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Spectral Applications of Vertex-Clique Incidence Matrices Associated with a Graph

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Abstract: Using the notions of clique partitions and edge clique covers of graphs, we consider the corresponding incidence structures. This connection furnishes lower bounds on the negative eigenvalues and their multiplicities associated with the adjacency matrix, bounds on the incidence energy, and on the signless Laplacian energy for graphs. For the more general and well-studied set $S(G)$ of all real symmetric matrices associated with a graph G, we apply an extended version of an incidence matrix tied to an edge clique cover to establish several classes of graphs that allow two distinct eigenvalues.

Keywords: clique partition; edge clique cover; vertex-clique incidence matrix; eigenvalues of graphs; graph energy; minimum number of distinct eigenvalues

MSC: 05C50; 15A29

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph with n vertices and m edges. A *clique* in G is a subset $C \subseteq V$ such that all vertices in C are pairwise adjacent. An *edge clique cover* F of G is a set of cliques $F = \{C_1, C_2, \ldots, C_k\}$ that together contain each edge of G at least once. The smallest size of an edge clique cover of G is called the *edge clique cover number of* G and is denoted by $cc(G)$. An edge clique cover of G with size $cc(G)$ is called a *minimum edge clique cover* of G. A special case of an edge clique cover in which every edge belongs to exactly one clique is called a *clique partition* of G. The size of the smallest clique partition of G is called the *clique partition number* of G, and is denoted by $cp(G)$. A clique partition of G with size $cp(G)$ is referred to as a *minimum clique partition* of G. It is clear that both $cc(G)$ and $cp(G)$ exist as E forms a clique partition (and hence an edge clique cover) of G. Further, note that any minimum clique partition does not contain any cliques of order one, and, by convention, the clique partition number of the empty graph is defined to be zero. Information concerning clique partitions and edge clique covers of a graph can be found in the works [1–4].

Before defining the various matrices associated with a graph, we make note of standard matrix notations: I_n denotes the $n \times n$ identity matrix; O denotes the zero matrix (size determined by context); J denotes the all ones matrix (size determined by context); and $\mathbb{1}$ denotes the all ones vector (size determined by context).

Given a graph G with $V = \{1, 2, \ldots, n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, the *(vertex-edge) incidence matrix* M of G is the $n \times m$ matrix defined as follows: the rows and the columns of M are indexed by V and E, respectively, and the (i, j)-entry of M is 0 if $i \notin e_j$ and 1 otherwise. Similarly, the adjacency matrix $\mathcal{A} = \mathcal{A}(G) = (a_{ij})$ is a $(0, 1)$-matrix of G such that $a_{ij} = 1$ if $ij \in E(G)$ and 0 otherwise. It is well known that [5]

$$MM^T = Q(G), \quad \text{and} \quad M^T M = \mathcal{A}(L_G) + 2I_m, \tag{1}$$

where $D(G) = (d_{ij})$ is the diagonal matrix of vertex degrees ($d_{i,i} = \deg(i) := d_i$, $i = 1, 2, \ldots, n$) and the matrix $Q(G) = D(G) + \mathcal{A}(G)$ is known as the *signless Laplacian matrix* of the graph G; the *line graph*, L_G, of the graph G is the graph whose vertex set is in one-to-one correspondence with the set of edges of G, where two vertices of L_G are adjacent if and only if the corresponding edges in G have a vertex in common [6]. Finally, the equations in (1) imply an important spectral relation between the signless Laplacian matrix $Q(G)$ and $\mathcal{A}(L_G)$, see Lemma 6.

As we are also interested in studying more general symmetric matrices associated with a graph on n vertices, we let $S(G)$ denote the collection of real symmetric matrices $A = (a_{ij})$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if $ij \in E(G)$. The main diagonal entries of any such A in $S(G)$ are not constrained. Observe that for any graph G, both $Q(G)$ and $\mathcal{A}(G)$ belong to $S(G)$.

We denote the spectrum of A, i.e., the multiset of eigenvalues of A, by $\mathrm{Spec}(A)$. In particular, $\mathrm{Spec}(A) = \{\lambda_1^{[m_1]}, \lambda_2^{[m_2]}, \ldots, \lambda_q^{[m_q]}\}$, where the distinct eigenvalues of A are given by $\lambda_1 < \lambda_2 < \cdots < \lambda_q$ with corresponding multiplicities of these eigenvalues are $m_1, m_2, \ldots, m_q$, respectively. Further, we consider the ordered multiplicity list of A as the sequence $m(A) = (m_1, m_2, \ldots, m_q)$. For brevity, a simple eigenvalue $\lambda_k^{[1]}$ is simply denoted by λ_k.

With respect to the set $S(G)$, the parameter $q(G)$ is defined by $q(G) = \min\{q(A) : A \in S(G)\}$, where $q(A)$ is the number of distinct eigenvalues of A (see [7,8]). The number $q(G)$ is known as the *minimum number of distinct eigenvalues of the graph* G. The class of matrices $S(G)$ has been of recent interest (see [9–11] and the references therein), and there has been considerable development on the inverse eigenvalue problem for graphs (see [12]) which continues to receive considerable and deserved attention, as it remains one of the most interesting unresolved issues in combinatorial matrix theory.

Using the notions of clique partitions and edge clique covers of a graph we generalize the conventional vertex-edge incidence matrix M by considering an incidence matrix called the *vertex-clique incidence matrix* of a graph. Eigenvalues of graphs and clique partitions have arisen previously, see, for example, the works [13,14], and for other types of graph decompositions see [15]. Suppose $F = \{C_1, C_2, \ldots, C_k\}$ is an edge clique cover of a graph G with $V = \{1, 2, \ldots, n\}$. The vertex-clique incidence matrix M_F of G associated with the edge clique cover F is defined as follows: the (i,j)-entry of M_F is real and nonzero if and only if the vertex i belongs to the clique $C_j \in F$. In the particular case when F is actually a clique partition, the vertex-clique incidence matrix is denoted by $\mathcal{M}_F$, and the (i,j)-entry of $\mathcal{M}_F$ is equal to one if and only if the vertex i belongs to the clique $C_j \in F$. We observe that for any graph G, the vertex-clique incidence matrix corresponding to a clique partition F preserves several main properties of its vertex-edge incidence matrix. For instance, in Section 3, $\mathcal{M}_F\mathcal{M}_F^T = \mathcal{D}_F + \mathcal{A}$, where $\mathcal{D}_F = diag(t_1^F, t_2^F, \ldots, t_n^F)$, where t_i^F is the number of cliques in F containing the vertex i (this parameter is discussed in more detail in Section 3). Note that for each i, $t_i^F \leq d_i$. This fact enables us to determine lower bounds for the negative eigenvalues of the graph.

This paper is organized as follows. In Section 2, we provide the necessary notions, notations, and known results that are needed in the sections containing our main observations. In Section 3, using the notion of a clique partition F of a graph G, we define a signless Laplacian matrix of the graph G associated with the clique partition F. A graph P_G is introduced as a generalization for the line graph of G. In Section 3.1, applying this theory of a vertex-clique incidence matrix, we produce lower bounds for the negative eigenvalues of the graph. Moreover, we present lower bounds for the negative inertia $\nu^-(G)$ of a graph G in terms of its order n and the rank of its vertex-clique incidence matrix. We also provide a sufficient condition under which the well-known inequality $\nu^-(G) \leq n - \alpha(G)$ holds with equality, where $\alpha(G)$ is the independence number of G. In Section 3.2, we introduce graph energies associated with a clique partition F of the graph G and study several associated properties. Moreover, upper bounds for the energies of the graph G and its clique partition graph and line graph are determined. In Section 4, studies on the vertex-clique incidence matrix of a graph associated with an edge clique cover lead to a derivation of some new classes of graphs with $q(G) = 2$ (see also Section 4.1).

2. Notations and Preliminaries

In this section, we provide known notions, notations, and results that are used later in this work.

We begin by introducing the notion of the eigenvalues of a graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the adjacency matrix $\mathcal{A}(G)$ (or shortened to $\mathcal{A}$ when reference to the graph G is clear from context) of the graph G are also called the *eigenvalues of* G. The number of positive (negative) eigenvalues in the spectrum of the graph G is called the *positive (negative) inertia* of the graph G, and is denoted by $\nu^+(G)$ $(\nu^-(G))$. The *energy* of the graph G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| \,. \tag{2}$$

Further details on various properties of graph energy can be found in [16–20]. Suppose $q_1, q_2, \ldots, q_n$ be the eigenvalues of the matrix $Q(G)$. Then, the *signless Laplacian energy* of the graph G is defined as

$$LE^+ = LE^+(G) = \sum_{i=1}^{n} |q_i - \frac{2m}{n}|. \tag{3}$$

More information on properties of the signless Laplacian energy can be found in [21], and the energy of a line graph and its relations with other graph energies are studied in [22,23].

A *subgraph* H of a graph G is a graph whose vertex set and edge set are subsets of those of G. If H is a subgraph of G, then G is said to be a *supergraph* of H. The subgraph of G obtained by deleting either a vertex v of G or an edge e of G is denoted by $G - v$ and $G - e$, respectively. Suppose H is a graph on n vertices. Then, we let $K_n \backslash H$ denote the graph obtained from the complete graph, K_n, by removing the edges from H (this graph is also known as the *complement of the graph* H). An *independent set* in the graph G is a set of vertices in G, no two of which are adjacent. The *independence number* $\alpha(G)$ of G is the number of vertices in the largest independent set of G. A *matching* in a graph G, is a collection of independent edges from G (i.e., no two edges in a matching share a common vertex from G). Additionally, a matching is referred to as *perfect* if each vertex from G is incident with one edge from the matching.

An $n \times n$ real symmetric matrix B is a *positive semi-definite matrix* if all of its eigenvalues are nonnegative. In this case, we denote $B \geq 0$. For real symmetric matrices B and C, if $B - C \geq 0$, then we write $B \geq C$.

Lemma 1 ([24])**.** *Let A and B be real symmetric matrices of order n, and assume that $A \leq B$. Then, for all $i = 1, 2, \ldots, n$, $\lambda_i(A) \leq \lambda_i(B)$, where $\lambda_i(M)$ is the ith largest eigenvalue of a square matrix M.*

The following result was obtained in [5].

Lemma 2 ([5])**.** *If B and C are matrices such that BC and CB are both defined, then BC and CB have the same nonzero eigenvalues with the same multiplicity.*

The Schur product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size is defined to be $A \circ B = (a_{ij}b_{ij})$. An $n \times n$ symmetric matrix A is said to have the *Strong Spectral Property* (or A has the SSP for short) if the only symmetric matrix X satisfying $A \circ X = O$, $I \circ X = O$ and $[A, X] = AX - XA = O$ is $X = O$ (see [25]). The following result is given in Theorem. 10 [25].

Lemma 3 ([25])**.** *If $A \in S(G)$ has the SSP, then every supergraph of G with the same vertex set has a matrix realization that has the same spectrum as A and has the SSP.*

Given two graphs G and H, the *join* of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$, by adding all possible edges between G and H. Suppose G is a graph with $q(G) = 2$. Then, among all matrix realizations A in $S(G)$ with two distinct eigenvalues, we define the multiplicity bi-partition $[n-k, k]$ associated with A if the two eigenvalues of A have respective multiplicities $n-k$ and k. Further, we define the minimal multiplicity bi-partition $MB(G)$ to be the least integer $k \leq \lfloor \frac{n}{2} \rfloor$ such that G achieves the multiplicity bi-partition $[n-k, k]$. We close this section with two useful results concerning specific classes of graphs realizing two distinct eigenvalues with respect to the set $S(G)$.

Lemma 4 ([26,27])**.** *Let G be a connected graph on n vertices. Then,*

(1) $MB(G) = 1$ if and only if G is the complete graph, K_n.

(2) $MB(G) = 2$ if and only if $G = (K_{p_1} \cup K_{q_1}) \vee (K_{p_2} \cup K_{q_2}) \vee \cdots \vee (K_{p_k} \cup K_{q_k})$ for non-negative integers $p_1, \ldots, p_k, q_1, \ldots, q_k$ with $k > 1$, and G is not isomorphic to either one of a complete graph or $G = (K_{p_1} \cup K_{q_1}) \vee K_1$.

Lemma 5 ([28])**.** *If G is a connected graph of order $n \in \{l, l+1, l+2\}$ and $n_1, \ldots, n_l \in \mathbb{N}$, then $q(G \vee \cup_{j \in [l]} K_{n_j}) = 2$, where $[l] := \{1, 2, \ldots, l\}$.*

3. Matrices Associated with a Clique Partition

In this section, we consider the vertex-clique incidence matrix associated with a clique partition of a graph G. Recall from the introduction that for a given clique partition $F = \{C_1, C_2, \ldots, C_k\}$ of G, the matrix $\mathcal{M}_F$ has (i, j)-entry is equal to one if and only if the vertex i belongs to the clique $C_j \in F$ (see also [13,14]). Observe that when $F = E$, $\mathcal{M}_F$ is simply the conventional incidence matrix of the graph G. For each vertex $i \in [n]$ of the graph G, we define the parameter $t_i^F = t_i^F(G)$ to be the number of cliques in F containing the vertex i, that is, $t_i^F = |\{j \in [k] : C_j \in F, i \in C_j\}|$. We call $t_i^F(G)$ *the clique-degree* of the vertex i in graph G associated with F, and, without loss of generality, after a re-labelling of the vertices if necessary, we assume that $t_1^F \geq t_2^F \geq \ldots \geq t_n^F$ (see also [13]). Given clique partition $F = \{C_1, C_2, \ldots, C_k\}$ of G, we consider different possible classes of graphs as follows:

(i) The graph G is t *clique-regular* if $t_1^F = \cdots = t_n^F = t$, for some positive integer t;

(ii) The graph G is s *clique-uniform* if $|C_1| = \cdots = |C_k| = s$, for some positive integer s;

(iii) The graph G is (s, t) *regular* if $t_1^F = \cdots = t_n^F = t$ and $|C_1| = \cdots = |C_k| = s$, for positive integers s, t. Any graph is 2 clique-uniform and any d-regular graph is also d clique-regular using the trivial clique partition $F = E$.

Let $\mathcal{D}_F$ be the $n \times n$ diagonal matrix with row and column indexed by the vertex set V with (i, i)-entry equal to t_i^F, that is, $\mathcal{D}_F = diag(t_1^F, \ldots, t_n^F)$. It follows that the inner product of row i and row j (with $i \neq j$) of $\mathcal{M}_F$ equals the number of cliques in F containing the vertices i and j. By definition of the clique partition F, if i and j are adjacent, then this number is equal to 1 and otherwise 0. This leads to the following result:

Theorem 1. *Let $\mathcal{M}_F$ be the vertex-clique incidence matrix of G associated with a given clique partition F. Then $\mathcal{M}_F \mathcal{M}_F^T = \mathcal{D}_F + \mathcal{A}$, where $\mathcal{D}_F = diag(t_1^F, \ldots, t_n^F)$ and $\mathcal{A}$ is the adjacency matrix of G.*

As mentioned above, in the case of $F = E$, the matrix $\mathcal{M}_F$ is the incidence matrix M of G and consequently, $\mathcal{M}_F \mathcal{M}_F^T = MM^T$ is the signless Laplacian matrix of G, where we assume, after possibly re-labelling, that the sequence of vertex degrees is ordered as $d_1 \geq d_2 \geq \cdots \geq d_n$. Notice that in this case, $t_i^F = d_i$ for $1 \leq i \leq n$. Motivated by this observation, for any clique partition F we call $\mathcal{Q}_F = \mathcal{M}_F \mathcal{M}_F^T$ the *signless Laplacian matrix of the graph G associated with the clique partition F*. Since we always have $D \geq \mathcal{D}_F$, it follows

$Q = D + \mathcal{A} \geq \mathcal{D}_F + \mathcal{A} = \mathcal{Q}_F \geq 0$. Now define *the clique partition graph* P_G with k vertices, where each vertex i corresponds to each clique C_i in F such that each pair of vertices of P_G are adjacent if and only if the corresponding cliques in F have a vertex in common. If $F = E$, then $P_G = L_G$ the line graph of G. The inner product of two columns of $\mathcal{M}_F$ is nonzero if and only if the corresponding cliques have a common vertex. From the definition of a clique partition, this nonzero value must be 1. These facts immediately yield the following result:

Theorem 2. *Let $\mathcal{M}_F$ be the incidence matrix of G associated with a clique partition F. Then, $\mathcal{M}_F^T\mathcal{M}_F = \mathcal{S}_F + \mathcal{A}(P_G)$, where $\mathcal{S}_F = diag(s_1^F, \ldots, s_k^F)$ and $s_i^F = |C_i|$ and $\mathcal{A}(P_G)$ denote the adjacency matrix of the graph P_G.*

For the case of $F = E$, we have $\mathcal{M}_F^T\mathcal{M}_F = M^TM = 2I_m + \mathcal{A}(L_G)$, and $P_G = L_G$ so $s_i^F = 2$ for $1 \leq i \leq k = m$.

3.1. Applications of the Vertex-Clique Incidence Matrix to Graph Spectrum

In this section, we develop several results on the spectrum of the graph G and its clique partition graph P_G by the vertex-clique incidence matrix of a graph. Considering $\mathcal{R}_F = \mathcal{M}_F^T\mathcal{M}_F$ with Lemma 2 we conclude that the nonzero eigenvalues of matrices $\mathcal{Q}_F$ and $\mathcal{R}_F$ are the same. This fact leads to the following basic results.

Theorem 3. *We have the following.*

(i) *If $1 \leq i \leq \min\{n,k\}$, then $\lambda_i(\mathcal{Q}_F) = \lambda_i(\mathcal{R}_F)$.*
(ii) *If $\min\{n,k\} = n$, then $\lambda_i(\mathcal{R}_F) = 0$ for $n+1 \leq i \leq k$.*
(iii) *If $\min\{n,k\} = k$, then $\lambda_i(\mathcal{Q}_F) = 0$ for $k+1 \leq i \leq n$.*

Recall that if $F = E$, then $\mathcal{Q}_F = Q$ and $\mathcal{R}_F = 2I_m + \mathcal{A}(L_G)$. Combining these equations with Theorem 3 leads to the following well-known result [22,29]:

Lemma 6. *Let G be a graph of order n with m edges. Then, $q_i(G) = 2 + \lambda_i(L_G)$ for $1 \leq i \leq \min\{n,m\}$. In particular if $m > n$ then $\lambda_i(L_G) = -2$ for $i > n$, and if $n > m$ then $q_i(G) = 0$ for $i > m$.*

The following result is obtained by applying Theorem 3 for a (s,t) regular graph G with the clique partition F.

Theorem 4. *Let G be a (s,t) regular graph of order n with a clique partition F of size k.*

(i) *If $1 \leq i \leq \min\{n,k\}$, then $\lambda_i(G) - \lambda_i(P_G) = s - t$.*
(ii) *If $\min\{n,k\} = n$, then $\lambda_i(P_G) = -s$ for $n+1 \leq i \leq k$.*
(iii) *If $\min\{n,k\} = k$, then $\lambda_i(G) = -t$ for $k+1 \leq i \leq n$.*

Proof. (i) By Theorem 3 *(i)*, if $1 \leq i \leq \min\{n,k\}$, then $\lambda_i(\mathcal{Q}_F) = \lambda_i(\mathcal{R}_F)$, that is, $\lambda_i(\mathcal{D}_F + \mathcal{A}(G)) = \lambda_i(\mathcal{S}_F + \mathcal{A}(P_G))$, that is, $\lambda_i(tI_n + \mathcal{A}(G)) = \lambda_i(sI_k + \mathcal{A}(P_G))$, that is, $t + \lambda_i(G) = s + \lambda_i(P_G)$.

(ii) By Theorem 3 *(ii)*, if $\min\{n,k\} = n$ then $\lambda_i(\mathcal{R}_F) = 0$ for $n+1 \leq i \leq k$, that is, $\lambda_i(sI_k + \mathcal{A}(P_G)) = 0$ for $n+1 \leq i \leq k$, that is, $\lambda_i(P_G) = -s$ for $n+1 \leq i \leq k$.

(iii) By Theorem 3 *(iii)*, if $\min\{n,k\} = k$ then $\lambda_i(\mathcal{Q}_F) = 0$ for $k+1 \leq i \leq n$, that is, $\lambda_i(tI_n + \mathcal{A}(G)) = 0$ for $k+1 \leq i \leq n$, that is, $\lambda_i(G) = -t$ for $k+1 \leq i \leq n$.

□

Example 1. *(i) Considering the complete graph K_n and its minimum clique partition F with only one clique, we have $\mathcal{M}_F = \mathbb{1}_n$, $\mathcal{M}_F\mathcal{M}_F^T = J_n$ and $\mathcal{M}_F^T\mathcal{M}_F = [n]$. Applying Theorem 4 here we have $t_i^F = 1$ for $1 \leq i \leq n$, $k = 1$ and $s_1^F = n$, that is, K_n is a $(n,1)$ regular graph. From this with Theorem 4 (i) we arrive at $1 + \lambda_1(K_n) = n + \lambda_1(K_1)$, that is, $\lambda_1(K_n) = n - 1$, and by Theorem 4 (iii), $\lambda_i(K_n) = -1$ for $2 \leq i \leq n$.*

(ii) Considering the clique partition $F = \Big\{C_1 = \{1,2,6\}, C_2 = \{2,3,4\}, C_3 = \{1,3,5\}, C_4 = \{4,5,6\}\Big\}$ *for G isomorphic to the complete tripartite graph* $K_{2,2,2}$ *(or* $G \cong K_{2,2,2}$*) in Figure 1, we have* $s_i^F = 3$ *for* $i \in [4]$ *and* $t_j^F = 2$ *for* $j \in [6]$*. Then, G is a* $(3,2)$ *regular graph. Moreover,*

$$\mathcal{M}_F = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \ \mathcal{Q}_F = \begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}, \ \mathcal{R}_F = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

and by Theorem 4, we have $\lambda_i(G) = 1 + \lambda_i(P_G)$ *for* $1 \le i \le 4$ *and* $\lambda_i(G) = -2$ *for* $i = 5, 6$*. From these facts with* $P_G \cong K_4$*, we arrive at* $\mathrm{Spec}(G) = \{4, 0, 0, 0, -2, -2\}$*.*

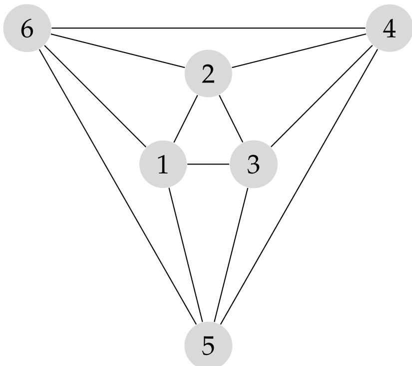

Figure 1. The graph $G \cong K_{2,2,2}$.

Now applying the theory of clique partitions and vertex-clique incidence matrices, we obtain a lower bound for the smallest eigenvalue of a graph. During the review of a previous version, we were made aware of the work in [15] where the first statement in the next result can also be found in Corollary 3.1 [15]. We include a proof here for completeness.

Theorem 5. *Let G be a graph of order n and let* t_1^F *be the largest clique-degree of G with a given clique partition F. Then*

$$\lambda_n(G) \ge -t_1^F. \tag{4}$$

Moreover, if equality holds in (4), then $rank(\mathcal{M}_F) < n$ *and if* $rank(\mathcal{M}_F) < n$ *and G is clique-regular, then equality holds in (4).*

Proof. Since $\mathcal{Q}_F = \mathcal{D}_F + \mathcal{A}$ is a positive semi-definite matrix, we have $\mathcal{D}_F \ge -\mathcal{A}$ and by Lemma 1 we arrive at

$$\lambda_i(\mathcal{D}_F) \ge \lambda_i(-\mathcal{A}) \ \text{ for } 1 \le i \le n. \tag{5}$$

Considering $i = 1$ we arrive at $-\lambda_n(G) = \lambda_1(-\mathcal{A}) \le \lambda_1(\mathcal{D}_F) = t_1^F$, which gives the required result in (4).

For the second part of the proof, suppose that $\lambda_n(G) = -t_1^F$. Then $\lambda_n(t_1^F I + \mathcal{A}) = 0$. This with the relation $0 \le \mathcal{Q}_F = \mathcal{D}_F + \mathcal{A} \le t_1^F I + \mathcal{A}$, gives $\lambda_n(\mathcal{Q}_F) = 0$, that is, $rank(\mathcal{M}_F) = rank(\mathcal{Q}_F) < n$. Now we assume that $t_1^F = \cdots = t_n^F$. If $rank(\mathcal{M}_F) < n$ then $rank(\mathcal{Q}_F) < n$, that is, $\lambda_i(\mathcal{Q}_F) = 0$ for $1 + k \le i \le n$, that is, $t_1^F + \lambda_i(G) = 0$ as $\mathcal{Q}_F = t_1^F I + \mathcal{A}$, that is, $\lambda_n(G) = -t_1^F$ with the multiplicity at least $n - k$. □

Corollary 1. *All regular bipartite graphs and all clique-regular graphs with* $n > |F|$ *satisfy the equality in (4).*

Proof. First we assume that G is a regular bipartite graph. Since G is bipartite, we have $t_i^F = d_i$ for $i \in [n]$ and $q_n = \lambda_n(Q) = 0$. On the other hand, since G is regular, we have $t_1^F = \cdots = t_n^F$. These facts with Theorem 5 give the fact that all regular bipartite graphs satisfies the equality in (4).

Next, assume that G is a clique-regular graph with $n > k = |F|$. Since $rank(\mathcal{M}_F) \leq \min\{n, k\} \leq k < n$, the required result is obtained by Theorem 5. □

Theorem 5 holds for any clique partition F of G, which leads to the following. However, during the review of a previous version, we were made aware of the work [13] where a version of the next result can also be found in Corollary 3.2 [13]. We include a proof here for completeness.

Corollary 2. *Let G be a graph of order n and let t_1^F be the largest clique-degree of G with a given clique partition F. Then,*

$$\lambda_n(G) \geq -\min_F t_1^F,$$

where the minimum is over all clique partitions F of G.

The following example shows that for the equality $\lambda_n(G) = -t_1^F$ the graph G need not be clique-regular.

Example 2. *For the graph G given in Figure 2, we have $F = \Big\{\{1,2\}, \{2,3\}, \{1,3,4,6,7\}, \{4,5\}, \{5,6\}\Big\}$. This gives $t_i^F = 2$ for $i \in [6]$ and $t_7^F = 1$. The graph is the line graph of the graph $H \cong K_1 \vee (2K_2 \cup K_1)$ of order 6 with 7 edges. Then the smallest eigenvalue of G is $\lambda_7(G) = \lambda_7(L_H) = -2 = -t_1^F$ while $t_1^F \neq t_7^F$.*

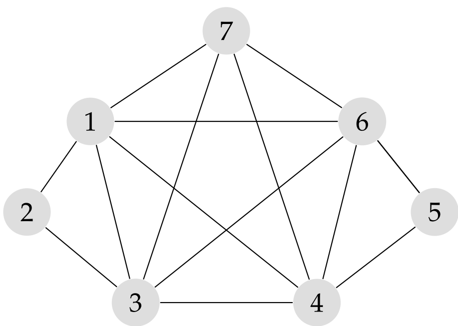

Figure 2. The Graph G.

In the following we provide a lower bound for the negative inertia of a graph G of order n.

Theorem 6. *Let G be a graph of order n. Then,*

$$\nu^-(G) \geq n - \min_F rank(\mathcal{M}_F), \tag{6}$$

where minimum is over all clique partitions F of G. Moreover, if $\min_F rank(\mathcal{M}_F) < n$, then $-t_1^F \leq \lambda_i(G) \leq -t_n^F$ for $1 + \min_F rank(\mathcal{M}_F) \leq i \leq n$.

Proof. If $\min_F rank(\mathcal{M}_F) = n$, then the result in (6) is obvious. Assume that F_1 is a clique partition of G with $rank(\mathcal{M}_{F_1}) = \min_F rank(\mathcal{M}_F) < n$. In this case, since $rank(\mathcal{Q}_{F_1}) = rank(\mathcal{M}_{F_1})$ and $\mathcal{Q}_{F_1}$ is positive semi-definite matrix, we have $\lambda_i(\mathcal{Q}_{F_1}) = 0$ for $1 + rank(\mathcal{M}_{F_1}) \leq i \leq n$. From this and the fact that $t_n^F + \lambda_i(G) \leq \lambda_i(\mathcal{Q}_{F_1}) \leq t_1^F + \lambda_i(G)$, we have $-t_1^F \leq \lambda_i(G) \leq -t_n^F < 0$ for $1 + rank(\mathcal{M}_{F_1}) \leq i \leq n$, which gives the required results. □

The following result is obtained by Theorem 6 and the fact $rank(\mathcal{M}_F) \leq |F|$.

Corollary 3. *Let G be a graph of the order n and a clique partition F such that $n > |F|$. Then,*
(i) $-t_1^F \leq \lambda_i(G) \leq -t_n^F$ *for* $|F| + 1 \leq i \leq n$.
(ii) $\nu^-(G) \geq n - |F|$.

Furthermore, if F is a minimum clique partition of G the next consequence immediately follows from Corollary 3.

Corollary 4. *Let G be a graph of the order n and clique partition number $cp(G)$. If $cp(G) < n$, then*

(i) $-t_1^F \leq \lambda_i(G) \leq -t_n^F$ *for* $cp(G) + 1 \leq i \leq n$.
(ii) $\nu^-(G) \geq n - cp(G)$.

For any graph G of order n we have [29]

$$\alpha(G) \leq \min\{n - \nu^-(G), n - \nu^+(G)\}, \tag{7}$$

where ν^- and ν^+ are the negative and positive parts of the inertia, respectively of the graph G. This implies that

$$\nu^-(G) \leq n - \alpha(G). \tag{8}$$

In the following we present a sufficient condition for which equality in (8) holds.

Theorem 7. *Let G be a graph of order n with the independence number $\alpha(G)$ and the clique partition number $cp(G)$. If F is a clique partition with $rank(\mathcal{M}_F) = \alpha(G)$, then $\nu^-(G) = n - \alpha(G)$. In particular, if $cp(G) = \alpha(G)$, then $\nu^-(G) = n - \alpha(G)$.*

Proof. By Theorem 6 we have $\nu^-(G) \geq n - rank(\mathcal{M}_F) = n - rank(\mathcal{Q}_F) = \eta(\mathcal{Q}_F)$. This fact along with (8) gives

$$\eta(\mathcal{Q}_F) \leq \nu^-(G) \leq n - \alpha(G). \tag{9}$$

The assumption that $\mathrm{rank}(\mathcal{M}_\mathrm{F}) = \alpha(G)$ is equivalent to $\eta(\mathcal{Q}_F) = n - \alpha(G)$. This with (9) gives the first required result.

Without loss of generality, we may assume that the vertex set $[\alpha]$ is a maximum independent set in G and C_i is a clique of a minimum clique partition F_m containing the vertex $i \in [\alpha]$. Now in $\mathcal{M}_{F_m}$ we consider the submatrix induced by the rows and columns corresponding to the vertex set $[\alpha]$ and the clique set $\{C_i : i \in [\alpha]\}$, respectively. Obviously, this square principal submatrix is equivalent to the identity matrix of size α and hence $\mathrm{rank}(\mathcal{Q}_{F_m}) \geq \mathrm{rank}(I_\alpha) = \alpha$. Since $\mathrm{rank}(\mathcal{Q}_{F_m}) \leq cp(G)$ and using the assumption $cp(G) = \alpha(G)$ we arrive at $\mathrm{rank}(\mathcal{Q}_{F_m}) = \alpha = \mathrm{rank}(\mathcal{M}_{F_m})$ and, therefore, $\nu^-(G) = n - \alpha(G)$ by the first part of the theorem. □

The following result is obtained from (5). During the review of a previous version, we were made aware of the work [13] where the next result can also be found in Theorem 3.1 [13]. We include a proof here for completeness.

Theorem 8. *Let G be a graph of order n and the negative inertia ν^-. Let t_i^F be the ith largest clique-degree of G with a clique partition F. Then, for $1 \leq i \leq \nu^-$, we have*

$$\lambda_{n-i+1}(G) \geq -t_i^F. \tag{10}$$

Equality holds in (10) if G is a clique-regular graph with $\nu^- = n - |F|$.

Since $\mathcal{R}_F$ is a positive semi-definite matrix, using a similar argument as in the proof of Theorem 5, we obtain the following result.

Theorem 9. *Let G be a graph of order n with a clique partition $F = \{C_1, \dots, C_k\}$ and let $|C_i| = s_i^F$ for $1 \le i \le k$ such that $s_1^F \ge s_2^F \ge \dots \ge s_k^F$. Then,*

$$\lambda_k(P_G) \ge -s_1^F. \tag{11}$$

Equality holds in (11) if G is a s_1^F clique-uniform graph with $k > n$.

Proof. Since $\mathcal{R}_F = \mathcal{S}_F + \mathcal{A}(P_G)$ is a positive semi-definite matrix, we have $\mathcal{S}_F \ge -\mathcal{A}(P_G)$ and by Lemma 1, it follows that

$$\lambda_i(\mathcal{S}_F) \ge \lambda_i(-\mathcal{A}(P_G)) \text{ for } 1 \le i \le k. \tag{12}$$

Considering $i = 1$ we have $-\lambda_k(P_G) = \lambda_1(-\mathcal{A}(P_G)) \le \lambda_1(\mathcal{S}_F) = s_1^F$, which gives the required result in (11).

Now assume that G is a s_1^F clique-uniform graph with $k > n$. By Theorem 3 (ii) with $k > n$, we arrive at $\lambda_i(\mathcal{R}_F) = 0$ for $n+1 \le i \le k$. On the other hand, since $s_1^F = \cdots = s_k^F$ we have $\mathcal{R}_F = s_1^F I_k + \mathcal{A}(P_G)$, and consequently $\lambda_i(\mathcal{R}_F) = s_1^F + \lambda_i(P_G) = 0$. That is, $\lambda_i(P_G) = -s_1^F$ for $n+1 \le i \le k$, that is, $\lambda_k(P_G) = -s_1^F$ with multiplicity at least $k - n$. □

Theorem 9 holds for any clique partition F of G, which gives the following result.

Corollary 5. *Let G be a graph of order n with a clique partition $F = \{C_1, \dots, C_k\}$ and let $|C_i| = s_i^F$ for $1 \le i \le k$ such that $s_1^F \ge s_2^F \ge \cdots \ge s_k^F$. Then,*

$$\lambda_k(P_G) \ge -\min_F s_1^F, \tag{13}$$

where minimum is over all clique partitions F of G.

In the case of $k > n$, we have $\lambda_i(\mathcal{R}_F) = 0$ for $1 + n \le i \le k$ by Theorem 3. Since $s_k^F + \lambda_i(P_G) \le \lambda_i(\mathcal{R}_F) \le s_1^F + \lambda_i(P_G)$, we get $-s_1^F \le \lambda_i(P_G) \le -s_k^F < 0$. We summarize this in the next result.

Theorem 10. *Let G be a graph of order n and a clique partition F with $|F| = k > n$. Then,*

(i) $-s_1^F \le \lambda_i(P_G) \le -s_k^F$ *for* $1 + n \le i \le k$.
(ii) $\nu^-(P_G) \ge k - n$.

The following result follows from (12).

Theorem 11. *Let G be a graph of order n with a clique partition $F = \{C_1, \dots, C_k\}$ and let $|C_i| = s_i^F$ for $1 \le i \le k$ such that $s_1^F \ge s_2^F \ge \cdots \ge s_k^F$. If P_G is the corresponding clique partition graph of G, then for $1 \le i \le \nu^-(P_G)$,*

$$\lambda_{k-i+1}(P_G) \ge -s_i^F. \tag{14}$$

Equality in (14) holds if G is a s_1^F clique-uniform graph with $\nu^-(P_G) = k - n$.

The following concerns the signless Laplacian eigenvalues of a graph.

Theorem 12. *Let G be a graph of order n and having a clique partition F with $|F| = k$ and assume $1 \le i \le \min\{n, k\}$.*

(i) *If G is a t clique-regular graph, then $q_i(G) - \lambda_i(G) \ge t$.*
(ii) *If G is a s clique-uniform graph, then $q_i(G) - \lambda_i(P_G) \ge s$.*

Proof. From Section 3, the signless Laplacian matrix Q of G satisfies $Q \ge \mathcal{Q}_F$. This fact with Lemma 1 gives $q_i(G) \ge \lambda_i(\mathcal{Q}_F)$, where $q_i(G)$ and $\lambda_i(\mathcal{Q}_F)$ are, respectively, the ith largest signless Laplacian eigenvalue of G and the ith largest eigenvalue of matrix $\mathcal{Q}_F$. Using the above analysis combined with Theorem 3 and the facts $\lambda_i(\mathcal{Q}_F) = t + \lambda_i(G)$ and $\lambda_i(\mathcal{R}_F) = s + \lambda_i(P_G)$ implies the required results in (i) and (ii). □

3.2. Applications to Energy of Graphs and Matrices

In this section, using the theory of vertex-clique incidence matrices of a graph, we introduce notions of graph energies, as a generalization of the incidence energy and the signless Laplacian energy of the graph. Finally, we present upper bounds on energies of a graph, its clique partition graph and line graph.

The energy $\mathcal{E}(G)$ of the graph G defined in (2) has the equivalent expressions as follows [22]:

$$\mathcal{E}(G) = 2\sum_{i=1}^{\nu^+} \lambda_i = 2\sum_{i=1}^{\nu^-} -\lambda_{n-i+1} = 2\max_{1\le k\le n}\sum_{i=1}^{k} \lambda_i = 2\max_{1\le k\le n}\sum_{i=1}^{k} -\lambda_{n-i+1} \tag{15}$$

where ν^+ and ν^- are, respectively, the positive and the negative inertia of G. Nikiforov [30–32] proposed a significant extension and generalization of the graph energy concept. The energy of an $r \times s$ matrix B is the summation of its singular values, that is,

$$\mathcal{E}(B) = \sum_{i=1}^{s} \sigma_i(B), \tag{16}$$

where $\sigma_i(B)$ denotes the ith singular value of B which is equal to $\sqrt{\lambda_i(B^TB)}$.

Consonni and Todeschini [33] introduced an entire class of matrix-based quantities, defined as

$$\sum_{i=1}^{n} |x_i - \overline{x}|, \tag{17}$$

where $x_1, x_2, \ldots, x_n$ are the eigenvalues of the respective matrix, and $\overline{x}$ is their arithmetic mean.

According to (16) and (17), two types of energies can then be defined for any matrix B. The incidence energy $IE(G)$ of a graph G is defined to be the energy of the incidence matrix of G of the type (16), i.e.,

$$IE(G) = \mathcal{E}(M) = \sum_{i=1}^{m} \sigma_i(M) = \sum_{i=1}^{m} \sqrt{\lambda_i(M^TM)} = \sum_{i=1}^{n} \sqrt{\lambda_i(MM^T)} = \sum_{i=1}^{n} \sqrt{q_i}.$$

Similarly, *the vertex-clique incidence energy $IE_F(G)$ of G associated with the clique partition* F is defined as the energy of the vertex-clique incidence matrix $\mathcal{M}_F$, i.e.,

$$\begin{aligned} IE_F(G) = \mathcal{E}(\mathcal{M}_F) &= \sum_{i=1}^{k} \sigma_i(\mathcal{M}_F) = \sum_{i=1}^{k} \sqrt{\lambda_i(\mathcal{M}_F^T\mathcal{M}_F)} \\ &= \sum_{i=1}^{n} \sqrt{\lambda_i(\mathcal{M}_F\mathcal{M}_F^T)} = \sum_{i=1}^{n} \sqrt{\lambda_i(\mathcal{Q}_F)}. \end{aligned}$$

Observe

$$Q - \mathcal{Q}_{\mathcal{F}} = (D + \mathcal{A}) - (\mathcal{D}_F + \mathcal{A}) = D - \mathcal{D}_F = diag(d_1 - t_1^F, d_2 - t_2^F, \ldots, d_n - t_n^F) \ge 0.$$

From the above and using Lemma 1 we have $q_i = \lambda_i(Q) \ge \lambda_i(\mathcal{Q}_F)$ and, consequently, we have

$$IE_F(G) = \sum_{i=1}^{n} \sqrt{\lambda_i(\mathcal{Q}_F)} \le \sum_{i=1}^{n} \sqrt{q_i} = IE(G)$$

with equality if and only if $F = E$.

Moreover,

$$\sum_{i=1}^{n} \lambda_i(\mathcal{Q}_F) = \sum_{i=1}^{n} t_i^F, \quad \sum_{i=1}^{n} \lambda_i^2(\mathcal{Q}_F) = \sum_{i=1}^{n}((t_i^F)^2 + t_i^F).$$

Applying the fact that the diagonal entries are majorized by the eigenvalues of $\mathcal{Q}_F$ and by a similar method given in [34] it can be shown that $\sum_{i=1}^{n} \sqrt{\lambda_i(\mathcal{Q}_F)} \leq \sum_{i=1}^{n} \sqrt{t_i^F}$.

Considering the energy of the matrix $\mathcal{Q}_F$ of the type (17) gives

$$\mathcal{E}(\mathcal{Q}_F) = \sum_{i=1}^{n} |\lambda_i(\mathcal{Q}_F) - \bar{t}|, \tag{18}$$

where $\bar{t} = \frac{\sum_{i=1}^{n} t_i^F}{n}$. The energy $\mathcal{E}(\mathcal{Q}_F)$ can be viewed as a generalization of the signless Laplacian energy $LE^+(G)$ of G which is defined in [21] as follows:

$$LE^+(G) = \mathcal{E}(Q) = \sum_{i=1}^{n} |q_i - \frac{2m}{n}|.$$

Due to the similarity of the definitions for signless Laplacian energy $LE^+(G)$ and $\mathcal{E}(\mathcal{Q}_F)$ it follows that in most cases, results derived about $LE^+(G)$ can be generalized to $\mathcal{E}(\mathcal{Q}_F)$. For example, from Lemma 2.12 in [22] for $LE^+(G)$, we obtain the following:

$$\mathcal{E}(\mathcal{Q}_F) = \max_{1 \leq j \leq n} \left\{ 2 \sum_{i=1}^{j} \lambda_i(\mathcal{Q}_F) - 2j\,\bar{t} \right\} = 2 \sum_{i=1}^{\tau} \lambda_i(\mathcal{Q}_F) - 2\bar{t}\,\tau, \tag{19}$$

where τ is the largest positive integer such that $\lambda_\tau(\mathcal{Q}_F) > \bar{t}$.

Using a method similar to the proof of Corollary 5 in [35] for $\mathcal{Q}_F - \bar{t}\, I = \mathcal{D}_F - \bar{t}\, I + \mathcal{A}$, we have $\mathcal{E}(\mathcal{Q}_F) - \mathcal{E}(G) \leq \sum_{i=1}^{n} |t_i^F - \bar{t}|$.

In the next result, we show that for a clique-regular graph G associated with a clique partition F, $\mathcal{E}(\mathcal{Q}_F) = \mathcal{E}(G)$.

Theorem 13. *If G is a clique-regular graph associated with a clique partition F, then $\mathcal{E}(\mathcal{Q}_F) = \mathcal{E}(G)$.*

Proof. Suppose that G is t clique-regular. Then,

$$\begin{aligned}
\mathcal{E}(\mathcal{Q}_F) &= \sum_{i=1}^{n} \left| \lambda_i(\mathcal{Q}_F) - \frac{\sum_{i=1}^{n} t_i^F}{n} \right| = \sum_{i=1}^{n} |\lambda_i(\mathcal{Q}_F) - t| \\
&= \sum_{i=1}^{n} |\lambda_i(\mathcal{D}_F + \mathcal{A}(G)) - t| = \sum_{i=1}^{n} |\lambda_i(tI + \mathcal{A}(G)) - t| \\
&= \sum_{i=1}^{n} |\lambda_i(G)| = \mathcal{E}(G).
\end{aligned}$$

□

Note that for any t clique-regular graph G, we have $IE_F(G) = \sum_{i=1}^{n} \sqrt{\lambda_i(\mathcal{Q}_F)} = \sum_{i=1}^{n} \sqrt{t + \lambda_i}$. Next, we show that for a clique-uniform graph G we have $\mathcal{E}(\mathcal{R}_F) = \mathcal{E}(P_G)$.

Theorem 14. *If G is a clique-uniform graph with the clique partition graph P_G, then $\mathcal{E}(\mathcal{R}_F) = \mathcal{E}(P_G)$.*

Proof. Suppose that G is a s clique-uniform graph. Then,

$$\begin{aligned}\mathcal{E}(\mathcal{R}_F) &= \sum_{i=1}^{k}\left|\lambda_i(\mathcal{R}_F) - \frac{\sum_{i=1}^{k} s_i^F}{k}\right| = \sum_{i=1}^{k}|\lambda_i(\mathcal{R}_F) - s| \\ &= \sum_{i=1}^{k}|\lambda_i(\mathcal{S}_F + \mathcal{A}(P_G)) - s| = \sum_{i=1}^{k}|\lambda_i(sI + \mathcal{A}(P_G)) - s| \\ &= \sum_{i=1}^{k}|\lambda_i(P_G)| = \mathcal{E}(P_G).\end{aligned}$$

□

Note that for any s clique-uniform graph G with the clique partition graph P_G, we have

$$IE_F(G) = \sum_{i=1}^{n}\sqrt{\lambda_i(\mathcal{Q}_F)} = \sum_{i=1}^{k}\sqrt{\lambda_i(\mathcal{R}_F)} = \sum_{i=1}^{k}\sqrt{s + \lambda_i(P_G)}.$$

In [22] Theorem 3.3, a relation between the energy of the line graph $\mathcal{E}(L_G)$ and the signless Laplacian energy $LE^+(G)$ of G is given. In the following, we generalize this result by using the notion of clique partition of a graph and we provide a comparison between the energy of the clique partition graph $\mathcal{E}(P_G)$ of P_G and $\mathcal{E}(\mathcal{Q}_F)$. For this, we need the following lemma, which is obtained from Theorem 3 and is a generalization of Lemma 6.

Lemma 7. *Let G be an s clique-uniform graph of order n associated with a clique partition F where $|F| = k$. Then, $\lambda_i(\mathcal{Q}_F) = \lambda_i(P_G) + s$, for $i \in \{1, \dots, \min\{n, k\}\}$.*

Theorem 15. *Let G be an s clique-uniform graph of order n associated with a clique partition F where $|F| = k$.*

(i) If $k < n$, then $\mathcal{E}(P_G) \leq \mathcal{E}(\mathcal{Q}_F) + \frac{2ks}{n} - 2s$.
(ii) If $k > n$, then $\mathcal{E}(P_G) \geq \mathcal{E}(\mathcal{Q}_F) + \frac{2ks}{n} - 2s$.
(iii) If $k = n$, then $\mathcal{E}(P_G) = \mathcal{E}(\mathcal{Q}_F)$.

Proof. (i) Let $\nu^+ = \nu^+(P_G) \leq k < n$. By Lemma 7 we have

$$\sum_{i=1}^{\nu^+}\lambda_i(P_G) = \sum_{i=1}^{\nu^+}(\lambda_i(\mathcal{Q}_F) - s) = \sum_{i=1}^{\nu^+}\lambda_i(\mathcal{Q}_F) - s\nu^+.$$

On the other hand, from (15) we have

$$\begin{aligned}\mathcal{E}(P_G) = 2\sum_{i=1}^{\nu+}\lambda_i(P_G) &= 2\sum_{i=1}^{\nu+}\lambda_i(\mathcal{Q}_F) - 2s\nu^+ - 2\nu^+\frac{\sum_{i=1}^{n} t_i^F}{n} + 2\nu^+\frac{\sum_{i=1}^{n} t_i^F}{n} \\ &\leq \mathcal{E}(\mathcal{Q}_F) - 2s\nu^+ + 2\nu^+\frac{ks}{n} \quad \text{as } (19)\,,\ \sum_{i=1}^{n} t_i^F = ks \\ &= \mathcal{E}(\mathcal{Q}_F) + 2\nu^+\left(\frac{ks}{n} - s\right) \quad \text{as } \nu^+ \geq 1,\ k < n \\ &\leq \mathcal{E}(\mathcal{Q}_F) + \frac{2ks}{n} - 2s.\end{aligned}$$

(ii) Recall that τ is the largest positive integer such that $\lambda_\tau(\mathcal{Q}_F) \geq \bar{t} = \frac{ks}{n}$ and let $\tau < n < k$. Again by Lemma 7 we have $\sum_{i=1}^{\tau} \lambda_i(\mathcal{Q}_F) = \sum_{i=1}^{\tau}(\lambda_i(P_G)) + s\tau$. On the other hand, by (19) and Lemma 7 we have

$$\mathcal{E}(\mathcal{Q}_F) = 2\sum_{i=1}^{\tau} \lambda_i(\mathcal{Q}_F) - \frac{2ks\tau}{n} = 2\sum_{i=1}^{\tau} \lambda_i(P_G) + 2s\tau - \frac{2ks\tau}{n}.$$

From (15) with the above equation we have

$$\mathcal{E}(P_G) \geq 2\sum_{i=1}^{\tau} \lambda_i(P_G) = \mathcal{E}(\mathcal{Q}_F) + 2\tau\left(\frac{ks}{n} - s\right) \geq \mathcal{E}(\mathcal{Q}_F) + \frac{2ks}{n} - 2s.$$

(iii) If $k \neq n$, then $\mathcal{E}(P_G) \neq \mathcal{E}(\mathcal{Q}_F)$ by (i) and (ii), i.e., if $\mathcal{E}(P_G) = \mathcal{E}(\mathcal{Q}_F)$, then $k = n$. It suffices to show that if $k = n$, then $\mathcal{E}(P_G) = \mathcal{E}(\mathcal{Q}_F)$. Indeed, if $k = n$, then

$$\mathcal{E}(\mathcal{Q}_F) = \sum_{i=1}^{n} |\lambda_i(\mathcal{Q}_F) - \frac{\sum_{i=1}^{n} t_i^F}{n}| = \sum_{i=1}^{n} |\lambda_i(\mathcal{Q}_F) - \frac{ks}{n}|.$$

Since $k = n$ with Lemma 7 we have $\mathcal{E}(\mathcal{Q}_F) = \sum_{i=1}^{n} |\lambda_i(P_G)| = \mathcal{E}(P_G)$. $\square$

In the following, we present an upper bound for the energy of a graph G.

Theorem 16. *Let G be a graph of order n and the negative inertia $\nu^- = \nu^-(G)$ and let t_i^F be the i^{th} largest clique degree of G associated with the clique partition F, for $1 \leq i \leq n$. Then,*

$$\mathcal{E}(G) \leq 2\min_F \sum_{i=1}^{\nu^-} t_i^F, \tag{20}$$

where the minimum is given over all clique partitions F of G. Equality holds if G is a clique-regular graph associated with a minimum clique partition of size $cp(G) = n - \nu^-$.

Proof. From (15) and (10) we have $\mathcal{E}(G) = 2\sum_{i=1}^{\nu^-} -\lambda_{n-i+1} \leq 2\sum_{i=1}^{\nu^-} t_i^F$, where t_i^F is i^{th} largest clique-degree of G associated with a clique partition F. Since this upper bound is valid for any clique partition of G, we select the optimal value, namely, $\min_F 2\sum_{i=1}^{\nu^-} t_i^F$. The second part of the proof follows directly from Theorem 8. $\square$

The following result provides an upper bound on the energy of G in terms of the vertex degrees.

Theorem 17. *Let G be a graph of order n with the vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n$. Then*

$$\mathcal{E}(G) \leq 2\sum_{i=1}^{h} d_i,$$

where $h = \min\{\nu^+, \nu^-\}$.

Proof. Since $t_i^F \leq d_i$ for $i \in [n]$ along with Theorem 16 gives

$$\mathcal{E}(G) \leq 2\sum_{i=1}^{\nu^-} d_i. \tag{21}$$

On the other hand, the Laplacian matrix $L = D - \mathcal{A}$ of G is a positive semi-definite matrix, so $\mathcal{A} \leq D$. From this with Lemma 1 we obtain $\lambda_i \leq d_i$ for $1 \leq i \leq n$. Then, $\mathcal{E}(G) = 2\sum_{i=1}^{\nu^+} \lambda_i \leq 2\sum_{i=1}^{\nu^+} d_i$. Using the previous inequality with (21) completes the proof. $\square$

From Theorem 16 with (7) we obtain the following upper bound for the energy of G:

$$\mathcal{E}(G) \leq 2\sum_{i=1}^{n-\alpha} t_i^F \leq 2\sum_{i=1}^{n-\alpha} d_i,$$

where α is the independence number of the graph G. By (15) and (14) applying a similar method carried out for the proof of Theorem 16, we obtain the next result.

Theorem 18. *Let G be a graph of order n with a clique partition $F = \{C_1, \ldots, C_k\}$ and let $|C_i| = s_i^F$ for $1 \leq i \leq k$ such that $s_1^F \geq s_2^F \geq \cdots \geq s_k^F$. For the clique partition graph P_G of G, we have*

$$\mathcal{E}(P_G) \leq 2\min_F \sum_{i=1}^{\nu^-(P_G)} s_i^F.$$

Equality holds if G is a clique-uniform graph associated with a minimum clique partition of size $cp(G) = n + \nu^-(P_G)$.

Next, we present an upper bound on the energy $\mathcal{E}(L_G)$ of the line graph L_G with a full characterization of the corresponding extremal graphs.

Theorem 19. *Let G be a graph with the line graph L_G. Then,*

$$\mathcal{E}(L_G) \leq 4\,\nu^-(L_G). \tag{22}$$

Equality holds if and only if G is a graph with connected components $G_i = (V_i, E_i)$ for $i \geq 1$ with $n_i = |V_i|$ and $|E_i| \geq 2$, and possibly some isolated vertices or single edges. Further, each non-bipartite connected component G_i satisfies $|E_i| > |V_i|$ and $q_{n_i} \geq 2$, and each bipartite connected component G_i is either a 4-cycle or satisfies $|E_i| > |V_i|$ and $q_{n_i-1} \geq 2$.

Proof. As previously noted, if the clique partition F of G is as same as the edge set E of G, then $s_i^F = 2$ for $i \in [n]$ and $P_G \cong L_G$. Using Theorem 18, we have

$$\mathcal{E}(L_G) = 2\sum_{i=1}^{\nu^-(L_G)} -\lambda_{m-i+1}(P_G) \leq 2\sum_{i=1}^{\nu^-(L_G)} 2 = 4\nu^-(L_G), \tag{23}$$

which gives the required result in (22).

To characterize these extreme graphs in (22), we assume equality holds in (23). Then, all negative eigenvalues of P_G must be equal to -2 by (23). We then consider the following two cases:

Case (1) G is connected. First, assume that $m > n$. If G is non-bipartite, then by Lemma 6, $\lambda_i(L_G) = -2$ for $n+1 \leq i \leq m$ and $\lambda_n(L_G) = q_n - 2 \neq -2$ as $q_n \neq 0$. Since $\lambda_n(L_G)$ must be nonnegative, we have $q_n \geq 2$. Otherwise G is bipartite and by Lemma 6 along with $q_n(G) = 0$, $\lambda_i(L_G) = -2$ for $n \leq i \leq m$ and $\lambda_{n-1}(L_G) = q_{n-1} - 2 \neq -2$ as $q_{n-1} \neq 0$. Since $\lambda_{n-1}(L_G)$ must be nonnegative, it follows that $q_{n-1} \geq 2$. Next, assume that $m = n$. Since all negative eigenvalues of L_G are equal to -2, we have $\lambda_m(L_G) = \lambda_n(L_G) = -2$. If $\nu^- = 1$, then $\mathrm{Spec}(L_G) = \{2, 0, 0, -2\}$ and L_G is the cycle C_4 of order 4. Otherwise $\nu^- \geq 2$, and $\lambda_{n-1} = -2$, that is, $q_{n-1} = 0$, which is a contradiction as G is connected. Finally, assume that $m < n$. Since G is connected it must be a tree and hence $m = n-1$. In this case we have $\lambda_m(L_G) = \lambda_{n-1}(L_G) = -2$, that is, $q_{n-1} = 0$, which again leads to a contradiction.

Case (2) Assume G is disconnected. Since isolated vertices and single edges do not affect the negative inertia of L_G, we may assume that G has connected components along with the possibility of some isolated vertices and single edges. Now each connected component of G can be characterized by the first case, and the proof is complete. □

4. Vertex-Clique Incidence Matrix of a Graph Associated with an Edge Clique Cover

In this section, we consider the vertex-clique incidence matrix, denoted by M_F, associated with an edge clique cover F of a graph G. Recall that the (i, j)-entry of M_F is real and nonzero if and only if the vertex i belongs to the clique $C_j \in F$. A strategy for minimizing the number of distinct eigenvalues of $M_F\, M_F^T \in S(G)$, is to minimize the number of distinct eigenvalues of the related matrix $M_F^T\, M_F$. Consequently, we obtain an upper bound on the parameter $q(G)$. A key technique used here is to consider an extended version of M_F, by considering arbitrary real entries in the matrix M_F, but simultaneously paying careful attention to preserving the condition that $M_F\, M_F^T \in S(G)$.

4.1. Applications to the Minimum Number of Distinct Eigenvalues of a Graph

In this section, applying the tool of the vertex-clique incidence matrix of a graph associated with its edge clique cover, we characterize a few new classes of graphs with $q(G) = 2$.

If G and H are graphs then the *Cartesian product* of G and H denoted by $G\square H$, is the graph on the vertex set $V(G) \times V(H)$ with $\{g_1, h_1\}$ and $\{g_2, h_2\}$ adjacent if and only if either $g_1 = g_2$ and h_1 and h_2 are adjacent in H or g_1 and g_2 are adjacent in G and $h_1 = h_2$. The first statement in the next theorem can also be found in [7], however, we include a proof here to aid in establishing the second claim.

Theorem 20. *Let $G \cong K_s\square K_2$ with $s \geq 3$. Then, $q(G) = 2$ and G has an SSP matrix realization with two distinct eigenvalues.*

Proof. Let $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$, where $M_1 = J_s - (s-1)I_s$ and $M_2 = J_s - I_s$. Then, we have

$$A = MM^T = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \begin{pmatrix} M_1^T\; M_2^T \end{pmatrix} = \left(\begin{array}{c|c} M_1M_1^T & M_1M_2^T \\ \hline M_2M_1^T & M_2M_2^T \end{array}\right) = \left(\begin{array}{c|c} A_1 & (s-1)I_s \\ \hline (s-1)I_s & A_2 \end{array}\right), \quad (24)$$

where

$$A_1 = M_1M_1^T = M_1^2 = (s-1)^2 I_s + (2-s)J_s\,, \quad A_2 = M_2M_2^T = M_2^2 = I_s + (s-2)J_s\,,$$
$$M_1M_2^T = M_1M_2 = (s-1)I_s.$$

From the structure of A, we have $A \in S(G)$. On the other hand,

$$M^TM = \begin{pmatrix} M_1^T\; M_2^T \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = M_1^TM_1 + M_2^TM_2 = (2-s)J_s + (s-1)^2 I_s + I_s + (s-2)J_s = cI_s,$$

where $c = s^2 - 2s + 2$. Hence $\mathrm{Spec}(MM^T) = \{c^{[s]}, 0^{[s]}\}$ and $q(G) = 2$.

Now, we show that the matrix A has SSP. We need to prove that the only symmetric matrix satisfying $A \circ X = O$, $I \circ X = O$, and $[A, X] = AX - XA = O$ is $X = O$.

From the two equations $A \circ X = O$, $I \circ X = O$, X must have the following form: $X = \left(\begin{array}{c|c} O & X_1 \\ \hline X_1^T & O \end{array}\right)$, where $X_1 = \begin{pmatrix} 0 & x_{12} & \dots & x_{1s} \\ x_{21} & 0 & & x_{2s} \\ \vdots & \ddots & \ddots & \vdots \\ x_{s1} & x_{s2} & \dots & 0 \end{pmatrix}$. The equality $AX = XA$ gives $X_1 = X_1^T$. Also, we have $A_1X_1 = X_1A_2$, i.e., $[(s-1)^2 I_s + (2-s)J_s]X_1 = X_1[I_s + (s-2)J_s]$.

Hence $sX_1 = X_1J_s + J_sX_1$. Then $(sX_1)_{ij} = (X_1J_s + J_sX_1)_{ij}$ for $i, j \in [s]$. Considering $i = j = 1$, we have $(sX_1)_{11} = 0$ and $(X_1J_s + J_sX_1)_{ii} = 2\sum_{j=1}^{s} x_{1j}$, and then $\sum_{j=1}^{s} x_{1j} = 0$. Considering $(i, j) = (k, k)$ for $2 \leq k \leq s$ we arrive at $\sum_{j=1}^{s} x_{kj} = 0$ for $2 \leq k \leq s$. This means that the row and column sums in X_1 are equal to zero. Now, consider $i, j \in [s]$ where $i \neq j$.

We have

$$sx_{ij} = (sX_1)_{ij} = (X_1J_s + J_sX_1)_{ij} = (X_1J_s)_{ij} + (J_sX_1)_{ij} = \sum_{k=1}^{s} x_{ik} + \sum_{k=1}^{s} x_{jk} = 0.$$

Thus, $X_1 = O_s$ and, consequently, $X = O$. Hence, the proof is complete. □

Corollary 6. *For even n, we have $q(\overline{C_n}) = 2$.*

Proof. Let $G \cong K_n \backslash H$ and let H be the graph obtained from the complete bipartite graph $K_{n/2,n/2}$ by removing a perfect matching. Then, by Theorem 20 and Lemma 3, for H or any subgraph of H, $q(G) = 2$. Considering this and that C_n is a subgraph of H, the result is obtained. □

Theorem 21. *Let G be a graph obtained from $(K_s \square K_2) \vee sK_1$ by removing a perfect matching between sK_1 and a copy of K_s. Then $q(G) = 2$ and G has an SSP matrix realization with two distinct eigenvalues.*

Proof. Let $M = \begin{pmatrix} M_1 \\ M_2 \\ I_s \end{pmatrix}$, where $M_1 = J_s - (s-1)I_s$ and $M_2 = J_s - I_s$. Considering the fact that M_1 and M_2 are symmetric, we have

$$A = MM^T = \begin{pmatrix} M_1 \\ M_2 \\ I_s \end{pmatrix} \begin{pmatrix} M_1^T \, M_2^T \, I_s \end{pmatrix} = \left(\begin{array}{c|c|c} M_1M_1^T & M_1M_2^T & M_1I_s \\ \hline M_2M_1^T & M_2M_2^T & M_2I_s \\ \hline M_1 & M_2 & I_s \end{array}\right) = \left(\begin{array}{c|c|c} A_1 & (s-1)I_s & M_1 \\ \hline (s-1)I_s & A_2 & M_2 \\ \hline M_1 & M_2 & I_s \end{array}\right),$$

where

$$A_1 = M_1M_1^T = M_1^2 = (s-1)^2I_s + (2-s)J_s\,, \quad A_2 = M_2M_2^T = M_2^2 = I_s + (s-2)J_s\,,$$

$$M_1M_2^T = M_1M_2 = (s-1)I_s.$$

From the structure of A, we have $A \in S(G)$. On the other hand,

$$M^TM = \begin{pmatrix} M_1^T \, M_2^T \, I_s \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ I_s \end{pmatrix} = M_1^TM_1 + M_2^TM_2 + I_s^2 = (2-s)J_s + (s-1)^2I_s + I_s + (s-2)J_s + I_s = cI_s,$$

where $c = s^2 - 2s + 3$. This gives $\mathrm{Spec}(MM^T) = \{c^{[s]}, 0^{[2s]}\}$, which proves $q(G) = 2$.

Now, we show that the matrix A has SSP. We need to prove that the only symmetric matrix satisfying $A \circ X = O$, $I \circ X = O$, and $[A, X] = AX - XA = O$ is $X = O$.

From the two equations $A \circ X = O$, $I \circ X = O$, X must have the following form: $X = \left(\begin{array}{c|c|c} O & X_1 & O \\ \hline X_1^T & O & X_2 \\ \hline O & X_2 & X_3 \end{array}\right)$, where $X_1 = \begin{pmatrix} 0 & x_{12} & \dots & x_{1s} \\ x_{21} & 0 & & x_{2s} \\ \vdots & \ddots & \ddots & \vdots \\ x_{s1} & x_{s2} & \dots & 0 \end{pmatrix}$, $X_2 = diag(y_1, \dots, y_s)$ and $X_3 = \begin{pmatrix} 0 & z_{12} & \dots & z_{1s} \\ z_{12} & 0 & & z_{2s} \\ \vdots & \ddots & \ddots & \vdots \\ z_{1s} & z_{2s} & \dots & 0 \end{pmatrix}$. The matrix equation

$$AX = XA \tag{25}$$

gives $X_1 = X_1^T$. From (25) we also have $M_2X_2 + X_3 = X_2M_2 + X_3$, i.e., $(J_s - I_s)X_2 = X_2(J_s - I_s)$, i.e., $J_sX_2 = X_2J_s$. This gives $y_1 = y_2 = \dots = y_s$, i.e., $X_2 = y_1I_s$.

Again from (25), we have $A_1X_1 + M_1X_2 = X_1A_2$, that is, $M_1X_2 = X_1A_2 - A_1X_1$, that is, $(J_s - (s-1)I_s)(y_1I_s) = X_1(I_s + (s-2)J_s) - ((s-1)^2I_s + (2-s)J_s)X_1$, i.e.,

$$y_1(2-s)I_s + y_1J_s = (2s - s^2)X_1 + (s-2)X_1J_s + (s-2)J_sX_1.$$

Considering a main diagonal entry, say (i,i), in the above matrix equation, we obtain

$$\sum_{j=1}^{s} x_{ij} = -\frac{y_1}{2}. \tag{26}$$

Considering the (i,j)-entry in the above matrix equation, we obtain $x_{ij} = -y_1\frac{s-1}{s-2}$. From the above and (26), $y_1 = 0$, that is, $X_2 = O$. Using the equation $A_1X_1 + M_1X_2 = X_1A_2$, we arrive at the matrix equation $A_1X_1 = X_1A_2$. Following a similar argument as in the proof of Theorem 20 we obtain $X_1 = O$.

Again from (25), we have $M_1X_1 + X_2 = X_2A_2 + X_3M_2$. Since $X_1 = X_2 = O$, we get $X_3M_2 = O$, i.e., $X_3 = X_3J_s$. Considering both the (i,i) and (i,j) entries from the matrix equation, we arrive at $\sum_{k=1}^{s} z_{ik} = 0$ and $z_{ij} = \sum_{k=1}^{s} z_{ik} = 0$, that is, $X_3 = O$, which gives $X = O$. □

Corollary 7. *Consider the complete bipartite graph $K_{s,s}$ by removing a perfect matching. Define a new graph H by adding a copy of K_s to this graph such that each vertex in K_s is adjacent to the corresponding vertex in a copy of sK_1. Then, $q(\overline{H}) = 2$. Moreover, the result holds for any subgraph of H on the same vertex set.*

In [36], the authors studied the problem of graphs requiring property $p(r,s)$. A graph G has $p(r,s)$ if it contains a path of length r and every path of length r is contained in a cycle of length s. They prove that the smallest integer m so that every graph on n vertices with m edges has $p(2,4)$ (or each path of length 2 is contained either in a 3-cycle, or a 4-cycle) is $\binom{n}{2} - (n-4)$ for all $n \geq 5$. Using this, it was noted in [37] that the above equation from [36] implies that the smallest number of edges required to guarantee that all graphs G on n vertices satisfy $q(G) = 2$ is at least $\binom{n}{2} - (n-3)$. For small values of n, it is known that in fact, equality holds in the previous claim. Namely, if at most $n-3$ edges are removed from the complete graph K_n with $n \leq 7$, then the resulting graph has a matrix realization with two distinct eigenvalues. Along these lines and based on [37] the following is a natural conjecture:

Conjecture 1. *Removing up to $n-3$ edges from K_n does not change the number of distinct eigenvalues of K_n. That is, for any subgraph H of K_n with $|E(H)| \leq n-3$, $q(K_n \backslash H) = 2$.*

We confirm Conjecture 1 for $n = 7, 8$ and note that our analysis of the case $n = 7$ differs slightly from [37]. For this, we need the next few lemmas.

Lemma 8. *Let T_1 be the tree given in Figure 3. We have $q(\overline{T_1}) = 2$ and $\overline{T_1}$ has an SSP matrix realization with two distinct eigenvalues.*

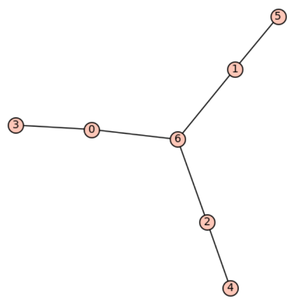

Figure 3. Tree T_1.

Proof. Consider the 7×4 matrix M_1 as follows:

$$M_1 = \begin{pmatrix} 1 & -2 & 2 & 1 \\ 2 & -1 & -2 & 2 \\ 2 & 2 & 1 & 2 \\ 1 & 2 & 2 & 0 \\ -2 & -1 & 2 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Using the Gram–Schmidt method we arrive at a column orthonormal matrix M_2. In this case, we have $A = M_2M_2^T \in S(\overline{T_1})$. In addition, $M_2^T M_2 = I_4$ and $\mathrm{Spec}(A) = \{1^{[4]}, 0^{[3]}\}$. This proves that $q(\overline{T_1}) = 2$. Furthermore, A has SSP (this can be confirmed using SageMath [38]), and by Lemma 3, the complement of any subgraph of T_1 on the same vertex set also has a matrix realization with two distinct eigenvalues. $\square$

Lemma 9. *Let $G \cong K_{1,3} \cup K_3$. Then, $q(\overline{G}) = 2$ and $\overline{G}$ has an SSP matrix realization with two distinct eigenvalues.*

Proof. Consider the 7×3 matrix M_1 corresponding to the labeled graph G given in Figure 4 as follows:

$$M_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

$A = M_1M_1^T \in S(\overline{G})$. Also $M_1^T M_1 = 13I_3$ and $\mathrm{Spec}(A) = \{13^{[3]}, 0^{[4]}\}$. This proves that $q(\overline{G}) = 2$. Furthermore, A has SSP (a computation that can be verified by SageMath [38]), and by Lemma 3, the complement of any subgraph of G on the same vertex set also has a matrix realization with two distinct eigenvalues. $\square$

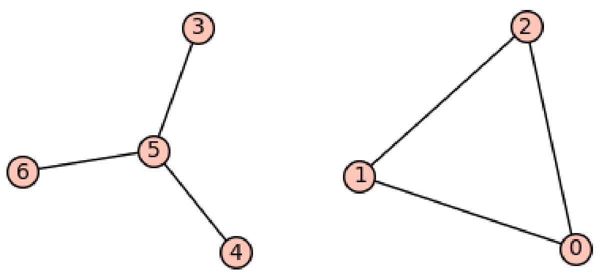

Figure 4. The graph G.

We now verify that Conjecture 1 holds for $n = 7$.

Theorem 22. *Removing up to four edges from K_7 does not change the number of distinct eigenvalues of K_7, i.e., for any subgraph H of K_7 on seven vertices, with $|E(H)| \leq 4$ we have $q(K_7 \backslash H) = 2$.*

Proof. To establish this result, it is sufficient to prove the complement of any graph H in Figure 5 has a matrix realization with two distinct eigenvalues. Suppose that the graphs in Figure 5 are denoted by H_i for $i \in [10]$ from left to right in each row. Then, the graphs H_i for $i = 1,3,7,8,10$ are the union of complete bipartite graphs with some isolated vertices. By Lemma 4 (2), the complements of these graphs and any subgraph of these graphs have a matrix realization with two distinct eigenvalues. Additionally, $q(\overline{H_i}) = 2$ for $i = 4,5,9$

and for any subgraph H_i' of H_i, $q(\overline{H_i'}) = 2$ by Lemma 8. Moreover, $q(\overline{H_6}) = 2$ and for any subgraph H_6' of H_6, $q(\overline{H_6'}) = 2$ by Lemma 9. Additionally, from Lemmas 8 and 9 such realizations exist with the SSP. Hence any subgraph of these graphs has a matrix realization with two distinct eigenvalues. To complete the proof, we need to show the complement graph of H_2 has a matrix realization with two distinct eigenvalues with the SSP. To this end, consider the 7×3 matrix M_1 as follows:

$$M_1 = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -1 & 2 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \\ -2 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Using the Gram–Schmidt method we arrive at a column orthonormal matrix M_2. We have $A = M_2 M_2^T \in S(\overline{H_2})$. In addition, $M_2^T M_2 = I_3$ and $\mathrm{Spec}(A) = \{1^{[3]}, 0^{[4]}\}$. Hence, $q(\overline{H_2}) = 2$. Furthermore, A has SSP (a computation that can be verified by SageMath [38]), and by Lemma 3, the complement of any subgraph of H_2 on the same vertex set also has a matrix realization with two distinct eigenvalues. $\square$

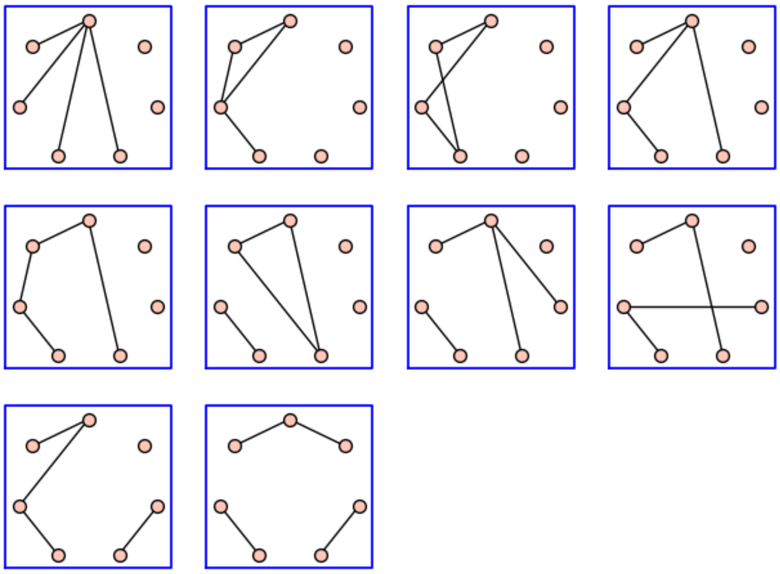

Figure 5. All graphs with 7 vertices and 4 edges.

We require the following results to confirm Conjecture 1 for $n = 8$.

Lemma 10. *Let $G \cong H_1 \cup 2K_1$, where H_1 is the graph on the left given in Figure 6. Then $q(\overline{G}) = 2$ and $\overline{G}$ has an SSP matrix realization with two distinct eigenvalues.*

Proof. Given G as assumed it can be shown without too much difficulty that $\overline{G} \cong (H_2 \vee K_3) - e$, where H_2 is the graph on the right given in Figure 6 minus an edge e with one endpoint in K_3 and the other endpoint in H_2 with degree three.

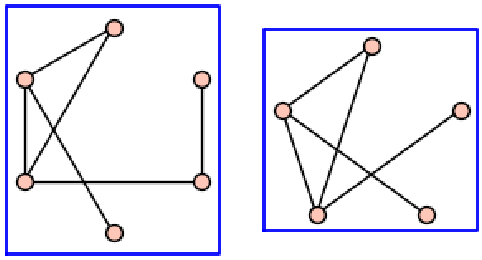

Figure 6. The graphs H_1 (**left**) and H_2 (**right**).

Suppose $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$, is a vertex-clique incidence matrix of $\overline{G}$, where the blocks M_1 and M_2 are vertex-clique incidence matrices corresponding to graphs H_2 and K_3, that is, $MM^T \in S(\overline{G})$. From (24) we have $M_1 M_1^T \in S(H_2)$ and $M_2 M_2^T \in S(K_3)$. On the other hand, we have

$$M^T M = M_1^T M_1 + M_2^T M_2. \tag{27}$$

Consider a vertex-clique incidence matrix M_1 as follows:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

Then we have $M_1 M_1^T \in S(H_2)$ and $M_1^T M_1 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$. Given M_1 above, the remainder of the proof is devoted to constructing a matrix M_2 so that following (27) we have $M^T M = cI_3$, for some scalar c. Consider a matrix M_2 so that

$$M_2^T M_2 = \begin{pmatrix} a & -1 & -1 \\ -1 & a & 0 \\ -1 & 0 & a \end{pmatrix}, \tag{28}$$

where a is a constant. Suppose the matrix $M_2 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$. This with (28) leads to the following equations:

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 = z_1^2 + z_2^2 + z_3^2 = a,$$

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = -1, \;\; x_1 z_1 + x_2 z_2 + x_3 z_3 = -1, \;\; y_1 z_1 + y_2 z_2 + y_3 z_3 = 0.$$

Solving this system of non-linear equations, we have a candidate matrix M_2: $M_2 = \begin{pmatrix} 1 & -1 & z_1 \\ -1 & 2 & z_2 \\ 2 & 1 & z_3 \end{pmatrix}$, where $z_1 = \frac{1}{7}(2\sqrt{51} - 1)$, $z_2 = \frac{1}{35}(6\sqrt{51} + 4)$, and $z_3 = \frac{-1}{35}(2\sqrt{51} + 13)$.

Thus

$$M = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ \hline 1 & -1 & z_1 \\ -1 & 2 & z_2 \\ 2 & 1 & z_3 \end{array}\right).$$

It is obvious that $MM^T \in S(\overline{G})$ and $M^T M = 9I_3$. Since the matrices AB and BA have same nonzero eigenvalues, we have $\text{Spec}(MM^T) = \{9^{[3]}, 0^{[5]}\}$, and then $q(\overline{G}) = 2$. Moreover, applying a basic computation from SageMath [38], we can confirm that MM^T has SSP and this completes the proof. □

By Lemma 10, $\overline{G}$ has an SSP realization $A = MM^T$ with two distinct eigenvalues. Then by Lemma 3, any supergraph on the same vertex set as G has a realization with the same spectrum as A. In particular, $q(H_2 \vee K_3) = 2$. This is stated in the following corollary.

Corollary 8. *Let $G \cong H_2 \cup 3K_1$, where H_2 is the right graph given in Figure 6. Then, $q(\overline{G}) = 2$ and $\overline{G}$ has an SSP matrix realization with two distinct eigenvalues.*

Lemma 11. *Let $G \cong H_3 \cup 3K_1$, where H_3 is obtained from C_5 by joining a vertex to any vertex in C_5. Then, $q(\overline{G}) = 2$ and $\overline{G}$ has an SSP matrix realization with two distinct eigenvalues.*

Proof. We know that $\overline{G} \cong (C_5 \vee K_3) - e$, where e is an edge with one endpoint in K_3 and the other in C_5. Suppose $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$, is a vertex-clique incidence matrix of $\overline{G}$, where blocks M_1 and M_2 are vertex-clique incidence matrices corresponding to graphs C_5 and K_3, that is, $MM^T \in S(\overline{G})$. From (24) we have $M_1 M_1^T \in S(C_5)$ and $M_2 M_2^T \in S(K_3)$. On the other hand, we also have the equations in (27). Now, we consider a vertex-clique incidence matrix M_1 as follows:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $M_1 M_1^T \in S(C_5)$ and $M_1^T M_1 = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}$. Given M_1 above, the remainder of the proof is devoted to constructing a matrix M_2 so that following (27) we have $M^T M = cI_3$, for some scalar c. We need to create a matrix M_2 so that

$$M_2^T M_2 = \begin{pmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 1 & 0 & a \end{pmatrix}, \tag{29}$$

where a is a constant. Suppose $M_2 = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$. This with (29) leads to the following equations:

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 = z_1^2 + z_2^2 + z_3^2 = a,$$

$$x_1 y_1 + x_2 y_2 + x_3 y_3 = 0, \;\; x_1 z_1 + x_2 z_2 + x_3 z_3 = 1, \;\; y_1 z_1 + y_2 z_2 + y_3 z_3 = 0.$$

Solving these non-linear equations we have $M_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$. Thus, we have

$$M = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ \hline \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{array}\right).$$

It is clear that $MM^T \in S(\overline{G})$ and $M^TM = 4I_3$. Since the matrices AB and BA have same nonzero eigenvalues, we have $\mathrm{Spec}(MM^T) = \{4^{[3]}, 0^{[5]}\}$, and $q(\overline{G}) = 2$. Moreover, applying a basic computation from SageMath [38], it follows that MM^T has SSP and this completes the proof. $\square$

By Lemma 11, $\overline{G}$ has an SSP realization $A = MM^T$ with two distinct eigenvalues. By Lemma 3, any supergraph on the same set of vertices as G has a matrix realization with the same spectrum as A. Thus, $q(C_5 \vee K_3) = 2$. This is stated in the following corollary.

Corollary 9. *Let $G \cong C_5 \cup 3K_1$. Then, $q(\overline{G}) = 2$ and $\overline{G}$ has an SSP matrix realization with two distinct eigenvalues.*

Proposition 1. *Let $G \cong K_3 \cup K_{1,n-4}$, where $n \geq 7$. Then $q(\overline{G}) = 2$ and $\overline{G}$ has an SSP matrix realization with two distinct eigenvalues.*

Proof. We show that the complement of G has a matrix realization with two distinct eigenvalues with the SSP. Consider $n \times 3$ matrix M_1 with rows labeled as given in Figure 7 for $n = 8$:

$$M_1 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{\frac{2}{n-4}} & 0 \\ \vdots & \vdots & \vdots \\ 0 & \sqrt{\frac{2}{n-4}} & 0 \end{pmatrix}.$$

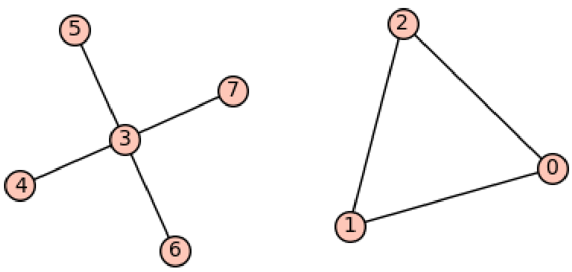

Figure 7. The graph G.

We have $A = M_1M_1^T \in S(\overline{G})$. Also $M_1^TM_1 = 11\, I_3$ and $\mathrm{Spec}(A) = \{11^{[3]}, 0^{[n-3]}\}$. This proves that $q(\overline{G}) = 2$. To verify that A has SSP, suppose X is a symmetric matrix such that

$A \circ X = O$, $I \circ X = O$, and $[A, X] = AX - XA = O$. Note to verify $[A, X] = AX - XA = O$ it is equivalent to prove that AX is symmetric. Now assume that X has the form:

$$X = \left(\begin{array}{c|c|c} 0 & O & x^T \\ \hline O & X_1 & O \\ \hline x & O & O \end{array}\right), \text{ where } X_1 = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix},$$

and x is a (possibly) nonzero vector of size $n-4$. Since AX is symmetric, comparing the (1,3) and (3,1) blocks of AX we note that $\alpha J x = 4x$. So if we set $\beta = \mathbb{1}^T x$, then $x = \frac{\alpha}{4}\beta\mathbb{1}$. Comparing the (1,2) and (2,1) blocks of AX gives

$$2\sqrt{\alpha}\beta = -4\sqrt{2}a - \sqrt{2}b = -\sqrt{2}b + 4\sqrt{2}c, \text{ and } \sqrt{\alpha}\beta = \sqrt{2}a - \sqrt{2}c.$$

Hence, it follows that $a = -c$ and $\beta = \frac{2\sqrt{2}a}{\sqrt{\alpha}}$. Finally, comparing the (2,3) and (3,2) blocks of AX, we have

$$a\sqrt{\alpha} - 2b\sqrt{\alpha} = 2a\sqrt{\alpha} - 2c\sqrt{\alpha} = 2b\sqrt{\alpha} + c\sqrt{\alpha} = \left(\frac{\alpha}{4}\beta\right)^2 = \frac{a^2}{2\alpha}.$$

From the above equations we deduce that $b = -\frac{3}{2}a$. Substituting the equations $a = -c$, $\beta = \frac{2\sqrt{2}a}{\sqrt{\alpha}}$, and $b = -\frac{3}{2}a$ into the equation $2\sqrt{\alpha}\beta = -\sqrt{2}b + 4\sqrt{2}c$, yields $4\sqrt{2}a = \frac{3}{\sqrt{2}}a - 4\sqrt{2}a$. Assuming $a \neq 0$, implies an immediate contradiction. Thus $a = 0$, and it follows, based on the analysis above that $X = 0$. Hence A has the SSP. Using the fact that this matrix realization has the SSP together with Lemma 3, it follows that the complement of any subgraph of G on the same vertex set also realizes distinct eigenvalues. □

Lemma 12. *Let G be the graph given in Figure 8. Then, $q(\overline{G}) = 2$ and $\overline{G}$ has an SSP matrix realization with two distinct eigenvalues.*

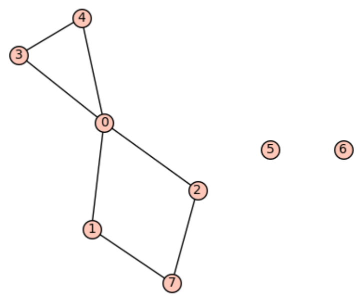

Figure 8. The graph G.

Proof. We show that the complement graph of G has a matrix realization with two distinct eigenvalues with the SSP. To do this, first we consider 8×3 matrix M as follows:

$$M = \begin{pmatrix} \sqrt{\frac{15}{2}} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \\ \sqrt{\frac{2}{2}} & \sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

We have $A = MM^T \in S(\overline{G})$. Also $M^TM = 10\,I_3$ so $\mathrm{Spec}(A) = \{10^{[3]}, 0^{[5]}\}$. This proves that $q(\overline{G}) = 2$. Furthermore, A has SSP (observed using SageMath [38]) and by

Lemma 3, the complement of any subgraph of G on the same vertex has a matrix realization having two distinct eigenvalues. □

Now we are in a position to establish that Conjecture 1 holds for $n = 8$.

Theorem 23. *Removing up to five edges from K_8 does not change the number of distinct eigenvalues of K_8, i.e., for any subgraph H on eight vertices of K_8 with $|E(H)| \le 5$, $q(K_8 \backslash H) = 2$.*

Proof. To establish this result, it is sufficient to prove the complement of any graph H in Figure 9 has a matrix realization with two distinct eigenvalues. Suppose that the graphs in Figure 9 are denoted by H_i for $i \in [24]$ from left to right in each row. The graphs H_i for $i = 1, 2, 9, 10, 15, 22, 23$ are the union of complete bipartite graphs with some isolated vertices. By Lemma 4 (2), the complements of these graphs and any subgraph of these graphs have a matrix realization with two distinct eigenvalues. Additionally, $q(\overline{H_i}) = 2$ for $i = 5, 11, 12, 16, 17, 18, 19, 20, 24$ and for any subgraph H_i' of H_i, $q(\overline{H_i'}) = 2$ by Theorem 20. For $i = 3, 7, 8, 13, 14$, we have $q(\overline{H_i}) = 2$ and for any subgraph H_i' of H_i, $q(\overline{H_i'}) = 2$ by Lemma 12. Additionally, from Theorem 20 and Lemma 12 such realizations exist with the SSP. Hence, any subgraph of these graphs has a matrix realization with two distinct eigenvalues.

Further, $q(\overline{H_{21}}) = q(\overline{(2K_2 \cup K_1) \cup K_3}) = q(G \vee 3K_1) = 2$ by Lemma 5, where the graph $G = \overline{2K_2 \cup K_1} = K_{2,2} \vee K_1$ is connected. If we remove any edges in H_{21} from the triangle, then the complement of the result graph has at least two distinct eigenvalues by Lemma 4 (2), and if we remove any edges in H_{21} from out of the triangle, again by Lemma 5, we have that the complement of the result graph has a matrix realization with at least two distinct eigenvalues. We have $q(\overline{H_4}) = 2$ and the complement of any subgraph of this graph has a matrix realization with two distinct eigenvalues, by Corollary 8. Moreover, $q(\overline{H_6}) = 2$, and the complement of any subgraph of this graph also has a matrix realization with two distinct eigenvalues, by Corollary 9. This completes the proof of the theorem. □

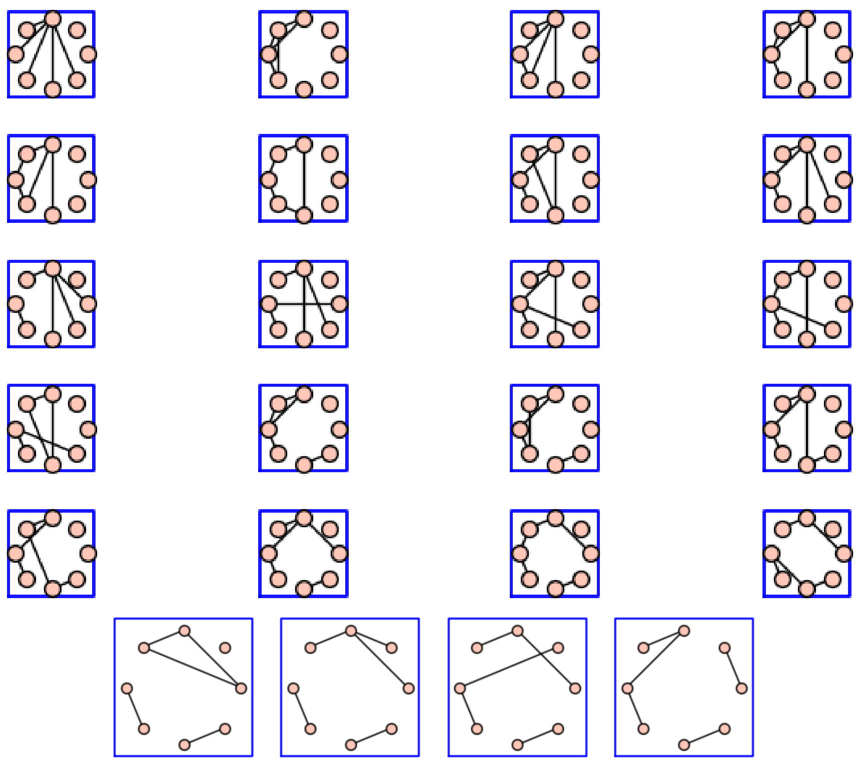

Figure 9. All graphs with 8 vertices and 5 edges.

5. Concluding Remarks and Open Problems

In this work, we utilized the notions of a clique partition and an edge clique cover of a graph to introduce and explore the various properties of a vertex-clique incidence matrix of the graph, which can be viewed as a generalization of the vertex-edge incidence matrix. Using these incidence matrices, we obtained sharp interesting lower bounds concerning the negative eigenvalues and thus the negative inertia of a graph, and we generalized the notion of the line graph of a graph by introducing the clique partition graph of the given graph. Additionally, we determined the relations between the spectrum of a graph and its clique partition graph. Further, we generalized the notion of incidence energy and signless Laplacian energy of a graph and provided some novel upper bounds for the energies of a graph, its clique partition graph, and the line graph. Finally, applying a general version of a vertex-clique incidence matrix of a graph associated with its edge clique cover, we were able to characterize a few classes of graphs with $q(G) = 2$. To close, we list two important and unresolved issues related to some of the content of the current work.

Problem 1: Characterize the corresponding extremal graphs for which the inequalities given in (4), (6), (10), (13), and (14) hold with equality.

Problem 2: Prove that Conjecture 1 is valid for any graph G of an order of at least nine.

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