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Optimization Models for the Vehicle Routing Problem under Disruptions

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Abstract: In this paper, we study the role of disruptions in the multi-period vehicle routing problem (VRP), which naturally arises in humanitarian logistics and military applications. We assume that at any time during the delivery phase, each vehicle could have chance to be disrupted. When a disruption happens, vehicles will be unable to continue their journeys and supplies will be unable to be delivered. We model the occurrence of disruption as a given probability and consider the multi-period expected delivery. Our objective is to either minimize the total travel cost or maximize the demand fulfillment, depending on the supply quantity. This problem is denoted as the multi-period vehicle routing problem with disruption (VRPMD). VRPMD does not deal with disruptions in real-time and is more focused on the long-term performance of a single routing plan. We first prove that the proposed VRPMD problems are NP-hard. We then present some analytical properties related to the optimal solutions to these problems. We show that Dror and Trudeau's property does not apply in our problem setting. Nevertheless, a generalization of Dror and Trudeau's property holds. Finally, we present efficient heuristic algorithms to solve these problems and show the effectiveness of the proposed models and algorithms through numerical studies.

Keywords: vehicle routing problem; disruption; expected delivery; multi-period; split delivery; mixed-integer programming

MSC: 90-10

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1. Introduction

In the face of natural or man-made disasters, an important task is to distribute vital emergency supplies. Following the earthquake in Nepal [1], or the ongoing armed conflict in Syria [2], numerous efforts have been made to provide humanitarian relief in a manner that minimizes human suffering, the cost of distributing emergency supplies, and the risk of further loss of life. Typically, we wish to plan out a route for which vehicles may supply those that are in need; however, one problem that often arises in these situations is the possibility that a supply vehicle becomes damaged or broken down. In natural disasters, these vehicles may be caught by resultant avalanches or tsunamis arising in the aftershocks of an earthquake. Bridges, underpasses, or tracks can be severely weakened from the strains of these disasters and may be on the brink of collapse. In armed conflicts, this can occur due to enemy action, land mines, or crossfire. Even in day-to-day business operations, supply trucks may break down or encounter other traffic obstructions that prohibit them from completing their delivery. Thus, we are interested in situations where the roads are highly unstable and have the potential to be disrupted during use, which may destroy the vehicles that are traveling on them. Although the chances for most of these scenarios to happen can be quite low, when these highly unstable networks are repeatedly traveled upon, it is very important to have a routing plan that addresses these long-term uncertainties of disruption.

The goal of our paper is to address the high risks that may be presented within a vehicle routing application. We introduce the multi-period vehicle routing problem with

disruption (VRPMD). That is, the vehicle routing problem (VRP) with the possibility of disruptions on any arc and at any period in a network. The classic VRP aims to satisfy the demands of all customer nodes while optimizing for the cost of delivery. In that problem, we are given a limited number of vehicles and each vehicle must begin and end its journey at the depot. This is the most common form of the VRP that has been widely studied in the fields of transportation, logistics, and operations research. We use the term *disruption* to describe when a vehicle in our network is damaged, blocked, or destroyed. Disruption may occur in many scenarios, including military supply distributions, disaster relief systems, communications networks, etc. When a disruption occurs on a path, all supplies on the vehicle traveling on that path are considered lost. As a result, vehicles will be unable to supply their target nodes in a single delivery period. Our problem setting emerges when it is possible to hedge against this uncertainty across multiple delivery periods to meet long-term delivery goals. This means that disruptions will not be dealt with in real time. Instead, we are only concerned about the long-term expected delivery goals. As a consequence, only the multi-period long-term demand goals can be met and it will be possible for nodes to have insufficient supply in a single delivery period.

We assume that it is impossible to guarantee successful delivery, especially in high-risk areas where we might encounter destructive natural forces or acts of violence. Instead, it is possible to estimate or predict the chance of disruption. Our problem setting focuses on customers (or locations requiring supplies) that have specified *expected* demand requirements, which receive deliveries over multiple periods. Therefore, in the VRPMD, we are interested in the expected supply rate of nodes in a network. We consider two major situations in this paper: When the total supply is greater than the total demand and when the total supply is less than the total demand. In the former case, we consider cost minimization, which is similar to the typical goal of VRP models. It is noted that a feasible solution is not always guaranteed when the total supply is greater than the total demand. In the case where the total supply is greater than the total demand, it becomes critical to optimize the fulfillment of the customers as much as possible and cost is no longer the priority. This situation often prevails in the midst of a disaster, where the number of affected people is high and available resources (supplies/vehicles) are limited.

In this paper, we present two models, the VRPMD (with the objective to minimize cost) and the VRPMD (with the objective to maximize demand fulfillment). These models build upon the split delivery vehicle routing problem (SDVRP), which comes with some unique analytical properties. These properties are examined in the VRPMD models. We also prove that the proposed VRPMD problems are NP-hard and we designed heuristic algorithms to solve these models. Finally, we conduct several computational studies to test our models and algorithms.

The remainder of this paper is organized as follows. Section 2 begins with a review of the literature that is relevant to our problem. Section 3 introduces the two models of VRPMD and provides some insights into the difficulty of our problem. Section 4 presents some analytical properties of the optimal solution, some of which are closely related to SDVRP. In Section 5, we describe two heuristic algorithms that are used to solve the VRPMD. The experimental results are presented in Section 6. The concluding remarks and future research directions are presented in Section 7.

2. Literature Review

Uncertainty in the vehicle routing problem has been studied extensively in the literature (e.g., [3,4]). However, most studies focus on stochastic customers [5], stochastic demands [6], and stochastic travel times [7]. These models are classified under chance-constrained programming, stochastic programming with recourse, or the Markov decision process framework. There are also studies using fuzzy logic (e.g., [8]) or robust optimization (e.g., [9]) to model uncertainty. The uncertainty in our model arises from disruption probability. We are interested in the long-term average performance and we studied a

deterministic model. In this section, we review streams of literature that are closely related to our research.

Firstly, we must look at the concept of disruption when applied to VRP models. Ref. [10] was the first paper in which the problem of vehicle disruption was directly tackled in real time. They introduced disruptions in the execution of a single instance of a VRP and generated a new routing solution each time the disruption occurred. In our research, we apply the same concept of disruption to VRP; however, we will not be rerouting the vehicles in real-time. Instead, our models have already accounted for the expected loss incurred by disruption and are able to hedge against the demand uncertainty through the oversupply in other periods. Although this creates some inflexibility in the planning of routes, this model is useful, especially in networks where there is a high volume of deliveries, or where rerouting is costly or impossible. Furthermore, we believe that in our situations, it is not unusual for multiple vehicles to break down over a long time span.

By using the expectation of disruption, we are able to apply the concept of maximum reliability, which was used by [11]. Their study focused on finding a path that has the maximum probability of traversing through a stochastic network. They also showed that the problem of finding maximum reliability could be transformed into a shortest path problem. We used the concepts in [11] as a starting point for the development of effective solutions to our models.

Another important branch of literature is the split delivery vehicle routing problem (SDVRP), which was proposed by [12]. In the SDVRP, a demand node can be served by multiple vehicles, contrary to the classical VRP, where a demand node can only be served by a single vehicle. They showed that the SDVRP could lead to potential cost savings. In our study, similar savings are possible. Moreover, having multiple vehicles visit a single demand node will help in the case of disruption because of the flexibility in route planning. Ref. [12] presented the SDVRP in a compact formulation and proved some structural properties of the optimal solution. A key property is that in the presence of triangle inequality, no two routes in the optimal solution can have more than one split demand point in common. Although we show that this property does not hold in the VRPMD, we can show that a generalization of this property holds. The computational complexity of the SDVRP was analyzed by [13] by solving the SDVRP on a specialized network. In their study, they concluded that SDVRP with a limited fleet is NP-hard.

Following the work by [12], researchers have been able to map several applications of the SDVRP to humanitarian logistics, which serves as an important tool in delivering aid to those in disaster or emergency situations. In particular, the work by [14] focuses on the distribution of relief supplies according to a few key performance metrics, commonly used in humanitarian logistics. The models in [14] address the routing problem in a purely deterministic fashion, while we build upon these models to introduce the concept of disruption across arcs.

Other works in disaster management that present uncertainty do not take into consideration the possibility that vehicles may be lost. For example, Ref. [15] presents uncertainty from the perspectives of travel time, demand, or service time. Ref. [16] studied the inventory positioning problem in the preparedness phase of disaster management, where they located an optimal set of service facilities in the event of arc failures. In a recent study, Ref. [17] discussed a vehicle routing problem, where arcs could be "destroyed". However, in their model, vehicles were not lost when this disruption occurred. Instead, they were rerouted according to a two-stage model, called the multi-vehicle path decision problem. Furthermore, in contrast to our assumption, they only considered the probabilistic disruption of a single path, while we consider the disruption of multiple possible paths.

3. Models

In the VRPMD, we begin with a network $G = (V, A)$, where V is the set of demand nodes plus the depot, and A is the set of arcs connecting two nodes. We assume that G is a complete graph. Each demand node, i, apart from the depot, has an associated demand, d_i, and each arc (i, j) has an associated cost, c_{ij}. Our situation arises when demand nodes require a fixed *expected* amount of supplies over multiple delivery periods. The optimal solution will be a routing plan that is executed over multiple periods. In this model, we assume that in any single delivery period, vehicles are subject to disruption. Every vehicle in our network has a chance to be disrupted, and when disruption occurs, the vehicle will be unable to continue its journey.

We denote the probability that a vehicle is disrupted on arc (i, j) as q_{ij} $(0 \leq q_{ij} < 1)$; thus, the probability that it is not disrupted is $p_{ij} = 1 - q_{ij}$. The values q_{ij} and p_{ij} are given for each arc (i, j), or these values can be estimated. Moreover, we assume that the disruption probability of each arc is independent of any other. Since we are optimizing for multiple periods, we use the expectation of delivery across all nodes of our network. This implies that it is possible to encounter insufficient supply in a single period due to disruption; however, in our model, we are able to safeguard against that situation so the long-term demand requirements can be met.

Our models are based on route formulation, similar to the one used by [14]. That is, we let Ω be an input parameter containing the set of all feasible routes that begin and end at the depot. Each route, $r \in \Omega$, is identified by the set of nodes that it visits. More formally, $r = \{0, n_1, \cdots, n_k, 0\}$, where $n_1, \cdots, n_k \in V$ and 0 represents the depot. Each route is associated with a cost that can be computed by $c_r = \sum_{(i,j)\in r} c_{ij}$. We can also compute the probability of reaching n_1 without disruption in route r as $\phi_{n_1 r} = p_{0n_1}$, the probability of reaching n_2 without disruption as $\phi_{n_2 r} = p_{0n_1} p_{n_1 n_2}$, and so on. Therefore, we define ϕ_{ir} as the probability of arriving at node i without disruption in route r (probability of a successful delivery). The route costs and success probabilities are computed for each route as input parameters. A summary of notations is given in Table 1.

Table 1. Notations for the VRPMD.

Indices	
0 :	index of the depot.
i :	index of demand nodes (customers); $i \in V = \{1, \cdots, N\}$.
(i, j) :	index of arcs; $(i, j) \in A$.
r, r_k :	index of routes; $r, r_k \in \Omega$.
Parameters	
d_i :	demand of node i.
c_r :	transportation cost of route r.
ϕ_{ir} :	probability of arriving at node i without disruption in route r.
a_{ir} :	$a_{ir} = 1$ if node i is in route r; otherwise $a_{ir} = 0$.
K :	number of vehicles.
Q :	vehicle capacity.
Decision Variables	
y_{ir} :	amount of supply delivered to node i in route r.
x_r:	$x_r = 1$ if route r is selected; otherwise $x_r = 0$.

Next, we examine two performance measures for the VRPMD: Cost minimization and fulfillment maximization. The option to use one model over the other depends primarily on the context of the situation and the amount of supply that is available to us.

When the total supply is greater than the total demand, we have an excess of supply, in which case, it is preferable to minimize with respect to cost. When the total supply is less than the total demand, we have a shortage of supplies. In this situation, our goal will be to optimize the distribution of our supplies. We want to make sure that those in need will receive as many supplies as we could possibly give. Here, cost is no longer a primary concern, as mentioned by [18], and cost can be treated as a constraint rather than an objective.

Cost minimization

In the first formulation, we will aim to minimize the cost that is required to satisfy the demands of all nodes. This formulation assumes that the total supply is larger than the total demand.

$$\begin{aligned}
\text{(VRPMD–1)} \quad \min \quad & \sum_{r\in\Omega} c_r x_r && && (1)\\
\text{s.t.} \quad & \sum_{r\in\Omega} x_r \le K && && (2)\\
& \sum_{i\in V} y_{ir} \le Q && \forall r \in \Omega && (3)\\
& y_{ir} \le Q a_{ir} x_r && \forall i \in V, \forall r \in \Omega && (4)\\
& \sum_{r\in\Omega} \phi_{ir} y_{ir} \ge d_i && \forall i \in V && (5)\\
& x_r \in \{0,1\} && \forall r \in \Omega && (6)\\
& y_{ir} \ge 0 && \forall i \in V, \forall r \in \Omega. && (7)
\end{aligned}$$

Objective (1) aims to minimize the total cost of all selected routes. Ω is the set of all feasible routes. Constraint (2) enforces the maximum number of vehicles selected. Constraint (3) ensures that the total supplies on a vehicle cannot exceed its capacity Q. Constraint (4) enforces the necessary logical requirement that delivery on a route can only be positive if the route is selected. Constraint (5) guarantees that the expected amount delivered to every node meets their demand. Note that we can add a safety stock, s_i, if necessary, so that the right-hand side of constraint (5) becomes $d_i + s_i$. Finally, we require that the decision variables x_r be binary, and y_{ir} be nonnegative in (6) and (7), respectively.

It is important to note that we chose to keep x_r as a binary variable to remain consistent with the models used in [14]. In Section 4, we will see the possibility of multiple repeated routes. This is a problem we have addressed using multiple sets of the same routes in the set Ω.

Even though this model assumes an excess of supply, we cannot always guarantee feasibility. Because we are looking at the expected delivery to nodes, having a risky routing plan (with high disruption probabilities) could produce a model that is infeasible. Additionally, if the total supply is less than the total demand, this model will be infeasible. In those cases, we should no longer be optimizing for cost. Instead, it is recommended that we optimize with respect to fulfillment (VRPMD–2), presented below.

Fulfillment maximization

We also propose a VRPMD formulation that aims to maximize demand fulfillment, which assumes that the total supply is less than the total demand. In humanitarian operations, we can use this model when the performance measure depends on how well we can

satisfy demands with limited resources. This model can also be used when (VRPMD–1) is infeasible since this directly implies that there is a shortage of supplies.

$$\text{(VRPMD–2)} \quad \max \quad \sum_{i\in V}\sum_{r\in\Omega} \phi_{ir} y_{ir} \tag{8}$$

$$\text{s.t.} \quad \sum_{r\in\Omega} x_r \le K \tag{9}$$

$$\sum_{i\in V} y_{ir} \le Q \qquad \forall r \in \Omega \tag{10}$$

$$y_{ir} \le Q a_{ir} x_r \qquad \forall i \in V, \forall r \in \Omega \tag{11}$$

$$\sum_{r\in\Omega} \phi_{ir} y_{ir} \le d_i \qquad \forall i \in V \tag{12}$$

$$x_r \in \{0,1\} \qquad \forall r \in \Omega \tag{13}$$

$$y_{ir} \ge 0 \qquad \forall i \in V, \forall r \in \Omega. \tag{14}$$

Objective (8) aims to maximize the expected demand fulfillment for all nodes. Constraints (9)–(11), (13), and (14) are the same as (2)–(4), (6), and (7), respectively. Constraint (12) guarantees that we do not oversupply a node so that we can satisfy the other demands as much as possible.

In (VRPMD–2), we assume that there is a lack of supply. If, on the contrary, there is an excess of supply, using the demand fulfillment model will give us multiple optimal solutions. In such situations, the solution of (VRPMD–2) becomes meaningless since it would be better to optimize over cost using (VRPMD–1). Alternatively, we can add a term $-\epsilon \sum_{r\in\Omega} c_r x_r$ in the objective function of (VRPMD–2), where ϵ is a small positive constant. This treatment could allow us to find the most economic route among the routes that have the highest fulfillment.

One important feature of the VRPMD formulations is that for the final leg of the journey (from node i back to depot 0), the disruption probability has no impact on the solution. Thus, this problem could be formulated as the open vehicle routing problem (OVRP), where vehicles do not need to return to the depot after servicing the last customer (see [19]). We chose not to adopt this model for the purpose of possibly reusing vehicles; another reason is that an empty vehicle has very little reason to be disrupted and, therefore, has no impact on the fulfillment of nodes.

Complexity

The classical VRP is shown to be NP-hard through a reduction to the traveling salesman problem by [20]. It was much later that [13] proved the NP-hardness of SDVRP with a limited fleet. Knowing the complexity of SDVRP, we can use simple reduction to show the complexity of (VRPMD–1).

Theorem 1. *(VRPMD–1) is NP-hard.*

Proof. The VRPMD for cost minimization is NP-hard since the SDVRP is a special case of (VRPMD–1), where the probability of disruption is 0 across all arcs. □

The complexity of (VRPMD–2) is not as obvious, since the objective of this model is very different from classical VRP models. However, we show that it is still a difficult problem using a reduction from the Hamiltonian cycle problem (HCP), which is known to be NP-complete [20]. The objective of the HCP is to find a Hamiltonian cycle (a cycle that visits each vertex exactly once in a given network).

Theorem 2. *(VRPMD–2) is NP-hard.*

Proof. We use a reduction from the Hamiltonian cycle problem (HCP). Consider any arbitrary graph G' with n nodes. A corresponding instance of the (VRPMD–2) can be constructed on a complete graph, as follows: We set $p_{ij} = 1$ if $(i, j) \in G$; otherwise, $p_{ij} = 0.01$ (or an arbitrary very small positive number). We optimize (VRPMD–2) using a single vehicle, $K = 1$ with capacity $Q = n$, and n customers each with $d_i = 1$. If and only if the optimal routing to this instance of the (VRPMD–2) gives us an objective value of n, the corresponding graph in G' has a Hamiltonian cycle. □

4. Analytical Properties

In this section, we characterize the properties of optimal solutions to the VRPMD models.

We begin by looking at a generalization of the theorem in [21]

Lemma 1. *For (VRPMD–1) and (VRPMD–2), if $q_{ij} \equiv \xi$ for a constant $0 \leq \xi < 1$ for all $(i, j) \in A$, then there exists an optimal solution where two routes will have, at most, one shared demand node.*

This conclusion is true because when q_{ij} is constant, the triangle inequality will hold, and taking a less direct route will not be cost-efficient. We refer to [21] for the proof.

Property 1. *For (VRPMD–1) and (VRPMD–2), two demand nodes may have multiple routes in common in an optimal solution.*

We notice that Property 1 is a direct violation of Lemma 1, i.e., Dror and Trudeau's property. An important assumption behind Dror and Trudeau's property is the fact that arc costs in the network satisfy the triangle inequality. While this is also true in the VRPMD, disruption probabilities do not satisfy this assumption. In the VRPMD, some overlapping routes may have a better probability of success (i.e., probability of no disruption) since the chance that each arc is disrupted is independent of one another. The *expected delivery* in our models (i.e., probability of no disruption multiplied by delivery quantity) does not follow the triangle inequality either, so additional routes covering the same nodes may be present in the optimal solution because of the higher probability of success. While a more direct route that bypasses nodes with fulfilled demands could appear to save costs, the expected delivery from taking this route could be small due to disruption. In this scenario, an additional vehicle would be required to cover any unmet demands, which is ultimately not optimal. In other words, the cost associated with adding an additional vehicle to our solution to cover these unmet demands could outweigh the cost of making a less direct route to avoid disruption. In some instances, adding an additional vehicle can even be infeasible (due to the constraint of vehicle number K). In general, as disruption probabilities become larger, there is a better chance that we see overlaps between routes where this property applies.

In Figure 1, we present an example where the objective is to minimize the cost. The optimal solution consists of $r_1 = \{0, 1, 2, 0\}$ and $r_2 = \{0, 1, 2, 3, 0\}$, both covering demand nodes 1 and 2. Although r_2 is not supplying node 1 and does not need to travel on arcs $(0, 1)$ and $(1, 2)$, the vehicle still makes that detour because, otherwise, our solution would require 3 vehicles ($p_{0,3} \cdot Q = 2 < 3 = d_3$ and $p_{0,2} \cdot p_{2,3} \cdot Q = 1.8 < 3 = d_3$), which is infeasible since $K = 2$.

For (VRPMD–2), it can be easily shown that Property 1 is also true. Naturally, arcs with a higher probability of success are more favorable in maximizing fulfillment because the expected delivery will be greater.

Figure 1 illustrates a critical property that is different from Dror and Trudeau's property.

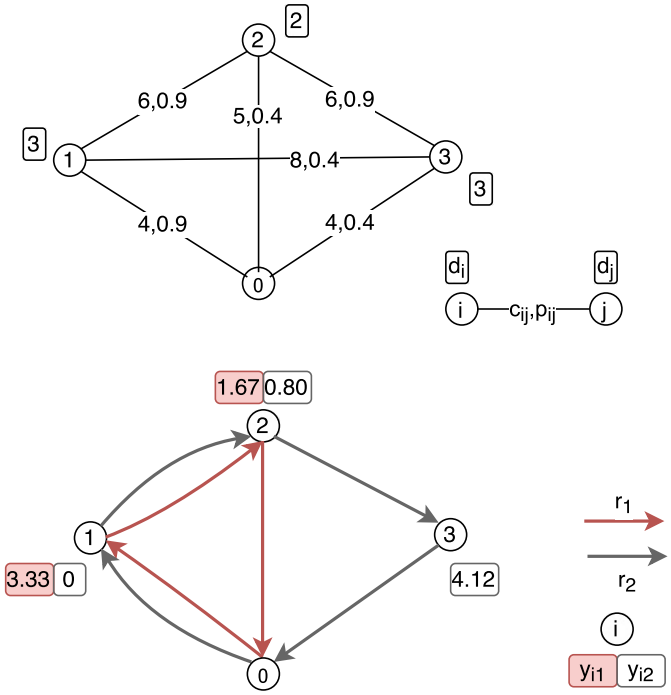


Figure 1. (VRPMD–1) with 3 demand nodes, $Q = 5$, $K = 2$.

Property 2. *For (VRPMD–1) and (VRPMD–2), there exists an optimal solution, where for every pair of nodes, at most one route delivers to both nodes.*

Proof. First, we prove the result for (VRPMD–1). We consider routes 1 and 2, both of which visit two arbitrary demand nodes, i and j. For simplicity, we use the following notation:

$$\phi_{i1} = \phi_1, \quad \phi_{j1} = \phi_2, \quad \phi_{i2} = \phi_3, \quad \phi_{j2} = \phi_4,$$
$$y_{i1} = y_1, \quad y_{j1} = y_2, \quad y_{i2} = y_3, \quad y_{j2} = y_4.$$

We assume that demands for i and j are both satisfied, i.e., $\phi_1 y_1 + \phi_3 y_3 \geq d_i$ and $\phi_2 y_2 + \phi_4 y_4 \geq d_j$. The capacity constraints are also satisfied, i.e., $y_1 + y_2 \leq Q$ and $y_3 + y_4 \leq Q$. Assume that in this optimal solution, both nodes have split demands, i.e., $y_1, y_2, y_3, y_4 > 0$.

We consider four cases:

Case 1: if $\phi_4 \geq \frac{\phi_2 \phi_3}{\phi_1}$ and $y_2 > \frac{\phi_3}{\phi_1} y_3$, we can make the following substitution:

$$y_1^* = y_1 + \frac{\phi_3}{\phi_1} y_3 \qquad y_2^* = y_2 - \frac{\phi_3}{\phi_1} y_3$$
$$y_3^* = 0 \qquad y_4^* = y_4 + y_3.$$

Case 2: if $\phi_4 < \frac{\phi_2 \phi_3}{\phi_1}$ and $y_4 > \frac{\phi_1}{\phi_3} y_1$, we can make the following substitution:

$$y_1^* = 0 \qquad y_2^* = y_2 + y_1$$
$$y_3^* = y_3 + \frac{\phi_1}{\phi_3} y_1 \qquad y_4^* = y_4 - \frac{\phi_1}{\phi_3} y_1.$$

Case 3: if $\phi_4 < \frac{\phi_2 \phi_3}{\phi_1}$ and $y_4 \leq \frac{\phi_1}{\phi_3} y_1$, we can make the following substitution:

$$y_1^* = y_1 - \frac{\phi_4}{\phi_2} y_4 \qquad y_2^* = y_2 + \frac{\phi_4}{\phi_2} y_4$$
$$y_3^* = y_3 + y_4 \qquad y_4^* = 0.$$

Case 4: if $\phi_4 \geq \frac{\phi_2\phi_3}{\phi_1}$ and $y_2 \leq \frac{\phi_3}{\phi_1}y_3$, we can make the following substitution:

$$y_1^* = y_1 + y_2 \qquad y_2^* = 0$$
$$y_3^* = y_3 - \frac{\phi_2}{\phi_4}y_2 \qquad y_4^* = y_4 + \frac{\phi_2}{\phi_4}y_2.$$

We can verify that all four cases provide feasible solutions. For example, in case 4, the capacity constraints are satisfied: $y_1^* + y_2^* = y_1 + y_2 \leq Q$ and $y_3^* + y_4^* = y_3 - \frac{\phi_2}{\phi_4}y_2 + y_4 + \frac{\phi_2}{\phi_4}y_2 = y_3 + y_4 \leq Q$. The demand constraints are also satisfied:

$$\begin{aligned}\phi_1 y_1^* + \phi_3 y_3^* &= \phi_1(y_1 + y_2) + \phi_3(y_3 - \frac{\phi_2}{\phi_4}y_2)\\ &= \phi_1 y_1 + \phi_3 y_3 + (\phi_1 - \frac{\phi_3\phi_2}{\phi_4})y_2\\ &\geq \phi_1 y_1 + \phi_3 y_3\\ &\geq d_i\end{aligned}$$

and

$$\begin{aligned}\phi_2 y_2^* + \phi_4 y_4^* &= \phi_2(0) + \phi_4(y_4 + \frac{\phi_2}{\phi_4}y_2)\\ &= \phi_2 y_2 + \phi_4 y_4\\ &\geq d_j.\end{aligned}$$

A similar argument applies to the other three cases. In all cases, after applying the substitution, we are left with a solution where no supply is delivered on one of the nodes from one of the routes.

Finally, we check to ensure that the adjusted deliveries do not negatively affect the objective value. In (VRPMD–1), it is clear that the objective value $\sum_{r\in\Omega} c_r x_r$ remains unchanged since the selection of routes is still the same. The only modification is the distribution of supplies between the vehicles.

The proof for (VRPMD–2) is similar. The only adjustment that must be made is when a node receives too much supply and, thus, violates constraint (12). To account for this ($\phi_1 y_1 + \phi_3 y_3 \leq d_i$ and $\phi_2 y_2 + \phi_4 y_4 \leq d_j$), we can simply remove the excess supply from the vehicles. In case 4:

$$y_1^* = y_1 + y_2 \qquad y_2^* = 0$$
$$y_3^* = min\{y_3 - \frac{\phi_2}{\phi_4}y_2, \frac{d_i - \phi_1 y_1^*}{\phi_3}\} \qquad y_4^* = min\{y_4 + \frac{\phi_2}{\phi_4}y_2, \frac{d_j}{\phi_4}\}.$$

The new objective value for this case is $\sum_{i\in V}\sum_{r\in\Omega}\phi_{ir}y_{ir}$. Other parts of the network remain unchanged, so we focus on the following four entries:

$$\begin{aligned}&\phi_1 y_1^* + \phi_2 y_2^* + \phi_3 y_3^* + \phi_4 y_4^*\\ &= \phi_1 y_1^* + \phi_2(0) + \phi_3(y_3 - \frac{\phi_2}{\phi_4}y_2) + \phi_4(y_4 + \frac{\phi_2}{\phi_4}y_2)\\ &\geq \phi_1 y_1^* + \phi_2(0) + \phi_3(\frac{d_i - \phi_1 y_1^*}{\phi_3}) + \phi_4(\frac{d_j}{\phi_4})\\ &= \phi_1 y_1^* + d_i - \phi_1 y_1^* + d_j\\ &\geq \phi_1 y_1^* + \phi_1 y_1 + \phi_3 y_3 - \phi_1 y_1^* + \phi_2 y_2 + \phi_4 y_4\\ &= \phi_1 y_1 + \phi_2 y_2 + \phi_3 y_3 + \phi_4 y_4.\end{aligned}$$

With the new delivery values, the objective value for (VRPMD–2) is non-decreasing. A similar argument applies to the other cases as well. Thus we have shown that there will always exist an optimal solution where for every pair of nodes, at most one route delivers to both nodes. □

Property 2 differs from Property 1 in that we acknowledge multiple routes may visit several common nodes; however, there exists an optimal solution, such that demand split only occurs on one of these nodes. For example, routes r_1 and r_2 may share nodes i, j; if $y_{i2} > 0$ and $y_{j2} > 0$, then either $y_{i1} = 0$ or $y_{j1} = 0$. On the other hand, Property 2 can be understood as a generalization of Dror and Trudeau's property. Indeed, in the classic SDVRP, Property 2 automatically implies Property 1, because when the triangle inequality holds, it is unnecessary to make a detour and visit nodes that have already received deliveries.

Next, we formally define the concept of "detour point".

Definition 1. *A route r contains a detour point at node i, if i is visited by r without receiving any supply.*

The use of a detour point is shown in Figure 1 and they are frequently used when there are large disruption probabilities. In fact, one distinguishing property of the VRPMD is the following.

Property 3. *For (VRPMD–1) and (VRPMD–2), an optimal solution may contain several detour points; a single node can be used as a detour point several times.*

An example of Property 3 is shown in Figure 2. Although the disruption probabilities in the network are quite extreme, such an instance may also occur when the disruption probabilities on arcs $(2,3)$ and $(3,4)$ are much larger.

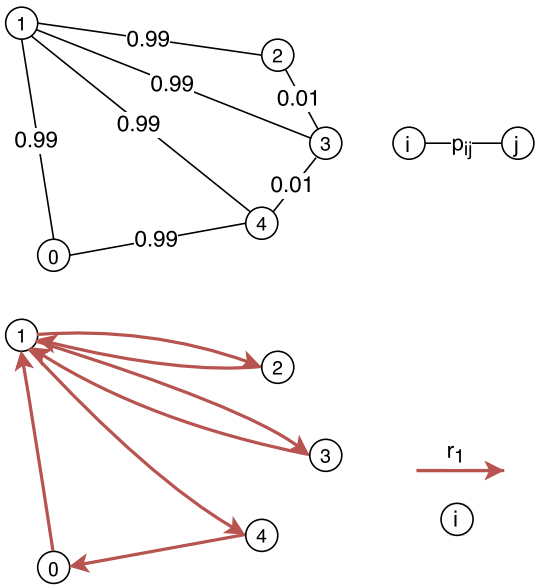


Figure 2. Multiple detours.

Another observation of the VRPMD comes from the fact that a single trip to a node can still be insufficient to satisfy the node demand. This can happen in both the case of $Q \geq d_i$ and the case of $Q < d_i$, since the disruption probability may make the expected delivery unable to satisfy the node demand completely. This observation brings us to our next property:

Property 4. *For (VRPMD–1) and (VRPMD–2), an optimal solution may use a single route multiple times.*

The presence of multiple routes is illustrated in Figure 3. In Figure 3, the optimal solution is to use route $r_1 = \{0, 1, 2, 0\}$ twice (i.e., send two vehicles along the route r_1). The supply distributions for the two routes are different and satisfy Properties 1 and 2. Route r_1 is chosen twice because it provides us with the highest probability of supplying enough stock to demand node 2. Indeed, if we were to send the second vehicle directly from the depot to node 2 (i.e., $r_2 = \{0, 2, 0\}$), then our solution would be infeasible since $p_{02}Q + y_{21} < d_2$.

Because of this property, it is also possible to model (VRPMD–1) and (VRPMD–2), defining x_r as an integer variable instead of binary. We chose to define x_r as binary because repeated routes will be represented in Ω and these models will allow us to make better comparisons with the original LMDP proposed by [14].

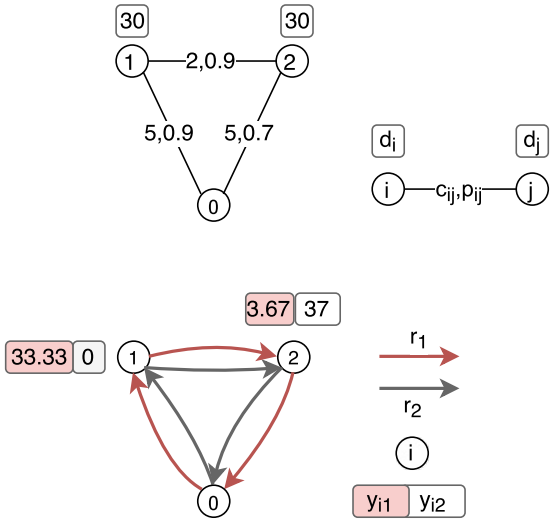


Figure 3. (VRPMD–1) with 2 demand nodes, $Q = 37$, $K = 2$.

Throughout this paper, we have assumed that the probability of disruption for each arc is independent of one another. Therefore, with respect to the probability of success, the triangle inequality does not hold. One consequence is that we could travel upon the same arc twice and revisit a node whose demand has already been satisfied (Properties 1–3), or the usage of multiple highly reliable routes (Property 4). Another interesting consequence is the existence of an optimal solution, where a vehicle uses the depot as a detour point. This scenario is shown in Figure 4.

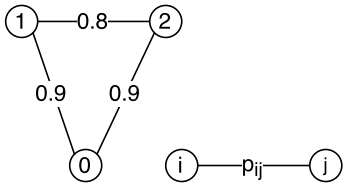


Figure 4. Depot as a detour point.

In Figure 4, we can see that a vehicle that just visits node 1 has a choice of going directly to node 2 (with $p_{12} = 0.8$), or making a detour via the depot to reach node 2 (with $p_{10}p_{02} = 0.81$). Obviously, making the direct trip would incur a higher disruption probability. So, in (VRPMD–2), we would choose the detour trip. The routing decision for (VRPMD–1) will only choose the direct trip from node 1 to node 2 if the supply left on the vehicle is enough to satisfy the demand at node 2 (i.e., $Q - y_{1r} \geq \frac{d_2}{\phi_{2r}}$). To account for this detour in our model, we could add an auxiliary node with demand 0 to act as the depot. In

practice, if such an instance occurs, another option could be to send the required supply to node 1, then replenish the vehicle at the depot before rerouting it. The option to replenish vehicles may come with an additional cost.

The same reasoning applies, in general, to any sequence with an arbitrary number of nodes. In (VRPMD–2), we consider sequence $i_1, i_2, \ldots, i_k$, and we use the path $\{i_1, i_2, \ldots, i_k\}$ over $\{i_1, i_k\}$ if $p_{i_1 i_2} \ldots p_{i_{k-1} i_k} > p_{i_1 i_k}$, although it would be more advantageous in terms of travel costs to make the direct trip from i_1 to i_k due to the triangle inequality on travel costs.

Finally, we have a property describing the number of vehicles in our optimal solution.

Property 5. *For (VRPMD–1), we may not minimize the number of vehicles used. However, for (VRPMD–2), we are always able to find an optimal solution, where we use all vehicles that are available to us.*

We observe that in some instances of (VRPMD–1), it is much more cost-effective to route a new vehicle from the depot than to use each vehicle to its maximum capacity. Consider, for example, a relaxed variation of Figure 1, where $K = 5$ (with all the other data unchanged). The current two-route solution yields an objective value of $z = c_{r_1} + c_{r_2} = 15 + 20 = 35$. Now, consider the following solution $r_1 = \{0, 1, 0\}$, $r_2 = \{0, 2, 0\}$, $r_3 = r_4 = \{0, 3, 0\}$. The objective value of this solution is $c_{r_1} + c_{r_2} + c_{r_3} + c_{r_4} = 8 + 10 + 8 + 8 = 34 < z$.

The second part of Property 5 is more obvious since the use of additional vehicles does not decrease our objective value in (VRPMD–2), so we are always able to find an optimal solution that uses the maximum allowable number of vehicles (this behavior changes if we introduce a cost constraint in the model).

In the VRPMD models, we optimize the long-term average performance of a single routing plan that is executed repeatedly. Thus, in a single delivery period, it is possible for us to oversupply nodes in a network. The amount of additional stock from the oversupply could be used as a safety stock to account for uncertainties in future deliveries due to disruption.

In the next section, we discuss the solution methodologies for VRPMD models.

5. Algorithm Design

To model the VRPMD problems, we use route formulations. Therefore, to solve a VRPMD problem to optimality, we have to enumerate all possible routes in Ω. However, it is time-consuming to do this. Moreover, according to Properties 3 and 4, there may exist multiple repeated routes or detour points in the optimal solution, which implies that a much larger set is required for Ω. On the other hand, although we must consider all possible routes, only a small subset of those routes will appear in the optimal solution, so an efficient procedure would only require a well-picked set of routes in Ω. To solve practical size problems, we must use heuristic algorithms to generate the set of routes that we will use, which are widely used for VRP problems (e.g., [22–24]).

For (VRPMD–1), we wish to minimize costs. Our algorithm starts with a generic solution to the VRP and we improve that solution based on how the disruptions may be located in the network. For (VRPMD–2), we attempt to maximize fulfillment; thus, we start our initial set of routes by using a maximum reliability framework and perform further improvements from there. In both algorithms, we consider repeated routes, revisited nodes, and detour points, as illustrated in Section 4. The algorithms are described in more detail below.

Cost minimization

Solving (VRPMD–1) by directly using a typical VRP algorithm will most likely produce an infeasible solution because, in many cases, a single route will be unable to sufficiently supply the later demand nodes due to the multiplicative disruption probabilities. As a result, we must perform some modifications to existing VRP algorithms so that a feasible solution to the VRPMD can be obtained.

In Algorithm 1, we begin with a feasible solution U' by solving the corresponding capacitated vehicle routing problem (CVRP), which is formulated by removing all disruption probabilities from the network. This CVRP model can be solved using any of the common solution methods, such as branch-and-bound, Tabu search, or genetic algorithms (in our experiments, we opted to use Tabu search because of its speed and multiple applications in VRP literature). This optimal set of routes U' is copied into the set of all unprocessed routes U.

Algorithm 1 Route generation for (VRPMD–1)

1: Initialization: Start with a feasible solution U' from solving the equivalent CVRP. Set $\Omega = \emptyset$; $U = U'$.
2: **while** $U \neq \emptyset$ **do**
3: Pick any $r_k = \{0, \ldots, 0\} \in U$
4: Set $Q' = Q$
5: **for** v in r_k **do**
6: **if** $Q' < 0$ **then**
7: Let path $p_u = \{v, \ldots, 0\} \subseteq r_k$
8: Find $p_d = \{0, \ldots, v\}$ with Min disruption. Add $\{p_d, p_u\}$ to U
9: Find $p_c = \{0, \ldots, v\}$ with Min cost. Add $\{p_c, p_u\}$ to U.
10: Find detour from [v-1] to [v]. Add to U
11: Break
12: **end if**
13: $Q' = Q' - \phi_{vr_k} d_v$
14: Add $r'_k = \{0, i, \ldots, v, 0\}$ to Ω
15: **end for**
16: Remove r_k from U
17: **end while**
18: Re-optimize Ω.

In the main loop of the algorithm, we process each route from U individually to locate the "stopping point" (v) at which the chance of disruption is too large for this route. That is, if we were to solve the (VRPMD–1) model, all demand nodes past this point would receive insufficient supply. Each node within the route is processed individually until this point is identified and the route will then be split into two parts. The piece pertaining to the first half of the route will be added directly to Ω. The piece pertaining to the second half of the route will be reprocessed in set U as a new route using the minimum disruption path and the minimum cost path. This main loop will be repeated until all the demands of all nodes have been fulfilled according to the disruption probabilities of the network.

The step to find detours consists of finding the most reliable path that links the stopping node to the node prior to that point. A more detailed explanation of the most reliable path is given below as part of Algorithm 2. A detour is found if the most reliable path differs from the current path. This step reliably improves our solution based on the properties that have been outlined in Section 4. If multiple routes are necessary, line 10 in our algorithm will create a branch out of the current path, which may select repeating routes.

The last step is to perform some local improvements and re-optimize Ω. This is line 18 of Algorithm 1, where we remove any routes that are dominated by another route. If any two routes contain the same demand nodes, we only keep the route that has a lower cost or higher probability of success. This step ensures a smoother runtime when we pass the set Ω into our mixed-integer programming (MIP) solver.

Fulfillment maximization

In (VRPMD–2), we notice that the solution, which maximizes our objective function, consists of the arcs yielding the highest value for $\phi_{ir} y_{ir}$. The choice of y_{ir} is greedy in the sense that a larger ϕ_{ir} leads to a larger y_{ir}. Thus, we turn to ϕ_{ir}, which is our probability of success. We use the idea of maximum reliability, introduced by [11], and solved using a

variation of the shortest path problem. In Algorithm 2, the initial loop begins by generating routes that maximize the probability of success (or minimize the chance of disruption). To find the path with a maximum probability of success, we compute $-log(p_{ij})$ for each arc (i, j) in our network and solve the shortest path problem (with Dijkstra's algorithm). These paths (computed in lines 2–5) serve as the initial routes from which we can make further improvements.

In the subsequent step, when additional capacity is available on a vehicle, we use a greedy route extension step: Picking a neighboring arc with unfulfilled demand that has the smallest chance of disruption to extend our current path. The route extension step continues until we systematically determine that the path can no longer contain any more demand nodes and/or when a certain threshold, representing the highest acceptable disruption chance, is reached. The set U is used to represent the set of unprocessed routes, which can be extended. Initial routes and routes that can no longer be extended are candidates to be placed in set Ω.

Algorithm 2 Route generation for (VRPMD–2)

1: Initialize $\Omega = \emptyset$, $U = \emptyset$, and set ϵ as the threshold value
2: **for** $i \in V$ **do**
3: Find the most reliable path $\{0, n_1, \ldots, n_k\}$ from 0 to n_k, where $n_k = i$
4: Add route $\{0, n_1, \ldots, n_k, 0\}$ to Ω and U
5: **end for**
6: **while** $U \neq \emptyset$ **do**
7: Select route $r = \{0, i, \ldots, m, 0\} \in U$
8: Find a neighboring node $u \notin r$ with $\text{Max}_u\{p_{mu}\}$
9: Let $r' = \{0, i, \ldots, m, u, 0\}$
10: **if** $\phi_{u,r'} > \epsilon$ and $\sum_{j \in r'} d_j < Q$ **then**
11: Add route r' to U
12: **end if**
13: Remove r from U and add to Ω
14: **end while**
15: Re-optimize Ω

This algorithm checks for the existence of repeated nodes or repeated routes in the first loop. By finding the most reliable paths for each of the demand nodes in the network, we may encounter multiple overlaps for nodes where there may be a low chance of disruption.

A threshold for stopping our route extension is added because routes need not be inclusive of all nodes. When looking at the general length of a route, optimally, we prefer a route that stops by the major nodes without being too constrained by the chances of disruption. We also observe that longer routes will have a quickly diminishing capacity unless the prior nodes in a route are detour points. Furthermore, the accumulation of disruption probabilities upon a longer route will quickly make such routes not optimal. Therefore, the capacity of a vehicle is diminished by approximately $\frac{d_i}{\phi_{ir}}$ for each subsequent service to demand node i in route r (except for routes with detour points). Another stopping condition that can be applied is a threshold based on a target service level that one must achieve. For example, in a network with a target service level of 90% and constant disruption probabilities of 0.01 across all arcs, we can expect the optimal routes to contain no more than 10 demand nodes (a route containing 11 nodes will have one customer with $\phi_{ir} = 0.99^{11} < 0.9$). The set of feasible routes is expected to be smaller for a higher service level.

6. Numerical Results

The numerical experiments serve two purposes: (1) we evaluate the effectiveness of the proposed models; (2) we test the efficiency of the proposed algorithms.

In addition to having the additional constraint of disruptions, Section 4 shows that the optimal solution to the VRPMD is very different from typical SDVRP problems. To better measure the effectiveness of the heuristic algorithms, we must construct networks that better suit the scenarios in which disruption can occur. In total, we designed four different types of networks:

- Small network with evenly distributed disruption rates;
- Large network with evenly distributed disruption rates;
- Small network with localized disruption rates;
- Large network with localized disruption rates.

Evenly distributed disruption rates are generated among a uniform distribution $U[0.01, 0.10]$. Localized disruption rates refer to a small subset of edges that incur higher than usual disruption $U[0.10, 0.25]$. All other edges will be set using a uniform disruption rate of $U[0.01, 0.10]$. We generate these localized disruption networks because in these networks there is a higher chance for detour nodes and multiple shared routes as described in Section 4.

In our small experiments, we tested networks of sizes 7, 20, and 50 demand nodes. Each node in the selected network was scattered randomly along a 200×200 square grid. The depot was located at the center of the grid and travel cost was determined by the Euclidean distance of the two nodes rounded to the nearest integer.

We set the parameters for each of the three network sizes as follows: We set the demands to be random values between 5 and 30. We used a homogeneous fleet consisting of $Q = 50$, $Q = 70$, or $Q = 100$ for small networks and $Q = 100$, $Q = 150$, or $Q = 250$ for larger networks. In each instance, we allotted the necessary amount of vehicles to the fleet size K, such that a feasible solution would always be possible. For each network type, we applied one of two types of disruption, as described above.

All small network instances are summarized in Table 2, where column "Ins. Num." presents the instance number. $U[0.01, 0.10]$ indicates that disruption probabilities are evenly distributed across all arcs in the network; otherwise, there is localized disruption; "2 Arcs" represents localized disruption on 2 arcs and "4 Arcs" represent localized disruption on 4 arcs. Larger instances are summarized in Table 3.

All of the models and algorithms were coded in C++ using the IBM ILOG CPLEX Concert Technology (CPLEX 12.2.0). The computations were performed on a 1.60 GHz 64-bit Intel Core i5 CPU with 8GB RAM.

Table 2. Selected small network instances for VRPMD models.

Ins. Num.	n	Q	K	Disruption
1	7	50	5	$U[0.01, 0.10]$
2	7	50	5	2 Arcs
3	7	50	5	4 Arcs
4	7	100	3	$U[0.01, 0.10]$
5	7	100	3	2 Arcs
6	7	100	3	4 Arcs

Table 3. Selected large network instances for VRPMD models.

Ins. Num.	n	Q	K	Disruption
7	20	70	10	$U[0.01, 0.10]$
8	20	70	10	2 Arcs
9	20	70	10	4 Arcs
10	20	150	5	$U[0.01, 0.10]$
11	20	150	5	2 Arcs
12	20	150	5	4 Arcs
13	50	100	17	$U[0.01, 0.10]$
14	50	100	17	2 Arcs
15	50	100	17	4 Arcs
16	50	250	7	$U[0.01, 0.10]$
17	50	250	7	2 Arcs
18	50	250	7	4 Arcs

6.1. VRPMD Route Structure

For (VRPMD–1), we created instances where the total supply was greater than the total demand, and for (VRPMD–2), we created instances where the total supply was less than the total demand. Small networks (7 nodes) were solved to optimality within a reasonable computational time; however, for larger networks (20, 50 nodes), the list of all possible permutations becomes too large. Computing all feasible routes in these large networks would take hours of computational time and massive amounts of computer memory. Thus, we approximated the lower bound of the solution using a linear relaxation.

Observation 1. *High disruption, low-cost arcs.*

Arcs with a relatively high disruption probability but low cost will appear near the rear of a route. The reasoning behind this is that, as a route traverses risky arcs, the accumulation of disruption probabilities makes the overall probability of success for reaching all subsequent nodes lower. If the risky arc in our network is selected much later in our route, then only nodes that are visited after the risky arc will be affected. Similarly, if we want to maximize the fulfillment in (VRPMD–2), then we will surely want to minimize the number of nodes that are affected by a single risky arc. Thus, if the selection of a risky arc is inevitable, it will be preferable to locate this arc near the rear of a route.

Observation 2. *Low-demand nodes.*

Nodes with relatively low demands are visited last in an optimal route. As our route lengthens, the cumulative disruption rate grows; therefore, the expected supply for the nodes served at the rear of a route will be less than if the node is one of the first ones served. Another way to look at it is to take into account the expected loss. When delivering a larger amount of supply through a risky path, the expected loss will be higher compared to delivering a smaller quantity through the same path. Correspondingly, nodes with higher demand will typically appear near the front of a route. The expected probability of success is much more likely to be higher at the front of a route. This observation holds for both (VRPMD–1) and (VRPMD–2).

Observation 3. *Split deliveries.*

A route experiences split delivery if the supply of a vehicle along that route cannot completely satisfy the demand of a node along the route. We notice that in all our experiments, the number of split deliveries is, at most, 2. This observation is recorded in both experiments with (VRPMD–1) and (VRPMD–2). It is conjectured that this observation

holds for all instances in general because the loss incurred by interdiction will deter the use of some routes, particularly routes where split deliveries occur.

Observation 4. *Repeated routes and detour points.*

Although we show with Property 4 that the presence of multiple routes is possible, in our generated instances of select networks, the optimal solution will rarely contain repeated routes in networks where there is evenly distributed disruption. Repeated routes become more apparent in networks with localized disruption. This is likely explained by the fact that it is more likely for networks to have detour points if the disruption is only focused on a small area.

Observation 5. *Size of routes.*

We observe that in the instances that we generated, a single route will often have a length of at most $\frac{Q}{\text{Min}_{i \in V}\{d_i : d_i > 0\}}$. In general, model (VRPMD–1) has a preference for shorter routes because the cumulative disruption probabilities along with the traveling costs make it less likely for demands to be completely satisfied on a long route. In (VRPMD–2), the route sizes tend to reach the maximum length. A reason for this is likely because of the absence of route costs in this model and having a longer route will allow a larger pool of candidate solutions. In these longer routes, there will often be many nodes that are visited but not serviced, and we can interpret solutions in a way such that these nodes are removed from the route. Applying this soft upper bound on the route sizes will allow us to find a good estimation of the optimal solution within a shorter time.

All of these observations allow us to have a better understanding of the optimal route structure of the VRPMD models as they are valid for most of the instances.

6.2. Solution of (VRPMD–1)

In (VRPMD–1), we generate the demands using a random uniform distribution for values between 5 and 30.

Three unique networks are created within the specifications for each instance and we record the averages of these three networks. We compare the computational time and objective value by our heuristic algorithm with those of the CPLEX solver. To compute the optimal solution in the small network instances, we generate all possible permutations of routes and run them through the CPLEX solver.

As mentioned above, for larger network instances, we must approximate the lower bound of the solution using a linear relaxation because of its high time requirements. An approximation to the CVRP problem was obtained using Tabu search. Below is a summary of the experiments.

In Table 4, column "Opt. Time" represents the solution time of the CPLEX solver. Column "Heu. Time" represents the solution time of the heuristic algorithm. All the times are counted in seconds. Column "Opt. Gap" represents the relative percentage gap of the solution found by the heuristic compared with the optimal objective value. Larger instances that use a linear approximation are summarized in Table 5.

From the numerical results for (VRPMD–1), we notice that there is generally a 1–6% gap between our heuristic solution and the optimal solution (or the lower bound for larger networks). It is possible that the quality of the heuristic solution may be affected by the type of CVRP solver that we use to initialize our algorithm. In our case, the Tabu search algorithm produced outputs in a way that was consistent with the path extension step in our heuristic Algorithm 1. Experiments with different CVRP solvers will be left for future work.

We observe that in our experiments, the performance of our algorithms was not significantly impacted by the type of networks; whether they have evenly distributed distribution or localized distribution. Although in some extreme cases, there may be

significant improvements from localized cases, on average, we see that the optimality gaps for both types of networks are very consistent.

Table 4. Computational results for small instances of (VRPMD–1).

Ins. Num.	Opt. Time	Heu. Time	Opt. Gap
1	636.10	3.29	2.61%
2	582.19	2.33	2.15%
3	613.23	3.63	1.60%
4	810.77	1.83	2.33%
5	599.18	1.50	3.12%
6	638.39	3.71	2.15%

Table 5. Computational results for large instances of (VRPMD–1).

Ins. Num.	Heu. Time	Opt. Gap
7	15.01	5.23%
8	15.03	5.68%
9	14.98	5.19%
10	16.53	5.73%
11	16.19	5.82%
12	15.63	5.71%
13	18.82	5.13%
14	17.61	4.28%
15	18.27	5.15%
16	18.15	4.33%
17	18.64	4.61%
18	18.73	4.74%

6.3. Solution of (VRPMD–2)

Experiments conducted for (VRPMD–2) are similar to those of (VRPMD–1). Note that we adjusted the demands slightly so that the total supply could be less than the total demand. We set $d_i = U[50, 70]$ for instances with $Q = 50$ (Instances 1–6) and $d_i = U[90, 110]$ when $Q = 100$ (Instances 7–18).

In some cases of the (VRPMD–2), we saw multiple optimal solutions, so we added $-\epsilon \sum_{r\in\Omega} c_r x_r$ in the objective function, as described in Section 3, and ϵ is set as 0.001.

Table 6 presents the computational results for (VRPMD–2) in small instances. In the first three instances (when the supply is much less than the demand), our heuristic algorithm is able to find the optimal solution in all generated configurations of the network. Table 7 presents the results for larger networks. Among all the instances, the optimality gap between the solution found by Algorithm 2 and the optimal solution never exceeds 4%, which demonstrates that the truncation of routes in Algorithm 2 tends not to dispose the optimal solution.

Similar to (VRPMD–1), the experimental results for (VRPMD–2) show that the performance of Algorithm 2 is consistent for both localized and evenly distributed networks. On average, the optimality gap is not affected by the network configuration.

Table 6. Computational results for small instances of (VRPMD–2).

Ins. Num.	Opt. Time	Heu. Time	Opt. Gap
1	591.83	3.70	0.00%
2	633.33	4.10	0.02%
3	612.31	3.92	0.00%
4	739.67	4.71	0.03%
5	731.52	4.49	0.01%
6	695.53	3.18	0.00%

Table 7. Computational results for large instances of (VRPMD–2).

Ins. Num.	Heu. Time	Opt. Gap
7	5.91	0.13%
8	5.68	0.17%
9	5.33	0.04%
10	6.03	0.93%
11	6.18	0.51%
12	6.63	1.01%
13	8.18	0.15%
14	7.49	0.23%
15	7.73	0.51%
16	8.33	1.23%
17	8.82	0.91%
18	7.82	0.37%

7. Conclusions and Future Research Directions

This paper establishes the general framework for the VRPMD. The proposed models are an extension of SDVRP with applications in humanitarian relief or military operations. VRPMD does not deal with real-time disruptions. Instead, we are more concerned about the long-term average performance of a single routing plan, which is important when the delivery volume is high.

Although the VRPMD is built upon the SDVRP, they do not share the same properties. For example, in the SDVRP, there is always an optimal solution where routes can have, at most, one shared demand node, whereas in the VRPMD, this property is relaxed, so routes can have multiple shared demand nodes. However, it is still true in the VRPMD that there is an optimal solution where only one of these routes will deliver to multiple nodes. The properties exhibited by the VRPMD provide us with key insights into the structures of optimal routes in these problems. In the VRPMD, it is necessary to introduce the concept of detour node and this is observed in the optimal solutions of many instances. We further introduced heuristic algorithms, which build upon the corresponding solution of the CVRP to solve small instances of the VRPMD. Compared with the commercial solver CPLEX, we found that our algorithm performs much more efficiently within the 6% optimality gap in (VRPMD–1) and within the 2% optimality gap in (VRPMD–2).

One direction for future research includes the linear combination of the two criteria, as well as an analysis of the optimal trade-off between cost and disruption probabilities. This will provide us with more insight on how to make an optimal problem reduction in larger cases. Additionally, exact algorithms based on the cutting plane and branch-and-price could be considered and different CVRP solvers can be analyzed for our models. Finally, in this paper, we only investigated the static models of the VRPMD. A dynamic version of the VRPMD is a good direction for future work to deal with disruptions in real time. Such a version would involve transmitting routing information in an ongoing manner and rerouting vehicles as disruption occurs.

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