

Article

Discrete and Continuous Operational Calculus in N-Critical Shocks Reliability Systems with Aging under Delayed Information

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Abstract: We study a reliability system subject to occasional random shocks of random magnitudes $W_0, W_1, W_2, \dots$ occurring at times $\tau_0, \tau_1, \tau_2, \dots$. Any such shock is harmless or critical dependent on $W_k \leq H$ or $W_k > H$, given a fixed threshold H . It takes a total of N critical shocks to knock the system down. In addition, the system ages in accordance with a monotone increasing continuous function δ , so that when $\delta(T)$ crosses some sustainability threshold D at time T , the system becomes essentially inoperational. However, it can still function for a while undetected. The most common way to do the checking is at one of the moments $\tau_1, \tau_2, \dots$ when the shocks are registered. Thus, if crossing of D by δ occurs at time $T \in (\tau_k, \tau_{k+1})$, only at time τ_{k+1} , can one identify the system's failure. The age-related failure is detected with some random delay. The objective is to predict when the system fails, through the N th critical shock or by the observed aging moment, whichever of the two events comes first. We use and embellish tools of discrete and continuous operational calculus ($\mathcal{D}$ -operator and Laplace–Carson transform), combined with first-passage time analysis of random walk processes, to arrive at fully explicit functionals of joint distributions for the observed lifetime of the system and cumulative damage to the system. We discuss various special cases and modifications including the assumption that D is random (and so is T). A number of examples and numerically drawn figures demonstrate the analytic tractability of the results.



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Keywords: reliability system with degradation; critical shocks; extreme shocks; fatal shocks; fluctuation theory; marked point process; position-dependent marking; marked Poisson process; time-to-failure; discrete operational calculus; continuous operational calculus; Laplace–Carson transform

MSC: 60G50; 60G55; 60K10; 90B25; 44A55; 44A45

1. Introduction

Consider a simple reliability system periodically bombarded by random hard shocks of magnitudes $W_1, W_2, \dots$ taking place at respective times $\tau_1, \tau_2, \dots$. Although some of these shocks are harmless, just one of them (*extreme*) is fatal. Their effect is binary with respect to a given threshold H , because with $W_k \leq H$, there is no impact felt by the system, but with $W_k > H$, the system is knocked down. We can regard a k th shock harmless if $W_k \leq H$, meaning that it is absorbed by one of its components such as suspension. When the system is unable to absorb a shock, which is when its magnitude exceeds H , the system fails.

This type of failure is called a *hard failure* as opposed to a commonly referred to *soft failure*, in which the system is continuously worn out by mere aging or by accumulating *soft shocks* (no matter how weak) until the wear process crosses some threshold M or when a soft shock escalates the wear to become *fatal*. Notice that in the former case, there is no damage to the device whenever $W_k \leq H$, whereas in the latter case, the overall damage is amalgamated through periodic *soft shocks* $X_1, X_2, \dots$ and cleaved to aging. A common

mixed system typically combines the two types of shocks, and the system fails when it is hit by one of the two fatal shocks, soft or hard, or by aging alone, whichever comes first.

1.1. System's Hard Failure

Regarding a hard failure, we specify it as follows. Because the impact of hard shocks is binary, we introduce an auxiliary sequence $Y_1, Y_2, \dots$ of Bernoulli RVs (random variables):

$$Y_i = \mathbf{1}_{\{W_i > H\}} \quad (1)$$

and

$$B_k = \sum_{i=1}^k Y_i \quad (2)$$

Then, obviously, the system fails when (at some epoch τ_k) $B_k = 1$, preceded by $B_1 = 0, \dots, B_{k-1} = 0$. System failure due to an extreme shock can be formalized as follows: let

$$\nu = \inf\{n \in \mathbb{N} : W_n > H\} = \inf\{n \in \mathbb{N} : Y_n = 1\} = \inf\{n \in \mathbb{N} : B_n = 1\} \quad (3)$$

Then, τ_ν is the *first passage time* of process $\{B_n\}$, that is, τ_ν is the *time-to-hard-failure* of the system.

1.2. N-Critical Shock System

We embellish this model by assuming that the system fails when at some τ_n , $B_n = N \geq 1$, meaning that only upon a total of N shocks landed at the device, which are deemed critical, does it fail. Thus, it occurs when $B_n = N$, while $B_{n-1} = N - 1$.

This system is referred to as the *N-critical shock* model. The above generalization is useful, in particular, when a system consists of N parallel-connected components, so that each critical shock becomes fatal to exactly one of the components, and when the process continues, it eventually knocks the entire system down. It is readily reducible to single fatal shock when $N = 1$. For example, in a car's suspension system assembled around each of the four wheels, which includes four shock absorbers, a critical shock that cannot be fully absorbed by one of the dampers or springs can damage the suspension of the underlying wheel. Suppose yet another such critical shock has landed upon the suspension unit of another wheel; now it can require a comprehensive repair of the car's suspension system before it becomes operational. In such a case $N = 2$. Notice that the vehicle will be hit by many noncritical shocks prior to the first critical shock and between the two critical shocks.

In Figure 1 below, we see a path of such stochastic process with $N = 4$ in a system being observed from time τ_0 and on.

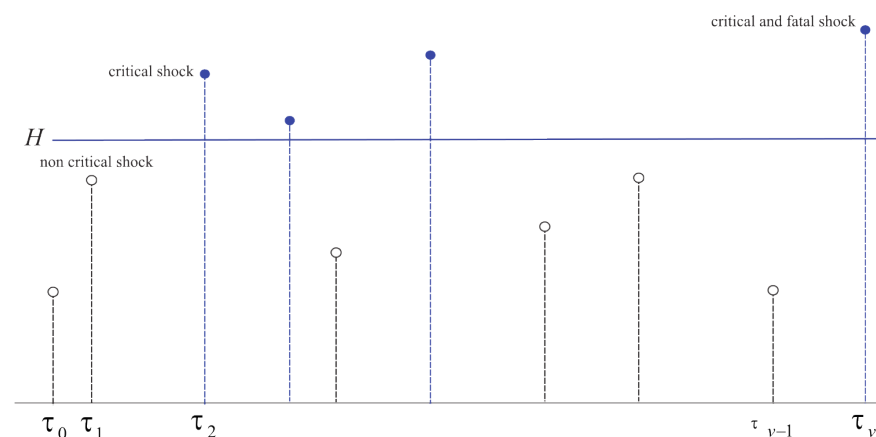


Figure 1. A total of 10 shocks of which 4 are critical of which the 4th ($N = 4$) is fatal.

1.3. *N*-Critical Shock System with Aging

We further embellish the above setting to model a more realistic but also analytically more challenging system. Assume that besides the hits exerted upon the system, the system also ages, so that without shocks alone, it deteriorates and eventually becomes inoperable. Suppose aging is formalized by a continuous, monotone increasing, deterministic function $t \mapsto \delta(t)$ of time, such that $\lim_{t \rightarrow \infty} \delta(t) = \infty$ (or under some other provisions to warrant crossing of some D by δ on a compact interval) and suppose $D \in \mathbb{R}_+$ is a sustainability threshold so that when it is crossed by δ , the system will be ordered to stop or suspend its operation.

In many practical situations, aging and shocks combined will cause an underlying system to fail sooner than just through aging or shocks alone. It is common that a homeowner's insurance company will cancel someone's policy after N large claims, or an automobile insurance company will drop its customer for N instances of traffic violations or small car accidents. On the other hand, an insurance company can drop an insured individual for reasons other than claims or violations. For example, dozens of insurance companies pulled out of the state of Florida because of recent major storms, and thus out of fear of being overwhelmed by more forthcoming claims. Also, quite a few automobile insurers significantly raised their premiums in 2023 forcing many customers to look for alternative providers. Among other reasons named by those insurers, such a policy was driven by an increasingly high volume of medical bills pertaining to car accidents in states like Florida, correlated with more intense traffic (possibly due to a record number of move-ins to Florida by residents from other states, especially California, Illinois, and New York that took place in the recent years). This situation can be identified with aging. Thus, an insured homeowner's policy can be terminated due to N large claims (N critical shocks) or simply by the insurer's relocation or high premium spikes (aging).

We notice, however, that system's wear is observed exclusively on the occurrences of shocks $W_1, W_2, \dots$ at times $\tau_1, \tau_2, \dots$. That is, if $\delta(T) = D$, and $T = \delta^{-1}(D)$ is the exact epoch of time when a system's wear function crosses its sustainability threshold, shutting off the system will not happen at exactly that time, but at a later moment closer to T , say some $\tau_\mu \geq T$ (the system can fail earlier at time τ_ν , as described above, if the N th critical shock occurs and instantly knocks the system down).

Here, we have an argument for delayed information. Unlike the common assumption in reliability literature, aging alone need not to have an instantaneous failure effect. In fact, a system can stay operational for a while, although without delivering a satisfactory performance. Consequently, an age-related failure is more difficult to detect and identify than is commonly assumed, let alone being hard to define it. This is because in many cases, systems give up slowly, and their glitches are not immediately manifested. Therefore, there is a degree of uncertainty on when the system malfunctions convincingly enough to be deemed inoperational. In real-world reliability systems, an age-related failure is often authenticated with a delay, for example, upon one of the easy-to-observe incidents of shocks when the system is rigorously tested. As it depicted in Figure 2. Thus, it stands to reason to model reliability systems with partially delayed information, although such modeling is analytically more challenging.

We first define the age-related *delayed exit index* (identifier)

$$\mu = \inf\{m = 1, 2, \dots : \tau_m \geq T\}.$$

Associated with μ is the delayed soft failure time τ_μ .

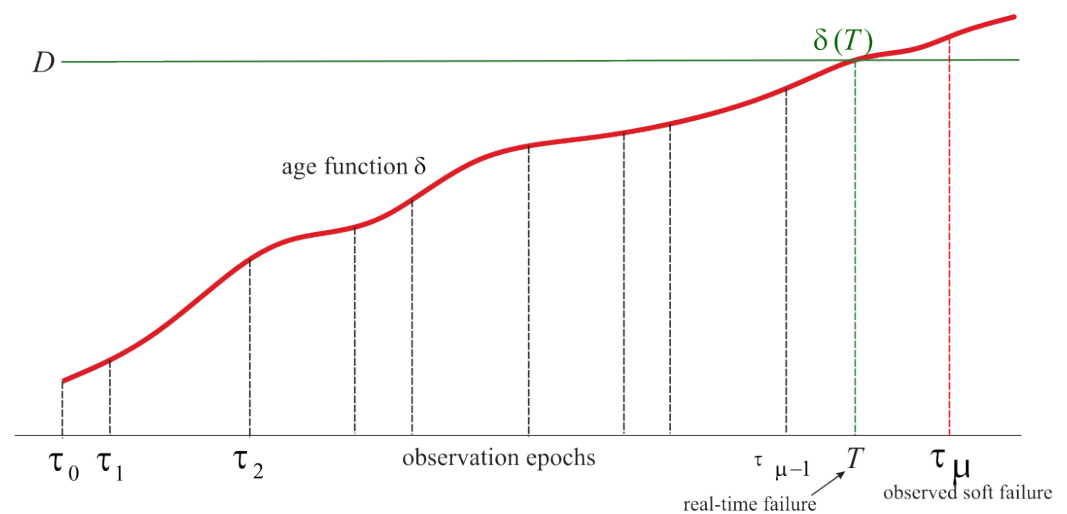


Figure 2. The age-related failure occurs at time T , but it is observed with a delay, at τ_μ .

As far as the hard shocks, $W_1, W_2, \dots$, are concerned, we stay under the same assumptions as above, namely, we assume that the system fails when at some τ_n , $B_n = N$, meaning that only upon a total of N critical shocks exerted on an underlying device does it fail. Hence, in this case,

$$\nu = \min\{n \in \mathbb{N} : B_n = N\}$$

As noted, the system can become inoperational at some earlier observed epoch τ_μ if $\mu < \nu$. Therefore, we define the cumulative exit index

$$\rho = \mu \wedge \nu$$

and the associated first passage time, τ_ρ , at which the system fails either through aging or on a fatal shock, whichever of the two events comes first.

Figure 3 depicts the situation when the fourth critical and, thus, fatal shock ($N = 4$) occurs earlier than the age-related failure at time T and, thus, at time τ_μ when it could have been first observed delaying the real-time crossing. Thus, $\tau_\rho = \tau_\nu$, while $\tau_\mu > \tau_\nu$.

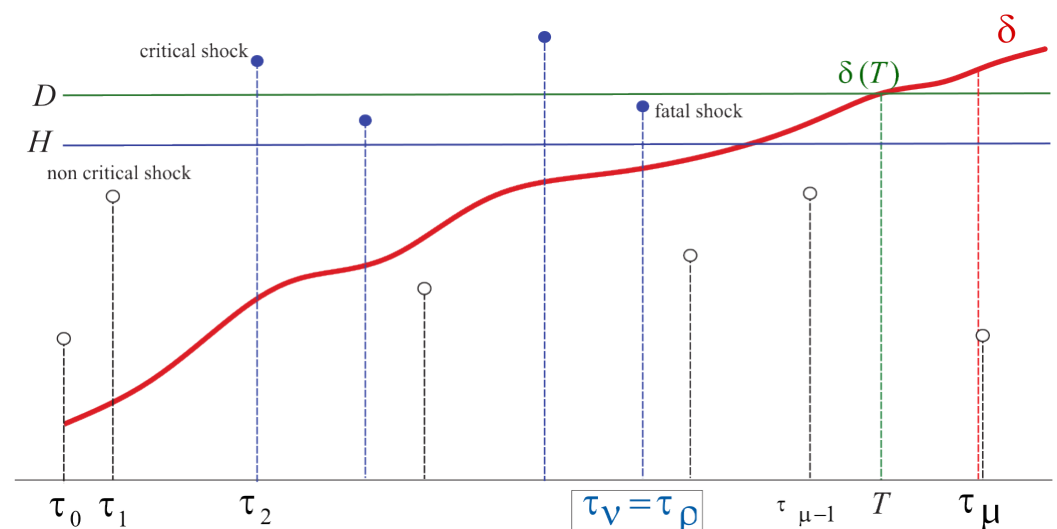


Figure 3. Of the total of 10 shocks, 4 are critical of which the 4th ($N = 4$) is fatal at $\tau_\nu < \tau_\mu$.

In Figure 4, we see three critical and five noncritical shocks. The 7th noncritical shock was observed at the epoch τ_μ when the system became inoperational due to aging, which makes the wear seen above D . The real crossing of D by δ occurred at an earlier time T . However, it was not detected in time. Furthermore, we see that the fourth critical

shock ($N = 4$) occurs at time $\tau_\nu > \tau_\mu$. However, the system failed earlier, at time τ_μ (on a noncritical shock), due to aging. Consequently, $\tau_\mu = \tau_\rho$.

Continuing with Figure 4, note that if a noncritical shock at time τ_μ became critical and, thus, extreme or fatal, the system would undergo the two fatalities: the age-related observed failure at time τ_μ and upon the fourth critical (thus fatal) shock at the same time implying that $\tau_\mu = \tau_\nu = \tau_\rho$. In this situation, even though the real time of the age-related fatality is at time $T < \tau_\rho$, it was not registered.

Our target functional is $\Phi_\rho(z, \theta) = \mathbb{E} z^{B_\rho} e^{-\theta \tau_\rho}$, which predicts the time τ_ρ of system failure and B_ρ —the total number of critical shocks at τ_ρ . Using B_ρ , we can find the total material damage to the system. The latter would make sense to predict, for example, if some components will remain intact after τ_ρ , thereby reducing the overhaul costs compared to the total replacement.

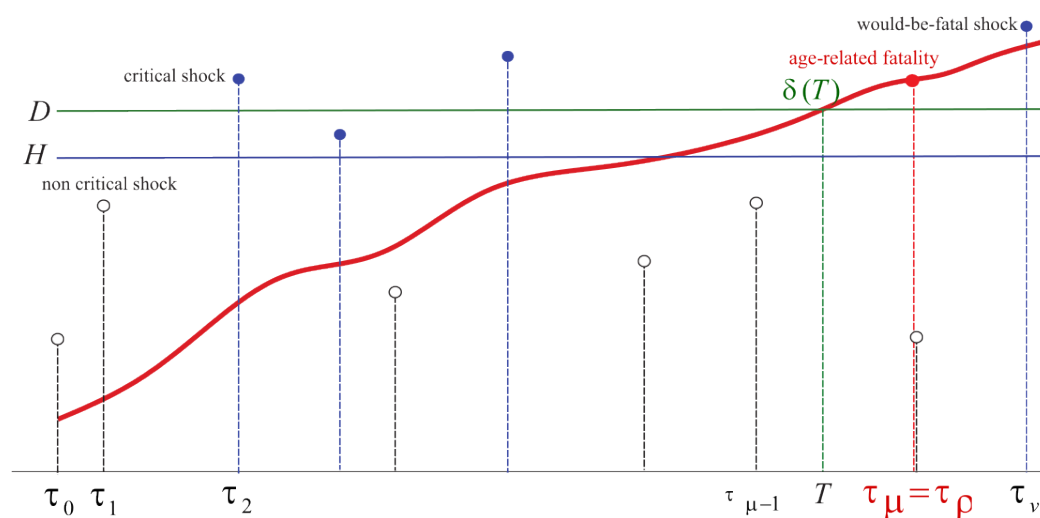


Figure 4. The system aged (as observed at τ_μ) earlier than the N th ($N = 4$) critical shock at τ_ν .

1.4. Analysis of Related Literature

To the best of our knowledge, the above setting is new, and the methods of the underlying system are novel and original. Note that some papers assume that the aging function is stochastic, such as the gamma or Brownian motion process. While the gamma process is a useful vehicle for modeling the aging function, in our case, this feature impedes the analytical tractability of the results and, thus, deviates from our objectives.

Apparently, an N -critical shocks system (with no aging) was first introduced by Cha and Finkelstein [1] in 2011 (the shock process was assumed to be nonhomogeneous Poisson). Wu et al. [2] in 2022 (with no aging included) resurrected this model under the assumption that shocks arrive according to a Markov renewal process.

In 2012, Jiang et al. [3] studied a variant of such a system with aging, soft shocks (cumulative shock model), and hard shocks. There are three thresholds, $H_0 < H_1 < H_2$, of which H_2 is “critical”. It takes just one shock of a magnitude above H_2 to knock the system down. However, once N shocks cross H_0 (but not H_2), the threshold H_2 is downgraded to H_1 , so that it now takes one H_1 -critical shock (that is, of a lesser magnitude) to knock the system down. Moreover, aging, along with soft shocks, takes its course, and if the aging curve crosses some D , the system soft fails, unless it fails earlier due to an extreme shock. Now, we see that while this system is not exactly an N -critical shock system, it carries some elements of the N -critical shock protocol.

Dshalalow and White [4] in 2022 studied a mixed system with aging, soft, and critical shocks. The aging process was defined as linear with a deterministic slope and it was combined with soft shocks that accelerated aging, and such a cumulative aging process sooner or later crossed a sustainability threshold. The projection of such a crossing point was referred to as a soft failure. After this random point, say η , the system was shut off.

The system could also fail if it was hit by one of the critical shocks, namely, by the N th critical shock, say at the instant τ_v . Thus, the system fails at time $\eta \wedge \tau_v$.

Related to the N -critical shocks system is a class of so-called run shock models in which an underlying system is knocked down by a total of N critical shocks landed in the system consecutively, rather than in an arbitrary order as in N -critical shock models. We note that a run shock model seems to be more rigid compared to an N -critical shock model, as the former seems to apply to more restricted classes of real-world reliability systems. Furthermore, the N -critical shock setting does not exclude consecutive runs of shocks and is, thus, more relaxed.

An apparent constraint of run models can be seen in the following situation. Assume that a system is hit by a run of $N - 1$ consecutive critical shocks followed by one noncritical shock, and then is followed by another run of $N - 1$ consecutive critical shocks and one noncritical shock, and so on. It seems likely that it takes a while (if ever) to come up against a run of N critical shocks before the system becomes inoperational, whereas in an N -critical shocks setting, a failure would have been detected much earlier. As argued in Mallor and Omei [5] when applied to insurance claims, only a series of N consecutive claims deemed large enough will raise flags. It seems as if N large claims in any order would be sufficiently concerning.

A run shock system was first introduced and studied in 2001 by Mallor and Omei [5] followed by Mallor and Santos [6] in 2003. Another run shock system, combined with other features, was studied by Lyu et al. [7]. More recently in 2019, Eryilmaz and Tekin [8] studied a run shock system that fails when the magnitudes of N consecutive critical shocks cross a threshold H_1 or the magnitude of one single extreme shock crosses threshold $H_2 > H_1$, whichever of the two events comes first. Further embellishments of a run shock model were introduced in 2018 by Gong et al. [9] and in 2022 by Wen et al. [10]. Here, we see two critical thresholds, $H_1 < H_2$, and two positive integers, $N_1 > N_2$. The system fails if N_1 consecutive critical shocks above H_1 or N_2 consecutive critical shocks above H_2 land in the system, whichever comes first. An even more complex run model with modifications was considered by Poursaeed [11] in 2021.

Somewhat related to our system are δ -shock models. A δ -shock policy applies to a system that fails when the time lag between two consecutive shocks becomes less than some fixed $\delta > 0$. A δ -shock policy is often implemented whenever shock damages are hard to observe. A δ -shock model was first introduced in 1999 by Li et al. [12] followed by Li and Kong [13] in 2007 with the same basic assumptions whereby the authors studied the asymptotic behavior when $\delta \rightarrow 0$. Another basic δ -shock model from the same period was analyzed by Tang and Lam [14].

Then, we see some modifications and embellishments in the δ -policy like in work of Parvardeh and Balakrishnan [15] from 2015 where their system fails when (a) the time lag between two consecutive δ -critical shocks is less than a δ , or (b) the magnitude of any single (H -critical) shock is larger than an H , whichever comes first. Eryilmaz [16] in 2012 studied a significantly upgraded variant of the δ -system. The author combined the idea of run shocks and δ -shocks in one system that fails when the time lags of each of N consecutive shocks is less than δ . Kus, Tuncel, and Eryilmaz [17], in 2022, studied a mixed shock model, which combines δ -shock and extreme shock models, that is, the system fails if (a) the time lag between two consecutive δ -critical shocks is less than a $\delta > 0$, or (b) a single extreme shock crosses a threshold H , whichever of the two comes first. A further embellishment comes under the assumption of matrix-exponential distributions for intershock times. Among other things, the authors showed that the lifetime (time-to-failure) of the system does not have a matrix-exponential distribution, but it is approximated by a matrix-exponential distribution.

An interesting embellishment of Eryilmaz's 2012 δ -run model was introduced by Jiang [18] in 2020. Such a system has N different failure thresholds $\delta_1 > \delta_2 > \dots > \delta_N > 0$. If the time lag between two consecutive critical shocks lies in (δ_{i+1}, δ_i) , $i = 1, \dots, N$ ($\delta_{N+1} = 0$),

the system is subject to the i th failure type. The N th type is irreparable and the whole system needs replacement. The first $N - 1$ types allow repair.

Most recently, Roozegar et al. [19] in 2023 combined δ -shock and extreme shock models. The system fails when (a) k interarrival times of noncritical shocks (each in interval (δ_1, δ_2)) land between two (nonconsecutive) critical shocks, that is, with magnitudes above H , (b), the time lag between two consecutive critical shocks is less than δ_1 , whichever comes first. Other mixed and complex δ -shock models were studied by Lorvand et al. [20] in 2020, Wu [21] 2022, and Doostmoradi [22] of 2023.

1.5. Our Results and Techniques Used

Our present system is different from all existing reliability models in the following way. First, it combines an N -critical shock model (only a few such models are in existence) and an aging process under an arbitrary monotone that increases the deterministic function δ . The crossing of a sustainability threshold D by δ occurs at some point $T = \delta^{-1}(D)$ observed with some random delay, where system failure can be verified, unless the system fails earlier after being hit by N critical shocks (i.e., those with magnitudes above some H), which can alternate with noncritical shocks in any order. Only after the N th critical shock is landed in the system does the system fail. Therefore, there is a competition (or game) between two failure processes with soft versus hard failure and delayed information about the soft crossing.

We apply tools of discrete and continuous operational calculus. This is novel in the reliability literature and was introduced by Dshalalow [23] in his earlier papers and has undergone further development throughout the years. The introduction of discrete operational calculus (including the $\mathcal{D}$ -operator) was motivated by the needs of fluctuation theory that have arisen from random walks (cf. Dshalalow and White [24]) and applied to queuing, stochastic games, and finance, to name a few. A similar utility of continuous operational calculus (driven by the Laplace–Carson transform) was also enhanced and included in applications to continuous-valued processes.

The objective of our methods is to bear fully explicit formulas, which offer numerous benefits over nonclosed forms (while algorithms are also useful, closed expressions offer many beneficial alternatives). We demonstrate the analytical tractability of our formulas on special cases and by using straightforward computations.

In Section 2, we start with the formalism of special cases solved using probabilistically straightforward tools and compare it with the same results obtained by means of discrete operational calculus. In Section 3, we formalize the general reliability problem and introduce various functionals of the time-to-failure, the overall damage to the system, and pre-time-to-failure moments, among other things, under the assumption that the stream of shocks forms an independent and stationary increment process with position-dependent marking. The main result is carried in Theorem 1. Sections 4 and 5 deal with the results for the model under special assumptions, one of which is that the input stream of shocks is a marked Poisson process. Section 6 deals with the distributions of τ_μ or τ_ν , assuming that one of them comes first. In Section 7, we discuss the case when the threshold D and, thus, the time T of crossing are random.

2. Motivation and Some Preliminaries

Throughout this paper, the principal reliability system and its variants are considered on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}_0}, \mathbb{P})$.

2.1. A System with Extreme Shocks ($N = 1$)

We start with the basic system introduced in Section 1.1 with only hard shocks landing in the system. It was specified by a sequence of Bernoulli RVs $\{Y_k\}$ of (1) and the sequence $\{B_n\}$ of partial sums in (2). The associated marked counting process

$$(\mathcal{B}, \mathcal{T}) = \sum_{k=1}^{\infty} Y_k \varepsilon_{\tau_k} (\varepsilon_a \text{ is the point mass}) \quad (4)$$

has stationary and independent increments and is assumed to have position-dependent marking. In this system, the first hard shock that is extreme, that is with $W_k > H$, knocks the system down.

This system can be analyzed through straightforward probabilistic tools, such as the double expectation, if the process $(\mathcal{B}, \mathcal{T})$ has a position-independent marking. Then, from (3), $\nu \in [\text{Geo1}(p)]$ (which means ν belongs to the equivalence class of all type 1-geometric RVs with parameter p), where

$$p = \mathbb{P}\{W > H\}$$

such that $W_i \in [W]$. Here, $[W]$ is a generic equivalence class of all RVs with the common distribution as W . Because ν is geometric, the first passage time (a system's lifetime) distribution can be expressed through the LST (Laplace–Stieltjes transform)

$$\mathbb{E}e^{-\theta\tau_\nu} = \mathbb{E}\left[\mathbb{E}\left[e^{-\theta\tau_\nu} | \nu\right]\right] = \mathbb{E}\gamma^\nu(\theta) = \frac{p\gamma(\theta)}{1 - (1-p)\gamma(\theta)} \quad (5)$$

where $\gamma(\theta) = \mathbb{E}e^{-\theta\Delta}$, $\Delta_1 = \tau_1, \Delta_2 = \tau_2 - \tau_1, \dots \in [\Delta]$.

In particular, let

$$\Delta \in [\text{Exp}(\gamma)], \gamma(\theta) = \frac{\gamma}{\gamma + \theta}.$$

To minimize the burden of a large number of characters throughout, and for convenience, we use $\frac{1}{\gamma}$ as the mean of Δ whose LST is $\gamma(\theta)$. In addition, γ is also the parameter of the distribution of Δ if it is exponential.

Then,

$$\mathbb{E}e^{-\theta\tau_\nu} = \frac{p\frac{\gamma}{\gamma+\theta}}{1 - (1-p)\frac{\gamma}{\gamma+\theta}} = \frac{p\gamma}{\gamma + \theta - (1-p)\gamma} = \frac{p\gamma}{p\gamma + \theta}$$

implying that $\tau_\nu \in [\text{Exp}(p\gamma)]$.

Note that the analysis of the above system becomes less straightforward if the marking is not position-independent.

2.2. N-Critical Shock System

With the aim of obtaining results that are as tractable as possible, we assume that the marks (W_k) are independent and identically distributed (iid) (meaning $W_k \in [W]$ —the equivalence class of all RVs sharing a common PDF—probability distribution function— F_W) and independent of (τ_k) . The lengths Δ_k of the intervals $(\tau_{k-1}, \tau_k], k = 1, 2, \dots$, are iid borrowed from the class $[\Delta]$, with a common PDF φ . On occasion, unless $\tau_0 = 0$ a.s. (almost surely), we assume that $\tau_0 \in [\Delta_0]$ with PDF φ_0 .

The lifetime (time-to-failure) of such a system is τ_ν , that is, when the N th critical shock lands in the system. If we operate with the above-introduced auxiliary sequence $\{B_n = \sum_{j=0}^n Y_j\}$ of partial sums, we see that $\{B_n\}$ is a monotone nondecreasing sequence of dependent binomial RVs with respective parameters (n, p) , $n = 0, 1, \dots$, with $p = \mathbb{P}\{W > H\}$ (unless we assume that $B_0 = r$ a.s. and then the parameters are $(n + r, p)$). A stopping time of the process relative to filtration $(\mathcal{F}_n)$ is τ_ν , that is, when $\{B_n\}$ hits N .

Under the assumed position independence, the LST of τ_ν becomes

$$\mathbb{E}e^{-\theta\tau_\nu} = \left(\frac{p\gamma(\theta)}{1 - (1-p)\gamma(\theta)} \right)^N, \quad (6)$$

which follows from (5) and that ν is negative binomial with parameters (N, p) . In the particular case, when $\gamma(\theta) = \frac{\gamma}{\gamma + \theta}$, it implies that

$$\mathbb{E}e^{-\theta\tau_\nu} = \left(\frac{\gamma p}{\gamma p + \theta}\right)^N \quad (7)$$

and thus $\tau_\nu \in [\text{Erlang}(N, \gamma p)]$.

2.3. Discrete Operational Calculus in the N-Critical Shock Model

The above system was straightforward, and the related Formulas (6) and (7) were an easy exercise presented as an illustration. To understand the tools employed in the forthcoming sections, we apply them for this very system as an alternative approach under the following formalism. Suppose that $(\mathcal{B}, \mathcal{T}) = \sum_{k=1}^{\infty} Y_k \varepsilon_{\tau_k}$ introduced in (1)–(3) and (4) is a marked delayed renewal process, with position-dependent marking and the functional

$$\Gamma(z, \theta) = \mathbb{E}z^Y e^{-\theta\Delta} \quad (8)$$

of the joint transform of the increments $(Y_1, \Delta_1), (Y_2, \Delta_2), \dots$ of random measure $(\mathcal{B}, \mathcal{T})$ assuming that

$$\tau_0 = \Delta_0 = 0 \text{ and } Y_0 = B_0 = r \text{ a.s.} \quad (9)$$

Using the results from Dshalalow [23] or Formula (4) from Theorem 1 [24] from Dshalalow and White, we obtain the joint transform of B_ν and τ_ν , where Y s (and their sums B s) are integer-valued RVs:

$$\Phi_\nu(z, \theta) = \mathbb{E}z^{B_\nu} e^{-\theta\tau_\nu} = z^r - z^r [1 - \Gamma(z, \theta)] \mathcal{D}_x^{N-r-1} \left(\frac{1}{1 - \Gamma(xz, \theta)} \right) \quad (10)$$

Here, Y and Δ are not independent, and $\mathcal{D}$ is the so-called $\mathcal{D}$ -operator introduced in [23] as

$$\mathcal{D}_x^k \psi(x, y) = \begin{cases} \lim_{x \rightarrow 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x} \psi(x, y) \right], & k \geq 0 \\ 0, & k < 0 \end{cases} \quad (11)$$

where ψ is a function analytic in x at zero.

We will make use of several pertinent properties of $\mathcal{D}$ (see some of them in Dshalalow [23] and Dshalalow and White [24]).

- (Di) $\mathcal{D}$ is a linear functional.
- (Dii) $\mathcal{D}_x^k(\mathbf{1}(x)) = 1$, where $\mathbf{1}(x) = 1$ for all $x \in \mathbb{R}$.
- (Diii) Let g be an analytic function at zero. Then, it holds true that

$$\mathcal{D}_x^k(x^j g(x)) = \mathcal{D}_x^{k-j} g(x).$$

- (Div) In particular, if $j = k$, we have

$$\mathcal{D}_x^k(x^k g(x)) = g(0).$$

- (Dv) $\mathcal{D}_x^k \left\{ \frac{1}{1-bx} \right\} = \frac{1-b^{k+1}}{1-b}$, $b \neq 1$, and $\mathcal{D}_x^k \frac{1}{1-x} = k+1$
- (Dvi) From (Div) and (Dv), $\mathcal{D}_x^k \frac{1}{a-bx} = \frac{1}{a-b} \left[1 - \left(\frac{b}{a} \right)^{k+1} \right]$
- (Dvii) $\mathcal{D}_x^j \frac{1}{a-bx} e^{\delta x} = \sum_{k=0}^{\infty} \frac{1}{k!} \delta^k \mathcal{D}_x^j x^k \frac{1}{a-bx} = \sum_{k=0}^{\infty} \frac{1}{k!} \delta^k \mathcal{D}_x^{j-k} \frac{1}{a-bx} = \sum_{k=0}^j \frac{1}{k!} \delta^k \mathcal{D}_x^{j-k} \frac{1}{a-bx}$

from (Dvi)

$$= \sum_{k=0}^j \frac{1}{k!} \delta^k \frac{1}{a-b} \left[1 - \left(\frac{b}{a} \right)^{k+1} \right] = \frac{1}{a-b} \sum_{k=0}^j \frac{1}{k!} \delta^k \left[1 - \left(\frac{b}{a} \right)^{j-k+1} \right].$$

In the event that Y s and Δ s are independent pertaining to our N-critical shock model in Section 2.2 (notation (8) and assumptions (9)), we show how the marginal functional of (10) can be reduced to (6)–(7).

First of all, because Y is Bernoulli,

$$\mathbb{E}z^Y = \mathbb{E}\left(z\mathbf{1}_{\{W_j > H\}} + \mathbf{1}_{\{W_j \leq H\}}\right) = z\mathbb{E}\mathbf{1}_{\{W_j > H\}} + \mathbb{E}\mathbf{1}_{\{W_j \leq H\}} = zp + q,$$

where $p = \mathbb{P}\{W_j > H\}$ and $q = 1 - p$. As for Δ , we assume it is exponential with parameter γ , that is, the support counting measure $\mathcal{T}$ of $(\mathcal{B}, \mathcal{T})$ is ordinary Poisson of rate γ . Thus,

$$\mathbb{E}e^{-\theta\Delta} = \frac{\gamma}{\gamma + \theta}$$

and

$$\Gamma(z, \theta) = [zp + q] \frac{\gamma}{\gamma + \theta}.$$

With no initial damage to the system, or $r = 0$, the joint distribution of the time-to-failure (first passage time) τ_ν and the cumulative damage to the system B_ν at time τ_ν is

$$\Phi_\nu(z, \theta) = \mathbb{E}z^{B_\nu} e^{-\theta\tau_\nu} = 1 - [1 - \Gamma(z, \theta)] \mathcal{D}_x^{N-1} \left(\frac{1}{1 - \Gamma(xz, \theta)} \right).$$

Under no restriction to Δ , with $\Gamma(z, \theta) = (pz + q)\gamma(\theta)$, and

$$\mathcal{D}_x^{N-1} \left(\frac{1}{1 - \Gamma(xz, \theta)} \right) = \mathcal{D}_x^{N-1} \left(\frac{1}{1 - (pxz + q)\gamma(\theta)} \right) = \mathcal{D}_x^{N-1} \left(\frac{1}{1 - \gamma(\theta)q - pxz\gamma(\theta)} \right)$$

then using (Dvi) and passing to the limit as $z \rightarrow 1 -$

$$= \frac{1}{1 - \gamma(\theta)q - pz\gamma(\theta)} \left[1 - \left(\frac{pz\gamma(\theta)}{1 - \gamma(\theta)q} \right)^N \right] \Big|_{z=1} = \frac{1}{1 - \gamma(\theta)} \left[1 - \left(\frac{p\gamma(\theta)}{1 - \gamma(\theta)q} \right)^N \right]$$

implying that the marginal functional $\mathbb{E}e^{-\tau_\nu\theta}$ is

$$\Phi_\nu(1, \theta) = \mathbb{E}e^{-\tau_\nu\theta} = 1 - \left[1 - \left(\frac{p\gamma(\theta)}{1 - \gamma(\theta)q} \right)^N \right] = \left(\frac{p\gamma(\theta)}{1 - (1-p)\gamma(\theta)} \right)^N$$

which agrees with (6) and (7) if $\gamma(\theta) = \frac{\gamma}{\gamma + \theta}$.

Note that the assumption that $(\mathcal{B}, \mathcal{T})$ is position-independent played an essential role in arriving at (6) and (7).

Remark 1. In the context of special case (7), that is, $\mathbb{E}e^{-\theta\tau_\nu} = \left(\frac{\gamma p}{\gamma p + \theta} \right)^N$, suppose N is random, then

$$\left(\frac{\gamma p}{\gamma p + \theta} \right)^N = \mathbb{E}\left[e^{-\theta\tau_\nu} | N\right].$$

If $N \in [\text{Geo1}(\pi)]$,

$$\mathbb{E}\left(\frac{\gamma p}{\lambda p + \theta} \right)^N = \frac{\pi \frac{\gamma p}{\gamma p + \theta}}{1 - (1 - \pi) \frac{\gamma p}{\gamma p + \theta}} = \frac{\gamma \pi p}{\gamma p + \theta - \gamma(1 - \pi)p} = \frac{\gamma \pi p}{\gamma \pi p + \theta},$$

implying that τ_ν is exponential with parameter $\gamma \pi p$.

The above discussion was a mere demonstration that the discrete operational calculus agrees with conventional techniques when it pertains to basic models. In the next subsection, we introduce a more complex model where it would be difficult to analyze the system using conventional probability tools.

3. The N -Critical Shock System under Aging General Case

For convenience, we summarize the above notations in Sections 1 and 2.

$$\mu = \inf\{m = 1, 2, \dots : \tau_m \geq T\} \quad (12)$$

$$\nu = \min\{n \in \mathbb{N} : B_n = N\} \quad (13)$$

$$\rho = \mu \wedge \nu \quad (14)$$

The following functionals are introduced:

$$\Phi_\rho(z, \theta; \xi, u, \vartheta; N, T) = \mathbb{E} \xi^\rho z^{B_\rho} u^{B_{\rho-1}} e^{-\theta \tau_\rho - \vartheta \tau_{\rho-1}} \quad (15)$$

$$\Phi_{\mu > \nu}(z, \theta; \xi, u, \vartheta; N, T) = \mathbb{E} \xi^\nu z^{B_\nu} u^{B_{\nu-1}} e^{-\theta \tau_\nu - \vartheta \tau_{\nu-1}} \mathbf{1}_{\mu > \nu} \quad (16)$$

$$\Phi_{\mu < \nu}(z, \theta; \xi, u, \vartheta; N, T) = \mathbb{E} \xi^\mu z^{B_\mu} u^{B_{\mu-1}} e^{-\theta \tau_\mu - \vartheta \tau_{\mu-1}} \mathbf{1}_{\mu < \nu} \quad (17)$$

$$\Phi_{\mu \geq \nu}(z, \theta; \xi, u, \vartheta; N, T) = \mathbb{E} \xi^\mu z^{B_\mu} u^{B_{\mu-1}} e^{-\theta \tau_\mu - \vartheta \tau_{\mu-1}} \mathbf{1}_{\mu \geq \nu} \quad (18)$$

$$\Phi_{\mu \leq \nu}(z, \theta; \xi, u, \vartheta; N, T) = \mathbb{E} \xi^\mu z^{B_\mu} u^{B_{\mu-1}} e^{-\theta \tau_\mu - \vartheta \tau_{\mu-1}} \mathbf{1}_{\mu \leq \nu} \quad (19)$$

We will use the following:

1. The Laplace–Carson transform is defined as

$$\mathcal{LC}_q(\cdot)(y) = y \int_{q=0}^{\infty} e^{-yq}(\cdot) dq, \Re(y) > 0 \quad (20)$$

with the inverse

$$\mathcal{LC}_y^{-1}(\cdot)(q) = \mathcal{L}_y^{-1}(\cdot/y)(q) \quad (21)$$

where $\mathcal{L}_y^{-1}$ is the inverse of the Laplace transform;

2. The $\mathcal{D}$ -operator, already introduced in (11).

Theorem 1. Let $(\mathcal{B}, \mathcal{T}) = \sum_{k=1}^{\infty} Y_k \varepsilon_{\tau_k}$ be a marked random measure with position-dependent marking representing a delayed marked renewal process, such that the joint transforms of the respective increments of $(\mathcal{B}, \mathcal{T})$ are

$$\begin{aligned} \Gamma(z, \theta) &= \mathbb{E} z^Y e^{-\theta \Delta} \text{ of } (Y_i, \Delta_i), i = 1, 2, \dots, \\ \Gamma_0(z, \theta) &= \mathbb{E} z^{Y_0} e^{-\theta \Delta_0} \end{aligned} \quad (22)$$

(since (Y_0, Δ_0) has a different distribution from (Y_i, Δ_i) s), with the respective components

$$\begin{aligned} \Delta_1 &= \tau_1, \Delta_2 = \tau_2 - \tau_1, \dots \in [\Delta] \\ \Delta_0 &= \tau_0 \notin [\Delta] \end{aligned} \quad (23)$$

$$\begin{aligned} Y_i &= \mathbf{1}_{\{W_i > H\}} \\ Y_0 &= B_0 \text{ is an integer-valued, non-negative, RV} \end{aligned} \quad (24)$$

Then, the functionals $\Phi_\rho, \Phi_{\mu > \nu}, \Phi_{\mu < \nu}, \Phi_{\mu \geq \nu}, \Phi_{\mu \leq \nu}$ of (15)–(19) satisfy the following formulas:

$$\begin{aligned} \Phi_\rho(z, \theta; \xi, u, \vartheta; N, T) \\ = \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \{ \Gamma_0(z, \theta) - \Gamma_0(zx, \theta + y) + \Psi[\Gamma(z, \theta) - \Gamma(zx, \theta + y)] \} (T) \end{aligned} \quad (25)$$

$$\begin{aligned} \Phi_{\mu > \nu}(z, \theta; \xi, u, \vartheta; N, T) \\ = \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \Gamma_0(z, \theta + y) - \Gamma_0(zx, \theta + y) + \Psi[\Gamma(z, \theta + y) - \Gamma(zx, \theta + y)] \right\} (T) \end{aligned} \quad (26)$$

$$\begin{aligned} \Phi_{\mu < \nu}(z, \theta; \xi, u, \vartheta; N, T) \\ = \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \Gamma_0(zx, \theta) - \Gamma_0(zx, \theta + y) + \Psi[\Gamma(zx, \theta) - \Gamma(zx, \theta + y)] \right\} (T) \end{aligned} \quad (27)$$

$$\begin{aligned} \Phi_{\mu \geq \nu}(z, \theta; \xi, u, \vartheta; N, T) \\ = \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \{ \Gamma_0(z, \theta) - \Gamma_0(zx, \theta) + \Psi[\Gamma(z, \theta) - \Gamma(zx, \theta)] \} (T) \end{aligned} \quad (28)$$

$$\begin{aligned} \Phi_{\mu \leq \nu}(z, \theta; \xi, u, \vartheta; N, T) \\ = \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \{ \Gamma_0(z, \theta) - \Gamma_0(z, \theta + y) + \Psi[\Gamma(z, \theta) - \Gamma(z, \theta + y)] \} (T) \end{aligned} \quad (29)$$

where

$$\Psi = \Psi(z, \theta; \xi, u, \vartheta; x, y) = \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} \quad (30)$$

Proof. Introduce the families of random indices extended from (12)–(14):

$$\begin{aligned} \{\mu(p) = \inf\{m : \tau_m > p\} : p \geq 0\} \\ \{\nu(q) = \inf\{n : B_n > q\} : q = 0, 1, \dots\} \\ \{\rho(p, q) = \mu(p) \wedge \nu(q) : (p, q) \in \mathbb{R}_+ \times \mathbb{N}_0\} \end{aligned}$$

and furthermore, the families of $\mathcal{F}$ -measurable sets:

$$\begin{aligned} H_{12} &= H_{12}(p, q) = \{\mu(p) > \nu(q)\}, (p, q) \in \mathbb{R}_+ \times \mathbb{N}_0 \\ H_{21} &= H_{21}(p, q) = \{\mu(p) < \nu(q)\}, (p, q) \in \mathbb{R}_+ \times \mathbb{N}_0 \\ H_{11} &= H_{11}(p, q) = \{\mu(p) = \nu(q)\}, (p, q) \in \mathbb{R}_+ \times \mathbb{N}_0. \end{aligned}$$

Note that $\mu = \mu(T)$, $\nu = \nu(N - 1)$, and $\rho = \rho(T, N - 1)$.

Next, the associated families of functionals are:

$$\begin{aligned} \Phi_{\rho(p, q)}(z, \theta; \xi, u, v) &= \mathbb{E} \xi^{\rho(p, q)} z^{B_{\rho(p, q)}} u^{B_{\rho(p, q)} - 1} e^{-\theta \tau_{\rho(p, q)} - \vartheta \tau_{\rho(p, q)} - 1} \\ &= \mathbb{E} \xi^{\rho(p, q)} z^{B_{\rho(p, q)}} u^{B_{\rho(p, q)} - 1} e^{-\theta \tau_{\rho(p, q)} - \vartheta \tau_{\rho(p, q)} - 1} (\mathbf{1}_{H_{12}} + \mathbf{1}_{H_{21}} + \mathbf{1}_{H_{11}}) \\ &= \Phi_{\mu(p) > \nu(q)}(z, \theta; \xi, u, \vartheta) + \Phi_{\mu(p) < \nu(q)}(z, \theta; \xi, u, \vartheta) + \Phi_{\mu(p) = \nu(q)}(z, \theta; \xi, u, \vartheta). \end{aligned} \quad (31)$$

The first term of (31) can be modified as

$$\begin{aligned} \Phi_{\mu(p) > \nu(q)}(z, \theta; \xi, u, \vartheta) &= \mathbb{E} \xi^{\rho(p, q)} z^{B_{\rho(p, q)}} u^{B_{\rho(p, q)} - 1} e^{-\theta \tau_{\rho(p, q)} - \vartheta \tau_{\rho(p, q)} - 1} \mathbf{1}_{\mu(p) > \nu(q)} \\ &= \mathbb{E} \xi^{\nu(q)} z^{B_{\nu(q)}} u^{B_{\nu(q)} - 1} e^{-\theta \tau_{\nu(q)} - \vartheta \tau_{\nu(q)} - 1} \mathbf{1}_{\mu(p) > \nu(q)} \\ &= \sum_{j=0}^{\infty} \sum_{k > j} \mathbb{E} \xi^j z^{B_j} u^{B_j - 1} e^{-\theta \tau_j - \vartheta \tau_j - 1} \mathbf{1}_{\{\mu(p) = k, \nu(q) = j\}} \end{aligned} \quad (32)$$

To continue, we introduce and use the following operators:

- (i) The operator D applied to function $[\mathbb{N}_0, \bar{B}(0, 1) \subseteq \mathbb{C}, f]$, where $B(0, 1)$ is a unit ball centered at zero:

$$D_p\{f(p)\}(x) = \sum_{p=0}^{\infty} x^p f(p)(1-x), \quad x \in B(0, 1).$$

Note that the dummy index p attached to D is being used for convenience only to indicate which variable (if more than one) it applies to. It can be readily shown that $\mathcal{D}^k$ of (11) is the inverse operator of D that restores f if we apply it for every k :

$$\mathcal{D}_x^k(D_p\{f(p)\}(x)) = f(k), k = 0, 1, \dots$$

- (ii) The Laplace–Carson transform $\mathcal{LC}_q(\cdot)(y)$ of (20) with the inverse $\mathcal{LC}_y^{-1}$ of (21). The dummy index q in $\mathcal{LC}_q$ plays the same role as p in D_p .

Then, the composition of the two operators reads

$$\mathfrak{D}_{pq}(\cdot)(x, y) = \mathcal{LC}_q \circ D_p(\cdot)(x, y).$$

Now, the application of operator $\mathfrak{D}_{pq}$ to $\mathbf{1}_{\{\mu(p)=k, \nu(q)=j\}}$ can be readily proven to yield

$$\begin{aligned} \mathfrak{D}_{pq}\mathbf{1}_{\{\mu(p)=k, \nu(q)=j\}}(x, y) &= (x^{B_{j-1}} - x^{B_j})(e^{-y\tau_{k-1}} - e^{-y\tau_k}) \\ &= x^{B_{j-1}}(1 - x^{Y_j})e^{-y\tau_{k-1}}(1 - e^{-y\Delta_k}), \quad j, k = 0, 1, \dots, \text{ where } B_{-1} = \tau_{-1} = 0 \end{aligned} \quad (33)$$

Using Fubini's theorem and noticing that $\mathfrak{D}_{pq}$ is a linear operator, we obtain from (31)–(33)

$$\begin{aligned} \Phi_{\mu>\nu}^*(z, \theta; \xi, u, \vartheta; x, y) &= \mathfrak{D}_{pq}\Phi_{\mu(p)>\nu(q)}(z, \theta; \xi, u, \vartheta)(x, y) \\ &= \sum_{j=0}^{\infty} \xi^j \sum_{k>j} \mathbb{E}[z^{B_j} u^{B_{j-1}} e^{-\theta\tau_j - \vartheta\tau_{j-1}} \mathfrak{D}_{pq}\mathbf{1}_{\{\mu(p)=k, \nu(q)=j\}}(x, y)] \\ &= \sum_{j=0}^{\infty} \xi^j \sum_{k>j} \mathbb{E}\left[(zux)^{B_{j-1}} e^{-(\theta+\vartheta+y)\tau_{j-1}} e^{-(\theta+y)\Delta_j} z^{Y_j} (1 - x^{Y_j})\right. \\ &\quad \left. \times e^{-y\sum_{i=j+1}^{k-1} \Delta_i} (1 - e^{-y\Delta_k})\right] \end{aligned}$$

and further by the independent increments property

$$\begin{aligned} &= \sum_{j=0}^{\infty} \xi^j \left[\mathbb{E}(zux)^{B_{j-1}} e^{-(\theta+\vartheta+y)\tau_{j-1}} \right] \mathbb{E}\left[e^{-(\theta+y)\Delta_j} z^{Y_j} (1 - x^{Y_j}) \right] \\ &\quad \times \sum_{k>j} \Gamma^{k-j-1}(1, y) [1 - \Gamma(1, y)] \end{aligned} \quad (34)$$

The expression $\sum_{k>j} \Gamma^{k-j-1}(1, y) [1 - \Gamma(1, y)] = 1$, because $\|\Gamma(1, y)\| < 1$ for $\Re(y) > 0$ as previously assumed.

From (8), (22), and (34),

$$\begin{aligned} \Phi_{\mu>\nu}^*(z, \theta; \xi, u, \vartheta; x, y) &= \mathbb{E}\left[e^{-(\theta+y)\Delta_0} z^{Y_0} (1 - x^{Y_0}) \right] \\ &+ \sum_{j=1}^{\infty} \xi^j \Gamma_0(zux, \theta + \vartheta + y) \Gamma^{j-1}(zux, \theta + \vartheta + y) [\Gamma(z, \theta + y) - \Gamma(zx, \theta + y)] \\ &= \Gamma_0(z, \theta + y) - \Gamma_0(zx, \theta + y) + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta + y) - \Gamma(zx, \theta + y)] \end{aligned} \quad (35)$$

after noticing that $\|\Gamma(zux, \theta + \vartheta + y)\| < 1$ as pointed out in [23,24] under similar circumstances.

Finally, we apply the inverse of $\mathfrak{D}_{pq}$,

$$\mathfrak{D}_{pq}^{-1} = \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1}$$

to $\Phi_{\mu>\nu}^*(z, \theta; \xi, u, \vartheta; x, y)$ of (35), thus arriving at (26):

$$\begin{aligned} \Phi_{\mu>\nu}(z, \theta; \xi, u, \vartheta; N, T) &= \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \Gamma_0(z, \theta + y) - \Gamma_0(zx, \theta + y) \right. \\ &\quad \left. + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta + y) - \Gamma(zx, \theta + y)] \right\} (T) \end{aligned}$$

A similar expression for $\Phi_{\mu<\nu}$ of (27) can be readily obtained by interchanging the roles of μ and ν :

$$\begin{aligned} \Phi_{\mu<\nu}(z, \theta; \xi, u, \vartheta; N, T) &= \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \Gamma_0(zx, \theta) - \Gamma_0(zx, \theta + y) \right. \\ &\quad \left. + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(zx, \theta) - \Gamma(zx, \theta + y)] \right\} (T). \end{aligned}$$

The expression for $\Phi_{\mu=\nu}(z, \theta; \xi, u, \vartheta; N, T)$ can be obtained using similar routines as in the derivation of $\Phi_{\mu>\nu}(z, \theta; \xi, u, \vartheta; N, T)$ to obtain

$$\begin{aligned} \Phi_{\mu=\nu}(z, \theta; \xi, u, \vartheta; N, T) &= \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \left[\Gamma_0(z, \theta) - \Gamma_0(zx, \theta) \right. \right. \\ &\quad \left. + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta) - \Gamma(zx, \theta)] \right] - \left[\Gamma_0(z, \theta + y) - \Gamma_0(zx, \theta + y) \right. \\ &\quad \left. + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta + y) - \Gamma(zx, \theta + y)] \right] \right\} (T). \end{aligned}$$

Finally, for (25),

$$\begin{aligned} \Phi_\rho(z, \theta; \xi, u, \vartheta; N, T) &= \Phi_{\mu>\nu}(z, \theta; \xi, u, \vartheta; N, T) + \Phi_{\mu<\nu}(z, \theta; \xi, u, \vartheta; N, T) \\ &\quad + \Phi_{\mu=\nu}(z, \theta; \xi, u, \vartheta; N, T) \\ &= \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \Gamma_0(z, \theta + y) - \Gamma_0(zx, \theta + y) + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta + y) - \Gamma(zx, \theta + y)] \right. \\ &\quad + \Gamma_0(zx, \theta) - \Gamma_0(zx, \theta + y) + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(zx, \theta) - \Gamma(zx, \theta + y)] \\ &\quad + \Gamma_0(z, \theta) - \Gamma_0(zx, \theta) + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta) - \Gamma(zx, \theta)] \\ &\quad \left. - \left[\Gamma_0(z, \theta + y) - \Gamma_0(zx, \theta + y) + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta + y) - \Gamma(zx, \theta + y)] \right] \right\} (T) \end{aligned}$$

after cancellation

$$= \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \Gamma_0(z, \theta) - \Gamma_0(zx, \theta + y) + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta) - \Gamma(zx, \theta + y)] \right\} (T).$$

The other useful combinations (28) and (29) are

$$\begin{aligned}
& \Phi_{\mu \geq \nu}(z, \theta; \xi, u, \vartheta; N, T) \\
&= \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \Gamma_0(z, \theta) - \Gamma_0(zx, \theta) + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta) - \Gamma(zx, \theta)] \right\} (T) \\
& \Phi_{\mu \leq \nu}(z, \theta; \xi, u, \vartheta; N, T) \\
&= \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \Gamma_0(z, \theta) - \Gamma_0(z, \theta + y) + \frac{\xi \Gamma_0(zux, \theta + \vartheta + y)}{1 - \Gamma(zux, \theta + \vartheta + y)} [\Gamma(z, \theta) - \Gamma(z, \theta + y)] \right\} (T).
\end{aligned}$$

□

Remark 2. Because we will focus on the marginal functional

$$\Phi_\rho(z, \theta; N, T) = \mathbb{E} z^{B_\rho} e^{-\theta \tau_\rho} = \mathbb{E} \xi^\rho z^{B_\rho} u^{B_{\rho-1}} e^{-\theta \tau_\rho - \vartheta \tau_{\rho-1}} \Big|_{\xi=u=1, \vartheta=0}$$

in (15) and in the other variants (16)–(19), setting $\xi = u = 1$ and $\vartheta = 0$, we drop $\rho, B_{\rho-1}, \tau_{\rho-1}$ for now. The latter can be of some interest though, if we want to predict the time and the wear of the system around the time of the $N - 1$ st critical shock for the benefit of early warning. The only part of all formulas that undergoes changes will be

$$\Psi(z, \theta; x, y) = \frac{\Gamma_0(zx, \theta + y)}{1 - \Gamma(zx, \theta + y)}$$

of (30). However, some formulas of Theorem 1 can be modified accordingly and simplified, such as

$$\Phi_\rho(z, \theta) := \Phi_\rho(z, \theta; N, T) = \Gamma_0(z, \theta) - [1 - \Gamma(z, \theta)] \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{\Gamma_0(zx, \theta + y)}{1 - \Gamma(zx, \theta + y)} \right\} (T) \quad (36)$$

after using the linearity of the inverse operator $\mathfrak{D}_{pq}^{-1} = \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1}$ and the property

$$\mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \mathbf{1}(x, y) = 1$$

which is due to the similar properties of the $\mathcal{D}$ -operator and the Laplace–Carson inverse.

In summary,

Corollary 1. Under the assumptions in Remark 2, the associated marginal functional Φ_ρ satisfies Formula (36).

4. Some Special Assumptions about the System

Besides dropping $\rho, B_{\rho-1}, \tau_{\rho-1}$ from the functional Φ_ρ in Remark 2, we specify the initial conditions for our model that are most commonly adapted to real-world reliability systems.

Initial Conditions. In what follows, we assume in (22)–(24) that

$$\tau_0 = \Delta_0 = 0 \text{ and } B_0 = r < N \quad (37)$$

as an initial wear of the system after it underwent r critical shocks prior to the beginning of observations at time zero, and thus,

$$\Gamma_0(z, \theta) = z^r, r < N \quad (38)$$

and further

$$\Gamma(z, \theta) = \mathbb{E} z^Y e^{-\theta \Delta} = \mathbb{E} z^Y \mathbb{E} e^{-\theta \Delta} = (pz + q)\gamma(\theta) \quad (39)$$

considering the binary nature of Y s and its assumed position independence of Δ s. From (36) and (37)–(39),

$$\Phi_\rho(z, \theta) = z^r - z^r[1 - (pz + q)\gamma(\theta)]\mathcal{D}_x^{N-r-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (pzx + q)\gamma(\theta + y)} \right\}^{(T)} \quad (40)$$

Consider two marginals.

Marginal 1. With $\theta = 0$,

$$\mathbb{E}z^{B_\rho} = \Phi_\rho(z, 0) = z^r - z^r p(1 - z)\mathcal{D}_x^{N-r-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (pzx + q)\gamma(y)} \right\}^{(T)} \quad (41)$$

implying that the mean quantity of critical shocks upon the system's failure is

$$\mathbb{E}B_\rho = \lim_{z \rightarrow 1-} \frac{d}{dz} \Phi_\rho(z, 0) = r + p\mathcal{D}_x^{N-r-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (px + q)\gamma(y)} \right\}^{(T)} \quad (42)$$

Marginal 2. With $z = 1$,

$$\mathbb{E}e^{-\theta\tau_\rho} = \Phi_\rho(1, \theta) = 1 - [1 - \gamma(\theta)]\mathcal{D}_x^{N-r-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (px + q)\gamma(\theta + y)} \right\}^{(T)} \quad (43)$$

implying that the mean time-to-failure is

$$\mathbb{E}\tau_\rho = -\frac{d}{d\theta} \Phi_\rho(1, \theta)|_{\theta=0} = \frac{1}{\gamma} \mathcal{D}_x^{N-r-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (px + q)\gamma(y)} \right\}^{(T)} \quad (44)$$

where $\mathbb{E}\Delta = \frac{1}{\gamma}$ is the mean time between two successive shocks.

From (42) and (44),

$$\frac{1}{\gamma p} (\mathbb{E}B_\rho - r) = \frac{1}{\gamma} \mathcal{D}_x^{N-r-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (px + q)\gamma(y)} \right\}^{(T)} = \mathbb{E}\tau_\rho$$

Consequently, we arrive at an interesting and surprising relationship between the two marginal expectations.

Corollary 2.

1. $\mathbb{E}B_\rho$ and $\mathbb{E}\tau_\rho$ are related through an affine transformation. $\mathbb{E}B_\rho = \gamma p \mathbb{E}\tau_\rho + r$
2. $\mathbb{E}B_\rho$ and $\mathbb{E}\tau_\rho$ are related through a linear transformation if and only if $r = 0$. In this case

$$\mathbb{E}B_\rho = \gamma p \mathbb{E}\tau_\rho$$

The surprising part of the linear relationship between $\mathbb{E}B_\rho$ and $\mathbb{E}\tau_\rho$ is because B_ρ and τ_ρ are not independent.

Remark 3. In Formula (40),

$$\Phi_\rho(z, \theta) = z^r \left[1 - [1 - (pz + q)\gamma(\theta)]\mathcal{D}_x^{N-r-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (pzx + q)\gamma(\theta + y)} \right\}^{(T)} \right]$$

we see that, with $r = 0$, (40) reduces to

$$\Phi_\rho(z, \theta) = 1 - [1 - (pz + q)\gamma(\theta)]\mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (pzx + q)\gamma(\theta + y)} \right\}^{(T)} \quad (45)$$

Observe that the functional of the simplified system can be easily restored to its original version under an arbitrary r via multiplication by z^r and by replacing N with $N - r$ in (45), which gives us the green light to proceed with $r = 0$.

Thus, with $r = 0$ in (38), $\Gamma_0(z, \theta) = 1$, and with (39), Ψ of (30) reduces to

$$\Psi = \frac{1}{1 - (pzx + q)\gamma(\theta + y)}. \quad (46)$$

Altogether, under the initial conditions specified in (37), Remark 3, and (46), the associated special case of Theorem 1 reads as follows:

Corollary 3. *Under the initial conditions*

$$\tau_0 = \Delta_0 = B_0 = 0 \quad (47)$$

and taking into account Remark 3, the functionals Φ_ρ and $\Phi_{\mu > \nu}$ satisfy the following formulas:

$$\begin{aligned} \Phi_\rho(z, \theta) &= \Phi_\rho(z, \theta; \xi = 1, u = 1, \vartheta = 0; N, T) \\ &= 1 - [1 - (pz + q)\gamma(\theta)] \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \left\{ \frac{1}{1 - (pzx + q)\gamma(\theta + y)} \right\} (T) \end{aligned} \quad (48)$$

$$\begin{aligned} \Phi_\rho(z, \theta) &= \Phi_{\mu > \nu}(z, \theta; \xi = 1, u = 1, \vartheta = 0; N, T) \\ &= pz \mathcal{D}_x^{N-1} \left[(1 - x) \mathcal{LC}_y^{-1} \left\{ \frac{\gamma(\theta + y)}{1 - (pzx + q)\gamma(\theta + y)} \right\} (T) \right]. \end{aligned} \quad (49)$$

5. Special Case Marked Poisson Stream of Shocks

In this section, we make the assumption that the stream of shocks hitting the system is marked Poisson with position-independent marking. More formally, the support counting measure of the stream $(\mathcal{B}, \mathcal{T}) = \sum_{k=1}^{\infty} Y_k \varepsilon_{\tau_k}$ is Poisson of rate γ . That being said, Formula (45) reduces to

$$\Phi_\rho(z, \theta) = 1 - \left[1 - a(z) \frac{\gamma}{\gamma + \theta} \right] \mathcal{D}_x^{N-1} \circ \mathcal{LC}_y^{-1} \{ \varphi(x, y) \} (T) \quad (50)$$

where $pz + q$ is tentatively abbreviated as $a(z)$ and φ stands for

$$\varphi(x, y) = \left[1 - a(zx) \frac{\gamma}{\gamma + \theta + y} \right]^{-1}. \quad (51)$$

Our first step is to apply the Laplace–Carson inverse of (50) to φ in (51). We point out the following properties of the Laplace–Carson inverse $\mathcal{LC}_y^{-1}$:

(LC1) $\mathcal{LC}_y^{-1}$ is linear;

(LC2) $\mathcal{LC}_y^{-1} \mathbf{1}(T) = \mathbf{1}$ (where $\mathbf{1}(y) = 1 \forall y \in \mathbb{C}$);

(LC3) $\mathcal{LC}_y^{-1} \frac{1}{\alpha + y}(T) = \frac{1}{\alpha} [1 - e^{-\alpha T}]$, where α is a function in variables, other than y .

Back to (51), φ can be rewritten as

$$\varphi(x, y) = 1 + \frac{\gamma a(zx)}{\gamma - \gamma a(zx) + \theta + y}.$$

Then, by LC1–LC3,

$$\psi(x) = \mathcal{LC}_y^{-1} \varphi(x, y)(T) = 1 + \frac{\gamma a(zx)}{\gamma - \gamma a(zx) + \theta} \left[1 - e^{-(\gamma - \gamma a(zx) + \theta)T} \right].$$

Next, we apply the $\mathcal{D}$ -operator to ψ . Rewriting ψ in its explicit form gives

$$\begin{aligned}\psi(x) &= 1 + \frac{\gamma p z x + \gamma q}{\gamma p - \gamma p z x + \theta} \left[1 - e^{-(\gamma p + \theta)T} e^{\gamma p z T x} \right] \\ &= \frac{\gamma + \theta}{\gamma p + \theta - \gamma p z x} + e^{-(\gamma p + \theta)T} e^{\gamma p z T x} - \frac{\gamma + \theta}{\gamma p + \theta - \gamma p z x} e^{-(\gamma p + \theta)T} e^{\gamma p z T x}\end{aligned}\quad (52)$$

Then, from (52), by property (Dvii),

$$\begin{aligned}\delta(z, \theta) = \mathcal{D}_x^{N-1} \psi(x) &= \frac{\gamma + \theta}{\gamma p(1-z) + \theta} \left[1 - \left(\frac{\gamma p z}{\gamma p + \theta} \right)^N \right] + e^{-(\gamma p + \theta)T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p z T)^k \\ &\quad - \frac{\gamma + \theta}{\gamma p(1-z) + \theta} e^{-(\gamma p + \theta)T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p z T)^k \left[1 - \left(\frac{\gamma p z}{\gamma p + \theta} \right)^{N-k} \right]\end{aligned}\quad (53)$$

Finally,

$$\mathbb{E} z^{B_p} e^{-\theta \tau_p} = \Phi_\rho(z, \theta) = 1 - \left[1 - (pz + q) \frac{\gamma}{\gamma + \theta} \right] \delta(z, \theta) \quad (54)$$

In particular, with $N = 1$, (53) reduces to

$$\delta(z, \theta) = \sigma(z, \theta) \left(1 - e^{-(\gamma p + \theta)T} \right) + e^{-(\gamma p + \theta)T}$$

or in the form

$$= \sigma(z, \theta) + e^{-(\gamma p + \theta)T} \{1 - \sigma(z, \theta)\} \quad (55)$$

where

$$\sigma(z, \theta) = \frac{\gamma + \theta}{\gamma p(1-z) + \theta} \left[1 - \frac{\gamma p z}{\gamma p + \theta} \right] \quad (56)$$

Marginal Transform of τ_ρ . Of further interest is the marginal transform of τ_ρ where the latter is the time-to-failure of the system. From (53)–(56),

$$\mathbb{E} e^{-\theta \tau_p} = 1 - \frac{\theta}{\gamma + \theta} \delta(1, \theta)$$

which can be reduced to the following expression:

$$\begin{aligned}\mathbb{E} e^{-\theta \tau_p} &= \left(\frac{\gamma p}{\gamma p + \theta} \right)^N + \frac{\gamma}{\gamma + \theta} e^{-(\gamma p + \theta)T} \sum_{k=0}^{N-1} \frac{1}{k!} T^k (\gamma p)^k \\ &\quad - \left(\frac{\gamma p}{\gamma p + \theta} \right)^N e^{-(\gamma p + \theta)T} \sum_{k=0}^{N-1} \frac{1}{k!} T^k (\gamma p + \theta)^k\end{aligned}\quad (57)$$

Now with $\mathbb{E} \tau_\rho(N) = (-1) \lim_{\theta \rightarrow 0} \frac{d}{d\theta} \mathbb{E} e^{-\theta \tau_p}$, we obtain

$$\mathbb{E} \tau_\rho(N) = \frac{N}{\gamma p} + \left(\frac{1}{\gamma} - \frac{N}{\gamma p} + T \right) e^{-\gamma p T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p T)^k - T e^{-\gamma p T} \frac{(\gamma p T)^{N-1}}{(N-1)!} \quad (58)$$

or in the form

$$\begin{aligned}\mathbb{E}\tau_p(N) &= \frac{N}{\gamma p} + \left(\frac{1}{\gamma} - \frac{N}{\gamma p}\right)e^{-\gamma p T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p T)^k \\ &\quad + T e^{-\gamma p T} \left[e^{\gamma p T} - \sum_{k=N-1}^{\infty} \frac{1}{k!} (\gamma p T)^k \right]\end{aligned}\quad (59)$$

The second term becomes zero for $N = 1$ and it is then simplified to

$$\mathbb{E}\tau_p(1) = \frac{1}{\gamma p} \left(1 - q e^{-\gamma p T}\right), \text{ with } q = 1 - p. \quad (60)$$

With $T \rightarrow \infty$ in (57), the system will be knocked down through the N th critical and, thus, the fatal shock alone (with no aging involved), so that

$$\mathbb{E}e^{-\theta\tau_p} \rightarrow \left(\frac{\gamma p}{\gamma p + \theta}\right)^N \text{ as } T \rightarrow \infty \quad (61)$$

which agrees with Formula (7) as expected. There is an interesting asymptotics of $\mathbb{E}\tau_p(N)$ with respect to N . First, we show the validity of the following lemma:

Lemma 1. From Formula (59), $\lim_{N \rightarrow \infty} N \left[1 - e^{-\gamma p T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p T)^k\right] = 0$.

Proof. First,

$$1 - e^{-\gamma p T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p T)^k = e^{-\gamma p T} \sum_{k=N}^{\infty} \frac{1}{k!} (\gamma p T)^k$$

Let $0 < a < 1$ for some a . Then, $N \sum_{k=N}^{\infty} \frac{1}{k!} a^k < N \sum_{k=N}^{\infty} a^k = N \frac{a^N}{1-a} \rightarrow 0$, because for $b > 1$, the function

$$f(x) = \frac{x}{b^x} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

Thus, $\lim_{N \rightarrow \infty} N \left[1 - e^{-\gamma p T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p T)^k\right] = 0$ if $\gamma p T < 1$. Now, let $a = \gamma p T \geq 1$. Then, using Stirling's formula,

$$\frac{a^k}{k!} < \left(\frac{ae}{k}\right)^k < z^k \text{ for some } 0 < z < 1, \text{ some large } N \text{ and all } k \geq N.$$

Thus,

$$\sum_{k=N}^{\infty} \frac{1}{k!} a^k < N \sum_{k=N}^{\infty} z^k = N z^N \frac{1}{1-z} \rightarrow 0 \text{ as } N \rightarrow \infty$$

□

Using Lemma 1 and (58), we have

Corollary 4. The mean time-to-failure of the system if $N \rightarrow \infty$ is

$$\lim_{N \rightarrow \infty} \mathbb{E}\tau_p(N) = \frac{1}{\gamma} + T \quad (62)$$

Example 1. In reality, however, the value $\frac{1}{\gamma} + T$ is reached by $\mathbb{E}\tau_p(N)$ much sooner, in fact, at relatively small values of N . For example, with $\gamma = 2$, $p = 0.5$, and $T = 10$, $\mathbb{E}\tau_p(N) = 10.5$ at $N = 32, 33, \dots$ as we can see from Figure 5 below and printed Table 1 in the associated values of $\mathbb{E}\tau_p(N)$ for $N = 1, \dots, 40$.

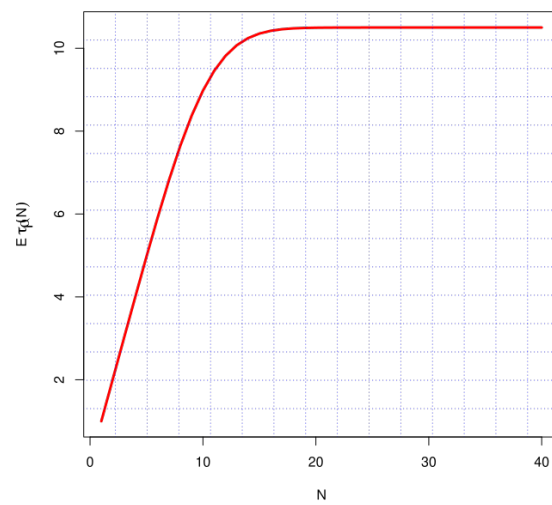


Figure 5. The values of $\mathbb{E}\tau_\rho(N)$, $N = 1, \dots, 40$, for $\gamma = 2$, $p = 0.5$, and $T = 10$.

Table 1. The values of $\mathbb{E}\tau_\rho(N)$, $N = 1, \dots, 40$.

[1]	0.9999773	1.9997049	2.9980705	3.9915178	4.9717234	5.9235541
[7]	6.8249404	7.6497594	8.3732392	8.9778645	9.4573798	9.8174718
[13]	10.0733055	10.2452951	10.3547921	10.4208915	10.4587825	10.4794421
[19]	10.4901742	10.4954946	10.4980159	10.4991599	10.4996576	10.4998655
[25]	10.4999490	10.4999814	10.4999934	10.4999977	10.4999993	10.4999998
[31]	10.4999999	10.5000000	10.5000000	10.5000000	10.5000000	10.5000000
[37]	10.5000000	10.5000000	10.5000000	10.5000000	10.5000000	10.5000000

Example 2. With $\gamma = 2$, $p = 0.5$, and $T = 40$, $\mathbb{E}\tau_\rho(N) = 40.5$ at $N = 73$, as seen in Figure 6, and further explained in Table 2 in a similar way as Table 1 explains Figure 5.

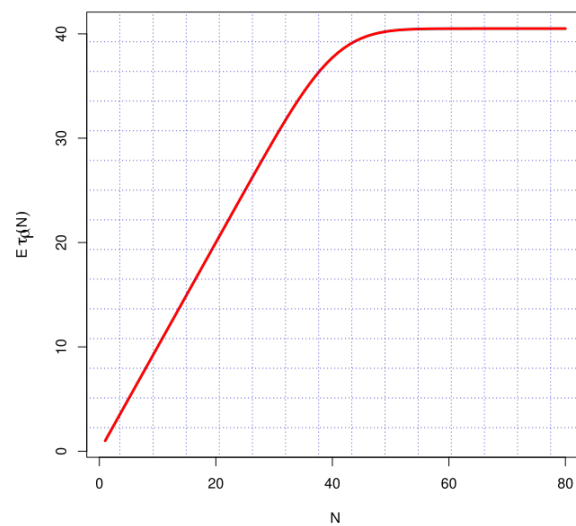


Figure 6. The values of $\mathbb{E}\tau_\rho(N)$, $N = 1, \dots, 80$, for $\gamma = 2$, $p = 0.5$, and $T = 40$.

Table 2. The values of $\mathbb{E}\tau_\rho(N)$, $N = 1, \dots, 80$.

[25]	24.99239	25.98636	26.97643	27.96060	28.93624	29.89994	30.84748	31.77387
[33]	32.67346	33.54010	34.36746	35.14932	35.87994	36.55444	37.16910	37.72160
[41]	38.21116	38.63854	39.00596	39.31694	39.57599	39.78832	39.95955	40.09540
[49]	40.20143	40.28283	40.34431	40.39000	40.42340	40.44742	40.46443	40.47628
[57]	40.48441	40.48990	40.49355	40.49593	40.49747	40.49845	40.49906	40.49944
[65]	40.49967	40.49981	40.49989	40.49994	40.49997	40.49998	40.49999	40.49999
[73]	40.50000	40.50000	40.50000	40.50000	40.50000	40.50000	40.50000	40.50000

Remark 4. Formula (62) shows that, with a very large number of critical shocks needed to knock the system down, it becomes inoperational due to aging (with a delay by $\frac{1}{\gamma}$ due to a delayed notice at some $\tau_n > T$ such that $\tau_{n-1} < T$).

On the other hand, for $T \rightarrow \infty$,

$$\mathbb{E}\tau_\rho(T) \rightarrow \frac{N}{\gamma p}$$

which is in agreement with (61).

Remark 5. From Formula (58), under $N = 1$ when one critical shock becomes fatal,

$$\mathbb{E}e^{-\theta\tau_\rho} = \frac{\gamma p}{\gamma p + \theta} + \frac{\gamma}{\gamma + \theta} e^{-(\gamma p + \theta)T} - \frac{\gamma p}{\gamma p + \theta} e^{-(\gamma p + \theta)T}$$

confirming Formula (60) for $\mathbb{E}\tau_\rho(1)$.

6. What Makes the System Fail Sooner

One worthwhile piece of information would be to learn which of the two events comes first: the fatal failure due to a fatal shock or through aging. The associated functional

$$\mathbb{E}z^{B_\rho} e^{-\theta\tau_\rho} \mathbf{1}_{\{\tau_\nu < \tau_\mu\}} = \Phi_{\mu>\nu}(z, \theta; N, T)$$

borrowed from Corollary 1 (Formula (23)) carries the joint distribution of (B_ρ, τ_ρ) on the confined probability space $(\Omega, \mathcal{F} \cap \{\mu > \nu\}, \mathbb{P})$. In particular, it will provide us with the likelihood $\mathbb{P}\{\mu > \nu\} = \Phi_\nu(1, 0)$ that the system becomes inoperational due to a fatal (N th critical) shock before it would degrade through aging.

Therefore, according to (23) of Corollary 1,

$$\Phi_{\mu>\nu}(z, \theta; N, T) = pz \mathcal{D}_x^{N-1} \left[(1-x) \mathcal{L}\mathcal{C}_y^{-1} \left\{ \frac{\gamma(\theta+y)}{1-(pzx+q)\gamma(\theta+y)} \right\} (T) \right].$$

Now, under the assumptions in Section 5 that shocks, with their magnitudes, arrive in accordance with a marked Poisson process, and thus $\gamma(\theta) = \frac{\gamma}{\gamma+\theta}$,

$$\Phi_{\mu>\nu}(z, \theta; N, T) = \gamma pz \mathcal{D}_x^{N-1} \left[(1-x) \mathcal{L}\mathcal{C}_y^{-1} \left\{ \frac{1}{\gamma p(1-zx) + \theta + y} \right\} (T) \right]$$

which after the use of property (LC3) $\mathcal{L}\mathcal{C}_y^{-1} \frac{1}{\alpha+y}(T) = \frac{1}{\alpha} [1 - e^{-\alpha T}]$, $\Phi_{\mu>\nu}(z, \theta; N, T)$ becomes

$$\begin{aligned} \Phi_{\mu>\nu}(z, \theta; N, T) &= \gamma pz \mathcal{D}_x^{N-1} \mathcal{L}\mathcal{C}_y^{-1} \Phi_{\mu>\nu}^*(z, \theta; x, y)(T) \\ &= \mathcal{D}_x^{N-1} \alpha(x) \left[1 - e^{-(\gamma p + \theta)T} e^{\gamma p T x} \right], \end{aligned}$$

where

$$\alpha(x) = \gamma pz \frac{1-x}{\gamma p(1-zx) + \theta}.$$

With the familiar routine, we reduce $\alpha(x)$ to a proper fraction in x before applying the $\mathcal{D}$ -operator. This straightforward operation leads to

$$\alpha(x) = 1 - \frac{\gamma p(1-z) + \theta}{\gamma p + \theta - \gamma pzx}.$$

Thus,

$$\begin{aligned} \mathbb{E}z^{B_\rho} e^{-\theta\tau_\rho} \mathbf{1}_{\{\mu > \nu\}} &= \Phi_{\mu > \nu}(z, \theta) = \mathcal{D}_x^{N-1} \circ \mathcal{L}\mathcal{C}_y^{-1} \Phi_{\nu < \mu}^*(z, \theta; x, y)(T) \\ &= \mathcal{D}_x^{N-1} \left[1 - \frac{\gamma p(1-z) + \theta}{\gamma p + \theta - \gamma pzx} - e^{-(\gamma p + \theta)T} e^{\gamma pzTx} + e^{-(\gamma p + \theta)T} e^{\gamma pzTx} \frac{\gamma p(1-z) + \theta}{\gamma p + \theta - \gamma pzx} \right] \end{aligned}$$

due to property (Dvi) that $\mathcal{D}_x^k \frac{1}{a-cx} = \frac{1}{a-c} \left[1 - \left(\frac{c}{a} \right)^{k+1} \right]$

$$\begin{aligned} &= 1 - \frac{\gamma p(1-z) + \theta}{\gamma p(1-z) + \theta} \left[1 - \left(\frac{\gamma pz}{\gamma p + \theta} \right)^N \right] - e^{-(\gamma p + \theta)T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma pzT)^k \\ &\quad + e^{-(\gamma p + \theta)T} \frac{\gamma p(1-z) + \theta}{\gamma p(1-z) + \theta} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma pzT)^k \left[1 - \left(\frac{\gamma pz}{\gamma p + \theta} \right)^{N-k} \right] \end{aligned}$$

finally arriving at

$$\begin{aligned} \mathbb{E}z^{B_\rho} e^{-\theta\tau_\rho} \mathbf{1}_{\{\mu > \nu\}} &= \Phi_{\mu > \nu}(z, \theta) \\ &= \left(\frac{\gamma pz}{\gamma p + \theta} \right)^N - e^{-(\gamma p + \theta)T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma pzT)^k \left(\frac{\gamma pz}{\gamma p + \theta} \right)^{N-k}. \end{aligned}$$

For $\theta = 0$, the associated marginal functional is

$$\mathbb{E}z^{B_\rho} \mathbf{1}_{\{\mu > \nu\}} = \Phi_{\nu < \mu}(z, 0; N, T) = z^N - z^N e^{-\gamma pT} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma pT)^k.$$

In particular, for $z = 1$,

$$\mathbb{P}\{\mu > \nu\}(N) = 1 - e^{-\gamma pT} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma pT)^k$$

which is a gamma probability distribution function with fixed parameters ($\alpha = N$ and $\beta = \gamma p$) and under a variable time T .

Example 3. Considering $\mathbb{P}\{\mu > \nu\} = \mathbb{P}\{\mu > \nu\}(N)$ as a function of variable N and with $\gamma = 2$, $p = 0.5$, and $T = 10$, we plot it for $N = 1, \dots, 40$. Figure 7 below shows that this probability is monotone decreasing in N as expected and then runs to zero.

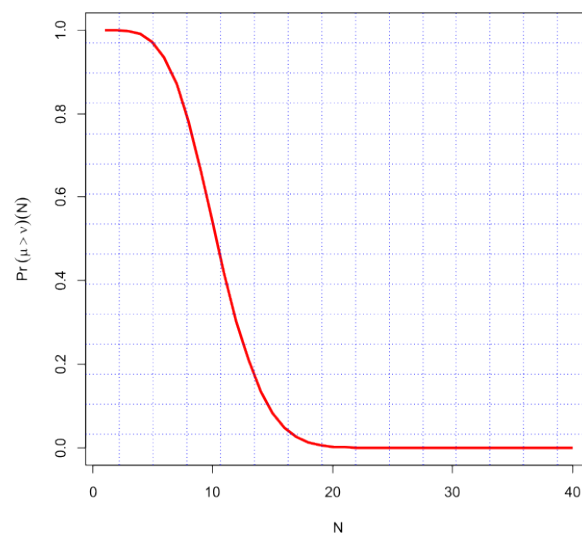


Figure 7. The values of $\mathbb{P}\{\mu > \nu\}(N)$, $N = 1, \dots, 40$, for $\gamma = 2$, $p = 0.5$, and $T = 10$.

7. The Case Where D Is Random

Suppose now that the sustainability threshold D is an RV. Then, $T = \delta^{-1}(D)$ is also random. We return to the special case with $\Delta \in [\text{Exp}(\gamma)]$. According to (59), we have

$$\begin{aligned} \mathbb{E}\tau_\rho = \mathbb{E}\tau_\rho(N) &= \frac{N}{\gamma p} + \left(\frac{1}{\gamma} - \frac{N}{\gamma p}\right) \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p)^k T^k e^{-\gamma p T} + \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p)^k T^{k+1} e^{-\gamma p T} \\ &\quad - e^{-\gamma p T} T^N \frac{(\gamma p)^{N-1}}{(N-1)!} \end{aligned} \quad (63)$$

under the assumption that T is deterministic. Given that T is random, we interpret $\mathbb{E}\tau_\rho$ of (63) as the conditional expectation $\mathbb{E}[\tau_\rho|T]$. Thus, from (63), under the assumption that the right-hand side represents $\mathbb{E}[\tau_\rho|T]$, we have

$$\begin{aligned} \mathbb{E}\tau_\rho = \mathbb{E}[\mathbb{E}[\tau_\rho|T]] &= \frac{N}{\gamma p} + \left(\frac{1}{\gamma} - \frac{N}{\gamma p}\right) \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p)^k \mathbb{E}[T^k e^{-\gamma p T}] \\ &\quad + \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p)^k \mathbb{E}[T^{k+1} e^{-\gamma p T}] - \mathbb{E}[e^{-\gamma p T} T^N] \frac{(\gamma p)^{N-1}}{(N-1)!} \end{aligned} \quad (64)$$

Now,

$$\mathbb{E}e^{-\theta T} T^j = \int_{x=0}^{\infty} e^{-\theta x} x^j f_T(x) dx = \mathcal{L}(x^j f(x))(\theta) = (-1)^j \frac{d^j}{d\theta^j} L_T(\theta)$$

where $L_T(\theta) = \mathbb{E}e^{-\theta T}$.

Example 4. Assuming

$$T \in [\text{Exp}(\xi)]$$

we have

$$L_T(\theta) = \frac{\xi}{\xi + \theta} \text{ and } L^{(j)}(\theta) = \xi(-1)^j \frac{j!}{(\xi + \theta)^{j+1}}.$$

Thus, $\mathbb{E}e^{-\theta T} T^j = \frac{\xi j!}{(\xi + \theta)^{j+1}}$ implying that

$$\mathbb{E}[e^{-\gamma p T} T^j] = \frac{\xi j!}{(\xi + \gamma p)^{j+1}}. \quad (65)$$

Then, from (64) and (65),

$$\begin{aligned}\mathbb{E}\tau_\rho &= \frac{N}{\gamma p} + \left(\frac{1}{\gamma} - \frac{N}{\gamma p}\right) \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p)^k \frac{\xi k!}{(\xi + \gamma p)^{k+1}} \\ &\quad + \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p)^k \frac{\xi(k+1)!}{(\xi + \gamma p)^{k+2}} - \frac{\xi N!}{(\xi + \gamma p)^{N+1}} \frac{(\gamma p)^{N-1}}{(N-1)!}\end{aligned}$$

or in the form

$$\begin{aligned}\mathbb{E}\tau_\rho &= \frac{N}{\gamma p} + \left(\frac{1}{\gamma} - \frac{N}{\gamma p}\right) \frac{\xi}{\xi + \gamma p} \sum_{k=0}^{N-1} \left(\frac{\gamma p}{\xi + \gamma p}\right)^k \\ &\quad + \frac{\xi}{(\xi + \gamma p)^2} \sum_{k=0}^{N-1} \left(\frac{\gamma p}{\xi + \gamma p}\right)^k (k+1) - \frac{\xi N}{(\xi + \gamma p)^2} \left(\frac{\gamma p}{\xi + \gamma p}\right)^{N-1}.\end{aligned}\quad (66)$$

Thus, if $T \in [\text{Exp}(\xi)]$, then $\mathbb{E}\tau_\rho$ satisfies Formula (66).

Example 5. We wonder what the aging function δ is supposed to be in order that $\delta(T) = D$, provided D is an RV. We give an example of such a function. Let $\beta > 1$ and $D = U \in [\text{Un}(0, \beta)]$. Then, if F_T is the probability distribution function of RV T ,

$$\begin{aligned}\mathbb{P}\left\{T = \delta^{-1}(D) = (\beta F_T)^{-1}(U) \leq t\right\} &= \mathbb{P}\left\{F_T^{-1}\left(\frac{1}{\beta}U\right) \leq t\right\} \\ &= \mathbb{P}\{U \leq \beta F_T(t)\} = \beta \frac{1}{\beta} F_T(t) = F_T(t)\end{aligned}$$

implying that $(\beta F_T)^{-1}(U) \in [T]$, that is, $(\beta F_T)^{-1}(U)$ has the same distribution as T , where $U \in [\text{Un}(0, \beta)]$ and, thus, δ is identified as βF_T .

In conclusion, when $D \in [\text{Un}(0, \beta)]$, the aging function δ is βF_T , where T is an RV with the probability distribution function F_T whatever F_T represents (note that $\beta > D$ a.s.).

For example, with $D = \delta(T) \in [\text{Un}(0, \beta)]$ and T exponential with parameter ξ , as assumed in Example 4, the aging function is $\delta(x) = \beta(1 - e^{-\xi x})$.

8. Summary

In this paper, we studied a reliability system on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}_0}, \mathbb{P})$ subject to occasional random shocks of random magnitudes $W_0, W_1, W_2, \dots$ occurring at times $\tau_0, \tau_1, \tau_2, \dots$, thus forming a marked point process $\sum_{k=0}^{\infty} W_k \varepsilon_{\tau_k}$. All shocks are either harmless or critical dependent on whether $W_k \leq H$ or $W_k > H$, given a fixed threshold H . It takes a total of N critical shocks to knock the system down. The very last of the N critical shocks is fatal. We form a sequence $\{Y_k\}$ of auxiliary Bernoulli RVs, where $Y_k = \mathbf{1}_{(H, \infty)}(W_k)$ and the associated sequence $\{B_n\}$ of its partial sums

$$B_n = \sum_{k=0}^n Y_k.$$

The RV $\nu = \min\{n = 0, 1, \dots : B_n = N\}$ is called the exit index, and the stopping time τ_ν relative to the filtration $(\mathcal{F}_n)$ is referred to as the first passage time of process $(B, \mathcal{T}) = \sum_{k=0}^{\infty} Y_k \varepsilon_{\tau_k}$. The first passage time (the term was borrowed from fluctuation theory) is also the lifetime of the system or, synonymously, the time-to-failure.

Besides the above shocks, the system also ages formalized by a monotone increasing continuous function δ , such that $\delta(t) \rightarrow \infty$ as $t \rightarrow \infty$. Suppose D is a fixed real number (failure threshold), thus, there is a compact interval $[0, T]$ such that $\delta(T) = D$. However, our aging process is restrictive, we embellish it arguing that in a real-world situation, the failure time T due to aging is hard to detect or even hard to define, because the system

does not really stop functioning, but it continues to operate, although not well enough. The latter is often difficult to diagnose even if watching all the time, and as a result, the system's aging is rather monitored at times $\{\tau_n\}$ when shocks occur, which undoubtedly is more cost-effective than watching it in real time. The shortcoming of this approach is that system failure associated with its aging will be identified with some delay, but this is either the only way or a preferred way to maintain such systems. In summary, this setting makes the aging-related failure random.

We define the age-related delayed exit index

$$\mu = \inf\{m = 1, 2, \dots : \tau_m \geq T\}.$$

Associated with μ is the delayed soft failure time τ_μ .

Now, combining the two causes for system failure, we define the cumulative exit index, which is the shock count for system failure

$$\rho = \mu \wedge \nu,$$

and the associated modified first passage time τ_ρ , which becomes the new lifetime of the system. Using the common terminology in the reliability literature, we deal with two competing processes: inducing a soft failure and a hard failure, and the system fails whenever one of the two failures comes first.

The goal of this research was to obtain the joint distribution of the time-to-failure τ_ρ and the total damage to the system upon its failure specified by the quantity B_ρ ($\leq N$), all in a closed form, rather than algorithmically or numerically. We treat the joint transform $\Phi_\rho = \mathbb{E}z^\rho z^{B_\rho} e^{-\theta\tau_\rho} u^{B_{\rho-1}} e^{-\theta\tau_{\rho-1}}$ that also carries the information about the system one shock prior to the fatal failure, and the total shock count ρ on the system's failure.

To obtain Φ_ρ , we use operational calculus that includes the $\mathcal{D}$ -operators introduced and developed in our past work, and modified for the current system. The results are self-contained, are based on Theorem 1, and were rigorously proved.

The main assumptions were that the stream of shocks $(\mathcal{B}, \mathcal{T}) = \sum_{k=0}^{\infty} Y_k \varepsilon_{\tau_k}$ was a process with independent increments, which is marked delayed renewal with position-dependent marking, that is, the increments $(Y_k, \Delta_k = \tau_k - \tau_{k-1})$ are mutually dependent for each $k = 0, 1, \dots$ ($\tau_{-1} = 0$). These assumptions we sustained throughout the proof of Theorem 1. Also, $(Y_0, \tau_0 = \Delta_0)$ are differently distributed from the rest of the sequence (Y_k, Δ_k) , because in some applications (Y_0, Δ_0) may contain a history of the process included as an initial condition.

In Section 4, the “terminated” version $\sum_{k=0}^{\rho} Y_k \varepsilon_{\tau_k}$ of $(\mathcal{B}, \mathcal{T})$ observed from its inception through the lifetime τ_ρ is not renewal, Markov, or even semi-Markov. Yet, one interesting property holds for the marginals of $B_\rho = \sum_{k=0}^{\rho} Y_k$ and τ_ρ when $Y_0 = B_0 = r$ and $\tau_0 = \Delta_0 = 0$ under the assumption of position independence. That is, when the system was observed at some moment when it underwent r critical shocks. It turns out that $\mathbb{E}B_\rho$ and $\mathbb{E}\tau_\rho$ are related through the affine transformation $\mathbb{E}B_\rho = \gamma p \mathbb{E}\tau_\rho + r$, where $\frac{1}{\gamma} = \mathbb{E}\Delta$ (the mean time between two successive shocks) and $p = \mathbb{P}\{W > H\}$ that a shock is critical. Or the relationship between $\mathbb{E}B_\rho$ (the mean number of critical shocks at time τ_ρ) and $\mathbb{E}\tau_\rho$ (the mean system's lifetime) is linear if and only if $r = 0$.

Fully explicit expressions (53) and (54) of the joint transform $\mathbb{E}z^{B_\rho} e^{-\theta\tau_\rho}$ and their marginals appear when the stream of shocks is a marked Poisson process (as per Section 5). Section 5 also discusses the asymptotics of $\mathbb{E}\tau_\rho(N)$ when $N \rightarrow \infty$, supported by numerical examples.

In Section 6, we deal with $\mathbb{E}z^{B_\rho} e^{-\theta\tau_\rho} \mathbf{1}_{\{\tau_\nu < \tau_\mu\}}$, that is, with the joint transform of B_ρ and τ_ρ on the confined σ -algebra $\mathcal{F} \cap \{\nu < \mu\}$. In particular, we arrive at $\mathbb{P}\{\nu < \mu\}$, i.e.,

the probability that the system fails due to shocks sooner than via aging. Very explicit and compact formulas such as

$$\mathbb{E}z^{B_p}e^{-\theta\tau_p}\mathbf{1}_{\{\mu>\nu\}} = \left(\frac{\gamma pz}{\gamma p + \theta}\right)^N - e^{-(\gamma p + \theta)T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma pzT)^k \left(\frac{\gamma pz}{\gamma p + \theta}\right)^{N-k}$$

are due to the assumption on the stream of shocks being marked Poisson (of rate γ , with $p = \mathbb{P}\{W > H\}$). From this expression,

$$\mathbb{P}\{\mu > \nu\}(N, T) = 1 - e^{-\gamma p T} \sum_{k=0}^{N-1} \frac{1}{k!} (\gamma p T)^k$$

which reads as the gamma probability distribution function with fixed parameters ($\alpha = N$ and $\beta = \gamma p$) if time T were a time variable (which, of course, is not).

Section 7 deals with the system under the assumption that D (the threshold of the aging function is supposed to cross) is random, thus also making the projection T of crossing random. Some special cases lead to explicit formulas. One worthwhile example discusses the case when D is a uniform RV in interval $(0, \beta)$ and the aging function is a multiple of the probability distribution function of the now RV T .

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