

```
#####
## PRELIMINARIES ##
#####

## packages
library("datasets")
library("graphics")
library("grDevices")

#####
## PROCEDURES ##
#####

## General Procedure for calculating the pdf of the product of two normal variables with limits
(a,b) and (c,d) with n points, a<b and c<d:

product.Normal <- function(n,mx,sx,my,sy){
  a<- mx-4*sx; b<-mx+4*sx; c<-my-4*sy; d<-my+4*sy;
  if (a>0 && b>0 && c>0 && d>0){
    product.1normal(n,a,b,c,d,mx,sx,my,sy)}
  else if (a<0 && b<0 && c<0 && d<0) {
    product.2normal(n,a,b,c,d,mx,sx,my,sy)}
  else if (a>0 && b>0 && c<0 && d<0){
    product.3normal(n,a,b,c,d,mx,sx,my,sy)}
  else if (a>0 && b>0 && c<0 && d>0){
    product.4normal(n,a,b,c,d,mx,sx,my,sy)}
  else if (a<0 && b>0 && c<0 && d<0){
    product.5normal(n,a,b,c,d,mx,sx,my,sy)}
  else {product.6normal(n,a,b,c,d,mx,sx,my,sy)}
}

## Different Scenarios considering values of a,b,c and d
# Procedure Scenario 1: $a,b,c,d >0$
product.1normal <- function(n,a,b,c,d,mx,sx,my,sy){
  if(a*d < b*c) {r.normal.p1(n,a,b,c,d,mx,sx,my,sy)}
  else if (a*d==b*c){r.normal.p2(n,a,b,c,d,mx,sx,my,sy)}
  else {r.normal.p3(n,a,b,c,d,mx,sx,my,sy)}
}

#Procedure Scenario 2: $a,b,c,d <0$
product.2normal <- function(n,a,b,c,d,mx,sx,my,sy){
  if(a*d < b*c) {r.normal.n1(n,a,b,c,d,mx,sx,my,sy)}
  else if (a*d==b*c){r.normal.n2(n,a,b,c,d,mx,sx,my,sy)}
  else {r.normal.n3(n,a,b,c,d,mx,sx,my,sy)}
}

#Procedure Scenario 3: $a,b >0; c,d <0$
product.3normal <- function(n,a,b,c,d,mx,sx,my,sy){
  if(a*c < b*d) {r.normal.np3a(n,a,b,c,d,mx,sx,my,sy)}
  else if (a*c==b*d){r.normal.np3c(n,a,b,c,d,mx,sx,my,sy)}
  else {r.normal.np3b(n,a,b,c,d,mx,sx,my,sy)}
}

#Procedure Scenario 4: $a < 0 ,b >0; c,d >0$
product.4normal <- function(n,a,b,c,d,mx,sx,my,sy){
  cs1<-product.3normal(n,a,b,c,-0.01,mx,sx,my,sy)
  cs2<-product.1normal(n,a,b,+0.01,d,mx,sx,my,sy)
  cst<-c(cs1,cs2)
}

#Procedure Scenario 5: $a < 0 ,b >0; c,d <0$
product.5normal <- function(n,a,b,c,d,mx,sx,my,sy){
  cs1<-r.normal.nn3(n,0.01,b,c,d,mx,sx,my,sy)
  cs2<-product.2normal(n,a,-0.01,c,d,mx,sx,my,sy)
  cst<-c(cs1,cs2)
}

#Procedure Scenario 6: $a < 0 ,b >0; c<0 ,d >0$
product.6normal <- function(n,a,b,c,d,mx,sx,my,sy){
  ssss<-sort(c(abs(a),abs(b),abs(c),abs(d)))
  if (a*c > b*d && a*d > b*c) {cs<-
c(r.normal.n61(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p61(n,a,b,c,d,mx,sx,my,sy,ssss))}
  else if (a*c < b*d && a*d > b*c) {cs<-
c(r.normal.n62(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p62(n,a,b,c,d,mx,sx,my,sy,ssss))}
  else if (a*c < b*d && a*d < b*c) {cs<-
c(r.normal.n63(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p63(n,a,b,c,d,mx,sx,my,sy,ssss))}
}
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else if (a*c < b*d && a*d == b*c) {cs<-
c(r.normal.n62(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p62(n,a,b,c,d,mx,sx,my,sy,ssss))}
else if (a*c > b*d && a*d < b*c) {cs<-
c(r.normal.n64(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p64(n,a,b,c,d,mx,sx,my,sy,ssss))}
else if (a*c > b*d && a*d == b*c) {cs<-
c(r.normal.n65(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p64(n,a,b,c,d,mx,sx,my,sy,ssss))}
else if (a*c == b*d && a*d > b*c) {cs<-
c(r.normal.n66(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p61(n,a,b,c,d,mx,sx,my,sy,ssss))}
else if (a*c == b*d && a*d < b*c) {cs<-
c(r.normal.n67(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p67(n,a,b,c,d,mx,sx,my,sy,ssss))}
else {cs<-
c(r.normal.n68(n,a,b,c,d,mx,sx,my,sy,ssss),r.normal.p68(n,a,b,c,d,mx,sx,my,sy,ssss))}
return(cs)
}

```

#Procedure for calculating numerical integrate between a and v/c:

```

l1 <-function(a,b,c,d,v,mx,sx,my,sy){
  integrandol <- function(x){dnorm(v/x,my,sy)*dnorm(x,mx,sx)*1/abs(x)}
  r1 <- integrate(integrandol,a,v/c)
  return(r1[[1]])
}

```

Procedure for calculating numerical integrate between v/d and v/c

```

l2 <- function(a,b,c,d,v,mx,sx,my,sy){
  integrandol <- function(x){dnorm(v/x,my,sy)*dnorm(x,mx,sx)*1/abs(x)}
  r2 <- integrate(integrandol,v/d,v/c)
  return(r2[[1]])
}

```

Procedure for calculating numerical integrate between v/d and b:

```

l3 <- function(a,b,c,d,v,mx,sx,my,sy){
  integrandol <- function(x){dnorm(v/x,my,sy)*dnorm(x,mx,sx)*1/abs(x)}
  r3 <- integrate(integrandol,v/d,b)
  return(r3[[1]])
}

```

Procedure for calculating numerical integrate between a and b:

```

l4 <-function(a,b,c,d,v,mx,sx,my,sy){
  integrandol <- function(x){dnorm(v/x,my,sy)*dnorm(x,mx,sx)*1/abs(x)}
  r4 <- integrate(integrandol,a,b)
  return(r4[[1]])
}

```

Procedures for calculating the product as a function of values a,b,c,d

```

r.normal.n1 <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(b*d,a*c,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]<b*c)
{x[i]<-l1(a,b,c,d,v=s[i],mx,sx,my,sy)}
else if (s[i]<a*d) {x[i]<-l2(a,b,c,d,v=s[i],mx,sx,my,sy)}
else {x[i]<-l3(a,b,c,d,v=s[i],mx,sx,my,sy)}
}
return(x)
}

```

```

r.normal.p1 <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(a*c,b*d,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]<a*d)
{x[i]<-l1(a,b,c,d,v=s[i],mx,sx,my,sy)}
else if (s[i]<b*c) {x[i]<-l2(a,b,c,d,v=s[i],mx,sx,my,sy)}
else {x[i]<-l3(a,b,c,d,v=s[i],mx,sx,my,sy)}
}
return(x)
}

```

```

r.normal.n2 <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(b*d,a*c,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]<b*c)
{x[i]<-l3(a,b,c,d,v=s[i],mx,sx,my,sy)}
else {x[i]<-l1(a,b,c,d,v=s[i],mx,sx,my,sy)}
}
return(x)
}

```

```

r.normal.p2 <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(a*c,b*d,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]<a*d)

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    {x[i]<-l1(a,b,c,d,v=s[i],mx,sx,my,sy)}
    else {x[i]<-l3(a,b,c,d,v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.n3 <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(b*d,a*c,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]<b*c)
{x[i]<-l1(a,b,c,d,v=s[i],mx,sx,my,sy)}
  else if (s[i]<a*d) {x[i]<-l4(a,b,c,d,v=s[i],mx,sx,my,sy)}
  else {x[i]<-l3(a,b,c,d,v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.p3 <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(a*c,b*d,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]<b*c)
{x[i]<-l1(a,b,c,d,v=s[i],mx,sx,my,sy)}
  else if (s[i]<a*d) {x[i]<-l4(a,b,c,d,v=s[i],mx,sx,my,sy)}
  else {x[i]<-l3(a,b,c,d,v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.np3a <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(b*c,a*d,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]>b*d)
{x[i]<-l1(a,b,d,c,v=s[i],mx,sx,my,sy)}
  else if (s[i]>a*c) {x[i]<-l4(a,b,c,d,v=s[i],mx,sx,my,sy)}
  else {x[i]<-l3(a,b,d,c,v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.np3b <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(b*c,a*d,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]>a*c)
{x[i]<-l1(a,b,d,c,v=s[i],mx,sx,my,sy)}
  else if (s[i]>b*d) {x[i]<-l2(a,b,d,c,v=s[i],mx,sx,my,sy)}
  else {x[i]<-l3(a,b,d,c,v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.np3c <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(b*c,a*d,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]>a*c)
{x[i]<-l1(a,b,d,c,v=s[i],mx,sx,my,sy)}
  else {x[i]<-l3(a,b,d,c,v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.nn3 <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(b*c,a*d,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){if (s[i]<a*c)
{x[i]<-l1(a,b,d,c,v=s[i],mx,sx,my,sy)}
  else if (s[i]<d*b) {x[i]<-l4(a,b,c,d,v=s[i],mx,sx,my,sy)}
  else {x[i]<-l3(a,b,d,c,v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.p41 <- function(n,a,b,c,d,mx,sx,my,sy){
  s <-seq(0.01*0.01,b*d,1/n)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]< -0.01*a)
      {x[i]<- (l1(0.01,b,0.01,d,v=s[i],mx,sx,my,sy)+l3(a,-0.01,c,-0.01,v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*b) {x[i]<-
(l1(0.01,b,0.01,d,v=s[i],mx,sx,my,sy)+l1(a,b,a,d,v=s[i],mx,sx,my,sy))}
    else if (s[i] < a*a) {x[i]<-
(l3(a,b,c,b,v=s[i],mx,sx,my,sy)+l1(a,b,a,d,v=s[i],mx,sx,my,sy))}
  }
}

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    else {x[i]<-l3(a,b,c,b,v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.n41 <- function(n,a,b,c,d,mx,sx,my,sy){s <-seq(a*b,0.01*-0.01,1/n)
x <-vector(length=length(s))
for (i in 1:length(x)){
  if (s[i] > 0.01*a) {x[i]<- -l1(-0.01,b,-0.01,d,v=s[i],mx,sx,my,sy)}
  else if (s[i]>-0.01*b) {x[i]<- -l2(a,b,b,0.01,v=s[i],mx,sx,my,sy)}
  else {x[i]<- -l3(a,a,c,b,v=s[i],mx,sx,my,sy)}
}
return(x)
}

r.normal.p42 <- function(n,a,b,c,d,mx,sx,my,sy){
s <-seq(0.01*0.01,a*c,1/n)
x <-vector(length=length(s))
for (i in 1:length(x)){
  if (s[i]< 0.01*b) {x[i]<- (-l1(0.01,b,0.01,d,v=s[i],mx,sx,my,sy)-
l3(a,-0.01,c,-0.01,v=s[i],mx,sx,my,sy))}
  else if (s[i] < -0.01*a) {x[i]<- (-l1(0.01,b,0.01,d,v=s[i],mx,sx,my,sy)-
l1(b,b,b,d,v=s[i],mx,sx,my,sy))}
  else if (s[i] < b*b) {x[i]<- (-l3(a,a,c,a,v=s[i],mx,sx,my,sy)-
l1(b,b,b,d,v=s[i],mx,sx,my,sy))}
  else {x[i]<-l3(a,a,c,a,v=s[i],mx,sx,my,sy)}
}
return(x)
}

r.normal.n42 <- function(n,a,b,c,d,mx,sx,my,sy){
s <-seq(a*b,0.01*-0.01,1/n)
x <-vector(length=length(s))
for (i in 1:length(x)){
  if (s[i] > 0.01*b) {x[i]<- l1(-0.01,b,-0.01,d,v=s[i],mx,sx,my,sy)}
  else if (s[i]>-0.01*a) {x[i]<- l2(a,b,a,-0.01,v=s[i],mx,sx,my,sy)}
  else {x[i]<- l3(a,b,c,a,v=s[i],mx,sx,my,sy)}
}
return(x)
}

r.normal.p61 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
s <-seq(0.01*0.01,max(a*c,b*d),1/n)
ss <-c(0.01,b,0.01,d,a,-0.01,c,-0.01)
x <-vector(length=length(s))
for (i in 1:length(x)){
  if (s[i]< 0.01*ssss[1]){x[i]<-
(l1(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+l1(ss[6],ss[7],ss[5],ss[8],v=s[i],mx,sx,my,sy))}
  else if (s[i] < 0.01*ssss[2]) {x[i]<-
(l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l1(ss[6],ss[7],ss[5],ss[8],v=s[i],mx,sx,my,sy))}
  else if (s[i] < 0.01*ssss[3]) {x[i]<-
(l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l4(ss[7],ss[8],ss[5],ss[6],v=s[i],mx,sx,my,sy))}
  else if (s[i] < 0.01*ssss[4]) {x[i]<- (l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
l3(ss[6],ss[5],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
  else if (s[i] < min(b*d,a*c)) {x[i]<- (l3(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)-
l3(ss[6],ss[5],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
  else {x[i]<-l3(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)}
}
return(x)
}

r.normal.n61 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
s <-seq(min(a*d,b*c),-0.01*0.01,1/n)
ss <-c(0.01,b,c,-0.01,a,-0.01,0.01,d)
x <-vector(length=length(s))
for (i in 1:length(x)){
  if (s[i]> -0.01*ssss[1]){x[i]<- (l1(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+
l1(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
  else if (s[i] > -0.01*ssss[2]){x[i]<-
(l1(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l2(ss[6],ss[8],ss[5],ss[7],v=s[i],mx,sx,my,sy))}
  else if (s[i] > -0.01*ssss[3]) {x[i]<-
(l2(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)+l2(ss[6],ss[8],ss[5],ss[7],v=s[i],mx,sx,my,sy))}
  else if (s[i] > -0.01*ssss[4]){x[i]<- (l2(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)+
l3(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
  else if (s[i] > max(a*d,b*c)){x[i]<- (l3(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+
l3(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
}

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    else {x[i]<-l3(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.p62 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(0.01*0.01,max(a*c,b*d),1/n)
  ss <-c(0.01,b,0.01,d,a,-0.01,c,-0.01)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]< 0.01*ssss[1]){x[i]<- (l1(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+
l1(ss[6],ss[7],ss[5],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[2]) {x[i]<-
(l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l1(ss[6],ss[7],ss[5],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[3]) {x[i]<-
(l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l4(ss[7],ss[8],ss[5],ss[6],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[4]) {x[i]<- (l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
l3(ss[6],ss[5],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] < min(a*c,b*d)) {x[i]<- (l3(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)-
l3(ss[6],ss[5],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else {x[i]<-l3(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.n62 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(min(a*d,b*c),-0.01*0.01,1/n)
  ss <-c(0.01,b,c,-0.01,a,-0.01,0.01,d)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]> -0.01*ssss[1]){x[i]<-
(l1(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l1(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[2]){x[i]<-
(l1(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l2(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[3]) {x[i]<-
(l2(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)+l2(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[4]) {x[i]<-
(l2(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)+l3(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > max(a*d,b*c)) {x[i]<-
(l3(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l3(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else {x[i]<-l3(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.p63 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(0.01*0.01,max(a*c,b*d),1/n)
  ss <-c(0.01,b,0.01,d,a,-0.01,c,-0.01)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]< 0.01*ssss[1]){x[i]<-
(l1(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+l1(ss[6],ss[7],ss[5],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[2]) {x[i]<-
(l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l1(ss[6],ss[7],ss[5],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[3]) {x[i]<-
(l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l4(ss[7],ss[8],ss[5],ss[6],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[4]) {x[i]<- (l2(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
l3(ss[6],ss[7],ss[8],ss[5],v=s[i],mx,sx,my,sy))}
    else if (s[i] < min(a*c,b*d)){x[i]<- (l3(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)-
l3(ss[6],ss[7],ss[8],ss[5],v=s[i],mx,sx,my,sy))}
    else {x[i]<-l3(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.n63 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(min(a*d,b*c),-0.01*0.01,1/n)
  ss <-c(0.01,b,c,-0.01,a,-0.01,0.01,d)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]> -0.01*ssss[1]) {x[i]<-
(l1(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l1(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[2]) {x[i]<-
(l1(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l2(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[3]) {x[i]<-
(l2(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)+l2(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}

```

```

    else if (s[i] > -0.01*ssss[4]) {x[i]<-
(12(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy))+13(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > max(a*d,b*c)) {x[i]<-
(13(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy))+13(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else {x[i]<-13(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.p64 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(0.01*0.01,max(a*c,b*d),1/n)
  ss <-c(0.01,b,0.01,d,a,-0.01,c,-0.01)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]< 0.01*ssss[1]){x[i]<- (11(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+
11(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[2]) {x[i]<-
(14(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy))+11(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[3]) {x[i]<-
(13(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy))+11(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[4]) {x[i]<-
(13(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy))+14(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < min(a*c,b*d)) {x[i]<-
(13(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy))+13(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else {x[i]<-13(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.n64 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(min(a*d,b*c),-0.01*0.01,1/n)
  ss <-c(0.01,b,c,-0.01,a,-0.01,0.01,d)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]> -0.01*ssss[1]){x[i]<- (11(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
11(ss[6],ss[5],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[2]) {x[i]<- (14(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)-
11(ss[6],ss[5],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[3]) {x[i]<- (14(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)-
12(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[4]) {x[i]<- (13(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
12(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > max(a*d,b*c)) {x[i]<- (13(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
13(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else {x[i]<--13(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.n65 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(min(a*d,b*c),-0.01*0.01,1/n)
  ss <-c(0.01,b,c,-0.01,a,-0.01,0.01,d)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]> -0.01*ssss[1]) {x[i]<- (-11(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)-
11(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[2]) {x[i]<- (-14(ss[3],ss[4],ss[1],ss[2],v=s[i],mx,sx,my,sy)-
11(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[3]) {x[i]<- (-14(ss[3],ss[4],ss[1],ss[4],v=s[i],mx,sx,my,sy)-
12(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[4]) {x[i]<- (-13(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
12(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > max(a*d,b*c)) {x[i]<- (-13(ss[1],ss[3],ss[4],ss[2],v=s[i],mx,sx,my,sy)-
13(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else {x[i]<--13(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.n66 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(min(a*d,b*c),-0.01*0.01,1/n)
  ss <-c(0.01,b,c,-0.01,a,-0.01,0.01,d)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]> -0.01*ssss[1]){x[i]<- (-11(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
11(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}

```

```

    else if (s[i] > -0.01*ssss[2]) {x[i]<- (-11(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)-
12(ss[5],ss[7],ss[6],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[3]) {x[i]<- (-12(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)-
12(ss[6],ss[8],ss[5],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[4]) {x[i]<- (-12(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)-
13(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > max(a*d,b*c)) {x[i]<-
(13(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+13(ss[6],ss[5],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else {x[i]<-13(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.p67 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(0.01*0.01,max(a*c,b*d),1/n)
  ss <-c(0.01,b,0.01,d,a,-0.01,c,-0.01)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]< 0.01*ssss[1]){x[i]<-
(11(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+11(ss[7],ss[6],ss[5],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[2]) {x[i]<- (-
12(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+11(ss[7],ss[6],ss[5],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[3]) {x[i]<- (-
12(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+14(ss[7],ss[8],ss[5],ss[6],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[4]) {x[i]<- (-12(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
13(ss[6],ss[7],ss[8],ss[5],v=s[i],mx,sx,my,sy))}
    else if (s[i] < min(a*c,b*d)) {x[i]<- (13(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)-
13(ss[6],ss[7],ss[8],ss[5],v=s[i],mx,sx,my,sy))}
    else {x[i]<-13(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.n67 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(min(a*d,b*c),-0.01*0.01,1/n)
  ss <-c(0.01,b,c,-0.01,a,-0.01,0.01,d)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]> -0.01*ssss[1]) {x[i]<-
(11(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+11(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[2]) {x[i]<-
(11(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+14(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[3]) {x[i]<- (-
12(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)+14(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[4]) {x[i]<- (-
12(ss[1],ss[3],ss[2],ss[4],v=s[i],mx,sx,my,sy)+13(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else if (s[i] > max(a*d,b*c)) {x[i]<-
(13(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+13(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    else {x[i]<-13(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy)}
  }
  return(x)
}

r.normal.p68 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(0.01*0.01,max(a*c,b*d),1/n)
  ss <-c(0.01,b,0.01,d,a,-0.01,c,-0.01)
  x <-vector(length=length(s))
  for (i in 1:length(x)){
    if (s[i]< 0.01*ssss[1]) {x[i]<-
(11(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+11(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[2]) {x[i]<-
(11(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+12(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[3]) {x[i]<- (14(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+
12(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] < 0.01*ssss[4]) {x[i]<-
(14(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+13(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
    else {x[i]<-
(13(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)+13(ss[5],ss[6],ss[7],ss[8],v=s[i],mx,sx,my,sy))}
  }
  return(x)
}

r.normal.n68 <- function(n,a,b,c,d,mx,sx,my,sy,ssss){
  s <-seq(min(a*d,b*c),-0.01*0.01,1/n)
  ss <-c(0.01,b,c,-0.01,a,-0.01,0.01,d)
  x <-vector(length=length(s))
  for (i in 1:length(x)){

```

```

    if (s[i]> -0.01*ssss[1]){x[i]<-
(l1(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l1(ss[7],ss[5],ss[6],ss[8],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[2]) {x[i]<- (l1(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)-
l2(ss[7],ss[8],ss[5],ss[6],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[3]) {x[i]<- (l4(ss[1],ss[2],ss[3],ss[4],v=s[i],mx,sx,my,sy)-
l2(ss[7],ss[8],ss[5],ss[6],v=s[i],mx,sx,my,sy))}
    else if (s[i] > -0.01*ssss[4]) {x[i]<-
(l3(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l2(ss[7],ss[8],ss[5],ss[6],v=s[i],mx,sx,my,sy))}
    else {x[i]<-
(l3(ss[1],ss[2],ss[4],ss[3],v=s[i],mx,sx,my,sy)+l3(ss[5],ss[6],ss[8],ss[7],v=s[i],mx,sx,my,sy))}
    }
    return(x)
  }
}

```

```

#####
##      GRAPHICAL EXAMPLES      ##
#####

```

Examples of products in the manuscript

Figure 1

```

xlm <- rnorm(10^6,5,0.5)
ylm <- rnorm(10^6,5,0.5)
xyl <- xlm*ylm
xyl.density<-density(xyl)
li <- (5-4*0.5)*(5-4*0.5)
ls <- (5+4*0.5)*(5+4*0.5)
xl <- seq(li,ls,1/100)
yl<-product.Normal(100,5,0.5,5,0.5)
xy2 <- data.frame(xl,yl)
plot(unlist(xy2$xl),unlist(xy2$yl),type="l",axes=TRUE,
xlab="Z=XY",ylab="Density",col="red",lwd=3)
par(new=TRUE)
plot(xyl.density,main="",axes=FALSE, xlab="",ylab="",xlim=c(li,ls),col="green",lwd=4)

```

Figure 2

```

xlm <- rnorm(10^6,0.5,2)
ylm <- rnorm(10^6,1,0.1)
xyl <- xlm*ylm
xyl.density<-density(xyl)
li <- (0.5-4*2)*(1+4*0.1)
ls <- (0.5+4*2)*(1+4*0.1)
xla <- seq(li,-0.01,1/100)
xlb <- seq(0.01,ls,1/100)
xl =c(xla,xlb)
yl<-product.Normal(100,1,0.1,0.5,2)
xy2 <- data.frame(xl,yl)
plot(unlist(xy2$xl),unlist(xy2$yl),type="l",axes=TRUE,
xlab="Z=XY",ylab="Density",col="red",lwd=3)
par(new=TRUE)
plot(xyl.density,main="",axes=FALSE, xlab="",ylab="",xlim=c(li,ls),col="green",lwd=4)

```

Figure 3

```

xlm <- rnorm(10^6,0.5,2)
ylm <- rnorm(10^6,-1,0.1)
xyl <- xlm*ylm
xyl.density<-density(xyl)
li <- (0.5+4*2)*(-1-4*0.1)
ls <- (0.5-4*2)*(-1-4*0.1)
xla <- seq(li,-0.01,1/100)
xlb <- seq(0.01,ls,1/100)
xl =c(xla,xlb)
yl<-product.Normal(100,0.5,2,-1,0.1)
xy2 <- data.frame(xl,yl)
plot(unlist(xy2$xl),unlist(xy2$yl),type="l",axes=TRUE,
xlab="Z=XY",ylab="Density",col="red",lwd=3)
par(new=TRUE)
plot(xyl.density,main="",axes=FALSE, xlab="",ylab="",xlim=c(li,ls),col="green",lwd=4)

```