d(h(x),l(x))<\varepsilon\},$$

where $A\in\mathfrak{K}(X)$ and $\varepsilon>0$.

Denote by $S(h,A,\varepsilon)$ the τ_{uc}-basic neighborhood of $h\in Y^X$. Thus,

$$S(h,A,\varepsilon)=\{l\in Y^X:\ d(h(x),l(x))<\varepsilon \text{ for every } x\in A\}.$$

Denote by τ_p the topology of pointwise convergence on Y^X.

The following theorem was proved in [7].

Theorem 1. *Let (X,d_X) be a metric space and (Y,d_Y) be a separable metric space. A function $g\colon X\to Y$ is Baire 1 if and only if for every $\varepsilon>0$ there is a countable family $(F_i)_{i\in\mathbb{N}}$ such that F_i is a closed set for every $i\in\mathbb{N}$, $\bigcup_{i\in\mathbb{N}}F_i=X$ and* $\operatorname{diam} g(F_i)\le\varepsilon$ *for every $i\in\mathbb{N}$.*

We will need the following result from [14].

Remark 1. *Let (X,d_X) and (Y,d_Y) be metric spaces and $g\colon X\to Y$. If for every $\varepsilon>0$ there is a countable family $(F_i)_{i\in\mathbb{N}}$ such that F_i is a closed set for every $i\in\mathbb{N}$, $\bigcup_{i\in\mathbb{N}}F_i=X$ and* $\operatorname{diam} g(F_i)\le\varepsilon$ *for every $i\in\mathbb{N}$, then g is Baire 1.*

In [5] [Thm 24.14], it was proved that if (X, d_X), (Y, d_Y) are metric spaces, (Y, d_Y) is separable, and $g\colon X \to Y$ is Baire 1, then the set of points of discontinuity of g is a F_σ set of the first Baire category. We will improve this result.

We say that a collection $\mathcal{H}$ of nonempty open subsets of a topological space X is a pseudo-base [23] for X if, for every nonempty open subset U of X, there is $H \in \mathcal{H}$ such that $H \subset U$.

The following result will be useful.

Lemma 1 ([24]). *Let X be a topological space. The following are equivalent:*

(1) A set M is of the first Baire category in X;

(2) There is a pseudo-base $\mathcal{H}$ for X such that for every $H \in \mathcal{H}$, $H \cap M$ is of the first Baire category in H.

Proposition 1. *Let (X, d_X), (Y, d_Y) be metric spaces and $g\colon X \to Y$ be Baire 1. Suppose that there is a dense set D in X such that for every $x \in D$, there is a neighborhood O of x with $g(O)$ separable. Then, the set of points of discontinuity of g is a F_σ set of the first Baire category.*

Proof. Put $M = \{x \in X : g \text{ is not continuous at } x\}$. Since the set of points of discontinuity of g has to be a F_σ set, it is sufficient to prove that M is of the first Baire category. Put $\mathcal{H} = \{V : V \text{ open and } V \cap M \text{ is of the first Baire category in } V\}$. We will prove that $\mathcal{H}$ is a pseudo-base for X. Let U be a nonempty open set in X. Let $x \in U \cap D$. There is an open set $O \subset U$ such that $x \in O$ and $g(O)$ is a separable subset of Y. The function $g|O : O \to g(O)$ satisfies conditions of Theorem 24.14 in [5]. Thus, $\{x \in O : g|O \text{ is not continuous at } x\}$ is of the first Baire category in O. Since the set $\{x \in O : g|O \text{ is not continuous at } x\}$ is equal to the set $\{x \in O : g \text{ is not continuous at } x\}$, the set $O \cap M$ is of the first Baire category in O. Then, $O \in \mathcal{H}$ and $O \subset U$. Thus, $\mathcal{H}$ is a pseudo-base for X. By Lemma 1, the set M is of the first Baire category in X. □

Corollary 1. *Let (X, d_X) be a Baire metric space, (Y, d_Y) be a metric space, and $g\colon X \to Y$ be Baire 1. Suppose there is a dense set D in X such that for every $x \in D$, there is a neighborhood O of x with $g(O)$ separable. Then, the set of points of continuity of g is a dense G_δ set in X.*

Proof. In every Baire space, the complement of a F_σ set of the first Baire category is a dense G_δ set. □

3. Metrizability of $(\mathcal{B}_1(X, Y), \tau_{UC})$

A topological space X is hemicompact [25] if there is a countable family $\{K_1, K_2, \ldots K_n, \ldots\}$ of compact sets in X such that for every $K \in \mathfrak{K}(X)$ there is $n \in \mathbb{N}$ with $K \subset K_n$. Every hemicompact metric space is locally compact [25]; however, σ-compact metric space need not be hemicompact.

Thus, if (X, d_X) is a hemicompact metric space, then it is locally compact and separable. Also, if (X, d_X) is a locally compact separable metric space, then it is a hemicompact metric space. It is also known that if (X, d_X) is a locally compact separable metric space, then there is a compatible metric ρ on X such that every closed ρ-bounded set is compact [26]. Such spaces are called boundedly compact metric spaces [27].

In what follows, let (X, d_X) and (Y, d_Y) be metric spaces with at least two different points and $\mathcal{B}_1(X, Y)$ be the family of Baire 1 functions from X to Y.

Lemma 2. *Let (X, d_X) and (Y, d_Y) be metric spaces. If X is hemicompact, then the uniformity $\mathfrak{U}_{uc}$ on Y^X is induced by a metric. Thus also the uniformity $\mathfrak{U}_{uc}$ on $\mathcal{B}_1(X, Y)$ is induced by a metric and $(\mathcal{B}_1(X, Y), \tau_{uc})$ is metrizable.*

Proof. Let $\{C_k : k \in \mathbb{N}\}$ be a countable family in $\mathfrak{K}(X)$ such that for every $C \in \mathfrak{K}(X)$ there is $l \in \mathbb{N}$ with $C \subset C_l$. Then, the family

$$\{S(C_k, 1/n) : k, n \in \mathbb{N}\}$$

is a countable base of $\mathfrak{U}_{uc}$. By the metrization theorem in [4], the uniformity $\mathfrak{U}_{uc}$ on Y^X is induced by a metric, and thus, $\mathfrak{U}_{uc}$ on $\mathcal{B}_1(X,Y)$ is also induced by a metric. □

Using a similar idea as in ([21], Theorem 2.8.1), it can be shown that $(\mathcal{B}_1(X,Y), \mathfrak{U}_{uc})$ is metrizable if and only if the space X is hemicompact.

The pointwise limit of a sequence of Baire 1 functions need not be Baire 1. A uniform limit of a sequence of Baire 1 functions is Baire 1 [7]. We also have the following proposition.

Proposition 2. *Let (X, d_X) be a σ-compact metric space and (Y, d_Y) be a metric space. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of Baire 1 functions from X to Y, which converges uniformly on compact sets to a function f from X to Y. Then f is a Baire 1 function.*

Proof. Let $K \in \mathfrak{K}(X)$. Then, the sequence $\{f_n|K : n \in \mathbb{N}\}$ converges uniformly to $f|K$. For every $n \in \mathbb{N}$, the function $f_n|K$ belongs to $\mathcal{B}_1(K,Y)$ [7] [p. 386]. Thus, by [7] [p. 395], the function $f|K$ also belongs to $\mathcal{B}_1(K,Y)$.

Let $\{K_n : n \in \mathbb{N}\}$ be a countable subfamily in $\mathfrak{K}(X)$ such that $X = \bigcup_{n\in\mathbb{N}} K_n$. Since $f|K_n \in \mathcal{B}_1(K_n, Y)$ for every $n \in \mathbb{N}$, by ([7], p. 385) $f \in \mathcal{B}_1(X,Y)$. □

Corollary 2. *Let (X, d_X) be a hemicompact metric space and (Y, d_Y) be a metric space. $\mathcal{B}_1(X,Y)$ is a closed set in (Y^X, τ_{UC}).*

Proof. If (X, d_X) is a hemicompact metric space, then by Lemma 2, $(Y^X, \mathfrak{U}_{uc})$ is a metrizable space. If $f \in Y^X$ is in the closure of $\mathcal{B}_1(X,Y)$ in (Y^X, τ_{uc}), then there is a sequence $\{f_n : n \in \mathbb{N}\}$ of Baire 1 functions from X to Y, which converges to f uniformly on compact sets. By Proposition 2 $f \in \mathcal{B}_1(X,Y)$. □

Corollary 3. *Let (X, d_X) be a hemicompact metric space and (Y, d_Y) be a complete metric space. Then, $(\mathcal{B}_1(X,Y), \tau_{uc})$ is a completely metrizable space.*

Proof. If (Y, d_Y) is a complete metric space, then by [4], $(Y^X, \mathfrak{U}_{uc})$ is a complete uniform space. By Lemma 2, the uniformity $\mathfrak{U}_{uc}$ on Y^X is induced by a metric. By Corollary 2, $\mathcal{B}_1(X,Y)$ is a closed set in (Y^X, τ_{uc}). □

If (X, d_X) is a hemicompact metric space, then using any countable cofinal subfamily in $\mathfrak{K}(X)$, we can define a natural metric on Y^X, which induces $\mathfrak{U}_{uc}$ and thus also on $\mathcal{B}_1(X,Y)$ [21].

4. Necessary Conditions for Compact Subsets of $(\mathcal{B}_1(X,Y), \tau_{uc})$

Lee, Thang, and Zhao, in their paper [8], found an interesting characterization of Baire 1 functions between Polish spaces. Fenecios and Cabral in [9] proved the following theorem.

Theorem 2 ([9])**.** *Let (X, d_X) and (Y, d_Y) be separable metric spaces. A function $g\colon X \to Y$ is Baire 1 if and only if for every $\varepsilon > 0$ there exists a function $\delta : X \to \mathbb{R}$ such that $\delta(x) > 0$ for every $x \in X$ and if $d_X(x,y) < \min\{\delta(x), \delta(y)\}$, then $d_Y(g(x), g(y)) < \varepsilon$.*

In his paper [10], Lecomte introduced the notion of equi-Baire 1 family of functions.

Definition 1 ([10])**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. A family $\mathcal{E}$ of functions from X to Y is equi-Baire 1 if, for each $\varepsilon > 0$, there is a function $\delta\colon X \to \mathbb{R}$ such that $\delta(x) > 0$ for every $x \in X$ and $d_X(x,y) < \min\{\delta(x), \delta(y)\}$ implies $d_Y(f(x), f(y)) \leq \varepsilon$ for every $f \in \mathcal{E}$.*

Notice that the notion of equi-Baire 1 family of functions can also be found in the paper of Alighani-Koopaei [11], who showed its applications in dynamical systems [11,12].

In [14], the authors introduced the notion of equi-Lebesgue family.

Definition 2. *Let (X, d_X) and (Y, d_Y) be metric spaces. A family $\mathcal{E}$ of functions from X to Y is called equi-Lebesgue if, for every $\varepsilon > 0$, there is a countable family $(F_i)_{i\in\mathbb{N}}$ such that F_i is a closed set for every $i \in \mathbb{N}$, $\bigcup_{i\in\mathbb{N}} F_i = X$ and* $\operatorname{diam} g(F_i) \leq \varepsilon$ *for every $i \in \mathbb{N}$ and for every $g \in \mathcal{E}$.*

Notice that every member of an equi-Lebesgue family of functions from X to Y is a Baire 1 function (see Remark 1).

It can be observed from the proof of Theorem 3.6 in [11] that every equi-Lebesgue family of functions from a metric space X to a metric space Y is equi-Baire 1, and if X is separable, then the notions of equi-Lebesgue and equi-Baire 1 families of functions from X to Y coincide.

Lecomte, in his paper [10], proved the following result.

Proposition 3. *Let (X, d_X) be a locally compact separable metric space and (Y, d_Y) be a separable metric space. If $\mathcal{E} \subset \mathcal{B}_1(X, Y)$, equipped with the topology of uniform convergence on compacta, is relatively compact in Y^X, then $\mathcal{E}$ is equi-Baire 1.*

We have the following generalization of Proposition 3 with a different proof.

Proposition 4. *Let (X, d_X) be a locally compact separable metric space and (Y, d_Y) be a separable metric space. Let $\mathcal{A} \subset \mathcal{B}_1(X, Y)$ be a totally bounded set in $(Y^X, \mathfrak{U}_{UC})$. Then, $\mathcal{A}$ is the equi-Lebesgue family, and thus is also equi-Baire 1.*

Proof. Of course, (X, d_X) is hemicompact. Let $\{K_n : n \in \mathbb{N}\}$ be a countable cofinal subfamily in $\mathfrak{K}(X)$. Let $k \in \mathbb{N}$ and $\epsilon > 0$. Since $\mathcal{A} \subset \mathcal{B}_1(X, Y)$ is totally bounded in $(Y^X, \mathfrak{U}_{UC})$, there are functions $f_1, f_2, \ldots, f_m \in \mathcal{A}$ such that

$$\mathcal{A} \subseteq \cup\{S(f_j, K_k, \epsilon/3) : j \in \{1, \ldots, m\}\}.$$

Let $\mathcal{V} = \{V_1, \ldots, V_n, \ldots\}$ be a countable family of open sets in Y, such that $diam V_i < \epsilon/3$ for every $i \in \mathbb{N}$ and $Y = \bigcup_{i\in\mathbb{N}} V_i$.

For each $j \in \{1, \ldots, m\}$ and $i \in \mathbb{N}$, put

$$D_i^j = f_j^{-1}(V_i).$$

Since f_j is a Baire 1 function, the set D_i^j is a F_σ set for every $j \in \{1, \ldots, m\}$, $i \in \mathbb{N}$. Put

$$\mathcal{D}' = \{K_k \cap D_{i_1}^1 \cap \ldots \cap D_{i_m}^m : i_j \in \mathbb{N}, j \in \{1, \ldots, m\}\}.$$

Every set $D \in \mathcal{D}'$ is a F_σ set. Let $\mathcal{D}$ be the family containing all nonempty sets from $\mathcal{D}'$. Clearly, $K_k = \cup\mathcal{D}$ and the family $\mathcal{D}$ is countable. Then for every $f \in \mathcal{A}$ and for every $D \in \mathcal{D}$, $\operatorname{diam} f(D) < \epsilon$.

Since $X = \bigcup\{K_l : l \in \mathbb{N}\}$, we are done. □

Corollary 4. *Let (X, d_X) be a hemicompact metric space and (Y, d_Y) be a separable metric space. If $\mathcal{A} \subset \mathcal{B}_1(X, Y)$ is a compact set in (Y^X, τ_{UC}), then $\mathcal{A}$ is a closed set in $(\mathcal{B}_1(X, Y), \tau_{uc})$, $\mathcal{A}$ is the equi-Lebesgue family, and $\mathcal{A}(x) = \{f(x) : f \in \mathcal{A}\}$ is compact in (Y, d_Y) for every $x \in X$.*

Lecomte in his paper [10] presented metric spaces (X, d_X), (Y, d_Y) and the equi-Baire 1 family $\mathcal{A} \subset \mathcal{B}_1(X, Y)$, such that $\mathcal{A}$ is closed in (Y^X, τ_{UC}), $\mathcal{A}(x)$ is compact in (Y, d_Y) for every $x \in X$ and $\mathcal{A}$ is not compact in (Y^X, τ_{UC}).

For the reader's convenience, we present an easier example.

Example 1. *Let $X = \mathbb{R} = Y$ equipped with the usual Euclidean metric. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of different points in $\mathbb{R}$ that converges to a point $x_0 \in \mathbb{R}$. Put $A = \{x_0\} \cup \{x_n : n \in \mathbb{N}\}$. Then, A is a compact set in $\mathbb{R}$; thus, the complement of A is an F_σ set in $\mathbb{R}$. Put $\mathcal{A} = \{\chi_a : a \in A\}$. It is easy to verify that $\mathcal{A}$ is equi-Lebesgue family. Put $\mathbb{R} \setminus A = \bigcup_{n\in\mathbb{N}} F_n$, where F_n is a closed set for every $n \in \mathbb{N}$. Then, the family $\mathcal{C} = \{\{a\} : a \in A\} \cup \{F_n : n \in \mathbb{N}\}$ is a countable cover of closed sets, such that $\operatorname{diam} \chi_a(C) < \epsilon$ for every $a \in A$, every $C \in \mathcal{C}$ and every $\epsilon < 1$. Of course, $\mathcal{A}(x)$ is compact for every $x \in \mathbb{R}$, $\mathcal{A}$ is a closed set in $(\mathbb{R}^{\mathbb{R}}, \tau_{uc})$, and $\mathcal{A}$ is not compact, since the sequence $\{\chi_{\{x_n\}} : n \in \mathbb{N}\}$ has no cluster point in $(\mathcal{A}, \tau_{uc})$.*

5. Ascoli-Type Theorem for Locally Bounded Baire 1 Functions

Definition 3. *Let (X, d_X) and (Y, d_Y) be metric spaces. A subset $\mathcal{A}$ of Y^X is finitely equi-Lebesgue at $x_0 \in X$ if, for every $\epsilon > 0$, there is a finite family $\mathcal{D}$ of nonempty subsets of X that are either open or F_σ nowhere dense such that $\cup\mathcal{D}$ is a neighborhood of x_0 and such that for every $g \in \mathcal{A}$, for every $D \in \mathcal{D}$, $\operatorname{diam} g(D) \leq \epsilon$. $\mathcal{A}$ is finitely equi-Lebesgue provided that it is finitely equi-Lebesgue at every point of X.*

Remark 2. *If (X, d_X) is a hemicompact metric space, then every finitely equi-Lebesgue subfamily of Y^X is equi-Lebesgue.*

Notice that in the study of Ascoli-type theorem for quasicontinuous functions in [15,16,21], the notion of a densely equiquasicontinuous family of functions was used.

Proposition 5. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{A}$ be a finitely equi-Lebesgue family of functions from X to Y. Then, the topologies τ_p and τ_{uc} coincide on $\mathcal{A}$.*

Proof. Let $\{h_i : i \in I\}$ be a net in $\mathcal{A}$ which τ_p-converges to $h \in \mathcal{A}$. We will show that $\{h_i : i \in I\}$ converges to h in (Y^X, τ_{uc}). Let $K \in \mathfrak{K}(X)$ and $\epsilon > 0$. Since $\mathcal{A}$ is finitely equi-Lebesgue and K is compact, there is a finite family $\mathcal{C}$ of F_σ subsets of K such that $K = \cup\mathcal{C}$ and $\operatorname{diam} g(C) \leq \epsilon/3$ for every $g \in \mathcal{A}$ and every $C \in \mathcal{C}$. For every $C \in \mathcal{C}$, pick $x_C \in C$. There is $i_0 \in I$ such that for every $i \geq i_0$ and every $C \in \mathcal{C}$, we have

$$d_Y(h(x_C), h_i(x_C)) < \epsilon/3.$$

Let $i \geq i_0$ and $x \in K$. There is $C \in \mathcal{C}$ such that $x \in C$. Then, we have

$$d_Y(h(x), h_i(x)) \leq d_Y(h(x), h(x_C)) + d_Y(h(x_C), h_i(x_C)) + d_Y(h_i(x_C), h_i(x)) < \epsilon.$$

□

Let (X, d_X) and (Y, d_Y) be metric spaces. Denote by $\mathcal{L}(X, Y)$ the set of all functions from Y^X such that for every $x \in X$, there is a neighborhood O_x of x with $f(O_x)$ totally bounded in (Y, d_Y).

Lemma 3. *Let (X, d_X) be a locally compact metric space and (Y, d_Y) be a metric space. $\mathcal{L}(X, Y)$ is a closed set in (Y^X, τ_{uc}).*

Proof. If $f \in \mathcal{L}(X, Y)$ and $K \in \mathfrak{K}(X)$, then $f(K)$ is a totally bounded set in (Y, d_Y). Let $g \in \overline{\mathcal{L}(X, Y)}$ in (Y^X, τ_{uc}). Let $x \in X$ and K_x be a compact neighborhood of x. We show that $g(K_x)$ is totally bounded. Let $\varepsilon > 0$. There is $h \in S(g, K_x, \varepsilon/2) \cap \mathcal{L}(X, Y)$. Then, $g(K_x) \subset B(h(K_x), \varepsilon/2)$. Since $h(K_x)$ is totally bounded, there is a finite set $L \subset Y$ such that $h(K_x) \subset B(L, \varepsilon/2)$. We have $g(K_x) \subset B(L, \epsilon)$. Since ε was arbitrary, $g(K_x)$ is a totally bounded set in (Y, d_Y). □

Put

$$\mathcal{LB}_1(X, Y) = \mathcal{L}(X, Y) \cap \mathcal{B}_1(X, Y).$$

Proposition 6. *Let (X, d_X) be a hemicompact metric space and (Y, d_Y) be a metric space. If $\mathcal{A} \subset (\mathcal{LB}_1(X,Y), \tau_{uc})$ is compact, then $\mathcal{A}$ is finitely equi-Lebesgue family.*

Proof. Let $x \in X$ and $\epsilon > 0$. Let L be an open subset of X with $x \in L$ and $\overline{L}$ compact. There are $g_1, g_2, \ldots, g_k \in \mathcal{A}$ such that

$$\mathcal{A} \subset \cup_{1 \leq j \leq k} S(g_j, \overline{L}, \epsilon/3).$$

Since $g_j \in \mathcal{L}(X,Y)$ for every $j \in \{1,2,\ldots,k\}$, $g_j(\overline{L})$ is a totally bounded set in (Y, d_Y) for every $j \in \{1,2,\ldots,k\}$. Thus, $\cup_{1 \leq j \leq k} g_j(\overline{L})$ is totally bounded in (Y, d_Y). Let $\{y_1, y_2, \ldots, y_l\}$ be a finite set in Y such that $\cup_{1 \leq j \leq k} g_j(\overline{L}) \subset \cup_{1 \leq i \leq l} B(y_i, \epsilon/6)$. For each $i \in \{1,\ldots,l\}$ and $j \in \{1,\ldots,k\}$, let

$$T_i^j = \{z \in L \cap C(g_j) : g_j(z) \in B(y_i, \epsilon/6)\},$$

where $C(g_j) = \{z \in X : \ g_j \text{ is continuous at } z\}$. Since X is a Baire space, by Corollary 1, $C(g_j)$ is dense in X for every $j \in \{1,\ldots,k\}$.

For each $z \in T_i^j$, let $S_i^j(z)$ be an open neighbourhood of z such that $S_i^j(z) \subset L$ and $g_j(S_i^j(z)) \subseteq B(y_i, \epsilon/6)$. Let

$$S_i^j = \bigcup \{S_i^j(z) : \ z \in T_i^j\}, \text{and}$$

$$\mathcal{G} = \{S_{i_1}^1 \cap \ldots \cap S_{i_k}^k : i_j \in \{1,\ldots,l\} \text{ for each } j \in \{1,\ldots,k\}\}.$$

For each $G \in \mathcal{G}$, let

$$H_i^j(G) = \{z \in \overline{G} \setminus G : \ g_j(z) \in B(y_i, \epsilon/6)\}, \text{and}$$

$$\mathcal{H} = \{H_{i_1}^1(G) \cap \ldots \cap H_{i_k}^k(G) : i_j \in \{1,\ldots,l\} \text{ for each } j \in \{1,\ldots,k\}, G \in \mathcal{G}\}.$$

The set $H_i^j(G)$ is a F_σ set for every $j \in \{1,\ldots,k\}$, $i \in \{1,\ldots,l\}$ and $G \in \mathcal{G}$. If $H \in \mathcal{H}$, then H is a F_σ set and, of course, nowhere dense.

We show that $\overline{L} = (\cup\mathcal{G}) \cup (\cup\mathcal{H})$. The set $(\cup\mathcal{G})$ is open and dense in $\overline{L}$. Let U be a nonempty open set in $\overline{L}$. There is a nonempty open set U_1 in X such that $U = U_1 \cap \overline{L}$. Choose some $u \in U_1 \cap L \cap C(g_1) \cap \ldots \cap C(g_k)$. Then, for every $j \in \{1,\ldots,k\}$, $u \in S_{i_j}^j$ for some $i_j \in \{1,\ldots,l\}$. Thus, $u \in U_1 \cap H \cap S_{i_1}^1 \cap \ldots \cap S_{i_k}^k$, i.e., $U \cap (\cup\mathcal{G}) \neq \emptyset$.

Let $z \in \overline{L} \setminus (\cup\mathcal{G})$. Then, $z \in \overline{\cup\mathcal{G}}$. Let $G \in \mathcal{G}$ be such that $z \in \overline{G} \setminus G$. Thus, for every $j \in \{1,\ldots,k\}$, there is $i_j \in \{1,\ldots,l\}$ such that $z \in H_{i_j}^j(G)$ and then $z \in \cup\mathcal{H}$. $\overline{L} = (\cup\mathcal{G}) \cup (\cup\mathcal{P})$ is proved.

Let $\mathcal{D}$ be the family that contains all nonempty sets from $\mathcal{G}$ and $\mathcal{H}$. It is easy to see that for every $g \in \mathcal{A}$ and for every $D \in \mathcal{D}$, $\operatorname{diam} g(D) < \epsilon$. □

From the following Theorem, we obtain the main result of our paper for locally bounded Baire 1 functions with values in metric spaces in which every bounded set is totally bounded.

Theorem 3. *Let (X, d_X) be a hemicompact metric space and (Y, d_Y) be a metric space. $\mathcal{A} \subset \mathcal{LB}_1(X,Y)$ is compact in (Y^X, τ_{uc}) if and only if $\mathcal{A}$ is finitely equi-Lebesgue family, $\mathcal{A}$ is closed in $\mathcal{LB}_1(X,Y)$, and $\overline{\mathcal{A}(x)}$ is compact for every $x \in X$.*

Proof. If $\mathcal{A} \subset \mathcal{LB}_1(X,Y)$ is compact in (Y^X, τ_{uc}), then by Proposition 6, $\mathcal{A}$ is finitely equi-Lebesgue family. Of course, $\mathcal{A}$ has to be closed in $\mathcal{LB}_1(X,Y)$. Since for every $x \in X$ the evaluation map $e_x : (Y^X, \tau_{uc}) \to (Y, d_Y)$ defined by $e_x(f) = f(x)$ is continuous, $\mathcal{A}(x)$ is compact for very $x \in X$.

Suppose now that $\mathcal{A}$ is finitely equi-Lebesgue family, $\mathcal{A}$ is closed in $\mathcal{LB}_1(X,Y)$, and $\overline{\mathcal{A}(x)}$ is compact for every $x \in X$. We will prove that $\mathcal{A}$ is compact in (Y^X, τ_{uc}). Put

$$L = \prod_{x \in X} \overline{\mathcal{A}(x)}.$$

Then, the space L is compact with the relative product topology from Y^X and $\mathcal{A} \subset L$. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence in $\mathcal{A}$. There is $f \in L$, which is a cluster point in L. It is easy to verify that family $\mathcal{A} \cup \{f\}$ is finitely equi-Lebesgue family, and by Proposition 5, f is a cluster point of $\{f_n : n \in \mathbb{N}\}$ in the topology of uniform convergence on compacta. Without loss of generality, we can suppose that $\{f_n : n \in \mathbb{N}\}$ converges to f in the topology of uniform convergence on compacta. By Proposition 2, $f \in \mathcal{B}_1(X,Y)$. By Lemma 3 $f \in \mathcal{L}(X,Y)$. Since $\mathcal{A}$ is a closed set in $\mathcal{LB}_1(X,Y)$, $f \in \mathcal{A}$. □

Let (X, d_X) and (Y, d_Y) be metric spaces. Denote by $\mathcal{B}_1^{lb}(X,Y)$ the family of all locally bounded Baire 1 functions from X to Y. From Theorem 3, we obtain the following main results of our paper.

Corollary 5. *Let (X, d_X) be a hemicompact metric space and (Y, d_Y) be a metric space in which every bounded set is totally bounded. $\mathcal{A} \subset \mathcal{B}_1^{lb}(X,Y)$ is compact in (Y^X, τ_{uc}) if and only if $\mathcal{A}$ is finitely equi-Lebesgue family, $\mathcal{A}$ is closed in $\mathcal{B}_1^{lb}(X,Y)$, and $\overline{\mathcal{A}(x)}$ is compact for every $x \in X$.*

Corollary 6. *Let (X, d_X) be a hemicompact metric space and (Y, d_Y) be a boundedly compact metric space. $\mathcal{A} \subset \mathcal{B}_1^{lb}(X,Y)$ is compact in (Y^X, τ_{uc}) if and only if $\mathcal{A}$ is finitely equi-Lebesgue family, $\mathcal{A}$ is closed in $\mathcal{B}_1^{lb}(X,Y)$, and $\overline{\mathcal{A}(x)}$ is compact for every $x \in X$.*

Corollary 7. *$\mathcal{A} \subset \mathcal{B}_1^{lb}(\mathbb{R},\mathbb{R})$ is compact in $(\mathbb{R}^{\mathbb{R}}, \tau_{uc})$ if and only if $\mathcal{A}$ is finitely equi-Lebesgue family, $\mathcal{A}$ is closed in $\mathcal{B}_1^{lb}(\mathbb{R},\mathbb{R})$, and $\overline{\mathcal{A}(x)}$ is compact for every $x \in \mathbb{R}$.*

Example 2. *From Corollary 7 and Example 1, we can deduce that the family $\mathcal{A} = \{\chi_a : a \in A\}$ in Example 1 is not finitely equi-Lebesgue family. In fact, family $\mathcal{A}$ is not finitely equi-Lebesgue at the point x_0. Suppose that $\mathcal{A}$ is finitely equi-Lebesgue at x_0. Then for $\epsilon = 1/2$, there exists a finite family $\mathcal{B}$ of nonempty subsets of X that are either open or F_σ nowhere dense such that $\cup\mathcal{B}$ is a neighborhood of x_0 and such that for every $\chi_a \in \mathcal{A}$, for every $B \in \mathcal{B}$, $\operatorname{diam} \chi_a(B) \leq \epsilon$. Without loss of generality, we can suppose that $A \subset \cup\mathcal{B}$. There must exist $B \in \mathcal{B}$ such that $A \cap B$ is infinite. Then, for every $a \in A \cap B$, $\operatorname{diam} \chi_a(B) = 1$, a contradiction.*

Example 3. *Let $X = [0,\infty)$ and $Y = \mathbb{R} \setminus \mathbb{N}$ be equipped with the induced topology from $\mathbb{R}$. Y is a metric space in which every bounded set is totally bounded. Let $f : X \to Y$ be defined as follows: $f(x) = x$ for $x \in X \setminus \mathbb{N}$ and $f(x) = 0$ otherwise. Put $f_1 = f$, and for every $n \geq 2$, $n \in \mathbb{N}$, define $f_n : X \to Y$ as follows: $f_n(x) = 0$, if $x \geq n$, $f_n(x) = f(x) + 1/n$ if $x \in X \setminus \{k - 1/n : k \leq n, k \in \mathbb{N}\}$ and $f_n(x) = f(x)$ otherwise. It is easy to verify that the sequence $\{f_n : n \in \mathbb{N}\}$ converges to f in $(\mathcal{B}_1^{lb}(X,Y), \tau_{uc})$. By Corollary 5, the family $\{f_n : n \in \mathbb{N}\} \cup \{f\}$ is finitely equi-Lebesgue.*

6. Conclusions and Future Work

The main purpose of this paper is to study compact subsets of $\mathcal{B}_1(X,Y)$ equipped with the topology of uniform convergence on compacta for metric spaces X and Y and prove Ascoli-type theorem for locally bounded Baire 1 functions.

The theory developed in this paper could be of interest to mathematicians working in fields including topology [13], functional analysis [10], and dynamical systems [11,12].

Lecomte, in his paper [10], introduced the notion of equi-Baire 1 family of functions from X to Y and studied versions of Ascoli's theorem for $\mathcal{B}_1(X,Y)$. However, he did not prove a characterization of compact subsets of $\mathcal{B}_1(X,Y)$ equipped with the topology of uniform convergence on compacta.

Concerning a future investigation of the space of Baire 1 functions, we plan to study cardinal invariants of $\mathcal{B}_1(X,\mathbb{R})$ equipped with the topology of uniform convergence on compacta and compare them with the cardinal invariants of the space $C(X,\mathbb{R})$ of continuous real-valued functions equipped with the topology of uniform convergence on compacta, which were studied in [23].

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