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Article

Representations by Beurling Systems

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Abstract: We prove that a Beurling system with $F \in H^p(\mathbb{D}), 1 \leq p < \infty$ is an $M-$basis in $H^p(\mathbb{D})$ with an explicit dual system. Any function $f \in H^p(\mathbb{D}), 1 \leq p < \infty$ can be expanded as a series by the system $\{z^m F(z)\}_{m=0}^{\infty}$. For different summation methods, we characterize the outer functions F for which the expansion with respect to the corresponding Beurling system converges to f. Related results for weighted Hardy spaces in the unit disc are studied. Particularly we prove Rosenblum's hypothesis.

Keywords: summation basis; hardy spaces; outer function; Beurling system; kernels; representation of functions

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1. Introduction

The present study is related to the main problem posed in [1]: describe the subsystems of the trigonometric system that are complete and minimal in some weighted L^p space. If one deletes a finite number of elements from the trigonometric system, then the remaining system has the desired property. But we are unable to find a subsystem that had the mentioned property and is obtained after deleting an infinite number of elements from the trigonometric system. The progress in the mentioned problem will be helpful in advancing a much more important problem posed more than a century ago by N.N. Luzin [2]: given a measurable function, determine the coefficients of the trigonometric series that represents it.

The formulation of the Fourier–Luzin problem is vague enough to leave room for imagination. In the described approach, we need minimality for the definition of the coefficients. In [3], we have observed that in any weighted $L^p, 1 \leq p < \infty$ space, the subsystem $\{e^{ikt}\}_{k=0}^{\infty}$ is minimal or complete. The information that a system is minimal in a subspace in itself is not sufficient for the study of the expansions by the given system. One needs to have the dual system in a form that can be used for the study. In the present work, we find the system dual to the system $\{e^{ikt}\}_{k=0}^{\infty}$ when it is minimal in a weighted L^p space.

We say that a system $\{z^m F(z)\}_{m=0}^{\infty}$ is a Beurling system if F is an outer function. In his fundamental work [4], Beurling proved that if F is an outer function in $H^2(\mathbb{D})$, then the system $\{z^m F(z)\}_{m=0}^{\infty}$ is complete in the space $H^2(\mathbb{D})$. This result can be easily extended to the spaces $H^p(\mathbb{D}), 1 \leq p < \infty$ (see [5]). In the present paper, we study questions of representations of functions from the spaces $H^p(\mathbb{D}), 1 \leq p < \infty$ by series with respect to Beurling systems. Our study is based on the fact that any Beurling system in $H^p(\mathbb{D}), 1 \leq p < \infty$ is an M—basis with an explicit dual system. It is a natural question to characterize the outer functions F for which the system $\{z^m F(z)\}_{m=0}^{\infty}$ is a basis or a summation basis in $H^p(\mathbb{D})$. In the theory of $H^p(\mathbb{D})$ spaces the most interesting case is to characterize the functions F for which the corresponding Beurling system $\{z^m F(z)\}_{m=0}^{\infty}$ is an A—summation basis in $H^p(\mathbb{D}), 1 \leq p < \infty$.

The obtained results can be interpreted in terms of weighted H^p spaces with weights that we call admissible weight functions. A non-negative function w defined on the boundary, such that $\ln w$ is integrable, is called an admissible weight function. For the

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weighted norm spaces, we obtain theorems that can be considered extensions of important results known for the H^p spaces. Moreover, this approach can be helpful for the study of similar questions in more general domains.

We used the results obtained in [6] for the study of the systems $e^{ikt}\Psi_{L,M}$ in the spaces $L^p(\mathbb{T})$, where

$$\Psi_{L,M} = \{\overline{L(t)}e^{int}\}_{n=-\infty}^{-1} \cup \{M(t)e^{int}\}_{n=0}^{\infty}$$

and L and M are the boundary values of some outer functions defined in $\mathbb{D}$.

This paper is divided into two parts. In the first part, we provide the results for Beurling systems, and the second part is dedicated to the study of weighted H^p spaces.

1.1. Preliminary Results, Definitions, and Notations

We say that $w \geq 0$ is a weight function on a measurable set $E \subseteq \mathbb{R}$ if w is integrable on E. A function $\varphi \in L^p(E, w), 1 \leq p < \infty$ if $\varphi : E \to \mathbb{C}$ is measurable on E, and the norm is defined by

$$\|\varphi\|_{L^p(E,w)} := \left(\int_E |\varphi(t)|^p w(t) dt \right)^{\frac{1}{p}} < +\infty.$$

When $w \equiv 1$, we write $L^p(E)$. Denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and identify $\mathbb{T}$ with any 2π length semi-open interval on the real line. For $1 < p < \infty$, the conjugate number p' is defined by the equation $\frac{1}{p} + \frac{1}{p'} = 1$ and $p' = \infty$ if $p = 1$. The set of integers is denoted by $\mathbb{Z}$ and $\mathbb{N} = \{1, 2, \dots\}$.

By $S[f](t)$, we denote the Fourier series of a function $f \in L^1(\mathbb{T})$. For any $n \in \mathbb{N}$,

$$S_n[\phi](t) = \sum_{j=-n}^{n} c_j(\phi)e^{ijt}, \qquad c_j(\phi) = \frac{1}{2\pi} \int_{\mathbb{T}} \phi(\theta)e^{-ij\theta} d\theta.$$

The space of continuous functions on $\mathbb{T}$ with the maximum norm is denoted by $C(\mathbb{T})$. For $1 \leq p \leq \infty$, we write

$$H^p(\mathbb{T}) = \{\phi \in L^p(\mathbb{T}) : \int_{\mathbb{T}} \phi(t)e^{int} dt = 0 \text{ for all } n \in \mathbb{N}\}.$$

The spaces $H^p(\mathbb{T}), 1 \leq p \leq \infty$ are Banach spaces of functions defined on $\mathbb{T}$. The Cauchy kernel is defined as follows:

$$C_r(\theta) = \sum_{n=0}^{+\infty} r^n e^{in\theta} \quad 0 < r < 1, \theta \in \mathbb{T},$$

and the Poisson and conjugate Poisson kernels are defined as follows:

$$P_r(\theta) = \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2},$$

$$Q_r(\theta) = \operatorname{Im} H_r(\theta), \quad P_r(\theta) = \operatorname{Re} H_r(\theta),$$

where

$$H_r(\theta) = 2C_r(\theta) - 1 = \frac{1 + re^{i\theta}}{1 - re^{i\theta}} \quad (0 < r < 1, \theta \in \mathbb{T}).$$

We denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and its closure by $\overline{\mathbb{D}}$. The convolution of functions $g, \varphi \in L(\mathbb{T})$ is denoted by

$$g * \varphi(t) = \frac{1}{2\pi} \int_{\mathbb{T}} g(\theta)\varphi(t - \theta) d\theta.$$

Let

$$\ln^+ x = \begin{cases} \ln x, & \text{if} \quad x \geq 1 \\ 0, & \text{if} \quad x < 1. \end{cases}$$

A holomorphic function $f(z), z \in \mathbb{D}$ is said to be of class $\mathbf{N}$ if

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}} \ln^+ |f(re^{it})| dt < +\infty,$$

and $f \in H^p(\mathbb{D}), 1 \leq p < \infty$ if

$$\sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(re^{it})|^p dt < +\infty.$$

Moreover, $f \in H^\infty(\mathbb{D})$ if $\sup_{0 \leq r < 1} \|f(re^{it})\|_{L^\infty(\mathbb{T})} < +\infty$. We also have that $H^p(\mathbb{D}) \subset \mathbf{N}$ for all $1 \leq p \leq \infty$.

If $f \in \mathbf{N}$, according to a well-known theorem [7] (see also [5]), f is a quotient of two bounded holomorphic functions. Hence, according to Fatou's theorem, the non-tangential limit $f(e^{it})$ exists almost everywhere on the unit circle, and $\ln |f(e^{it})|$ is integrable unless f vanishes everywhere. Moreover, the map $T : f(z) \longrightarrow f(e^{it})$ establishes an isomorphism of $H^p(\mathbb{D}), 1 \leq p < \infty$ onto $H^p(\mathbb{T})$. Furthermore, facts related to metric properties in the space $H^p(\mathbb{D})$ are applicable in $H^p(\mathbb{T})$ and vice versa, without any special quotation.

The spaces $H^p(\mathbb{D})$ have been studied in several works (e.g., [5,8–10], among others). A holomorphic function F in $\mathbb{D}$ is an outer function if

$$F(re^{it}) = e^{i\alpha} e^{\varphi * H_r(t)}, \quad \alpha \in \mathbb{T},$$

where φ is a real-valued integrable function defined on $\mathbb{T}$ [4] (see also [11,12]). Evidently, F is a non-zero holomorphic function and $F \in H^1(\mathbb{D})$ if and only if $e^{\varphi(t)}$ is integrable. The function F has non-tangential limits almost everywhere on the unit circle: $F^*(t) := \lim_{z \to e^{it}} F(z)$, and

$$|F^*(t)| = e^{\varphi(t)}.$$

Moreover, $\ln |F(z)|$ is a harmonic function in $\mathbb{D}$ and

$$\frac{1}{2\pi} \int_{\mathbb{T}} \ln |F(re^{it})| dt = \ln |F(0)| \quad 0 \leq r < 1.$$

For a complex-valued integrable function g defined on $\mathbb{T}$, such that $\ln |g(t)|$ is integrable, we set

$$G_g(re^{it}) = e^{\ln |g| * H_r(t)}. \tag{1}$$

The following statement [8] holds.

Proposition 1. *Let $F \in H^1(\mathbb{D})$ be an outer function. Then*

$$G_{F^*}(z) = e^{i\alpha} F(z), z \in \mathbb{D} \quad \textit{for some} \quad \alpha \in \mathbb{T}.$$

If $F \in H^p(\mathbb{D}), 1 \leq p < \infty$, then

$$G_{F^*}(re^{it}) = e^{\ln |F^*| * P_r(t)} e^{i \ln |F^*| * Q_r(t)}$$

and according to Fatou's and Luzin–Privalov's theorems [10,13], we have that

$$\ln |F^*(t)| = \lim_{r \to 1-} \ln |F^*| * P_r(t) \qquad \text{a.e. on} \quad \mathbb{T},$$

$$\lim_{r \to 1-} \ln |F^*| * Q_r(t) := \widetilde{\ln |F^*|}(t) \qquad \text{a.e. on} \quad \mathbb{T},$$

where $\tilde{g}$ is denoted as the conjugate function of an integrable function g. Thus, we have that almost everywhere on $\mathbb{T}$

$$G^*_{F^*}(t) = \lim_{r\to 1-} G_{F^*}(re^{it}) = |F^*(t)|e^{i\widetilde{\ln|F^*|}(t)}, \tag{2}$$

According to Jensen's inequality, it follows that for $0 < r < 1$

$$\frac{1}{2\pi}\int_{\mathbb{T}} |G_{F^*}(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi}\int_{\mathbb{T}} |F^*(\theta)|^p d\theta,$$

which yields $G_{F^*} \in H^p(\mathbb{D})$ and $G^*_{F^*} \in H^p(\mathbb{T})$. We also have that

$$G_{F^*}(0) = e^{\frac{1}{2\pi}\int_{\mathbb{T}} \ln|F^*(t)|dt} \neq 0.$$

The function $\frac{1}{G_{F^*}(re^{it})}$ is holomorphic in $\mathbb{D}$, has no zeros, and belongs to $\mathbf{N}$. Clearly,

$$\lim_{r\to 1-} [G_{F^*}(re^{it})]^{-1} = |F^*(t)|^{-1} e^{-i\widetilde{\ln|F^*|}(t)}.$$

Let $\mathbf{B}$ be a separable Banach space with the dual space $\mathbf{B}^*$. The closed linear span in $\mathbf{B}$ of a system of elements $X = \{x_k\}_{x=0}^{\infty} \subset \mathbf{B}$ is denoted by $\overline{\mathrm{span}}_{\mathbf{B}}(X)$. A system $X = \{x_k\}_{x=0}^{\infty} \subset \mathbf{B}$ is complete in $\mathbf{B}$ if $\overline{\mathrm{span}}_{\mathbf{B}}(X) = \mathbf{B}$. A system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ is called minimal if there exists a system $X^* = \{\phi_n\}_{n=0}^{\infty} \subset \mathbf{B}^*$, such that

$$\phi_n(x_k) = \delta_{nk} \quad (n, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

where δ_{nk} is the Kronecker symbol ($\delta_{nk} = 0$ if $n \neq k$ and $\delta_{kk} = 1$). The system X^* is called dual to X. It is easy to observe that if X is a complete and minimal system in $\mathbf{B}$, then the dual system X^* is unique [14]. A set $\Psi \subset \mathbf{B}^*$ is called total if

$$\phi(x) = 0 \quad \text{for} \quad x \in \mathbf{B} \quad \text{and for all} \quad \phi \in \Psi$$

if and only if $x = \mathbf{0}$. A system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ is an M—basis in $\mathbf{B}$ if X is complete and minimal in $\mathbf{B}$ and its dual system X^* is total. A complete and minimal system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ with the dual system $X^* = \{\phi_k\}_{k=0}^{\infty} \subset \mathbf{B}^*$ is uniformly minimal if there exists $C > 0$, such that

$$\|x_k\|_{\mathbf{B}} \|\phi_k\|_{\mathbf{B}^*} \leq C \qquad \text{for all} \quad k \in \mathbb{N}_0.$$

We say that a system of elements, $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$, is an A—basis of the Banach space $\mathbf{B}$ if X is closed and minimal in $\mathbf{B}$ and for any $x \in \mathbf{B}$

$$\lim_{r\longrightarrow 1-} \|x - \sum_{k=1}^{\infty} r^k \phi^*_k(x) x_k\|_{\mathbf{B}} = 0,$$

where $X^* = \{\phi^*_k\}_{n=1}^{\infty} \subset B^*$ is the dual system.

1.2. Classes of Weight Functions

Furthermore, we consider only the weight functions on $\mathbb{T}$. For any $1 \leq p < \infty$, we denote by $\mathcal{W}_p$ the class of all weight functions $w \geq 0$ integrable on $\mathbb{T}$ and such that

$$\int_{\mathbb{T}} [w(t)]^{-\frac{1}{p-1}} dt < +\infty, \qquad \text{if} \quad p = 1 \qquad \text{then} \quad \frac{1}{w} \in L^\infty(\mathbb{T}).$$

Denote

$$e^{\mathcal{W}} := \{w \geq 0 : \ln w \in L^1(\mathbb{T})\}.$$

We say that $w(x) \geq 0$ is an admissible weight function if $w \in e^{\mathcal{W}}$. The class $\mathcal{A}_p, p \geq 1$ contains only the weights w that satisfy the following condition: there exists $C_p > 0$, such that

$$\frac{1}{|I|}\int_I w(t)dt\left[\frac{1}{|I|}\int_I w(t)^{-\frac{1}{p-1}}dt\right]^{p-1} \leq C_p$$

holds for any interval $I \subset \mathbb{T}$. Sometimes, it is called the Muckenhoupt condition [15,16]. We note that the class $\mathcal{A}_p \subset \mathcal{W}_p$ in an equivalent form appeared earlier in M. Rosenblum's article [17], where weighted H^p spaces were considered, maybe for the first time. In the same article, another class of weight functions $\mathcal{R}$ was studied. We say that $w \in \mathcal{R}$ if $w \in e^{\mathcal{W}}$, and there exists $C > 0$, such that

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{w(t)}{|G_w(re^{it})|} P_r(\theta - t)dt \leq C \qquad \forall\theta \in \mathbb{T} \quad \text{and} \quad 0 \leq r < 1. \tag{3}$$

Using (1), it is easy to observe that if $w \in e^{\mathcal{W}}$, then for $q > 0$,

$$|G_{w^{1/q}}(re^{it})|^q = |G_w(re^{it})| \qquad \forall t \in \mathbb{T} \quad \text{and} \quad 0 \leq r < 1. \tag{4}$$

Note that (see [17])

$$\mathcal{A}_\infty := \bigcup_{p\geq 1} \mathcal{A}_p \subseteq \mathcal{R} \subseteq e^{\mathcal{W}}. \tag{5}$$

2. On Beurling Systems

Let $p \in [1, +\infty)$ be a fixed number and suppose that $F \in H^p(\mathbb{D})$ is an outer function. Using Proposition 1, we have that $G^*_{F^*}(t) = e^{i\alpha}F^*(t)$ almost everywhere on $\mathbb{T}$. Furthermore, we suppose that

$$G_{F^*}(z) = F(z), \quad z \in \mathbb{D} \quad \text{and} \quad F(0) \in \mathbb{R},$$

for convenience. Note that if $\{e^{ijt}F^*(t)\}_{j=0}^\infty$ is a basis in one or another sense, then the system $\{e^{ijt}cF^*(t)\}_{j=0}^\infty$ for any constant c, $|c| = 1$ will be a basis in the same sense. We write the Fourier series of the function $F^* \in H^p(\mathbb{T})$ as

$$F^*(\theta) \sim a_0 + \sum_{n=1}^\infty a_n e^{in\theta}, \qquad a_0 \neq 0, a_0 \in \mathbb{R}.$$

Moreover,

$$F(z) = a_0 + \sum_{n=1}^\infty a_n z^n \quad z \in \mathbb{D}.$$

Let

$$[F(z)]^{-1} = c_0 + \sum_{n=1}^\infty c_n z^n \quad z \in \mathbb{D}. \tag{6}$$

Then, for $z \in \mathbb{D}$,

$$1 = F(z)[F(z)]^{-1} = 1 + \sum_{n=1}^\infty b_n z^n,$$

where

$$b_n = \sum_{j=0}^n a_j c_{n-j} = 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad c_0 = \frac{1}{a_0}. \tag{7}$$

2.1. A Remarkable System of Polynomials

Set $T_0(t) \equiv c_0$ and for $n \in \mathbb{N}$

$$T_n(t) = c_0 e^{in\theta} + \sum_{\nu=0}^{n-1} \bar{c}_{n-\nu} e^{i\nu t}, \tag{8}$$

where $\{c_j\}_{j=0}^{\infty}$ are the corresponding coefficients of the representation of $F^{-1}(z)$. Using (7), we can deduce that if $j \in \mathbb{N}_0$ and $j \leq n$,

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}} e^{ijt} F^*(t)\overline{T_n(t)}dt &= \frac{1}{2\pi}\int_{\mathbb{T}} \sum_{k=0}^{n-j} a_k e^{i(j+k)t}\overline{T_n(t)}dt \\ &= \sum_{k=0}^{n-j} a_k c_{n-j-k} = \delta_{jn}.\end{aligned}$$

It is clear that the above integral is equal to zero if $j > n$. The following theorem holds.

Theorem 1. *The system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is an M—basis in $H^p(\mathbb{T})$.*

Proof. We have checked that $\{T_n(t)\}_{n=0}^{\infty}$ is the system dual to $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$. Suppose that there exists $f \in H^p(\mathbb{T})$, such that

$$\int_{\mathbb{T}} f(t)\overline{T_n(t)}dt = 0 \quad \text{for all} \quad n \in \mathbb{N}_0.$$

From (8), it follows that

$$\int_{\mathbb{T}} e^{-int} f(t)dt = 0 \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Hence, $f(t) = 0$ almost everywhere on $\mathbb{T}$. □

Theorem 2. *The system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is uniformly minimal in $H^p(\mathbb{T})$, $1 < p < \infty$ if and only if $[F^*]^{-1} \in H^{p'}(\mathbb{T})$.*

Proof. If $[F^*]^{-1} \in H^{p'}(\mathbb{T}), 1 < p < \infty$, then using Proposition 1 and Jensen's inequality, we have that for $0 < r < 1$,

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}} |F(re^{it})|^{-p'}dt &= \frac{1}{2\pi}\int_{\mathbb{T}} e^{\ln|F^*|^{-p'} * P_r(t)}dt \\ &\leq \frac{1}{2\pi}\int_{\mathbb{T}} |F^*(\theta)|^{-p'}d\theta < +\infty,\end{aligned}$$

which yields $\frac{1}{F} \in H^{p'}(\mathbb{D})$. Set

$$S_n(t) = \sum_{k=0}^{n} c_k e^{ikt}, \qquad n \in \mathbb{N}. \tag{9}$$

Then, we have that $e^{int}\overline{S_n(t)} = T_n(t)$, which means that

$$\|T_n\|_{L^q(\mathbb{T})} = \|S_n\|_{L^q(\mathbb{T})} \qquad \text{for} \quad 1 \leq q \leq \infty, n \in \mathbb{N}. \tag{10}$$

Note that $S_n[\frac{1}{F^*}](t) = S_n(t)$ if $\frac{1}{F^*} \in H^{p'}(\mathbb{T})$. Hence, there exists $C_{p'} > 0$ (see [10], volume 2, chapter 7), such that

$$\sup_n \|S_n\|_{L^{p'}(\mathbb{T})} \leq C_{p'}\|[F^*]^{-1}\|_{L^{p'}(\mathbb{T})}.$$

For the proof of the necessity, suppose that the system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is uniformly minimal in $H^p(\mathbb{T})$. The norms in $H^p(\mathbb{T})$ of all elements of the system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ are equal to $\|F^*\|_{L^p(\mathbb{T})}$. On the other hand, using (10), it follows that

$$\sup_n \|S_n\|_{L^{p'}(\mathbb{T})} < +\infty,$$

where $S_n(t)$ is defined in (9). According to Banach's theorem (see [18]) on the weak* compactness of the closed unit ball in the dual space, it follows that there exists a subsequence of natural numbers $\{n_\nu\}_{\nu=1}^{\infty}$ and $\psi \in L^{p'}(\mathbb{T})$, such that for any $\varphi \in L^p(\mathbb{T})$,

$$\lim_{\nu\to\infty} \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(t)\overline{S_{n_\nu}(t)}dt = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(t)\overline{\psi(t)}dt.$$

Hence, $\psi \in H^{p'}(\mathbb{T})$ and

$$c_j(\psi) = c_j \qquad j \in \mathbb{N}_0.$$

This means that

$$\psi * P_r(t) = \frac{1}{F(re^{it})} \qquad 0 \le r < 1, t \in \mathbb{T}.$$

Thus, we find that ψ coincides with $[F^*]^{-1}$ almost everywhere on $\mathbb{T}$, and $[F^*]^{-1} \in H^{p'}(\mathbb{T})$. □

Theorem 3. *If the system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is uniformly minimal in $H^1(\mathbb{T})$, then $[F^*]^{-1} \in H^\infty(\mathbb{T})$. If $[F^*]^{-1} \in H^\infty(\mathbb{T})$ and the partial sums of its Fourier series are uniformly bounded in the $C(\mathbb{T})$ norm, then the system $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is uniformly minimal in $H^1(\mathbb{T})$.*

We omit the proof because it is similar to the proof of the previous theorem.

The following lemma is a useful tool for further exposition. The related statements can be found in [10,13].

Lemma 1. *Let $f \in H^p(\mathbb{T})$ and $g \in H^{p'}(\mathbb{T})$, where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $p' = \infty$ if $p = 1$. Then:*
(1) $S[fg] = S[f]S[g]$, and $c_n(fg) = \sum_{j=0}^{n} c_j(f)c_{n-j}(g)$;
(2) $S_n[fg](t) = \sum_{j=0}^{n} c_j(f)e^{ijt}S_{n-j}[g](t)$ for any $n = 0, 1, \ldots$.

Proof. We omit the proof of statement (1) because it should be well-known. We have

$$\begin{aligned}\sum_{j=0}^{n} c_j(f)e^{ijt}S_{n-j}[g](t) &= \sum_{j=0}^{n}\sum_{\nu=0}^{n-j} c_j(f)c_\nu(g)e^{i(j+\nu)t} \\ &= \sum_{k=0}^{n} e^{ikt}\sum_{j=0}^{n} c_j(f)c_{k-j}(g).\end{aligned}$$

□

Theorem 4. *Let $|F^*|^p \in \mathcal{A}_p$, $1 < p < \infty$. Then, $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is a Schauder basis in $H^p(\mathbb{T})$.*

Proof. We should check that the conditions of Banach's theorem [18] hold in our case. According to Theorem 1, we know that $\{e^{ijt}F^*(t)\}_{j=0}^{\infty}$ is complete and minimal in $H^p(\mathbb{T})$. Set

$$E_n(t,\theta) = F^*(t)\sum_{k=0}^{n} e^{ikt}\overline{T_k}(\theta) = F^*(t)\sum_{k=0}^{n} e^{ikt}\sum_{j=0}^{k} c_{k-j}e^{-ij\theta}$$
$$= F^*(t)\sum_{j=0}^{n} e^{-ij\theta}\sum_{k=j}^{n} c_{k-j}e^{ikt} = F^*(t)\sum_{j=0}^{n} e^{ij(t-\theta)}S_{n-j}(t).$$

For $g \in H^p(\mathbb{T})$, let

$$\sigma_n[g](t) = \frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)E_n(t,\theta)d\theta = F^*(t)\sum_{j=0}^{n} c_j(g)e^{ijt}S_{n-j}[[F^*]^{-1}](t)$$
$$= F^*(t)S_n[g[F^*]^{-1}](t),$$

where the last equality is obtained using Lemma 1. Let $w(t) = |F^*(t)|^p$. Then, using a well-known weighted norm inequality [19] (see also [20]), we finish the proof.

$$\int_{\mathbb{T}} |\sigma_n[g](t)|^p dt = \int_{\mathbb{T}} |S_n[g[F^*]^{-1}](t)|^p w(t)dt$$
$$\leq B_p \int_{\mathbb{T}} |g(t)[F^*(t)]^{-1}|^p w(t)dt = B_p \int_{\mathbb{T}} |g(t)|^p dt.$$

□

3. Weighted H^p Spaces

In this section, we consider that $p, 1 \leq p < \infty$ is fixed. Let w be an admissible weight function. We would like to use the notations of the previous section. Let $F = G_{w^{1/p}} \in H^p(\mathbb{D})$. Then, using (1) and Proposition 1, F is an outer function. From (2)

$$F^*(t) = \lim_{r\to 1-} F(re^{it}) = [w(t)]^{\frac{1}{p}} e^{\frac{i}{p}\varrho(t)}, \quad \text{where} \quad \varrho(t) = \widetilde{\ln w}(t). \tag{11}$$

Set $\mathrm{Y} = \{e^{int}\}_{n=0}^{\infty}$ and $\mathrm{Y}_0 = \{e^{int}\}_{n=1}^{\infty}$. We write

$$\overline{\mathrm{span}}_{L^p(\mathbb{T},w)}(\mathrm{Y}) := H^p(\mathbb{T},w) \quad \text{and} \quad \overline{\mathrm{span}}_{L^p(\mathbb{T},w)}(\mathrm{Y}_0) := H_0^p(\mathbb{T},w).$$

We consider that

$$H^\infty(\mathbb{T},w) := H^\infty(\mathbb{T}) = \{f \in L^\infty(\mathbb{T}) : \int_{\mathbb{T}} f(t)e^{int}dt = 0 \text{ for all } n \in \mathbb{N}\}.$$

Set

$$\overline{H^p}(\mathbb{T},w) = \{\overline{f(t)} : f \in H^p(\mathbb{T},w)\} \quad \text{and} \quad \overline{H_0^p}(\mathbb{T},w) = e^{-it}\overline{H^p}(\mathbb{T},w).$$

In [17], the weighted spaces $H^q(\mathbb{D},w)$ were defined for $w \in \mathcal{R}$. A function f holomorphic in $\mathbb{D}$ belongs to $H^q(\mathbb{D},w), 1 \leq q < \infty$ if

$$\|f\|_{H^q(\mathbb{D},w)} = \sup_{0\leq r<1}\left(\int_{\mathbb{T}} |f(re^{it})|^q w(t)dt\right)^{\frac{1}{q}} < \infty.$$

The results on weighted Hardy spaces can be found in [21]. The following statement was formulated by M.Rosenblum in the introduction to his article [17]. In the text, the reader can find indications for the proof, but the author did not formulate the statement as a theorem. This is why we prefer to formulate the statement as a hypothesis.

Hypothesis 1 (M. Rosenblum)**.** *Let $w \in \mathcal{R}$. Then, the operator $\Lambda : f(z) \to f(e^{it})$ is a vector space isomorphism mapping $H^q(\mathbb{D}, w)$ onto $H^q(\mathbb{T}, w)$, such that Λ and Λ^{-1} are bounded. If Λ is an isometry, then $w \equiv c, c > 0$.*

In the formulated statement, one considers that given $f \in H^q(\mathbb{D}, w)$, where

$$f(z) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k z^k,$$

then $f(e^{it}) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k e^{ikt}$ exists and $f(e^{it}) \in H^q(\mathbb{T}, w)$. Later in this paper, we show that it is also true when $w \in e^{\mathcal{W}}$(see Proposition 2). In the next section, we prove that Hypothesis 1 holds. Moreover, we provide the integral representation of the operator Λ^{-1}.

3.1. On the Dual Space of $H^p(\mathbb{T}, w)$

In this subsection, we provide the characterization of the dual space of $H^p(\mathbb{T}, w)$, $1 \leq p < \infty$ when w is an admissible weight function.

Lemma 2. *Let w be an admissible weight function, $1 \leq p < \infty$, and $F \in H^p(\mathbb{D})$ be the outer function defined as above. Then, $\phi F^* \in H^p(\mathbb{T})$ for $\phi \in H^p(\mathbb{T}, w)$, and if $\psi \in H^p(\mathbb{T})$, then $\psi[F^*]^{-1} \in H^p(\mathbb{T}, w)$.*

Proof. For the proof, we use Relation (11) and the fact that the system $\{e^{int} F^*(t)\}_{n=0}^{\infty}$ is complete in $H^p(\mathbb{T})$. If $\phi \in H^p(\mathbb{T}, w)$, there exists a sequence of trigonometric polynomials $P_n(t) = \sum_{j=0}^{N_n} b_j e^{ijt}$, such that

$$0 = \lim_{n\to\infty} \int_{\mathbb{T}} |\phi(t) - P_n(t)|^p w(t) dt = \lim_{n\to\infty} \int_{\mathbb{T}} |\phi(t)F^*(t) - P_n(t)F^*(t)|^p dt.$$

Thus, $\phi F^* \in H^p(\mathbb{T})$. On the other hand, if $\psi \in H^p(\mathbb{T})$ we find the trigonometric polynomials $\tilde{P}_n(t) = \sum_{j=0}^{\tilde{N}_n} \tilde{b}_j e^{ijt}$, such that

$$0 = \lim_{n\to\infty} \int_{\mathbb{T}} |\psi(t) - \tilde{P}_n(t)F^*(t)|^p dt = \lim_{n\to\infty} \int_{\mathbb{T}} \Big|\frac{\psi(t)}{F^*(t)} - \tilde{P}_n(t)\Big|^p w(t) dt.$$

□

We should describe the annihilator $H^p(\mathbb{T}, w)^{\perp}$ of $H^p(\mathbb{T}, w)$ in $L^{p'}(\mathbb{T}, w)$:

$$H^p(\mathbb{T}, w)^{\perp} = \left\{ \psi \in L^{p'}(\mathbb{T}, w) : \int_{\mathbb{T}} \phi(\theta)\overline{\psi(\theta)} w(\theta) d\theta = 0 \quad \forall \phi \in H^p(\mathbb{T}, w) \right\}.$$

Suppose that $1 < p < \infty$, and let $\psi \in H^p(\mathbb{T}, w)^{\perp}$. For any $\phi \in H^p(\mathbb{T}, w)$, we write

$$0 = \int_{\mathbb{T}} \phi(\theta)\overline{\psi(\theta)} w(\theta) d\theta = \int_{\mathbb{T}} \phi(\theta)F^*(\theta)\overline{\psi(\theta)}[w(\theta)]^{\frac{1}{p'}} e^{-\frac{i}{p}\varrho(\theta)} d\theta.$$

It is well-known that the annihilator of $H^p(\mathbb{T})$ is $\overline{H_0^{p'}(\mathbb{T})}$ (see, e.g., [5]). Hence, using Lemma 2, it follows that

$$\overline{\psi(\theta)}[w(\theta)]^{\frac{1}{p'}} e^{-\frac{i}{p}\varrho(\theta)} \in H_0^{p'}(\mathbb{T}) \quad \text{and} \quad \overline{\psi(\theta)} e^{-i\varrho(\theta)} \in H_0^{p'}(\mathbb{T}, w).$$

This yields $\psi(\theta) \in e^{-i\varrho(\theta)}\overline{H_0^{p'}}(\mathbb{T}, w)$. Conversely, if

$$\psi(\theta) \in e^{-i\varrho(\theta)}\overline{H_0^{p'}}(\mathbb{T}, w) = e^{-i\varrho(\theta)} e^{-it}\overline{H^{p'}}(\mathbb{T}, w)$$

the using (11) and Lemma 2, $\overline{\psi(\theta)}e^{-\frac{i}{p}\varrho(\theta)}[w(\theta)]^{\frac{1}{p'}} \in H^{p'}(\mathbb{T})$. Hence, for all $\phi \in H^p(\mathbb{T},w)$

$$\int_{\mathbb{T}} \phi(\theta)\overline{\psi(\theta)}w(\theta)d\theta = \int_{\mathbb{T}} \phi(\theta)F^*(\theta)\overline{\psi(\theta)}[w(\theta)]^{\frac{1}{p'}}e^{-\frac{i}{p}\varrho(\theta)}d\theta = 0.$$

In the case of $p = 1$, the proof is similar and we omit it. Thus,

$$H^p(\mathbb{T},w)^{\perp} = e^{-i\varrho(\theta)}\overline{H_0^{p'}}(\mathbb{T},w)$$

and from Theorem 7.1 in [5], it follows that $[H^p(\mathbb{T},w)]^*$ is isometrically isomorphic to $L^{p'}(\mathbb{T},w)/\overline{H_0^{p'}}(\mathbb{T},w)$, and for every $\psi \in [H^p(\mathbb{T},w)]^*$,

$$\sup_{\|\phi\|_{H^p(\mathbb{T},w)}\le 1}\left|\frac{1}{2\pi}\int_{\mathbb{T}} \phi(t)\overline{\psi(t)}w(t)dt\right| = \min_{h\in H^p(\mathbb{T},w)^{\perp}} \|\psi + h\|_{L^{p'}(\mathbb{T},w)}.$$

As above, we check that for $1 < p < \infty$,

$$H_0^p(\mathbb{T},w)^{\perp} = e^{-i\varrho(\theta)}\overline{H^{p'}}(\mathbb{T},w).$$

Thus, the following statement is proved.

Theorem 5. *For $1 < p < \infty$, the dual space $[H^p(\mathbb{T},w)]^*$ is a reflexive Banach space isometrically isomorphic to $L^{p'}(\mathbb{T},w)/\overline{H_0^{p'}}(\mathbb{T},w)$. Moreover, $[H^1(\mathbb{T},w)]^*$ is isometrically isomorphic to $L^{\infty}(\mathbb{T})/\overline{H_0^{\infty}(\mathbb{T})}$.*

3.2. Summation Basis

The following lemma is the analog of Banach's theorem for a given system to be an A—basis. The proof is similar to the proof of Banach's original theorem, and we do not provide it here. References for the summation bases can be found in [22].

Lemma 3. *Let $X = \{x_k\}_{x=1}^{\infty} \subset \mathbf{B}$ be complete and minimal in $\mathbf{B}$ with the dual system $X^* = \{\varphi_n\}_{n=1}^{\infty}$. Then, X is an A—basis of $\mathbf{B}$ if and only if there exists a constant $C > 0$, such that for any $x \in \mathbf{B}$,*

$$\sup_{0<r<1} \|\sum_{k=1}^{\infty} r^k \phi_k(x)x_k\|_{\mathbf{B}} \le C\|x\|_{\mathbf{B}}.$$

In this subsection, we suppose that w is an admissible weight function. We recall that in this case, $F^{-1} \in \mathbf{N}$, and we have Representation (6). Hence, for any $0 < R < 1$, there exists $N_R \in \mathbb{N}$, such that $|c_n| < \frac{1}{R^n}$ for $n > N_R$. For $0 \le r < 1$, set

$$K_r(t,\theta) = F^*(t)[c_0 + \sum_{n=1}^{\infty} r^n e^{int}\overline{T_n(\theta)}]. \tag{12}$$

Note that the following series

$$\sum_{n=1}^{\infty} r^n \sum_{j=0}^{n} |c_j|$$

converges uniformly on $[0,\rho]$ for any $0 < \rho < 1$. Indeed, for $\rho < R < 1$ and $n > N_R$, we have that

$$\sum_{j=0}^{n} |c_j| \le \sum_{j=0}^{N_R} |c_j| + R^{-N_R-1}\sum_{\nu=0}^{n-N_R-1} R^{-\nu} \le C_R + R^{-n}\frac{1}{1-R}.$$

Hence, Series (12) converges absolutely on $[0,1) \times \mathbb{T} \times \mathbb{T}$. The absolute convergence of the series permits us to write

$$\begin{aligned} K_r(t,\theta) &= F^*(t) \sum_{n=0}^{\infty} r^n e^{int} \sum_{j=0}^{n} c_{n-j} e^{-ij\theta} = F^*(t) \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} r^n c_{n-j} e^{int} e^{-ij\theta} \\ &= F^*(t) C_r(t-\theta) \sum_{\nu=0}^{\infty} r^\nu c_\nu e^{i\nu t} = F^*(t) C_r(t-\theta) F^{-1}(re^{it}). \end{aligned}$$

Thus, from Fatou's theorem, we obtain the following theorem.

Theorem 6. *Any function $\phi \in H^p(\mathbb{T})$ is the non-tangential limit of*

$$\Phi(re^{it}) := \frac{1}{2\pi} \int_{\mathbb{T}} \phi(\theta) K_r(t,\theta) d\theta.$$

Let $\phi(\theta) = u(\theta) + iv(\theta) \in H^p(\mathbb{T}), 1 \le p < \infty$ and $c_0(f) \in \mathbb{R}$. Then, it is well-known (see, e.g., [8]) that

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{T}} \phi(\theta) C_r(t-\theta) d\theta &= \frac{1}{2\pi} \int_{\mathbb{T}} u(\theta) H_r(t-\theta) d\theta \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} \phi(\theta) P_r(t-\theta) d\theta. \end{aligned} \tag{13}$$

Consider the following family of maps

$$\Lambda_r[\phi](t) = F^*(t) F^{-1}(re^{it}) \frac{1}{2\pi} \int_{\mathbb{T}} \phi(\theta) P_r(t-\theta) d\theta \quad 0 < r < 1,\ \phi \in H^p(\mathbb{T}).$$

Theorem 7. *The inequality*

$$\|\Lambda_r[\phi]\|_{L^p(\mathbb{T})} \le C_p \|\phi\|_{H^p(\mathbb{T})} \qquad 0 < r < 1, \tag{14}$$

holds for all $\phi \in H^p(\mathbb{T}), 1 \le p < \infty$ and $C_p > 0$ independent of ϕ, if and only if $w \in \mathcal{R}$.

Proof. Let $w \in \mathcal{R}$. If $p = 1$, then

$$\begin{aligned} \|\Lambda_r[\phi]\|_{L(\mathbb{T})} &= \frac{1}{4\pi^2} \int_{\mathbb{T}} \left| F^*(t) F^{-1}(re^{it}) \int_{\mathbb{T}} \phi(\theta) P_r(t-\theta) d\theta \right| dt \\ &\le \frac{1}{4\pi^2} \int_{\mathbb{T}} |\phi(\theta| \int_{\mathbb{T}} w(t) |F^{-1}(re^{it})| P_r(t-\theta) dt d\theta \le C \|\phi\|_{H^1(\mathbb{T})}, \end{aligned}$$

where $C > 0$ is the constant in the condition $\mathcal{R}$. If $1 < p < \infty$, then for any $\psi \in L^{p'}(\mathbb{T})$, we have

$$\begin{aligned} \left| \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t) \Lambda_r[\phi](t) dt \right| &= \frac{1}{4\pi^2} \left| \int_{\mathbb{T}} \psi(t) F^*(t) F^{-1}(re^{it}) \int_{\mathbb{T}} \phi(\theta) P_r(t-\theta) d\theta dt \right| \\ &\le \frac{1}{4\pi^2} \int_{\mathbb{T}} |\phi(\theta)| \int_{\mathbb{T}} |\psi(t)| |F^*(t) F^{-1}(re^{it})| P_r(t-\theta) dt d\theta. \end{aligned}$$

From the Hölder inequality, we deduce

$$\begin{aligned}
&\int_{\mathbb{T}} |\psi(t)||F^*(t)F^{-1}(re^{it})|P_r(t-\theta)dt \\
&\quad \le \left(\int_{\mathbb{T}} |\psi(t)|^{p'} P_r(t-\theta)dt\right)^{\frac{1}{p'}} \left(\int_{\mathbb{T}} |F^*(t)F^{-1}(re^{it})|^p P_r(t-\theta)dt\right)^{\frac{1}{p}} \\
&\quad = \left(\int_{\mathbb{T}} |\psi(t)|^{p'} P_r(t-\theta)dt\right)^{\frac{1}{p'}} \left(\int_{\mathbb{T}} \frac{w(t)}{|F(re^{it})|^p} P_r(t-\theta)dt\right)^{\frac{1}{p}} \\
&\quad \le (2\pi C)^{\frac{1}{p}} \left(\int_{\mathbb{T}} |\psi(t)|^{p'} P_r(t-\theta)dt\right)^{\frac{1}{p'}}.
\end{aligned}$$

The last inequality follows by using (3) and (4). Hence,

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{\mathbb{T}} \psi(t)\Lambda_r[\phi](t)dt\right| \\
&\quad \le \frac{1}{4\pi^2}(2\pi C)^{\frac{1}{p}} \left(\int_{\mathbb{T}} |\phi(\theta)|^p d\theta\right)^{\frac{1}{p}} \left(\int_{\mathbb{T}}\int_{\mathbb{T}} |\psi(t)|^{p'} P_r(t-\theta)d\theta dt\right)^{\frac{1}{p'}} \\
&\quad = C^{\frac{1}{p}} \left(\frac{1}{2\pi}\int_{\mathbb{T}} |\phi(\theta)|^p d\theta\right)^{\frac{1}{p}} \left(\frac{1}{2\pi}\int_{\mathbb{T}} |\psi(t)|^{p'} dt\right)^{\frac{1}{p'}},
\end{aligned}$$

which yields (14).

For the proof of necessity, fix some $\rho, 0 < \rho < 1$, and set $\varphi_\theta(t), \theta \in \mathbb{T}$ equal to any branch of $\frac{1}{(1-\rho e^{i(t-\theta)})^{\frac{2}{p}}}$. Then, we have that

$$\begin{aligned}
\|\Lambda_r[\varphi_\theta]\|^p_{L^p(\mathbb{T})} &= \frac{1}{2\pi}\int_{\mathbb{T}} \left|\frac{F^*(t)}{F(re^{it})}\right|^p \frac{1}{|1-\rho re^{i(t-\theta)}|^2} dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{w(t)}{|F(re^{it})|^p} \frac{1}{|1-\rho re^{i(t-\theta)}|^2} dt
\end{aligned}$$

If we suppose that the maps $\Lambda_r : H^p(\mathbb{T}) \to L^p(\mathbb{T})$ are uniformly bounded, then for $r = \rho$, we obtain

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{w(t)}{|F(re^{it})|^p} \frac{1}{|1-r^2 e^{i(t-\theta)}|^2} dt \le \frac{C_p^p}{2\pi}\int_{\mathbb{T}} \frac{1}{|1-re^{i(t-\theta)}|^2} dt = \frac{C_p^p}{1-r^2},$$

where the last relation follows using Parseval's equality. Using the inequality $|1-r^2e^{ix}|^2 \le 4|1-re^{ix}|^2$ for all $x \in \mathbb{T}$ and $0 < r < 1$, we finish the proof. $\square$

We would like to formulate the main result of this subsection from another point of view. Let $F \in H^p(\mathbb{D}), 1 \le p < \infty$ be an outer function, and let $F(e^{it})$ be the non-tangential limit of F on the unit circle. Note that Beurling's approximation theorem states that the system $\{F(e^{it})e^{ikt}\}_{k=0}^\infty$ is complete in $H^p(\mathbb{T})$. Set $w(t) = |F(e^{it})|^p$. Then, using Proposition 1, we can claim that Theorem 7 yields the following.

Theorem 8. *Let $F \in H^p(\mathbb{D}), 1 \le p < \infty$ be an outer function, and let $F(e^{it})$ be the non-tangential limit of F on the unit circle.*
Then, $\{F(e^{it})e^{ikt}\}_{k=0}^\infty$ is an $A-$basis in $H^p(\mathbb{T})$ if and only if $|F(e^{it})|^p \in \mathcal{R}$.

3.3. The System $\{e^{ijt}\}_{j=0}^\infty$ in the Space $H^p(\mathbb{T}, w)$

The following assertion holds.

Theorem 9. *For any admissible weight function w, the system $\{e^{ijt}\}_{j=0}^{\infty}$ is an $M-$basis in $H^p(\mathbb{T}, w)$, $1 \le p < \infty$.*

Proof. The completeness of the system $\{e^{ijt}\}_{j=0}^{\infty}$ in $H^p(\mathbb{T}, w)$ follows by definition. Set

$$\varphi_n(t) = \frac{\overline{F^*(t)}}{w(t)} T_n(t), \qquad n \in \mathbb{N}_0,$$

where the polynomials $T_n(t)$ are defined using (8). As in the proof of Theorem 1, it is easy to check that $\{\varphi_n(t)\}_{n=0}^{\infty}$ is the dual system of $\{e^{ijt}\}_{j=0}^{\infty}$ in $H^p(\mathbb{T}, w)$. Suppose that there exists $f \in H^p(\mathbb{T}, w)$, such that for all $n \in \mathbb{N}_0$

$$0 = \int_{\mathbb{T}} f(t)\overline{\varphi_n(t)} w(t) dt = \int_{\mathbb{T}} f(t) F^*(t) \overline{T_n(t)} dt.$$

Using Lemma 2, we have that $fF^* \in H^p(\mathbb{T})$. Hence, using Theorem 1, it follows that $f(t) = 0$ almost everywhere on $\mathbb{T}$. □

Theorem 10. *The system $\{e^{ijt}\}_{j=0}^{\infty}$ is uniformly minimal in $H^p(\mathbb{T}, w)$, $1 < p < \infty$ if and only if $w \in \mathcal{W}_p$.*

Proof. The statement is an immediate consequence of Theorem 2. Using (11), we deduce that

$$\frac{1}{2\pi} \int_{\mathbb{T}} |\varphi_n(t)|^{p'} w(t) dt = \frac{1}{2\pi} \int_{\mathbb{T}} |T_n(t)|^{p'} dt.$$

□

The following theorem is a direct consequence of Theorem 4.

Theorem 11. *Let $w \in \mathcal{A}_p, 1 < p < \infty$. Then, the system $\{e^{ijt}\}_{j=0}^{\infty}$ is a Schauder basis in $H^p(\mathbb{T}, w)$.*

Let w be an admissible weight function. We expand any $f \in H^p(\mathbb{T}, w)$ with respect to the system $\{e^{ijt}\}_{j=0}^{\infty}$ and consider the Abel means of the obtained expansion. Let

$$L_r(t, \theta) = \sum_{n=0}^{\infty} r^n e^{int} \overline{\varphi_n(\theta)} = \frac{F^*(\theta)}{w(\theta)} \sum_{n=0}^{\infty} r^n e^{int} \overline{T_n(\theta)}, \quad 0 < r < 1.$$

As in the case of the kernel $K_r(t, \theta)$, we deduce that

$$L_r(t, \theta) = \frac{F^*(\theta)}{w(\theta)} C_r(t - \theta) F^{-1}(re^{it}).$$

Set

$$\sigma_r[f](t) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) L_r(t, \theta) w(\theta) d\theta = \frac{F^{-1}(re^{it})}{2\pi} \int_{\mathbb{T}} f(\theta) F^*(\theta) C_r(t - \theta) d\theta.$$

Using Lemma 2, we have that $fF^* \in H^p(\mathbb{T})$. Hence, the following theorem holds.

Theorem 12. *Any function $f \in H^p(\mathbb{T}, w), 1 \le p < \infty$ is the non-tangential limit of*

$$\Psi(re^{it}) := \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) L_r(t, \theta) d\theta.$$

Afterward, we write $f(\theta)F^*(\theta) = u_1(\theta) + iv_1(\theta) \in H^p(\mathbb{T}), 1 \leq p < \infty$ and assume that $c_0(fF^*) \in \mathbb{R}$. Then, as in (13),

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)F^*(\theta)C_r(t-\theta)d\theta &= \frac{1}{2\pi}\int_{\mathbb{T}} u_1(\theta)H_r(t-\theta)d\theta \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta.\end{aligned}$$

Theorem 13. *Let $1 \leq p < \infty$. The inequality*

$$\left\| \frac{F^{-1}(re^{it})}{2\pi}\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta \right\|_{L^p(\mathbb{T},w)} \leq C_p^*\|f\|_{H^p(\mathbb{T},w)} \qquad 0 < r < 1,$$

holds for all $f \in H^p(\mathbb{T}, w), 1 \leq p < \infty$, and $C_p^* > 0$ independent of f if and only if $w \in \mathcal{R}$.

Proof. Let $w \in \mathcal{R}$. Then, for any $g \in L^{p'}(\mathbb{T}, w)$ we have

$$\begin{aligned}&\frac{1}{4\pi^2}\left|\int_{\mathbb{T}} g(t)F^{-1}(re^{it})\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta w(t)dt\right| \\ &\leq \frac{1}{4\pi^2}\int_{\mathbb{T}} |f(\theta)F^*(\theta)|\int_{\mathbb{T}} |g(t)||F^{-1}(re^{it})|P_r(t-\theta)w(t)dtd\theta.\end{aligned}$$

Afterward, using (11), we obtain

$$\begin{aligned}&\int_{\mathbb{T}} |g(t)||F^{-1}(re^{it})|P_r(t-\theta)w(t)dt \\ &\leq \left(\int_{\mathbb{T}} |g(t)|^{p'}P_r(t-\theta)w(t)dt\right)^{\frac{1}{p'}}\left(\int_{\mathbb{T}} |F^{-1}(re^{it})|^p w(t)P_r(t-\theta)dt\right)^{\frac{1}{p}} \\ &\leq (2\pi C)^{\frac{1}{p}}\left(\int_{\mathbb{T}} |g(t)|^{p'}P_r(t-\theta)w(t)dt\right)^{\frac{1}{p'}},\end{aligned}$$

where $C > 0$ is the constant in the condition $\mathcal{R}$. Hence,

$$\begin{aligned}&\frac{1}{4\pi^2}\left|\int_{\mathbb{T}} g(t)h_p^{-1}(re^{it})\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta w(t)dt\right| \\ &\leq \frac{1}{4\pi^2}(2\pi C)^{\frac{1}{p}}\left(\int_{\mathbb{T}} |f(\theta)|^p w(\theta)d\theta\right)^{\frac{1}{p}}\left(\int_{\mathbb{T}}\int_{\mathbb{T}} |g(t)|^{p'}w(t)P_r(t-\theta)d\theta dt\right)^{\frac{1}{p'}} \\ &= C^{\frac{1}{p}}\left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(\theta)|^p w(\theta)d\theta\right)^{\frac{1}{p}}\left(\frac{1}{2\pi}\int_{\mathbb{T}} |g(t)|^{p'}w(t)dt\right)^{\frac{1}{p'}}.\end{aligned}$$

For the proof of necessity, we fix some $r, 0 < r < 1$ and take $\frac{1}{F^*(t)}\varphi_\theta(t), \theta \in \mathbb{T}$, where $\varphi_\theta(t)$ is equal to any branch of $\frac{1}{(1-re^{i(t-\theta)})^{\frac{2}{p}}}$. We omit further details because they are similar to those given in the proof of Theorem 7. □

The following statement provides a representation of the inverse operator Λ^{-1} from Hypothesis 1.

Corollary 1. *Let $w \in \mathcal{R}$ and $f \in H^p(\mathbb{T}, w), 1 \leq p < \infty$. Then, the holomorphic function*

$$f(re^{it}) = \frac{F^{-1}(re^{it})}{2\pi}\int_{\mathbb{T}} f(\theta)F^*(\theta)P_r(t-\theta)d\theta$$

belongs to $H^p(\mathbb{D}, w)$.

Thus, using Lemma 3, we obtain

Theorem 14. *The system* $\{e^{ikt}\}_{k=0}^{\infty}$ *is an* $A-$*basis in* $H^p(w), 1 \leq p < \infty$ *if and only if* $w \in \mathcal{R}$.

The following proposition is true.

Proposition 2. *Let* $w \in e^{\mathcal{W}}$ *and* $1 \leq p < \infty$. *Let*

$$f(z) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k z^k$$

be a holomorphic function in $\mathbb{D}$, *such that*

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{\mathbb{T}} |f(re^{i\theta})|^p w(\theta) d\theta < +\infty.$$

Then, there exists $\varphi \in H^p(\mathbb{T}, w)$, *such that for all* $n \in \mathbb{N}_0$,

$$\alpha_n = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(t) \overline{\varphi_n(t)} w(t) dt. \tag{15}$$

Proof. Using Lemma 2, we have that $f(re^{it})F^*(t), r \in (0,1)$ is a uniformly bounded family of functions in $H^p(\mathbb{T})$. Thus, using Banach's theorem [18], we can find a sequence $0 < r_1 < r_2 < \cdots < 1$, such that $\lim_{j\to\infty} r_j = 1$ and $f(r_j e^{it})F^*(t)$ converges weakly in $H^p(\mathbb{T}), 1 < p < \infty$. In other words, there exists $\psi \in H^p(\mathbb{T})$, such that for any $h \in H^{p'}(\mathbb{T})$,

$$\lim_{j\to\infty} \frac{1}{2\pi} \int_{\mathbb{T}} f(r_j e^{it}) F^*(t) \overline{h(t)} dt = \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t) \overline{h(t)} dt.$$

If we fix $j \in \mathbb{N}$, then for any $n \in \mathbb{N}$,

$$\alpha_n r_j^n = \frac{1}{2\pi} \int_{\mathbb{T}} f(r_j e^{it}) \overline{\varphi_n(t)} w(t) dt = \frac{1}{2\pi} \int_{\mathbb{T}} f(r_j e^{it}) F^*(t) \overline{T_n(t)} dt.$$

By letting $j \to +\infty$, we obtain

$$\alpha_n = \frac{1}{2\pi} \int_{\mathbb{T}} \psi(t) \overline{T_n(t)} dt = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(t) \overline{\varphi_n(t)} w(t) dt,$$

where $\varphi(t) = \frac{\psi(t)}{F^*(t)} \in H^p(\mathbb{T}, w)$. The proof for the case of $p = 1$ is longer but its first part is well-known (see, e.g., [8,9]). The set $f(re^{it})F^*(t), r \in (0,1)$ is uniformly bounded in the $L^1(\mathbb{T})$ norm. Afterward, we consider $L^1(\mathbb{T})$ as a subspace of the space of Borel measures, the dual of $C(\mathbb{T})$. Thus, as above, one can pick an increasing sequence $\{r_j\}_{j=1}^{\infty}, \lim_{j\to\infty} r_j = 1$, such that for some analytic Borel measure μ,

$$\lim_{j\to\infty} \frac{1}{2\pi} \int_{\mathbb{T}} f(r_j e^{it}) F^*(t) \overline{g(t)} dt = \frac{1}{2\pi} \int_{\mathbb{T}} \overline{g(t)} d\mu \quad \text{for any} \quad g \in C(\mathbb{T}).$$

Using the Riesz Brothers theorem, we deduce that μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$, $d\mu(t) = \psi(t)dt$, where $\psi \in H^1(\mathbb{T})$. Afterward, we finish the proof as above. □

4. Discussion

The author feels obliged to explain some trivial things. Professional mathematicians may omit the following few lines. It is clear that any solved problem is no longer a problem. The key instrument for the present study is polynomials, as defined in Section 2.1. It is easy to check that they constitute a dual system for the corresponding Beurling system. The main difficulty is to determine that system.

These polynomials are remarkable because, by expanding elements of the Hardy spaces with a Beurling system, we obtain integral representations in terms of classical kernels. This fact permits us to use tools developed for weighted norm inequalities in our research. The obtained results belong to different topics, which can be classified as parallel.

On the one hand, we extend Beurling's approximation theorem, showing that any Beurling system is an $M-$basis in the corresponding Hardy space. Moreover, we characterize the outer functions for which the corresponding Beurling system is a uniformly bounded $M-$basis, Schauder basis, and summation basis. On the other hand, we can study weighted-norm Hardy spaces. Here, we should mention M. Rosenblum's important article [17]. In the introduction in [17], a statement was formulated related to weighted-norm H^p spaces. In my talks related to the present study, we formulated that statement as Rosenblum's theorem. We should note that in [17], the author did not formulate that statement as a theorem. Hence, after some reflection, it seems more adequate to formulate it as Rosenblum's hypothesis. Our study permits us to provide a complete proof of Hypothesis 1. Moreover, we determine the precise formula for representing the function from the space $H^p(\mathbb{D}, w)$ with its boundary value, which belongs to $H^p(\mathbb{T}, w)$, when $w \in \mathcal{R}$. It should be mentioned that the class $\mathcal{R}$ is large enough (see (5)). These relations need further study.

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