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Article

Cutting-Edge Analytical and Numerical Approaches to the Gilson–Pickering Equation with Plenty of Soliton Solutions

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Abstract: In this paper, the Gilson–Pickering (GP) equation with applications for wave propagation in plasma physics and crystal lattice theory is studied. The model with wave propagation in plasma physics and crystal lattice theory is explained. A collection of evolution equations from this model, containing the Fornberg–Whitham, Rosenau–Hyman, and Fuchssteiner–Fokas–Camassa–Holm equations is developed. The descriptions of new waves, crystal lattice theory, and plasma physics by applying the standard $\tan(\phi/2)$-expansion technique are investigated. Many alternative responses employing various formulae are achieved; each of these solutions is represented by a distinct plot. Some novel solitary wave solutions of the nonlinear GP equation are constructed utilizing the Paul–Painlevé approach. In addition, several solutions including soliton, bright soliton, and periodic wave solutions are reached using He's variational direct technique (VDT). The superiority of the new mathematical theory over the old one is demonstrated through theorems, and an example of how to design and numerically calibrate a nonlinear model using closed-form solutions is given. In addition, the influence of changes in some important design parameters is analyzed. Our computational solutions exhibit exceptional accuracy and stability, displaying negligible errors. Furthermore, our findings unveil several unprecedented solitary wave solutions of the GP model, underscoring the significance and novelty of our study. Our research establishes a promising foundation for future investigations on incompressible fluids, facilitating the development of more efficient and accurate models for predicting fluid behavior.

Keywords: Gilson–Pickering equation; standard $\tan(\phi/2)$-expansion technique; Paul–Painlevé approach; He's variational direct technique; soliton solutions

MSC: 35C08; 35Q51; 34A25

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1. Introduction

Wave propagation in plasma physics and crystal lattice theory according to the GP equation is described as follows [1]:

$$-q_1\psi_{xxt} - q_4\psi_x\psi_{xx} - \psi\psi_{xxx} - q_3\psi\psi_x + 2q_2\psi_x + \psi_t = 0, \tag{1}$$

where $q_i, i = 1$ to 4 are arbitrary parameters, while ψ describes wave propagation in plasma physics and crystal lattice theory [1]. Many solutions have been obtained using the modified extended mapping method [1], the tanh–coth method [2], the Jacobi elliptic function and exponential rational function approaches [3], the traveling wave transformation method [4], the localized meshless radial basis function method [5], and the finite difference technique [6] with the GP equation. Selecting special values of $q_1 = 1, q_2 = -1, q_3 = 0.5, q_4 = 3$, for the above-mentioned parameters such leads to the FW [7], $q_1 = 0, q_2 = 1, q_3 = 0, q_4 = 3$ leads to the Rosenau–Hyman [8,9], and $q_1 = 1, q_2 = -3, q_3 = 0, q_4 = 2$ leads to the Fuchssteiner–Fokas–Camassa–Holm equations [9,10]. Dynamical systems and nonlinear waves in plasmas have been studied in the areas of applied physics, applied mathematics, dynamical systems, and nonlinear waves in plasmas or other nonlinear media. Plasma is rich in wave phenomena. In any plasma system, plasma particles oscillate indiscriminately and interact using electrostatic or electromagnetic forces. A qualitatively different set of waveforms is available in plasma. The study of various kinds of nonlinear waves in plasmas is an important research topic because such waves are easy to observe and the theoretical background is well established. Important plasma waves include dust acoustic waves, ion acoustic waves, dust ion acoustic waves, hybrid waves, and electrostatic cyclotron waves. Plasma is a many-body system composed of very large numbers of charged particles whose dynamics are governed by long-range collective effects via electromagnetic forces. Intuitively, one can think of it as an intriguingly conductive fluid. Analytical wave solutions for these nonlinear evolution equations have been presented in a very simple way using the bifurcation theory of planar dynamical systems [11].

There are astrophysical plasmas in the accretion disks that surround stars and compact objects like white dwarfs, neutron stars, and black holes in binary star systems. Like M87's 5000-light-year jet, materials ejected by astrophysical jets have often been associated with plasma [12]. Much like its many applications, plasma has been manufactured in various methods [13]. Energy is required for the production and upkeep of anything [14]. When the voltage is high enough, the current stresses the material over its dielectric limit, producing an electrical breakdown indicated by an electric spark as the insulator transforms into a conductor [15]. The Townsend avalanche is triggered when an electron crashes into a neutral gas atom [16].

Many powerful approaches have been applied to study and discuss the explicit solutions to these models and their physical behavior. These techniques include the Gauss quadrature method [17], the eigenvalue problem for the elliptic operator with variable domain [18], the metaheuristic algorithms [19], linear spectral dynamic analysis [20], seismic wave attenuation anisotropy [21], the shear-wave splitting method [22], velocity variation in remote-controlled planes [23], a list–ranking algorithm method [24], the energy system models [25–27], the lump solution method [28], Hirota's bilinear method [29,30], a generalized trial equation scheme [31], the risk factors of mortality [32], high-efficiency sub-microscale uncertainty measurement method [33], the average method and Lyapunov's first method [34], the evidence theory and knowledge meta-theory method [35], a novel settling time solution method [36], a class of learning-based optimal control method [37], the adaptive sliding mode approach [38], the extended homoclinic technique [39], the optimal economic renewable energy methods [40–44], the Lie method [45], the adaptive memory event-triggered mechanism [46], the dynamic gain approach [47], a theoretical derivation approach [48], the inverse scattering technique [49], the finite element technique [50], the N-lump technique [51], Hirota's bilinear operator [52], the robust multi-objective optimal design method [53], correlation of random variables with Copula theory method [54], and the modulation instability scheme [55].

The authors of [56] developed a hybrid scheme for the Gilson–Pickering equation by extracting advantageous features of the collocation method and B-splines. The direct meshless local Petrov–Galerkin method has been employed to solve the stochastic Cahn–Hilliard–Cook and Swift–Hohenberg equations [57]. The shape functions of interpolating moving least squares approximation have been applied to the variational multiscale EFG technique to numerically study Navier–Stokes equations coupled with a heat transfer equation such as two-dimensional nonstationary Boussinesq equations [58].

Also, various nonlinear partial equations are being solved using the above-mentioned direct approaches. In this work, a nonlinear partial differential system, namely, the nonlinear GP equation, is discussed. The above mentioned three efficient and powerful analytical approaches, namely, the standard $\tan(\phi/2)$-expansion technique, the Paul–Painlevé approach, and He's variational direct technique, are utilized to develop some novel exact traveling wave solutions with the help of the computer software maple and mathematica. Some solutions including soliton, bright soliton, singular soliton, and periodic wave solutions by three methods are also obtained. This study aims to derive the solitary wave solutions to Equation (1) using the standard $\tan(\phi/2)$-expansion technique. The mentioned model is even explained by the PDE that intends to solve it to obtain plenty of soliton solutions. Although the $\tan(\phi/2)$-expansion, the Paul–Painlevé and He's variational direct algorithms are reliable, easy to implement and mathematically well-established, as far as we know, no one has considered the aforementioned methods to find the traveling wave solutions for the nonlinear GP equation. In this paper, the analytical traveling wave solutions to the GP equation have been obtained using the mentioned algorithms.

The strategy of the given article is given as: Section 2 elucidates for transforming PDE and its properties. Whereas Section 3 defines the Paul–Painlevé approach and its application. Section 4 contains the using of the standard $\tan(\phi/2)$-expansion scheme and its behavior. Section 5 demonstrates He's variational direct technique for the observed wave structures. At the end, the summary of this paper is illustrated in Section 7.

2. Transforming PDE to ODE

For Equation (1), both x and t represent the transverse coordinate [1]. Employing the wave transformation $\psi(x,t) = \psi(\eta), \eta = ax + bt$, where a and b are the arbitrary values to be determined through the algorithm's steps, rising to

$$\psi'(b - q_4 a^3 \psi'' - a q_3 \psi + 2 a q_2) - a^2 (b q_1 + a\psi)\psi''' = 0. \tag{2}$$

Integrating Equation (2) once with respect to η, one obtains the below ODE:

$$\psi(b + 2aq_2) - \frac{1}{2} a\Big(a(2(bq_1 + a\psi)\psi'' + a(q_4 - 1)(\psi')^2) + q_3\psi^2\Big) = 0. \tag{3}$$

Evaluating the positive integer using the balance method for the highest order derivative terms and the nonlinear terms, namely, between ψ'' and ψ^2, then by supposing $\psi(\eta) = \sum_{l=0}^{k} A_l f(\eta)$ one obtains $A_l' f^{k+2} + \cdots = \sum_{l=0}^{k} A_l \frac{d^2}{d\eta} f(\eta) = \psi''(\eta) = \psi^2(\eta) \simeq \left(\sum_{l=0}^{k} A_l f(\eta)\right)^2 = A_l' f^{2k}(\eta) + \ldots$. Hence, simplifying the mentioned computation obtains $k + 2 = 2k$, so $k = 2$.

3. Brief Description of Paul–Painlevé Approach

The Paul–Painlevé approach (PPA) is mainly described to solve the non-integrable NLPDEs. The important steps of this method are given as follows:

Step 1: Consider a nonlinear partial differential equation in ψ as a function of space variable x and time variable t as

$$\mathcal{W}_1(\psi, \psi_x, \psi_t, \psi_{xx}, \ldots) = 0, \tag{4}$$

where $\mathcal{W}_1$ is contained the nonlinear terms as well.

Step 2: Using the transformation $\psi(x,t) = \psi(\eta), \eta = ax + bt$ into Equation (4). It will convert Equation (4) into the ordinary differential equation as

$$\mathcal{W}_2(\psi, \psi', \psi'', \ldots) = 0. \tag{5}$$

Step 3: According to the PPA [59], the analytic solution of Equation (5) will be written in the form as

$$\psi(\eta) = \sum_{l=0}^{k} A_l G^l(X)\, e^{-lN\eta}, \tag{6}$$

where k is obtained by balancing the higher order term with the nonlinear term via homogeneous balance method, $X = f(\eta) = B_1 - \frac{e^{-N\eta}}{N}$, and $G(X)$ satisfies the Riccati-equation of the form $\frac{dG}{dX} - KG^2 = 0$.
Step 4: Solution of the Riccati-equation is obtained as

$$G(X) = \frac{1}{KX + c_0}, \tag{7}$$

where c_0 and K are the constants.
Step 5: One can obtain the value of constants K, N and $A_l's$ for $l = 1, \ldots, k$ substituting Equation (6) into Equation (5) and equating the different exponents of $G(X)\, e^{-N\eta}$ to zero.

Application of the PPA on GPE

This section will show the implementation of the PPA on GPE to derive the exact solutions. Balancing ψ'' and ψ^2 in Equation (3) employing the homogeneous balance method such that receive $k = 2$. Now, Equation (6) will be expanded up to $k = 2$ as

$$\psi(\eta) = A_0 + A_1 G(X)\, e^{-N\eta} + A_2 G^2(X)\, e^{-2N\eta}, \tag{8}$$

where A_0, A_1, A_2 and N are the unknown constants that have to be determined, and $X = f(\eta) = B_1 - \frac{e^{-N\eta}}{N}$. Also, $G(X)$ satisfies the Riccati equation in the form $\frac{dG}{dX} - KG^2 = 0$ and the solution of this Riccati equation is given using Equation (7).

Consequently,

$$\begin{cases}
\psi^2(\eta) = A_0^2 + \frac{2A_0 G(\xi) A_1}{e^{N\xi}} + \frac{2A_0 G(\xi)^2 A_2}{(e^{N\xi})^2} + \frac{G(\xi)^2 A_1^2}{(e^{N\xi})^2} + \frac{2G(\xi)^3 A_1 A_2}{(e^{N\xi})^3} + \frac{G(\xi)^4 A_2^2}{(e^{N\xi})^4}, \\
\psi^3(\eta) = A_0^3 + \frac{3A_0^2 G(\xi) A_1}{e^{N\xi}} + \frac{3A_0^2 G(\xi)^2 A_2}{(e^{N\xi})^2} + \frac{3A_0 G(\xi)^2 A_1^2}{(e^{N\xi})^2} + \frac{6A_0 G(\xi)^3 A_1 A_2}{(e^{N\xi})^3} + \frac{3A_0 G(\xi)^4 A_2^2}{(e^{N\xi})^4} + \frac{G(\xi)^3 A_1^3}{(e^{N\xi})^3} + \\
\frac{3G(\xi)^4 A_1^2 A_2}{(e^{N\xi})^4} + \frac{3G(\xi)^5 A_1 A_2^2}{(e^{N\xi})^5} + \frac{G(\xi)^6 A_2^3}{(e^{N\xi})^6} \\
\frac{d}{d\eta}\psi(\eta) = D(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right) A_1 e^{-N\xi} - G(X(\xi)) A_1 N\, e^{-N\xi} + 2G(X(\xi)) A_2 \left(e^{-N\xi}\right)^2 D(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right) - \\
2G(X(\xi))^2 A_2 \left(e^{-N\xi}\right)^2 N, \\
\frac{d^2}{d\eta^2}\psi(\eta) = D^{(2)}(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right)^2 A_1 e^{-N\xi} + D(G)(X(\xi))\left(\frac{d^2}{d\xi^2}X(\xi)\right) A_1 e^{-N\xi} - 2D(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right) A_1 N\, e^{-N\xi} \\
+ G(X(\xi)) A_1 N^2 e^{-N\xi} + 2D(G)(X(\xi))^2 \left(\frac{d}{d\xi}X(\xi)\right)^2 A_2 \left(e^{-N\xi}\right)^2 - 8G(X(\xi)) A_2 \left(e^{-N\xi}\right)^2 D(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right) N + \\
2G(X(\xi)) A_2 \left(e^{-N\xi}\right)^2 D^{(2)}(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right)^2 + 2G(X(\xi)) A_2 \left(e^{-N\xi}\right)^2 D(G)(X(\xi))\left(\frac{d^2}{d\xi^2}X(\xi)\right) + \\
4G(X(\xi))^2 A_2 \left(e^{-N\xi}\right)^2 N^2, \\
\left(\frac{d}{d\eta}\psi(\eta)\right)^2 = \frac{D(G)(X(\xi))^2 \left(\frac{d}{d\xi}X(\xi)\right)^2 A_1^2}{(e^{N\xi})^2} - \frac{2D(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right) A_1^2 G(X(\xi)) N}{(e^{N\xi})^2} + \\
\frac{4D(G)(X(\xi))^2 \left(\frac{d}{d\xi}X(\xi)\right)^2 A_1 G(X(\xi)) A_2}{(e^{N\xi})^3} - \frac{8D(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right) A_1 G(X(\xi))^2 A_2 N}{(e^{N\xi})^3} + \frac{G(X(\xi))^2 A_1^2 N^2}{(e^{N\xi})^2} + \\
\frac{4G(X(\xi))^3 A_1 N^2 A_2}{(e^{N\xi})^3} + \frac{4G(X(\xi))^2 A_2^2 D(G)(X(\xi))^2 \left(\frac{d}{d\xi}X(\xi)\right)^2}{(e^{N\xi})^4} - \frac{8G(X(\xi))^3 A_2^2 D(G)(X(\xi))\left(\frac{d}{d\xi}X(\xi)\right) N}{(e^{N\xi})^4} + \frac{4G(X(\xi))^4 A_2^2 N^2}{(e^{N\xi})^4}.
\end{cases} \tag{9}$$

Plugging the values of $\psi'(\eta), \psi''(\eta), (\psi(\eta))^2$ and $(\psi(\eta)')^2$ from Equation (9), respectively, into Equation (3) and collecting the coefficients of different exponents of $G(X)\, e^{-N\eta}$ equating to zero yield the system of equations including eight equations. The following solutions are obtained by solving the above system of nonlinear algebraic equations:

Theorem 1. *An analytical solution for the (1+1)-dimensional GP Equation is given using Equation (1) is generated as the first solution:*

$$\psi_1(x,t) = \frac{216KNq_1q_2\left(q_4^2-2q_4+1\right)}{\left(N^2q_4^2-2N^2q_4+N^2-9q_3\right)\left(N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9\right)} \times \frac{e^{-N\left((-q_4/3+1/3)x-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9}t\right)}}{K\left(B_1-\frac{e^{-N\left((-q_4/3+1/3)x-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9}t\right)}}{N}\right)+c_0}$$
$$+\frac{KA_1}{N}\frac{1}{\left[K\left(B_1-\frac{e^{-N\left((-q_4/3+1/3)x-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9}t\right)}}{N}\right)+c_0\right]^2} \times e^{-2N\left((-q_4/3+1/3)x-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9}t\right)}.$$

Proof. Using $\psi(x,t)=\psi(\eta), \eta = ax+bt$, Equation (1) will be converted into the following ODE as

$$\psi_1'(b-q_4a^3\psi_1''-aq_3\psi_1+2aq_2)-a^2(bq_1+a\psi_1)\psi_1'''=0,$$

next utilizing the integration respect to η, leads to

$$\psi_1(b+2aq_2)-\frac{1}{2}a\left(a(2(bq_1+a\psi_1)\psi_1''+a(q_4-1)(\psi_1')^2)+q_3\psi_1^2\right)=0, \tag{10}$$

where $\prime = \frac{d}{d\eta}$. Balancing reaches $k=2$, then the exact solution is

$$\psi_1(\eta) = A_0 + A_1G(X)\,e^{-N\eta} + A_2G^2(X)\,e^{-2N\eta}, \tag{11}$$

Using Equation (11) in Equation (10) and collecting the coefficients of different exponents of $G(X)\,e^{-N\eta}$ equating to zero yield the following system of equations, then solving the mentioned system of equations the parameter is reached as

$$a=-\frac{q_4}{3}+\frac{1}{3},\quad A_0=0,\quad A_1=A_1,\quad A_2=\frac{KA_1}{N},\quad A_1=\frac{216KNq_1q_2\left(q_4^2-2q_4+1\right)}{\left(N^2q_4^2-2N^2q_4+N^2-9q_3\right)\left(N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9\right)},$$

$$b=-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9},$$

provided that $(q_4-1)^2N^2-9q_3$ and $-9+q_1(q_4-1)^2N^2$ are non-zero. Therefore, Equation (1) gives the first solution as

$$\begin{cases} \psi_1(x,t) = \frac{216KNq_1q_2\left(q_4^2-2q_4+1\right)}{\left(N^2q_4^2-2N^2q_4+N^2-9q_3\right)\left(N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9\right)} \\ \times \frac{e^{-N\left((-q_4/3+1/3)x-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9}t\right)}}{K\left(B_1-\frac{e^{-N\left((-q_4/3+1/3)x-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9}t\right)}}{N}\right)+c_0} \\ +\frac{KA_1}{N}\frac{1}{\left[K\left(B_1-\frac{e^{-N\left((-q_4/3+1/3)x-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9}t\right)}}{N}\right)+c_0\right]^2} \\ \times e^{-2N\left((-q_4/3+1/3)x-\frac{6q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-9}t\right)}. \end{cases} \tag{12}$$

□

Theorem 2. *An analytical solution for the (1+1)-dimensional GP Equation is given using Equation (1) is generated as the second solution*

$$\psi_2(x,t) = A_0 + \frac{A_1}{K\left(-\frac{c_0}{K}-\frac{e^{-N\left(ax+\frac{a\left(K^2A_0q_3-KNA_1q_3+N^2A_2q_3-4K^2q_2\right)}{2K^2}t\right)}}{N}\right)+c_0} \times e^{-N\left(ax+\frac{a\left(K^2A_0q_3-KNA_1q_3+N^2A_2q_3-4K^2q_2\right)}{2K^2}t\right)}$$
$$+\frac{A_2}{\left[K\left(-\frac{c_0}{K}-\frac{e^{-N\left(ax+\frac{a\left(K^2A_0q_3-KNA_1q_3+N^2A_2q_3-4K^2q_2\right)}{2K^2}t\right)}}{N}\right)+c_0\right]^2} \times e^{-2N\left(ax+\frac{a\left(K^2A_0q_3-KNA_1q_3+N^2A_2q_3-4K^2q_2\right)}{2K^2}t\right)}.$$

Proof. Using $\psi(x,t) = \psi(\eta), \eta = ax + bt$, Equation (1) will be converted into the following ODE as

$$\psi_2'(b - q_4a^3\psi_2'' - aq_3\psi_2 + 2aq_2) - a^2(bq_1 + a\psi_2)\psi_2''' = 0,$$

Next, utilizing the integration respect to η leads to

$$\psi_2(b+2aq_2) - \frac{1}{2}a\Big(a(2(bq_1+a\psi_2)\psi_2'' + a(q_4-1)(\psi_2')^2) + q_3\psi_2^2\Big) = 0, \tag{13}$$

where $\prime = \frac{d}{d\eta}$. Balancing reaches $k = 2$, then the exact solution is

$$\psi_1(\eta) = A_0 + A_1G(X)\,e^{-N\eta} + A_2G^2(X)\,e^{-2N\eta}. \tag{14}$$

Using Equation (14) in Equation (13) and collecting the coefficients of different exponents of $G(X)\,e^{-N\eta}$ equating to zero yield the following system of equations, then by solving the mentioned system of equations, the parameter is reached as

$$A_0 = A_0,\quad A_1 = A_1,\quad A_2 = A_2,\quad b = \frac{a\left(K^2A_0q_3 - KNA_1q_3 + N^2A_2q_3 - 4K^2q_2\right)}{2K^2},\quad B_1 = B_1,\quad c_0 = c_0,\quad B_1 = -\frac{c_0}{K},$$

and K is non-zero. Therefore, Equation (1) gives the second solution as

$$\begin{cases} \psi_2(x,t) = A_0 + \dfrac{A_1}{K\left(-\frac{c_0}{K} - \frac{e^{-N\left(ax+\frac{a\left(K^2A_0q_3-KNA_1q_3+N^2A_2q_3-4K^2q_2\right)}{2K^2}t\right)}}{N}\right)+c_0} \\ \times e^{-N\left(ax+\frac{a\left(K^2A_0q_3-KNA_1q_3+N^2A_2q_3-4K^2q_2\right)}{2K^2}t\right)} \\ +\dfrac{A_2}{\left[K\left(-\frac{c_0}{K} - \frac{e^{-N\left(ax+\frac{a\left(K^2A_0q_3-KNA_1q_3+N^2A_2q_3-4K^2q_2\right)}{2K^2}t\right)}}{N}\right)+c_0\right]^2} \\ \times e^{-2N\left(ax+\frac{a\left(K^2A_0q_3-KNA_1q_3+N^2A_2q_3-4K^2q_2\right)}{2K^2}t\right)}. \end{cases} \tag{15}$$

□

Theorem 3. *An analytical solution for the (1+1)-dimensional GP Equation is given using Equation (1) and is generated as the third solution:*

$$\psi_3(x,t) = A_0 + \frac{A_1}{K\left(B_1 - \frac{e^{-N\left((-\frac{q_4}{3}+\frac{1}{3})x-(\frac{1}{6}A_0q_3q_4+\frac{1}{6}A_0q_3+\frac{2}{3}q_2q_4-\frac{2}{3}q_2)t\right)}}{N}\right)+c_0} \times e^{-N\left((-\frac{q_4}{3}+\frac{1}{3})x-(\frac{1}{6}A_0q_3q_4+\frac{1}{6}A_0q_3+\frac{2}{3}q_2q_4-\frac{2}{3}q_2)t\right)} + \frac{-\frac{A_1^2\left(N^2q_4^2-2N^2q_4+N^2-9q_3\right)}{54q_3A_0}}{\left[K\left(B_1 - \frac{e^{-N\left((-\frac{q_4}{3}+\frac{1}{3})x-(\frac{1}{6}A_0q_3q_4+\frac{1}{6}A_0q_3+\frac{2}{3}q_2q_4-\frac{2}{3}q_2)t\right)}}{N}\right)+c_0\right]^2} \times e^{-2N\left((-\frac{q_4}{3}+\frac{1}{3})x-(\frac{1}{6}A_0q_3q_4+\frac{1}{6}A_0q_3+\frac{2}{3}q_2q_4-\frac{2}{3}q_2)t\right)}.$$

Proof. Using $\psi(x,t) = \psi(\eta), \eta = ax + bt$, Equation (1) will be converted into the following ODE as

$$\psi_3'(b - q_4a^3\psi_3'' - aq_3\psi_3 + 2aq_2) - a^2(bq_1 + a\psi_3)\psi_3''' = 0,$$

Next, utilizing the integration respect to η leads to

$$\psi_3(b+2aq_2) - \frac{1}{2}a\left(a(2(bq_1+a\psi_3)\psi_3'' + a(q_4-1)(\psi_3')^2) + q_3\psi_3^2\right) = 0, \tag{16}$$

where $\prime = \frac{d}{d\eta}$. Balancing reaches $k = 2$, then the exact solution is

$$\psi_3(\eta) = A_0 + A_1G(X)\,e^{-N\eta} + A_2G^2(X)\,e^{-2N\eta}. \tag{17}$$

Using Equation (17) in Equation (16) and collecting the coefficients of different exponents of $G(X)\,e^{-N\eta}$ equating to zero yield the following system of equations, then by solving the mentioned system of equations the parameter is reached as

$$A_0 = A_0, \quad A_1 = A_1, \quad a = -\frac{q_4}{3} + \frac{1}{3}, \quad b = -\frac{1}{6}A_0q_3q_4 + \frac{1}{6}A_0q_3 + \frac{2}{3}q_2q_4 - \frac{2}{3}q_2,$$
$$B_1 = B_1, \quad A_2 = -\frac{A_1^2\left(N^2q_4^2 - 2N^2q_4 + N^2 - 9q_3\right)}{54q_3A_0}, \quad c_0 = c_0, \quad q_1 = -\frac{A_0\left(2N^2q_4^2 - 4N^2q_4 + 2N^2 + 9q_3\right)}{N^2(q_4-1)^2(A_0q_3 - 4q_2)},$$

provided that $N^2(q_4-1)^2(A_0q_3-4q_2)\neq 0$. Therefore, Equation (1) gives the third solution as

$$\left\{\begin{array}{l}\psi_3(x,t)=A_0+\dfrac{A_1}{K\left(B_1-\frac{e^{-N\left((-\frac{q_4}{3}+\frac{1}{3})x-(\frac{1}{6}A_0q_3q_4+\frac{1}{6}A_0q_3+\frac{2}{3}q_2q_4-\frac{2}{3}q_2)t\right)}}{N}\right)+c_0}\\ \times e^{-N\left((-\frac{q_4}{3}+\frac{1}{3})x-(\frac{1}{6}A_0q_3q_4+\frac{1}{6}A_0q_3+\frac{2}{3}q_2q_4-\frac{2}{3}q_2)t\right)}+\\ \dfrac{-\frac{A_1^2\left(N^2q_4^2-2N^2q_4+N^2-9q_3\right)}{54q_3A_0}}{\left[K\left(B_1-\frac{e^{-N\left((-\frac{q_4}{3}+\frac{1}{3})x-(\frac{1}{6}A_0q_3q_4+\frac{1}{6}A_0q_3+\frac{2}{3}q_2q_4-\frac{2}{3}q_2)t\right)}}{N}\right)+c_0\right]^2}\\ \times e^{-2N\left((-\frac{q_4}{3}+\frac{1}{3})x-(\frac{1}{6}A_0q_3q_4+\frac{1}{6}A_0q_3+\frac{2}{3}q_2q_4-\frac{2}{3}q_2)t\right)}.\end{array}\right. \tag{18}$$

□

Theorem 4. *An analytical solution for the (1+1)-dimensional GP Equation is given using Equation (1) is generated as the fourth solution:*

$$\psi_4(x,t)=-\frac{64q_1q_2}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16}+\frac{A_1}{K\left(B_1-\frac{e^{-N\left((-\frac{q_4}{4}+\frac{1}{4})x-(\frac{8q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16})t\right)}}{N}\right)+c_0}$$
$$\times e^{-N\left((-\frac{q_4}{4}+\frac{1}{4})x-(\frac{8q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16})t\right)}.$$

Proof. Using $\psi(x,t)=\psi(\eta),\eta=ax+bt$, Equation (1) will be converted into the following ODE as

$$\psi_4'(b-q_4a^3\psi_4''-aq_3\psi_4+2aq_2)-a^2(bq_1+a\psi_4)\psi_4'''=0,$$

Next, utilizing the integration with respect to η leads to

$$\psi_4(b+2aq_2)-\frac{1}{2}a\left(a(2(bq_1+a\psi_4)\psi_4''+a(q_4-1)(\psi_4')^2)+q_3\psi_4^2\right)=0, \tag{19}$$

where $\prime=\frac{d}{d\eta}$. Balancing reaches $k=2$, then the exact solution is

$$\psi_4(\eta)=A_0+A_1G(X)\,e^{-N\eta}+A_2G^2(X)\,e^{-2N\eta}. \tag{20}$$

Using Equation (20) in Equation (19) and collecting the coefficients of different exponents of $G(X)\,e^{-N\eta}$ equating to zero yield the following system of equations, then by solving the mentioned system of equations the parameter is reached as

$$A_0=-\frac{64q_1q_2}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16},\quad A_1=A_1,\quad a=-\frac{q_4}{4}+\frac{1}{4},\quad b=-\frac{8q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16},$$
$$B_1=B_1,\quad A_2=0,\quad K=-\frac{(N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16)NA_1}{64q_1q_2},\quad c_0=c_0,\quad q_3=-\frac{1}{16}N^2q_4^2+\frac{1}{8}N^2q_4-\frac{1}{16}N^2,$$

provided that $N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16\neq 0$. Therefore, Equation (1) gives the fourth solution as

$$
\begin{cases}
\psi_4(x,t) = -\frac{64q_1q_2}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16} + \frac{A_1}{K\left(B_1 - \frac{e^{-N\left((-\frac{q_4}{4}+\frac{1}{4})x-(\frac{8q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16})t\right)}}{N}\right)+c_0} \\
\times e^{-N\left((-\frac{q_4}{4}+\frac{1}{4})x-(\frac{8q_2(q_4-1)}{N^2q_1q_4^2-2N^2q_1q_4+N^2q_1-16})t\right)}.
\end{cases} \tag{21}
$$

□

4. Standard tan(ϕ/2)-Expansion Technique

Handling the investigated model through the standard $\tan(\phi/2)$-expansion method creates the following issues:

Step 1.

$$\mathcal{S}_1(\psi, \psi_x, \psi_t, \psi_{xx}, \psi_{tt}, \dots) = 0, \tag{22}$$

where $\mathcal{S}_1$ is a polynomial of ψ and its partial derivatives.

Step 2. Employing the traveling wave transformation

$$\eta = \kappa x + \omega t + \theta_0, \tag{23}$$

where κ and ω are the non-zero values, allows us to diminish Equation (22) to an ODE of $\psi = \psi(\eta)$ in the below form:

$$\mathcal{S}_2(\psi, \kappa\psi', \omega\psi', \kappa^2\psi'', \omega^2\psi'', \dots) = 0. \tag{24}$$

Step 3. The exact form of the solution (22) is given as

$$\psi(\eta) = \sum_{l=0}^{k} \lambda_l \tan(\phi/2)^l + \sum_{l=1}^{k} \mu_l (\tan(\phi/2))^{-l}, \tag{25}$$

where $\lambda_k, \mu_k \neq 0$, and $\phi = \phi(\eta)$ satisfies the following:

$$\phi = w_1 \sin(\phi) + w_2 \cos(\phi) + w_3. \tag{26}$$

The obtained solutions to Equation (26) are presented as

Product 1. *With $\Delta = w_1^2 + w_2^2 - w_3^2 < 0$ and $w_2 - w_3 \neq 0$, we can find the result*

$$\phi(\eta) = 2\arctan\left[\frac{w_1}{w_2 - w_3} - \frac{\sqrt{-\Delta}}{w_2 - w_3}\tan\left(\frac{\sqrt{-\Delta}}{2}\eta\right)\right].$$

Product 2. *With $\Delta = w_1^2 + w_2^2 - w_3^2 > 0$ and $s_2 - w_3 \neq 0$, we can find the result*

$$\phi(\eta) = 2\arctan\left[\frac{w_1}{w_2 - w_3} + \frac{\sqrt{\Delta}}{w_2 - w_3}\tanh\left(\frac{\sqrt{\Delta}}{2}\eta\right)\right].$$

Product 3. *With $w_1 = 0, w_3 = 0$, we can find the result*

$$\phi(\eta) = \arctan\left[\frac{e^{2w_2(\xi)} - 1}{e^{2w_2(\eta)} + 1}, \frac{2e^{w_2(\eta)}}{e^{2w_2(\eta)} + 1}\right].$$

Product 4. *With $w_1 = 0, w_2 = 0$, we can find the result $\phi(\eta) = w_3\eta + C$.*

Product 5. *With $w_2 = -w_3$, we can find the result* $\phi(\xi) = 2\arctan\left[\frac{w_1 e^{w_1\eta}}{1-w_3 e^{w_1\eta}}\right]$.

Product 6. *With $w_1 = w_3$, we then obtain* $\phi(\eta) = 2\arctan\left[\frac{(w_2+w_3)e^{w_2\eta}+1}{(w_2-w_3)e^{w_2\eta}-1}\right]$.

Product 7. *With $w_3 = w_1$, we can find the result* $\phi(\eta) = 2\arctan\left[\frac{(w_1+w_2)e^{w_2\eta}-1}{1-(w_1-w_2)e^{w_2\eta}}\right]$.

Product 8. *With $w_2 = 0, w_3 = w_1$, we can find the result* $\phi(\eta) = 2\arctan\left[\frac{w_1\eta+2}{w_1\eta}\right]$.

Product 9. *With $w_2 = 0, w_3 = 0$, we then obtain* $\phi(\eta) = \arctan\left(2\,\frac{e^{\eta\, w_1}}{e^{2\,\eta\, w_1}+1}, -\frac{e^{2\,\eta\, w_1}-1}{e^{2\,\eta\, w_1}+1}\right)$.

Product 10. *With $w_2 = w_3$, we can find the result* $\phi(\eta) = 2\ \arctan\left(\frac{e^{\eta\, w_1}-w_3}{w_1}\right)$.

Also, $\lambda_l, \mu_r (l = 0, 2, \ldots, k, r = 1, \ldots, k), w_1, w_2$ and w_3 are also the values to be explored later. The balance number can be obtained using of the linear and nonlinear terms of ODE. Balancing the terms of the above-equation through the homogeneous balance principle along with the above-mentioned computational scheme.
Step 4. Evaluate the positive integer k in Equation (25) using the balance method of the highest order derivative terms.
Step 5. Substitute the gained values with solutions of Equation (25) into Equation (24), the analytical solutions for Equation (1) are obtained.

4.1. The Soliton Solutions

It is supposed that Equation (25) has a formal solution of the form given below:

$$\psi(\eta) = \lambda_0 + \lambda_1 \tan(\phi/2) + \lambda_2 \tan^2(\phi/2) + \mu_1 \cot(\phi/2) + \mu_2 \cot^2(\phi/2). \quad (27)$$

Inserting (27) into (3), a collection of solutions with different given constants is obtained. After solving this system with a computer application like Mathematica, they obtain the corresponding results:

The set of category of solutions:

4.1.1. Set I

$$a = 1 - q_4,\ \ b = 2\,\frac{q_2(-1+q_4)}{q_1 q_3 + 1},\ \lambda_0 = \frac{-A_2 + \sqrt{-4\,A_1 A_3 + {A_2}^2}}{2A_1},\ \ \lambda_1 = \lambda_2 = \mu_2 = 0,\ \ w_1 = \frac{\sqrt{-q_3}}{-1+q_4},\ \ w_2 = -w_3, \quad (28)$$

$$A_1 = q_1 {q_3}^2 + q_3,\ \ A_2 = 2\,\sqrt{-q_3}\mu_1 w_3(-1+q_4)(q_1 q_3 + 1) - 4\,q_1 q_2 q_3,$$

$$A_3 = 2\,\sqrt{-q_3}\mu_1 w_3(-1+q_4)(q_1 q_3 + 1) - 4\,\sqrt{-q_3}\mu_1 q_1 q_2 w_3(-1+q_4),\ \ \Delta = -\frac{q_3}{(-1+q_4)^2},$$

$$\psi(\eta) = \frac{-A_2 + \sqrt{-4\,A_1 A_3 + {A_2}^2}}{2A_1} + \mu_1 \cot\left(\frac{\phi(\eta)}{2}\right).$$

As a consequence (Products 1, 2, 5), the periodic, soliton, and kink soliton solution are given by

$$\psi_1(x,t) = \frac{-A_2 + \sqrt{-4\,A_1 A_3 + A_2{}^2}}{2A_1} + 2\mu_1 w_3 \left[-\frac{\sqrt{-q_3}}{-1+q_4} + \frac{\sqrt{q_3}}{-1+q_4} \tan\left(\frac{\sqrt{q_3}}{2(-1+q_4)}\eta \right) \right]^{-1}, \tag{29}$$

$$\psi_2(x,t) = \frac{-A_2 + \sqrt{-4\,A_1 A_3 + A_2{}^2}}{2A_1} + 2\mu_1 w_3 \left[-\frac{\sqrt{-q_3}}{-1+q_4} - \frac{\sqrt{-q_3}}{-1+q_4} \tanh\left(\frac{\sqrt{-q_3}}{2(-1+q_4)}\eta \right) \right]^{-1}, \tag{30}$$

$$\psi_3(x,t) = \frac{-A_2 + \sqrt{-4\,A_1 A_3 + A_2{}^2}}{2A_1} + \mu_1 \frac{1 - w_3 e^{\frac{\sqrt{-q_3}}{-1+q_4}\eta}}{\frac{\sqrt{-q_3}}{-1+q_4} e^{\frac{\sqrt{-q_3}}{-1+q_4}\eta}}, \ \ \eta = (1-q_4)x + 2\,\frac{q_2(-1+q_4)}{q_1 q_3 + 1} t. \tag{31}$$

4.1.2. Set II

$$a = 1/4 - 1/4\,q_4,\ \ b = -\frac{q_2(-1+q_4)}{q_1 q_3 - 2},\ \ \lambda_1 = \lambda_2 = \mu_2 = 0,\ \ w_1 = \frac{2\sqrt{2\,q_3}}{-1+q_4},\ \ \lambda_0 = \frac{(-1+q_4)\sqrt{2\,q_3}\mu_1 w_3}{4q_3}, \tag{32}$$

$$w_2 = -w_3,\ \ \psi(\eta) = \frac{(-1+q_4)\sqrt{2\,q_3}\mu_1 w_3}{4q_3} + \mu_1 \cot\left(\frac{\phi(\eta)}{2}\right),\ \ \eta = (1/4 - 1/4\,q_4)x - \frac{q_2(-1+q_4)}{q_1 q_3 - 2} t. \tag{33}$$

As a consequence (Products 1, 2, 5), the periodic, soliton, and kink soliton solution are given by

$$\psi_1(x,t) = \frac{(-1+q_4)\sqrt{2\,q_3}\mu_1 w_3}{4q_3} - \mu_1 w_3 \left[\frac{\sqrt{2\,q_3}}{-1+q_4} - \frac{\sqrt{-2\,q_3}}{-1+q_4} \tan\left(\frac{\sqrt{-2\,q_3}}{-1+q_4}\eta \right) \right]^{-1}, \tag{34}$$

$$\psi_2(x,t) = \frac{(-1+q_4)\sqrt{2\,q_3}\mu_1 w_3}{4q_3} - \mu_1 w_3 \left[\frac{\sqrt{2\,q_3}}{-1+q_4} + \frac{\sqrt{2\,q_3}}{-1+q_4} \tanh\left(\frac{\sqrt{2\,q_3}}{-1+q_4}\eta \right) \right]^{-1}, \tag{35}$$

$$\psi_3(x,t) = \frac{(-1+q_4)\sqrt{2\,q_3}\mu_1 w_3}{4q_3} + \mu_1 \frac{1 - w_3 e^{\frac{2\sqrt{2\,q_3}}{-1+q_4}\eta}}{\frac{2\sqrt{2\,q_3}}{-1+q_4} e^{\frac{2\sqrt{2\,q_3}}{-1+q_4}\eta}}, \ \ \eta = (1/4 - 1/4\,q_4)x - \frac{q_2(-1+q_4)}{q_1 q_3 - 2} t. \tag{36}$$

Here, the graphical interpretation of the developed results are discussed. Figure 1 shows the behavior of analysis related to the periodic solution where graphs of ψ are added as follows:

$$q_1 = 2, q_2 = 3, q_3 = -3, q_4 = 3, \mu_1 = 2, w_3 = 1, \tag{37}$$

$$\psi = -1/3\,\sqrt{2}\sqrt{-3} - 4\left(\sqrt{2}\sqrt{-3} - \sqrt{6}\tan\left(1/2\,\sqrt{6}(-x/2 + 3/4\,t)\right)\right)^{-1}, \tag{38}$$

for Equation (34). The behavior of the general periodic solution received from the mentioned technique is investigated, which is presented in Figure 1. From the graph, it is ostensible that the periodic form solution presents a stable propagation to the generalized nonlocal nonlinearity as discussed in Figure 1. Also, Figure 2 shows the behavior of analysis related to soliton solution where plots of ψ are added to the following:

$$q_1 = 2, q_2 = 3, q_3 = 2, q_4 = 2, \mu_1 = 2, w_3 = 1, \tag{39}$$

$$\psi = 1/2 - 4\,(4 + 4\,\tanh(-x/2 - 3\,t))^{-1}, \tag{40}$$

for Equation (35). Moreover, Figure 3 shows the behavior of analysis related to the soliton solution where graphs of ψ are added to the following

$$q_1 = 2, q_2 = 3, q_3 = 2, q_4 = 2, \mu_1 = 2, w_3 = 1, \tag{41}$$

$$\psi = 1/2 + \frac{1 - \mathrm{e}^{-x-6\,t}}{\mathrm{e}^{-x-6\,t}}, \tag{42}$$

for Equation (36). For Figure 1, a 2D plot to $(x = -10, 0, 10)$; for Figure 2, a 2D plot to $(x = -2, 0, 2)$; and also for Figure 3, a 2D plot to $(x = -2, 0, 2)$ are included.

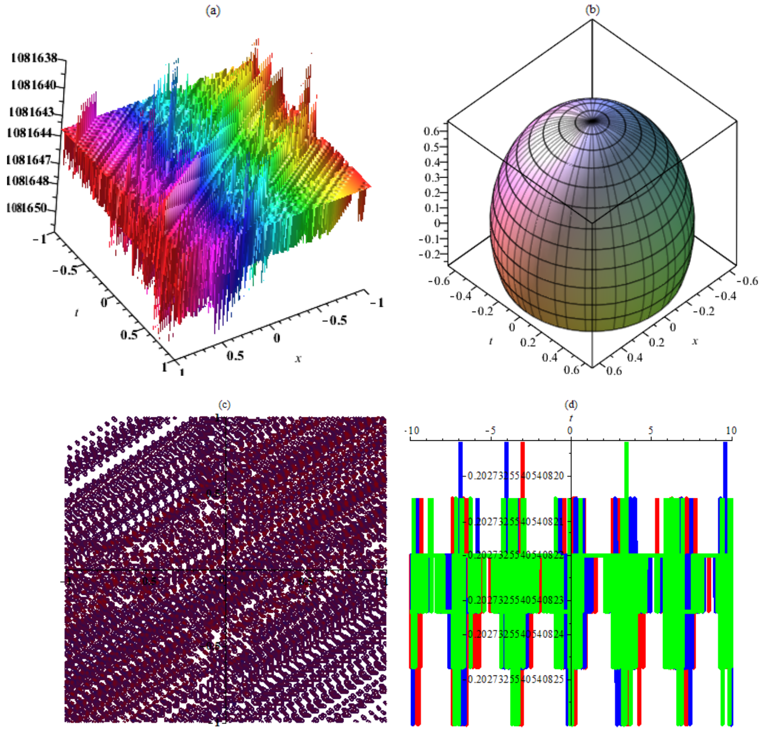

Figure 1. Plots of real ((**a**) (3D plot), (**b**) (spherical plot), (**c**) (contour plot), (**d**) (2D plot)) parts of solution (34) graph of ψ_1 for the parameter values $q_1 = 2, q_2 = 3, q_3 = -3, q_4 = 3, \mu_1 = 2, w_3 = 1$.

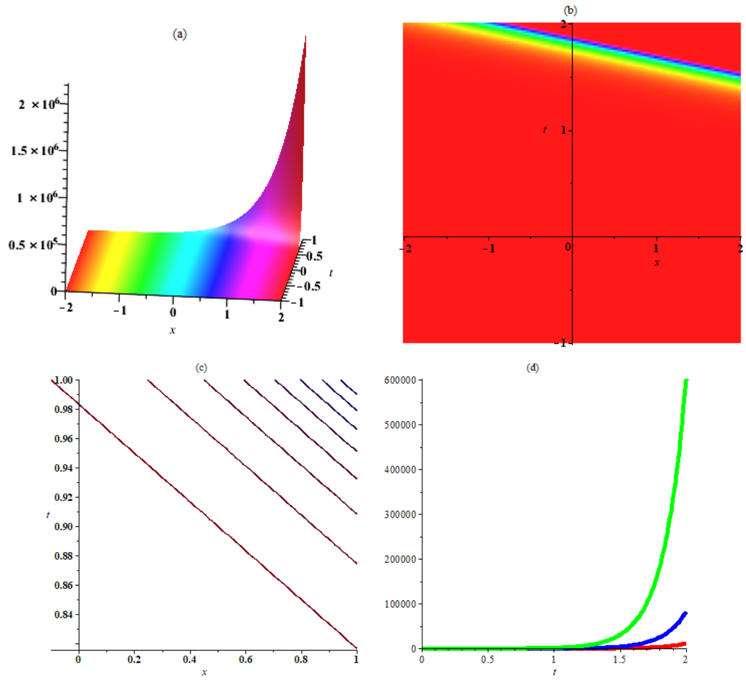

Figure 2. Plots of real ((**a**) (3D plot), (**b**) (density plot), (**c**) (contour plot), (**d**) (2D plot)) parts of solution (35) graph of ψ_2 for the parameter values $q_1 = 2, q_2 = 3, q_3 = 2, q_4 = 2, \mu_1 = 2, w_3 = 1$.

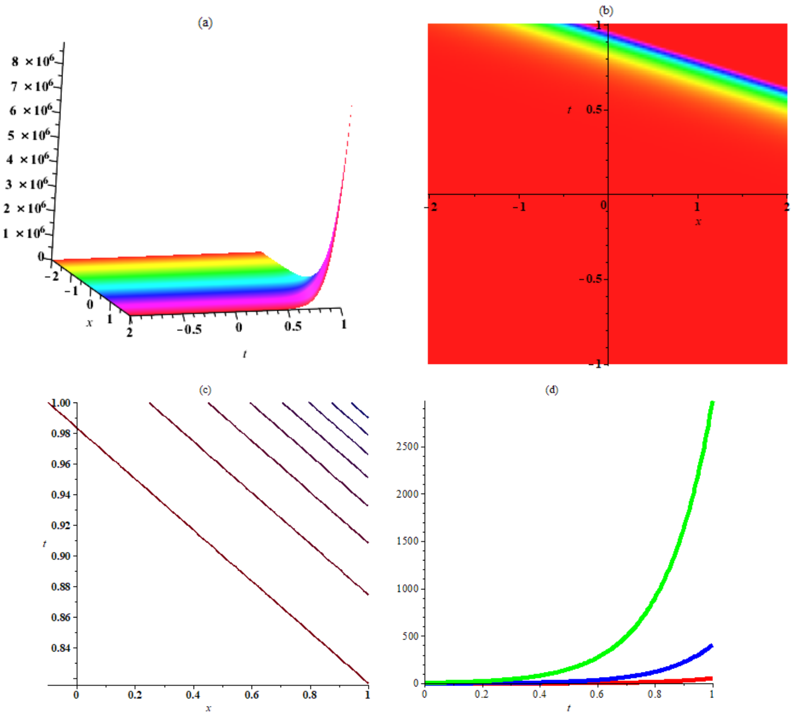

Figure 3. Plots of real ((**a**) (3D plot), (**b**) (density plot), (**c**) (contour plot), (**d**) (2D plot)) parts of solution (36) graph of ψ_3 for the parameter values $q_1 = 2, q_2 = 3, q_3 = 2, q_4 = 2, \mu_1 = 2, w_3 = 1$.

4.1.3. Set III

$$a = \frac{1}{4} - \frac{q_4}{4}, \;\; b = \frac{2q_2(-1+q_4)}{q_1q_3+4}, \;\; \lambda_1 = \lambda_2 = \mu_2 = 0, \;\; w_1 = \frac{2\sqrt{2\,q_3}}{-1+q_4}, \tag{43}$$

$$w_2 = -w_3, \;\; \lambda_0 = \frac{\sqrt{2q_3}\mu_1 w_3(-1+q_4)(q_1q_3+4) + 16\,q_1q_2q_3}{4(q_1q_3+4)q_3},$$

$$\psi(\eta) = \frac{\sqrt{2q_3}\mu_1 w_3(-1+q_4)(q_1q_3+4) + 16\,q_1q_2q_3}{4(q_1q_3+4)q_3} + \mu_1 \cot\left(\frac{\phi(\eta)}{2}\right), \;\; \eta = (\frac{1-q_4}{4})x + \frac{2q_2(-1+q_4)}{q_1q_3-2}t. \tag{44}$$

As a consequence (Products 1, 2, 5), the periodic, soliton, and kink soliton solution are given by

$$\psi_1(x,t) = \frac{\sqrt{2q_3}\mu_1 w_3(q_4-1)(q_1q_3+4) + 16\,q_1q_2q_3}{4(q_1q_3+4)q_3} - \mu_1 w_3\left[\frac{\sqrt{2\,q_3}}{q_4-1} - \frac{\sqrt{-2\,q_3}}{q_4-1}\tan\left(\frac{\sqrt{-2\,q_3}}{q_4-1}\eta\right)\right]^{-1}, \tag{45}$$

$$\psi_2(x,t) = \frac{\sqrt{2q_3}\mu_1 w_3(q_4-1)(q_1q_3+4) + 16\,q_1q_2q_3}{4(q_1q_3+4)q_3} - \mu_1 w_3\left[\frac{\sqrt{2\,q_3}}{q_4-1} + \frac{\sqrt{2\,q_3}}{q_4-1}\tanh\left(\frac{\sqrt{2\,q_3}}{q_4-1}\eta\right)\right]^{-1}, \tag{46}$$

$$\psi_3(x,t) = \frac{\sqrt{2q_3}\mu_1 w_3(q_4-1)(q_1q_3+4) + 16\,q_1q_2q_3}{4(q_1q_3+4)q_3} + \mu_1(q_4-1)\frac{1 - w_3 e^{\frac{2\sqrt{2\,q_3}}{q_4-1}\eta}}{2\sqrt{2\,q_3}e^{\frac{2\sqrt{2\,q_3}}{q_4-1}\eta}}, \;\; \eta = (\frac{1}{4} - \frac{q_4}{4})x + \frac{2q_2(q_4-1)}{q_1q_3+4}t. \tag{47}$$

4.1.4. Set IV

$$a = 1/4 - 1/4\,q_4, \;\; b = \frac{q_2(-1+q_4)}{2(q_1q_3+4)}, \;\; \lambda_1 = \lambda_2 = \mu_2 = 0, \;\; w_1 = -2\,\frac{q_3(\lambda_0 q_1 q_3 - 2\,q_1q_2 + \lambda_0)}{(-1+q_4)\sqrt{-q_3}q_1q_2}, \tag{48}$$

$$w_2 = \frac{\left(\lambda_0{}^2 q_1 q_3 - \mu_1{}^2 q_1 q_3 - 4\,\lambda_0 q_1 q_2 + \lambda_0{}^2 - \mu_1{}^2\right) q_3}{(-1+q_4)\mu_1 \sqrt{-q_3} q_1 q_2},\ w_3 = \frac{\sqrt{-q_3}\left(\lambda_0{}^2 q_1 q_3 + \mu_1{}^2 q_1 q_3 - 4\,\lambda_0 q_1 q_2 + \lambda_0{}^2 + \mu_1{}^2\right)}{(-1+q_4) q_2 q_1 \mu_1},$$

$$\psi(\eta) = \lambda_0 + \mu_1 \cot\left(\frac{\phi(\eta)}{2}\right),\ \eta = (1/4 - 1/4\,q_4)x + \frac{q_2(-1+q_4)}{2(q_1 q_3 + 4)} t. \tag{49}$$

As a consequence (Products 1, 2), the periodic and soliton solutions are given by

$$\psi_1(x,t) = \lambda_0 + \mu_1 \left[-\frac{(\lambda_0 q_1 q_3 - 2\,q_1 q_2 + \lambda_0)\mu_1}{\lambda_0(\lambda_0 q_1 q_3 - 4\,q_1 q_2 + \lambda_0)} + \frac{2\mu_1 \sqrt{-q_3} q_1 q_2}{\sqrt{q_3}\lambda_0(-\lambda_0 q_1 q_3 + 4\,q_1 q_2 - \lambda_0)} \tan\left(\frac{2\sqrt{q_3}}{-1+q_4}\eta\right)\right]^{-1}, \tag{50}$$

$$\psi_2(x,t) = \lambda_0 + \mu_1 \left[-\frac{(\lambda_0 q_1 q_3 - 2\,q_1 q_2 + \lambda_0)\mu_1}{\lambda_0(\lambda_0 q_1 q_3 - 4\,q_1 q_2 + \lambda_0)} - \frac{2\mu_1 \sqrt{q_3} q_1 q_2}{\sqrt{q_3}\lambda_0(-\lambda_0 q_1 q_3 + 4\,q_1 q_2 - \lambda_0)} \tanh\left(\frac{2\sqrt{-q_3}}{-1+q_4}\eta\right)\right]^{-1}, \tag{51}$$

$$\eta = (1/4 - 1/4\,q_4)x + \frac{q_2(-1+q_4)}{2(q_1 q_3 + 4)} t.$$

4.1.5. Set V

$$a = a,\ b = 2\,\frac{a q_2(-1+q_4)}{a q_1 q_3 - q_4 + 1},\ \lambda_1 = \lambda_2 = \mu_2 = 0,\ w_1 = \sqrt{-\frac{q_3}{2\,a^2 + a q_4 - a}}, \tag{52}$$

$$w_2 = \frac{(a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0) q_3}{\sqrt{-q_3 a(2\,a + q_4 - 1)}\mu_1(a q_1 q_3 - q_4 + 1)},\ w_3 = -\frac{(a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0) q_3}{\sqrt{-q_3 a(2\,a + q_4 - 1)}\mu_1(a q_1 q_3 - q_4 + 1)},$$

$$\psi(\eta) = \lambda_0 + \mu_1 \cot\left(\frac{\phi(\eta)}{2}\right),\ \eta = a x + 2\,\frac{a q_2(-1+q_4)}{a q_1 q_3 - q_4 + 1} t. \tag{53}$$

As a consequence (Products 1, 2), the periodic and soliton solutions are given by

$$\begin{cases} \psi_1(\eta) = \lambda_0 + \mu_1 \left[\frac{1}{2}\frac{\mu_1(a q_1 q_3 - q_4 + 1)}{a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0} - \frac{\sqrt{-1}}{2}\frac{\mu_1(a q_1 q_3 - q_4 + 1)}{a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0} \tan\left(\sqrt{\frac{q_3}{4a(2\,a + q_4 - 1)}}\eta\right)\right]^{-1}, \\ \psi_2(\eta) = \lambda_0 + \mu_1 \left[\frac{1}{2}\frac{\mu_1(a q_1 q_3 - q_4 + 1)}{a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0} + \frac{1}{2}\frac{\mu_1(a q_1 q_3 - q_4 + 1)}{a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0} \tanh\left(\sqrt{-\frac{q_3}{4a(2\,a + q_4 - 1)}}\eta\right)\right]^{-1}, \\ \eta = a x + 2\,\frac{a q_2(-1+q_4)}{a q_1 q_3 - q_4 + 1} t. \end{cases} \tag{54}$$

4.1.6. Set VI

$$a = 1 - q_4,\ b = 2\,\frac{q_2(-1+q_4)}{q_1 q_3 + 1},\ \lambda_2 = \mu_1 = \mu_2 = 0,\ \lambda_0 = \frac{-A_2 + \sqrt{-4\,A_1 A_3 + A_2{}^2}}{2A_1},\ w_1 = \frac{\sqrt{-q_3}}{-1+q_4}, \tag{55}$$

$$w_2 = w_3,\ A_1 = q_1 q_3{}^2 + q_3,\ A_2 = 2\,\sqrt{-q_3}\lambda_1 w_3(-1+q_4)(q_1 q_3 + 1) - 4\,q_1 q_2 q_3,$$

$$A_3 = -\lambda_1{}^2 w_3{}^2 (-1+q_4)^2 (q_1 q_3 + 1) - 4\,\sqrt{-q_3}\lambda_1 q_1 q_2 w_3(-1+q_4),$$

$$\psi(\eta) = \frac{-A_2 + \sqrt{-4\,A_1 A_3 + A_2{}^2}}{2A_1} + \lambda_1 \tan\left(\frac{\phi(\eta)}{2}\right),\ \eta = (1 - q_4)x + 2\,\frac{q_2(-1+q_4)}{q_1 q_3 + 1} t. \tag{56}$$

As a consequence (Product 10), the kink solution is given by

$$\psi_1(\eta) = \frac{-A_2 + \sqrt{-4\,A_1 A_3 + A_2{}^2}}{2A_1} + \lambda_1 \frac{e^{\frac{\sqrt{-q_3}}{-1+q_4}\eta} - w_3}{\frac{\sqrt{-q_3}}{-1+q_4}},\ \eta = (1 - q_4)x + 2\,\frac{q_2(-1+q_4)}{q_1 q_3 + 1} t. \tag{57}$$

4.1.7. Set VII

$$a=1/4-1/4\,q_4,\ \ b=-\frac{q_2(-1+q_4)}{q_1q_3-2},\ \ \lambda_2=\mu_1=\mu_2=0,\ \ \lambda_0=\frac{(-1+q_4)\sqrt{2\,q_3}\lambda_1w_3}{4q_3},\ \ w_1=\frac{2\sqrt{2\,q_3}}{-1+q_4},\tag{58}$$

$$w_2=w_3,\ \ \psi(\eta)=\frac{(-1+q_4)\sqrt{2\,q_3}\lambda_1w_3}{4q_3}+\lambda_1\tan\left(\frac{\phi(\eta)}{2}\right),\ \ \eta=(1/4-1/4\,q_4)x-\frac{q_2(-1+q_4)}{q_1q_3-2}t.\tag{59}$$

As a consequence (Product 10), the kink solution is given by

$$\psi_1(\eta)=\frac{(-1+q_4)\sqrt{2\,q_3}\lambda_1w_3}{4q_3}+\lambda_1(q_4-1)\frac{e^{\frac{2\sqrt{2\,q_3}}{-1+q_4}\eta}-w_3}{2\sqrt{2\,q_3}},\ \ \eta=(\frac{1-q_4}{4})x-\frac{q_2(-1+q_4)}{q_1q_3-2}t.\tag{60}$$

4.1.8. Set VIII

$$a=1/4-1/4\,q_4,\ \ b=1/2\,\frac{q_2(-1+q_4)}{q_1q_3+1},\ \ \lambda_2=\mu_1=\mu_2=0,\ \ w_1=-2\,\frac{q_3(\lambda_0q_1q_3-2\,q_1q_2+\lambda_0)}{(-1+q_4)\sqrt{-q_3}q_1q_2},\tag{61}$$

$$w_2=-\frac{({\lambda_0}^2q_1q_3-{\lambda_1}^2q_1q_3-4\,\lambda_0q_1q_2+{\lambda_0}^2-{\lambda_1}^2)q_3}{(q_4-1)\lambda_1\sqrt{-q_3}q_1q_2},\ \ w_3=\frac{\sqrt{-q_3}({\lambda_0}^2q_1q_3+{\lambda_1}^2q_1q_3-4\,\lambda_0q_1q_2+{\lambda_0}^2+{\lambda_1}^2)}{(q_4-1)q_2q_1\lambda_1},$$

$$\psi(\eta)=\lambda_0+\lambda_1\tan\left(\frac{\phi(\eta)}{2}\right),\ \ \eta=(1/4-1/4\,q_4)x+1/2\,\frac{q_2(-1+q_4)}{q_1q_3+1}t.\tag{62}$$

As a consequence (Products 1,2), the periodic and soliton solutions are given by

$$\psi_1(\eta)=\lambda_0-\frac{\lambda_0q_1q_3-2\,q_1q_2+\lambda_0}{(q_1q_3+1)}+\frac{2\sqrt{-1}q_1q_2}{(q_1q_3+1)}\tan\left(\frac{2\sqrt{q_3}}{-1+q_4}\eta\right),\tag{63}$$

$$\psi_2(\eta)=\lambda_0-\frac{\lambda_0q_1q_3-2\,q_1q_2+\lambda_0}{(q_1q_3+1)}-\frac{2q_1q_2}{(q_1q_3+1)}\tanh\left(\frac{2\sqrt{-q_3}}{-1+q_4}\eta\right),\ \ \eta=(\frac{1-q_4}{4})x+1/2\,\frac{q_2(-1+q_4)}{q_1q_3+1}t.\tag{64}$$

4.1.9. Set IX

$$a=a,\ \ b=-2\,\frac{aq_2(2\,a+q_4-1)}{aq_1q_3+2\,a+q_4-1},\ \ \lambda_2=\mu_1=\mu_2=0,\ \ w_1=\sqrt{-\frac{q_3}{2\,a^2+aq_4-a}},\tag{65}$$

$$w_2=-\frac{\lambda_0q_3}{\lambda_1\sqrt{-q_3a(2\,a+q_4-1)}},\ \ w_3=-\frac{\lambda_0q_3}{\lambda_1\sqrt{-q_3a(2\,a+q_4-1)}},$$

$$\psi(\eta)=\lambda_0+\lambda_1\tan\left(\frac{\phi(\eta)}{2}\right),\ \ \eta=ax-2\,\frac{aq_2(2\,a+q_4-1)}{aq_1q_3+2\,a+q_4-1}t.\tag{66}$$

As a consequence (Product 10), the kink solution is given by

$$\psi_1(\eta)=\lambda_0+\lambda_1\frac{e^{\sqrt{-\frac{q_3}{2\,a^2+aq_4-a}}\eta}-w_3}{\sqrt{-\frac{q_3}{2\,a^2+aq_4-a}}},\ \ \eta=ax-2\,\frac{aq_2(2\,a+q_4-1)}{aq_1q_3+2\,a+q_4-1}t.\tag{67}$$

4.1.10. Set X

$$b=\frac{a(a\lambda_0q_3+\lambda_0q_3q_4-4\,aq_2-\lambda_0q_3-2\,q_2q_4+2\,q_2)}{aq_1q_3+2\,a+q_4-1},\tag{68}$$

$$\mu_1=-1/4\,\frac{(a\lambda_0q_1q_3-4\,aq_1q_2-\lambda_0q_4+\lambda_0)\lambda_0(2\,a+q_4-1)}{(aq_1q_3q_4-aq_1q_3+2\,aq_4+{q_4}^2-2\,a-2\,q_4+1)\lambda_1},$$

$$a = a,\ \ \lambda_2 = \mu_2 = w_2 = w_3 = 0,\ \ w_1 = \sqrt{-\frac{q_3}{2\,a^2 + aq_4 - a}},$$

$$\psi(\eta) = \lambda_0 + \lambda_1 \tan\left(\frac{\phi(\eta)}{2}\right) - 1/4\,\frac{(a\lambda_0 q_1 q_3 - 4\,aq_1 q_2 - \lambda_0 q_4 + \lambda_0)\lambda_0(2\,a + q_4 - 1)}{(aq_1 q_3 q_4 - aq_1 q_3 + 2\,aq_4 + q_4{}^2 - 2\,a - 2\,q_4 + 1)\lambda_1} \cot\left(\frac{\phi(\eta)}{2}\right), \tag{69}$$

$$\eta = ax + \frac{a(a\lambda_0 q_3 + \lambda_0 q_3 q_4 - 4\,aq_2 - \lambda_0 q_3 - 2\,q_2 q_4 + 2\,q_2)}{aq_1 q_3 + 2\,a + q_4 - 1}t.$$

As a consequence (Product 9), the kink solution is given by

$$\psi_1(\eta) = \lambda_0 + \lambda_1 \tan\left(\frac{1}{2}\arctan\left(2\,\frac{e^{\eta\, w_1}}{e^{2\,\eta\, w_1} + 1}, -\frac{e^{2\,\eta\, w_1} - 1}{e^{2\,\eta\, w_1} + 1}\right)\right) \tag{70}$$

$$-1/4\,\frac{(a\lambda_0 q_1 q_3 - 4\,aq_1 q_2 - \lambda_0 q_4 + \lambda_0)\lambda_0(2\,a + q_4 - 1)}{(aq_1 q_3 q_4 - aq_1 q_3 + 2\,aq_4 + q_4{}^2 - 2\,a - 2\,q_4 + 1)\lambda_1} \cot\left(\frac{1}{2}\arctan\left(2\,\frac{e^{\eta\, w_1}}{e^{2\,\eta\, w_1} + 1}, -\frac{e^{2\,\eta\, w_1} - 1}{e^{2\,\eta\, w_1} + 1}\right)\right),$$

$$\eta = ax + \frac{a(a\lambda_0 q_3 + \lambda_0 q_3 q_4 - 4\,aq_2 - \lambda_0 q_3 - 2\,q_2 q_4 + 2\,q_2)}{aq_1 q_3 + 2\,a + q_4 - 1}t.$$

4.1.11. Set XI

$$a = -\frac{-1 + q_4}{q_1 q_3 + 2},\ \ b = -\frac{(\lambda_1 \mu_1 q_1{}^2 q_3{}^2 + 2\,\lambda_1 \mu_1 q_1 q_3 - q_1{}^2 q_2{}^2 + \lambda_1 \mu_1)(-1 + q_4)}{(q_1 q_3 + 1) q_1{}^2 q_2},\ \ w_1 = \frac{q_1 q_3 + 2}{(-1 + q_4)\sqrt{q_1}}, \tag{71}$$

$$\lambda_2 = \mu_2 = w_2 = w_3 = 0,\ \ \psi(\eta) = \lambda_0 + \lambda_1 \tan\left(\frac{\phi(\eta)}{2}\right) + \mu_1 \cot\left(\frac{\phi(\eta)}{2}\right), \tag{72}$$

$$\eta = -\frac{-1 + q_4}{q_1 q_3 + 2}x - \frac{(\lambda_1 \mu_1 q_1{}^2 q_3{}^2 + 2\,\lambda_1 \mu_1 q_1 q_3 - q_1{}^2 q_2{}^2 + \lambda_1 \mu_1)(-1 + q_4)}{(q_1 q_3 + 1) q_1{}^2 q_2}t.$$

As a consequence (Product 9), the kink solution is given by

$$\psi_1(\eta) = \lambda_0 + \lambda_1 \tan\left(\frac{1}{2}\arctan\left(2\,\frac{e^{\eta\, w_1}}{e^{2\,\eta\, w_1} + 1}, -\frac{e^{2\,\eta\, w_1} - 1}{e^{2\,\eta\, w_1} + 1}\right)\right) + \mu_1 \cot\left(\frac{1}{2}\arctan\left(2\,\frac{e^{\eta\, w_1}}{e^{2\,\eta\, w_1} + 1}, -\frac{e^{2\,\eta\, w_1} - 1}{e^{2\,\eta\, w_1} + 1}\right)\right), \tag{73}$$

$$w_1 = \frac{q_1 q_3 + 2}{(-1 + q_4)\sqrt{q_1}},\ \ \eta = -\frac{-1 + q_4}{q_1 q_3 + 2}x - \frac{(\lambda_1 \mu_1 q_1{}^2 q_3{}^2 + 2\,\lambda_1 \mu_1 q_1 q_3 - q_1{}^2 q_2{}^2 + \lambda_1 \mu_1)(-1 + q_4)}{(q_1 q_3 + 1) q_1{}^2 q_2}t.$$

4.1.12. Set XII

$$\begin{cases} a = 1/4 - 1/4\,q_4,\ \ b = b,\ \ \lambda_0 = 4/3\,\frac{bq_1 q_3 + q_2 q_4 - 2\,b - q_2}{q_3(-1 + q_4)}, \\ \lambda_1 = -2/9\,\frac{(bq_1 q_3 + q_2 q_4 - 2\,b - q_2)(bq_1 q_3 - 2\,q_2 q_4 + 4\,b + 2\,q_2)}{\mu_1 q_3{}^2(q_4{}^2 - 2\,q_4 + 1)}, \\ \psi(\eta) = 4/3\,\frac{bq_1 q_3 + q_2 q_4 - 2\,b - q_2}{q_3(-1 + q_4)} - 2/9\,\frac{(bq_1 q_3 + q_2 q_4 - 2\,b - q_2)(bq_1 q_3 - 2\,q_2 q_4 + 4\,b + 2\,q_2)}{\mu_1 q_3{}^2(q_4{}^2 - 2\,q_4 + 1)} \tan\left(\frac{\phi(\eta)}{2}\right) + \mu_1 \cot\left(\frac{\phi(\eta)}{2}\right), \\ w_1 = 2\,\frac{\sqrt{2\,q_3}}{-1 + q_4},\ \ \ \lambda_2 = \mu_2 = w_2 = w_3 = 0,\ \ \eta = (1/4 - 1/4\,q_4)x + bt. \end{cases} \tag{74}$$

As a consequence (Product 9), the kink solution is given by

$$\psi_1(\eta) = 4/3\,\frac{bq_1 q_3 + q_2 q_4 - 2\,b - q_2}{q_3(-1 + q_4)} - 2/9\,\frac{(bq_1 q_3 + q_2 q_4 - 2\,b - q_2)(bq_1 q_3 - 2\,q_2 q_4 + 4\,b + 2\,q_2)}{\mu_1 q_3{}^2(q_4{}^2 - 2\,q_4 + 1)} \tag{75}$$

$$\times \tan\left(\frac{1}{2}\arctan\left(2\,\frac{e^{\eta\, w_1}}{e^{2\,\eta\, w_1} + 1}, -\frac{e^{2\,\eta\, w_1} - 1}{e^{2\,\eta\, w_1} + 1}\right)\right) + \mu_1 \cot\left(\frac{1}{2}\arctan\left(2\,\frac{e^{\eta\, w_1}}{e^{2\,\eta\, w_1} + 1}, -\frac{e^{2\,\eta\, w_1} - 1}{e^{2\,\eta\, w_1} + 1}\right)\right),$$

$$w_1 = 2\,\frac{\sqrt{2\,q_3}}{-1 + q_4},\ \ \ \eta = (1/4 - 1/4\,q_4)x + bt.$$

4.1.13. Set XIII

$$a = 2/3 - 2/3\,q_4,\ \ b = 4\,\frac{q_2(-1+q_4)}{2\,q_1q_3+3},\ \ \lambda_0 = -\frac{8\,\mu_2 w_3{}^2(-1+q_4)^2(2\,q_1q_3+3) - 72\,q_1q_2q_3}{9\,q_3(2\,q_1q_3+3)}, \tag{76}$$

$$\mu_1 = -8/3\,\frac{(-1+q_4)w_3\mu_2\sqrt{-1/2\,q_3}}{q_3},\ \ w_1 = 3/2\,\frac{\sqrt{-1/2\,q_3}}{-1+q_4},\ \ \lambda_1 = \lambda_2 = 0,$$

$$\psi(\eta) = -\frac{8\,\mu_2 w_3{}^2(q_4-1)^2(2\,q_1q_3+3) - 72\,q_1q_2q_3}{9\,q_3(2\,q_1q_3+3)} - \frac{8}{3}\,\frac{(q_4-1)w_3\mu_2\sqrt{-1/2\,q_3}}{q_3}\cot\left(\frac{\phi(\eta)}{2}\right) + \mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right), \tag{77}$$

$$w_2 = -w_3,\ \ \eta = (2/3 - 2/3\,q_4)x + 4\,\frac{q_2(-1+q_4)}{2\,q_1q_3+3}t.$$

As a consequence (Products 1, 2, 5), the periodic, soliton, and kink soliton solutions are given by

$$\psi_1(x,t) = -\frac{8\,\mu_2 w_3{}^2(-1+q_4)^2(2\,q_1q_3+3) - 72\,q_1q_2q_3}{9\,q_3(2\,q_1q_3+3)} - \tag{78}$$

$$8/3\,\frac{(-1+q_4)w_3\mu_2\sqrt{-1/2\,q_3}}{q_3}\left[-3/2\,\frac{\sqrt{-1/2\,q_3}}{2w_3(-1+q_4)} + 3/2\,\frac{\sqrt{1/2\,q_3}}{2w_3(-1+q_4)}\tan\left(3/2\,\frac{\sqrt{1/2\,q_3}}{2(-1+q_4)}\eta\right)\right]^{-1} +$$

$$\mu_2\left[-3/2\,\frac{\sqrt{-1/2\,q_3}}{2w_3(-1+q_4)} + 3/2\,\frac{\sqrt{1/2\,q_3}}{2w_3(-1+q_4)}\tan\left(3/2\,\frac{\sqrt{1/2\,q_3}}{2(-1+q_4)}\eta\right)\right]^{-2},$$

$$\psi_2(x,t) = -\frac{8\,\mu_2 w_3{}^2(-1+q_4)^2(2\,q_1q_3+3) - 72\,q_1q_2q_3}{9\,q_3(2\,q_1q_3+3)} - \tag{79}$$

$$8/3\,\frac{(-1+q_4)w_3\mu_2\sqrt{-1/2\,q_3}}{q_3}\left[-3/2\,\frac{\sqrt{-1/2\,q_3}}{2w_3(-1+q_4)} + 3/2\,\frac{\sqrt{-1/2\,q_3}}{2w_3(-1+q_4)}\tanh\left(3/2\,\frac{\sqrt{-1/2\,q_3}}{2(-1+q_4)}\eta\right)\right]^{-1} +$$

$$\mu_2\left[-3/2\,\frac{\sqrt{-1/2\,q_3}}{2w_3(-1+q_4)} + 3/2\,\frac{\sqrt{-1/2\,q_3}}{2w_3(-1+q_4)}\tanh\left(3/2\,\frac{\sqrt{-1/2\,q_3}}{2(-1+q_4)}\eta\right)\right]^{-2},$$

$$\psi_3(x,t) = -\frac{8\,\mu_2 w_3{}^2(q_4-1)^2(2q_1q_3+3) - 72\,q_1q_2q_3}{9\,q_3(2\,q_1q_3+3)} - \frac{8}{3}\,\frac{(q_4-1)w_3\mu_2\sqrt{-1/2\,q_3}}{q_3}\left[\frac{1 - w_3 e^{3/2\,\frac{\sqrt{-1/2\,q_3}}{q_4-1}\eta}}{3/2\,\frac{\sqrt{-1/2\,q_3}}{q_4-1}e^{3/2\,\frac{\sqrt{-1/2\,q_3}}{-1+q_4}\eta}}\right] \tag{80}$$

$$+\mu_2\left[\frac{1 - w_3 e^{3/2\,\frac{\sqrt{-1/2\,q_3}}{-1+q_4}\eta}}{3/2\,\frac{\sqrt{-1/2\,q_3}}{-1+q_4}e^{3/2\,\frac{\sqrt{-1/2\,q_3}}{-1+q_4}\eta}}\right]^2,\ \ \eta = (2/3 - 2/3\,q_4)x + 4\,\frac{q_2(-1+q_4)}{2\,q_1q_3+3}t.$$

4.1.14. Set XIV

$$a = 2/3 - 2/3\,q_4,\ \ b = 4/3\,\frac{q_2(-1+q_4)}{2\,q_1q_3+1},\ \ \lambda_0 = -\frac{8\,\mu_2 w_3{}^2(q_4{}^2 - 2\,q_4+1)}{9\,q_3},\ \ \mu_1 = -4/3\,\frac{(-1+q_4)w_3\mu_2\sqrt{-2\,q_3}}{q_3}, \tag{81}$$

$$\psi(\eta) = -\frac{8\,\mu_2 w_3{}^2(q_4{}^2 - 2\,q_4+1)}{9\,q_3} - 4/3\,\frac{(-1+q_4)w_3\mu_2\sqrt{-2\,q_3}}{q_3}\cot\left(\frac{\phi(\eta)}{2}\right) + \mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right), \tag{82}$$

$$w_1 = 3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4},\ \ \lambda_1 = \lambda_2 = 0,\ \ w_2 = -w_3,\ \ \eta = (2/3 - 2/3\,q_4)x + 4/3\,\frac{q_2(-1+q_4)}{2\,q_1q_3+1}t.$$

As a consequence (Products 1, 2), the periodic and soliton solutions are given by

$$\psi_1(x,t) = -\frac{8\,\mu_2 {w_3}^2({q_4}^2 - 2\,q_4 + 1)}{9\,q_3} + 32/9 w_3\,\frac{(-1+q_4)w_3\mu_2\sqrt{-2\,q_3}}{q_3}\left[\frac{\sqrt{-2\,q_3}}{q_4-1} - \frac{\sqrt{2\,q_3}}{q_4-1}\tan\left(3/2\,\frac{\sqrt{2\,q_3}}{q_4-1}\eta\right)\right]^{-1} \tag{83}$$

$$+4\mu_2 w_3^2\left[3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4} - 3/4\,\frac{\sqrt{2\,q_3}}{-1+q_4}\tan\left(3/2\,\frac{\sqrt{2\,q_3}}{-1+q_4}\eta\right)\right]^{-2},$$

$$\psi_2(x,t) = -\frac{8\,\mu_2 {w_3}^2(q_4-1)^2}{9\,q_3} + 32/9 w_3\,\frac{(q_4-1)w_3\mu_2\sqrt{-2\,q_3}}{q_3}\left[\frac{\sqrt{-2\,q_3}}{q_4-1} + \frac{\sqrt{-2\,q_3}}{q_4-1}\tanh\left(3/2\,\frac{\sqrt{-2\,q_3}}{q_4-1}\eta\right)\right]^{-1} \tag{84}$$

$$+4\mu_2 w_3^2\left[3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4} + 3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}\tanh\left(3/2\,\frac{\sqrt{-2\,q_3}}{-1+q_4}\eta\right)\right]^{-2},\ \ \eta = (2/3 - 2/3\,q_4)x + 4/3\,\frac{q_2(-1+q_4)}{2\,q_1q_3+1}t.$$

4.1.15. Set XV

$$a = 1/3 - 1/3\,q_4,\ \ b = -2/3\,\frac{q_2(-1+q_4)}{q_1q_3-1},\ \ \lambda_0 = 4/9\,\frac{{w_3}^2\mu_2({q_4}^2 - 2\,q_4 + 1)}{q_3},\ \ \mu_1 = 4/3\,\frac{w_3\mu_2\sqrt{q_3}(-1+q_4)}{q_3}, \tag{85}$$

$$\psi(\eta) = 4/9\,\frac{{w_3}^2\mu_2({q_4}^2 - 2\,q_4 + 1)}{q_3} + 4/3\,\frac{w_3\mu_2\sqrt{q_3}(-1+q_4)}{q_3}\cot\left(\frac{\phi(\eta)}{2}\right) + \mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right), \tag{86}$$

$$w_1 = 3/2\,\frac{\sqrt{q_3}}{-1+q_4},\ \ \lambda_1 = \lambda_2 = 0,\ \ w_2 = -w_3,\ \ \eta = (1/3 - 1/3\,q_4)x - 2/3\,\frac{q_2(-1+q_4)}{q_1q_3-1}t.$$

As a consequence (Products 1, 2), the periodic and soliton solutions are given by

$$\psi_1(x,t) = 4/9\,\frac{{w_3}^2\mu_2(q_4-1)^2}{q_3} + 16/9 w_3\,\frac{w_3\mu_2\sqrt{q_3}(-1+q_4)}{q_3}\left[\frac{\sqrt{q_3}}{-1+q_4} - \frac{\sqrt{-q_3}}{-1+q_4}\tan\left(3/4\,\frac{\sqrt{-q_3}}{-1+q_4}\eta\right)\right]^{-1} \tag{87}$$

$$+4\mu_2 w_3^2\left[3/2\,\frac{\sqrt{q_3}}{-1+q_4} - 3/2\,\frac{\sqrt{-q_3}}{-1+q_4}\tan\left(3/4\,\frac{\sqrt{-q_3}}{-1+q_4}\eta\right)\right]^{-2},$$

$$\psi_2(x,t) = 4/9\,\frac{{w_3}^2\mu_2(q_4-1)^2}{q_3} + 16/9 w_3\,\frac{w_3\mu_2\sqrt{q_3}(-1+q_4)}{q_3}\left[\frac{\sqrt{q_3}}{-1+q_4} + \frac{\sqrt{q_3}}{-1+q_4}\tanh\left(3/4\,\frac{\sqrt{q_3}}{-1+q_4}\eta\right)\right]^{-1} \tag{88}$$

$$+4\mu_2 w_3^2\left[3/2\,\frac{\sqrt{q_3}}{-1+q_4} + 3/2\,\frac{\sqrt{q_3}}{-1+q_4}\tanh\left(3/4\,\frac{\sqrt{q_3}}{-1+q_4}\eta\right)\right]^{-2},\ \ \eta = (1/3 - 1/3\,q_4)x - 2/3\,\frac{q_2(-1+q_4)}{q_1q_3-1}t.$$

4.1.16. Set XVI

$$a = \frac{1-q_4}{3},\ \ b = 2\,\frac{q_2(q_4-1)}{q_1q_3+3},\ \ \lambda_0 = 4/9\,\frac{{w_3}^2\mu_2(q_4-1)^2(q_1q_3+3) + 9\,q_1q_2q_3}{q_3(q_1q_3+3)},\ \ \mu_1 = 4/3\,\frac{w_3\mu_2\sqrt{q_3}(q_4-1)}{q_3}, \tag{89}$$

$$\psi(\eta) = 4/9\,\frac{{w_3}^2\mu_2({q_4}^2 - 2\,q_4 + 1)}{q_3} + 4/3\,\frac{w_3\mu_2\sqrt{q_3}(-1+q_4)}{q_3}\cot\left(\frac{\phi(\eta)}{2}\right) + \mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right), \tag{90}$$

$$w_1 = 3/2\,\frac{\sqrt{q_3}}{-1+q_4},\ \lambda_1 = \lambda_2 = 0,\ w_2 = -w_3,\ \eta = (1/3 - 1/3\,q_4)x + 2\,\frac{q_2(-1+q_4)}{q_1q_3+3}t.$$

As a consequence (Products 1, 2), the periodic and soliton solutions are given by

$$\psi_1(x,t) = 4/9\,\frac{{w_3}^2\mu_2(q_4-1)^2}{q_3} - 16/9w_1\,\frac{w_3\mu_2\sqrt{q_3}(-1+q_4)}{q_3}\left[\frac{\sqrt{q_3}}{-1+q_4} - \frac{\sqrt{-q_3}}{-1+q_4}\tan\left(3/4\,\frac{\sqrt{-q_3}}{-1+q_4}\eta\right)\right]^{-1} \tag{91}$$
$$+4\mu_2w_3^2\left[3/2\,\frac{\sqrt{q_3}}{-1+q_4} - 3/2\,\frac{\sqrt{-q_3}}{-1+q_4}\tan\left(3/4\,\frac{\sqrt{-q_3}}{-1+q_4}\eta\right)\right]^{-2},$$

$$\psi_2(x,t) = 4/9\,\frac{{w_3}^2\mu_2(q_4-1)^2}{q_3} - 16/9w_1\,\frac{w_3\mu_2\sqrt{q_3}(-1+q_4)}{q_3}\left[\frac{\sqrt{q_3}}{-1+q_4} + \frac{\sqrt{q_3}}{-1+q_4}\tanh\left(3/4\,\frac{\sqrt{q_3}}{-1+q_4}\eta\right)\right]^{-1} \tag{92}$$
$$+4\mu_2w_3^2\left[3/2\,\frac{\sqrt{q_3}}{-1+q_4} + 3/2\,\frac{\sqrt{q_3}}{-1+q_4}\tanh\left(3/4\,\frac{\sqrt{q_3}}{-1+q_4}\eta\right)\right]^{-2},\ \eta = (1/3 - 1/3\,q_4)x + 2\,\frac{q_2(-1+q_4)}{q_1q_3+3}t.$$

4.1.17. Set XVII

$$a = 1/3 - 1/3\,q_4,\ b = -2\,\frac{aq_2(2\,a+q_4-1)}{aq_1q_3+2\,a+q_4-1},\ \lambda_0 = 1/4\,\frac{{\mu_1}^2}{\mu_2},\ \lambda_1 = \lambda_2 = 0, \tag{93}$$

$$\psi(\eta) = 1/4\,\frac{{\mu_1}^2}{\mu_2} + \mu_1\cot\left(\frac{\phi(\eta)}{2}\right) + \mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right),\ \eta = ax - 2\,\frac{aq_2(2\,a+q_4-1)}{aq_1q_3+2\,a+q_4-1}t, \tag{94}$$

$$w_1 = \sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}},\ w_2 = 1/16\,\frac{q_3\mu_1}{\mu_2\sqrt{-q_3a(2\,a+q_4-1)}},\ w_3 = -1/16\,\frac{q_3\mu_1}{\mu_2\sqrt{-q_3a(2\,a+q_4-1)}}.$$

As a consequence (Product 5), the kink soliton solution is given by

$$\psi_1(x,t) = 1/4\,\frac{{\mu_1}^2}{\mu_2} + \mu_1\,\frac{1 - w_3e^{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}\eta}}{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}e^{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}\eta}} + \mu_2\left(\frac{1 - w_3e^{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}\eta}}{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}e^{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}\eta}}\right)^2, \tag{95}$$
$$\eta = ax - 2\,\frac{aq_2(2\,a+q_4-1)}{aq_1q_3+2\,a+q_4-1}t.$$

4.1.18. Set XVIII

$$a = \frac{4\,\lambda_0\mu_2q_4 - {\mu_1}^2q_4 - 4\,\lambda_0\mu_2 + {\mu_1}^2}{(4\,\lambda_0\mu_2q_3 - {\mu_1}^2q_3 - 16\,\mu_2q_2)q_1},\ b = 1/8\,\frac{4\,\lambda_0\mu_2q_4 - {\mu_1}^2q_4 - 4\,\lambda_0\mu_2 + {\mu_1}^2}{\mu_2q_1},\ \lambda_0 = 1/4\,\frac{{\mu_1}^2}{\mu_2}, \tag{96}$$
$$w_1 = \frac{sq_1\left(4\,\lambda_0\mu_2q_3 - {\mu_1}^2q_3 - 16\,\mu_2q_2\right)}{-1+q_4},\ \lambda_1 = \lambda_2 = 0,$$
$$w_2 = -w_3 = \frac{\mu_1\left(4\,\lambda_0\mu_2q_3 - {\mu_1}^2q_3 - 16\,\mu_2q_2\right)q_1q_3}{8(q_1q_3(4\,\lambda_0\mu_2 - {\mu_1}^2) - 16\,\mu_2q_2q_1 + 8\,\lambda_0\mu_2 - 2\,{\mu_1}^2)s\mu_2(4\,\lambda_0\mu_2q_4 - {\mu_1}^2q_4 - 4\,\lambda_0\mu_2 + {\mu_1}^2)},$$
$$s = \sqrt{-\frac{q_3}{4\,q_1(4\,\lambda_0\mu_2 - {\mu_1}^2)(4\,\lambda_0\mu_2q_3 - {\mu_1}^2q_3 - 16\,\mu_2q_2) + 8\,(4\,\lambda_0\mu_2 - {\mu_1}^2)^2}},$$

$$\psi(\eta) = \lambda_0 + \mu_1\cot\left(\frac{\phi(\eta)}{2}\right) + \mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right), \tag{97}$$
$$\eta = \frac{4\,\lambda_0\mu_2q_4 - {\mu_1}^2q_4 - 4\,\lambda_0\mu_2 + {\mu_1}^2}{(4\,\lambda_0\mu_2q_3 - {\mu_1}^2q_3 - 16\,\mu_2q_2)q_1}x + \frac{4\,\lambda_0\mu_2q_4 - {\mu_1}^2q_4 - 4\,\lambda_0\mu_2 + {\mu_1}^2}{8\mu_2q_1}t.$$

As a consequence (Product 5), the kink soliton solution is investigated by

$$\psi_1(\eta) = \lambda_0 + \mu_1 \frac{1 - w_3 e^{\frac{sq_1\left(4\lambda_0\mu_2q_3 - \mu_1{}^2q_3 - 16\,\mu_2q_2\right)}{-1+q_4}\eta}}{\frac{sq_1(4\lambda_0\mu_2q_3 - \mu_1{}^2q_3 - 16\,\mu_2q_2)}{-1+q_4} e^{\frac{sq_1\left(4\lambda_0\mu_2q_3 - \mu_1{}^2q_3 - 16\,\mu_2q_2\right)}{-1+q_4}\eta}} \tag{98}$$

$$+\mu_2\left(\frac{1 - w_3 e^{\frac{sq_1\left(4\lambda_0\mu_2q_3 - \mu_1{}^2q_3 - 16\,\mu_2q_2\right)}{-1+q_4}\eta}}{\frac{sq_1(4\lambda_0\mu_2q_3 - \mu_1{}^2q_3 - 16\,\mu_2q_2)}{-1+q_4} e^{\frac{sq_1\left(4\lambda_0\mu_2q_3 - \mu_1{}^2q_3 - 16\,\mu_2q_2\right)}{-1+q_4}\eta}}\right)^2,$$

$$\eta = \frac{4\lambda_0\mu_2q_4 - \mu_1{}^2q_4 - 4\lambda_0\mu_2 + \mu_1{}^2}{(4\lambda_0\mu_2q_3 - \mu_1{}^2q_3 - 16\,\mu_2q_2)q_1}x + \frac{4\lambda_0\mu_2q_4 - \mu_1{}^2q_4 - 4\lambda_0\mu_2 + \mu_1{}^2}{8\mu_2q_1}t.$$

4.1.19. Set XIX

$$a = a,\ \ b = 2\,\frac{aq_2(q_4-1)}{aq_1q_3 - q_4 + 1},\ \ \lambda_0 = \frac{4aq_1q_2}{aq_1q_3 - q_4 + 1},\ \ w_1 = \sqrt{-\frac{q_3}{8\,a^2 + 4\,aq_4 - 4\,a}}, \tag{99}$$

$$\lambda_1 = \mu_1 = \mu_2 = w_2 = w_3 = 0,\ \ \psi(\eta) = \frac{4aq_1q_2}{aq_1q_3 - q_4 + 1} + \lambda_2\tan^2\left(\frac{\phi(\eta)}{2}\right),\ \ \eta = ax + 2\,\frac{aq_2(-1+q_4)}{aq_1q_3 - q_4 + 1}t. \tag{100}$$

According to 9, the exact solution is obtained as

$$\psi_1(x,t) = \frac{4aq_1q_2}{aq_1q_3 - q_4 + 1} + \lambda_2\tan^2\left(\frac{1}{2}\arctan\left(2\,\frac{\mathrm{e}^{\eta\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}}}{\mathrm{e}^{2\,\eta\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}}+1}, -\frac{\mathrm{e}^{2\,\eta\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}}-1}{\mathrm{e}^{2\,\eta\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}}+1}\right)\right), \tag{101}$$

$$\eta = ax + 2\,\frac{aq_2(-1+q_4)}{aq_1q_3 - q_4 + 1}t.$$

4.1.20. Set XX

$$\begin{cases} a = 2/3 - 2/3\,q_4,\ \ b = 4\,\frac{q_2(q_4-1)}{2\,q_1q_3+3},\ \ \lambda_0 = -\frac{8\,\lambda_2w_3{}^2(q_4-1)^2(2\,q_1q_3+3) - 72\,q_1q_2q_3}{9\,q_3(2\,q_1q_3+3)},\ \ \lambda_1 = -\frac{4(q_4-1)w_3\lambda_2\sqrt{-2\,q_3}}{3q_3}, \\ \psi(\eta) = -\frac{8\,\lambda_2w_3{}^2(-1+q_4)^2(2\,q_1q_3+3) - 72\,q_1q_2q_3}{9\,q_3(2\,q_1q_3+3)} - \frac{4(-1+q_4)w_3\lambda_2\sqrt{-2\,q_3}}{3q_3}\tan\left(\frac{\phi(\eta)}{2}\right) + \lambda_2\tan^2\left(\frac{\phi(\eta)}{2}\right), \\ w_1 = 3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4},\ \ w_2 = w_3,\ \ \mu_1 = \mu_2 = 0,\ \ \eta = (2/3 - 2/3\,q_4)x + 4\,\frac{q_2(-1+q_4)}{2\,q_1q_3+3}t. \end{cases} \tag{102}$$

As a consequence (Product 10), the kink soliton solution is discussed by

$$\psi(\eta) = -\frac{8\,\lambda_2w_3{}^2(-1+q_4)^2(2\,q_1q_3+3) - 72\,q_1q_2q_3}{9\,q_3(2\,q_1q_3+3)} - \frac{4(-1+q_4)w_3\lambda_2\sqrt{-2\,q_3}}{3q_3}\left(\frac{\mathrm{e}^{3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}\eta} - w_3}{3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}}\right) \tag{103}$$

$$+\lambda_2\left(\frac{\mathrm{e}^{3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}\eta} - w_3}{3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}}\right)^2,\ \ \eta = (2/3 - 2/3\,q_4)x + 4\,\frac{q_2(-1+q_4)}{2\,q_1q_3+3}t.$$

Figure 4 shows the behavior of analysis related to the kink soliton solution where plots of ψ are added to the following:

$$q_1 = 2, q_2 = 3, q_3 = 2, q_4 = 2, \lambda_2 = 2, w_3 = 1, \tag{104}$$

$$\psi = -\frac{448}{9} + \frac{32\,\mathrm{e}^{3/4\,\sqrt{2}(-2/3\,x-12\,t)}}{9} + \frac{16\left(\mathrm{e}^{3/4\,\sqrt{2}(-2/3\,x-12\,t)} - 1\right)^2}{9}, \tag{105}$$

for Equation (103). For Figure 4, a 2D plot to $(x = -2, 0, 2)$ is included.

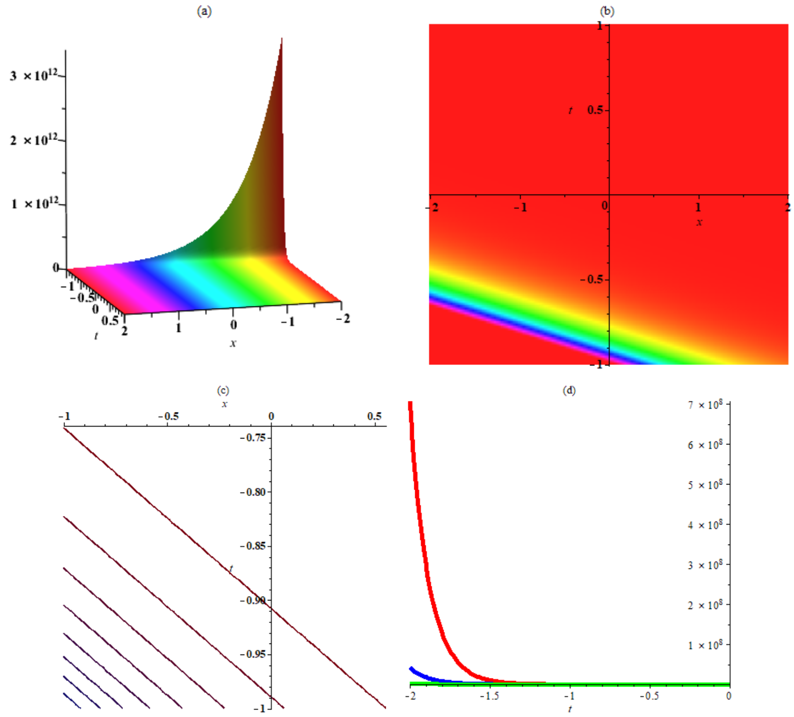

Figure 4. Plots of real ((**a**) (3D plot), (**b**) (spherical plot), (**c**) (contour plot), (**d**) (2D plot)) parts of solution (34) graph of ψ_1 for the parameter values $q_1 = 2, q_2 = 3, q_3 = 2, q_4 = 2, \lambda_2 = 2, w_3 = 1$.

4.1.21. Set XXI

$$a = 2/3 - 2/3\,q_4,\;\; b = 4/3\,\frac{q_2(-1+q_4)}{2\,q_1q_3+1},\;\; \lambda_0 = -\frac{8\,\lambda_2 w_3{}^2\left(q_4{}^2 - 2\,q_4 + 1\right)}{9\,q_3},\;\; \lambda_1 = -4/3\,\frac{(-1+q_4)w_3\lambda_2\sqrt{-2\,q_3}}{q_3}, \tag{106}$$

$$\psi(\eta) = -\frac{8\,\lambda_2 w_3{}^2\left(q_4{}^2 - 2\,q_4 + 1\right)}{9\,q_3} - 4/3\,\frac{(-1+q_4)w_3\lambda_2\sqrt{-2\,q_3}}{q_3}\tan\left(\frac{\phi(\eta)}{2}\right) + \lambda_2\tan^2\left(\frac{\phi(\eta)}{2}\right), \tag{107}$$

$$w_1 = 3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4},\;\; w_2 = w_3,\;\; \mu_1 = \mu_2 = 0,\;\; \eta = (2/3 - 2/3\,q_4)x + 4/3\,\frac{q_2(-1+q_4)}{2\,q_1q_3+1}t.$$

As a consequence (Product 10), the kink soliton solution is given by

$$\begin{aligned}\psi(\eta) = &-\frac{8\,\lambda_2 w_3{}^2\left(q_4{}^2 - 2\,q_4 + 1\right)}{9\,q_3} - 4/3\,\frac{(-1+q_4)w_3\lambda_2\sqrt{-2\,q_3}}{q_3}\left(\frac{e^{3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}\eta} - w_3}{3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}}\right) \\ &+\lambda_2\left(\frac{e^{3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}\eta} - w_3}{3/4\,\frac{\sqrt{-2\,q_3}}{-1+q_4}}\right)^2,\;\; \eta = (2/3 - 2/3\,q_4)x + 4/3\,\frac{q_2(-1+q_4)}{2\,q_1q_3+1}t.\end{aligned} \tag{108}$$

4.1.22. Set XXII

$$a = 1/3 - 1/3\,q_4,\;\; b = -2/3\,\frac{q_2(-1+q_4)}{q_1q_3-1},\;\; \lambda_0 = 4/9\,\frac{w_3{}^2\lambda_2\left(q_4{}^2 - 2\,q_4 + 1\right)}{q_3},\;\; \lambda_1 = 4/3\,\frac{w_3\lambda_2\sqrt{q_3}(-1+q_4)}{q_3}, \tag{109}$$

$$\psi(\eta) = 4/9\,\frac{{w_3}^2\lambda_2\left({q_4}^2 - 2\,q_4 + 1\right)}{q_3} + 4/3\,\frac{w_3\lambda_2\sqrt{q_3}(-1+q_4)}{q_3}\tan\left(\frac{\phi(\eta)}{2}\right) + \lambda_2\tan^2\left(\frac{\phi(\eta)}{2}\right), \tag{110}$$

$$w_1 = 3/2\,\frac{\sqrt{q_3}}{-1+q_4},\ w_2 = w_3,\ \mu_1 = \mu_2 = 0,\ \eta = (1/3 - 1/3\,q_4)x - 2/3\,\frac{q_2(-1+q_4)}{q_1q_3 - 1}t.$$

As a consequence (Product 10), the kink soliton solution is given by

$$\psi(\eta) = 4/9\,\frac{{w_3}^2\lambda_2\left({q_4}^2 - 2\,q_4 + 1\right)}{q_3} + 4/3\,\frac{w_3\lambda_2\sqrt{q_3}(-1+q_4)}{q_3}\left(\frac{e^{3/2\,\frac{\sqrt{q_3}}{-1+q_4}\eta} - w_3}{3/2\,\frac{\sqrt{q_3}}{-1+q_4}}\right) \tag{111}$$

$$+\lambda_2\left(\frac{e^{3/2\,\frac{\sqrt{q_3}}{-1+q_4}\eta} - w_3}{3/2\,\frac{\sqrt{q_3}}{-1+q_4}}\right)^2,\ \eta = (1/3 - 1/3\,q_4)x - 2/3\,\frac{q_2(-1+q_4)}{q_1q_3 - 1}t.$$

4.1.23. Set XXIII

$$\begin{cases} a = 1/3 - 1/3\,q_4,\ b = 2\,\frac{q_2(-1+q_4)}{q_1q_3+3},\ \lambda_0 = 4/9\,\frac{{w_3}^2\lambda_2(-1+q_4)^2(q_1q_3+3)+9\,q_1q_2q_3}{q_3(q_1q_3+3)},\ \lambda_1 = 4/3\,\frac{w_3\lambda_2\sqrt{q_3}(-1+q_4)}{q_3}, \\ \psi(\eta) = 4/9\,\frac{{w_3}^2\lambda_2(-1+q_4)^2(q_1q_3+3)+9\,q_1q_2q_3}{q_3(q_1q_3+3)} + 4/3\,\frac{w_3\lambda_2\sqrt{q_3}(-1+q_4)}{q_3}\tan\left(\frac{\phi(\eta)}{2}\right) + \lambda_2\tan^2\left(\frac{\phi(\eta)}{2}\right), \\ w_1 = 3/2\,\frac{\sqrt{q_3}}{-1+q_4},\ w_2 = w_3,\ \mu_1 = \mu_2 = 0,\ \eta = (1/3 - 1/3\,q_4)x + 2\,\frac{q_2(-1+q_4)}{q_1q_3+3}t. \end{cases} \tag{112}$$

As a consequence (Product 10), the kink soliton solution is obtained by

$$\psi(\eta) = 4/9\,\frac{{w_3}^2\lambda_2(-1+q_4)^2(q_1q_3+3)+9\,q_1q_2q_3}{q_3(q_1q_3+3)} + 4/3\,\frac{w_3\lambda_2\sqrt{q_3}(-1+q_4)}{q_3}\left(\frac{e^{3/2\,\frac{\sqrt{q_3}}{-1+q_4}\eta} - w_3}{3/2\,\frac{\sqrt{q_3}}{-1+q_4}}\right) \tag{113}$$

$$+\lambda_2\left(\frac{e^{3/2\,\frac{\sqrt{q_3}}{-1+q_4}\eta} - w_3}{3/2\,\frac{\sqrt{q_3}}{-1+q_4}}\right)^2,\ \eta = (1/3 - 1/3\,q_4)x + 2\,\frac{q_2(-1+q_4)}{q_1q_3+3}t.$$

4.1.24. Set XXIV

$$a = a,\ b = -2\,\frac{aq_2(2\,a + q_4 - 1)}{aq_1q_3 + 2\,a + q_4 - 1},\ \lambda_0 = 1/4\,\frac{{\lambda_1}^2}{\lambda_2},\ w_1 = \sqrt{-\frac{q_3}{8\,a^2 + 4\,aq_4 - 4\,a}}, \tag{114}$$

$$w_2 = w_3 = -1/16\,\frac{q_3\lambda_1}{\lambda_2\sqrt{-q_3a(2\,a + q_4 - 1)}},\ \psi(\eta) = 1/4\,\frac{{\lambda_1}^2}{\lambda_2} + \lambda_1\tan\left(\frac{\phi(\eta)}{2}\right) + \lambda_2\tan^2\left(\frac{\phi(\eta)}{2}\right), \tag{115}$$

$$\mu_1 = \mu_2 = 0,\ \eta = ax + 2\,\frac{q_2(-1+q_4)}{q_1q_3+3}t.$$

As a consequence (Product 10), the kink soliton solution is shown by

$$\psi(\eta) = \frac{{\lambda_1}^2}{4\lambda_2} + \lambda_1\left(\frac{e^{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}\eta} - w_3}{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}}\right) + \lambda_2\left(\frac{e^{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}\eta} - w_3}{\sqrt{-\frac{q_3}{8\,a^2+4\,aq_4-4\,a}}}\right)^2,\ \eta = ax + 2\,\frac{q_2(-1+q_4)}{q_1q_3+3}t. \tag{116}$$

4.1.25. Set XXV

$$a = a,\ b = \frac{a(a\lambda_0q_3 + q_3\lambda_0q_4 - 4\,aq_2 - q_3\lambda_0 - 2\,q_2q_4 + 2\,q_2)}{aq_1q_3 + 2\,a + q_4 - 1},\ \lambda_0 = \lambda_0,\ w_1 = \sqrt{-\frac{q_3}{8\,a^2 + 4\,aq_4 - 4\,a}}, \tag{117}$$

$$\lambda_2 = -1/4\,\frac{(a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0)\lambda_0(2\,a + q_4 - 1)}{(a q_1 q_3 q_4 - a q_1 q_3 + 2\,a q_4 + q_4{}^2 - 2\,a - 2\,q_4 + 1)\mu_2},$$

$$\psi(\eta) = \lambda_0 - 1/4\,\frac{(a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0)\lambda_0(2\,a + q_4 - 1)}{(a q_1 q_3 q_4 - a q_1 q_3 + 2\,a q_4 + q_4{}^2 - 2\,a - 2\,q_4 + 1)\mu_2}\tan^2\left(\frac{\phi(\eta)}{2}\right) + \mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right), \tag{118}$$

$$\lambda_1 = \mu_1 = w_2 = w_3 = 0,\ \ \eta = ax + \frac{a(a\lambda_0 q_3 + q_3\lambda_0 q_4 - 4\,a q_2 - q_3\lambda_0 - 2\,q_2 q_4 + 2\,q_2)}{a q_1 q_3 + 2\,a + q_4 - 1}t.$$

As a consequence (Product 9), the kink soliton solution is discussed by

$$\psi(\eta) = \lambda_0 - 1/4\,\frac{(a\lambda_0 q_1 q_3 - 4\,a q_1 q_2 - \lambda_0 q_4 + \lambda_0)\lambda_0(2\,a + q_4 - 1)}{(a q_1 q_3 q_4 - a q_1 q_3 + 2\,a q_4 + q_4{}^2 - 2\,a - 2\,q_4 + 1)\mu_2}\tan^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1} + 1}, -\frac{e^{2\,\eta\,w_1} - 1}{e^{2\,\eta\,w_1} + 1}\right)\right) \tag{119}$$

$$+\mu_2\cot^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1} + 1}, -\frac{e^{2\,\eta\,w_1} - 1}{e^{2\,\eta\,w_1} + 1}\right)\right),$$

$$w_1 = \sqrt{-\frac{q_3}{8\,a^2 + 4\,a q_4 - 4\,a}},\ \ \eta = ax + \frac{a(a\lambda_0 q_3 + q_3\lambda_0 q_4 - 4\,a q_2 - q_3\lambda_0 - 2\,q_2 q_4 + 2\,q_2)}{a q_1 q_3 + 2\,a + q_4 - 1}t.$$

4.1.26. Set XXVI

$$a = -\frac{-1 + q_4}{q_1 q_3 + 2},\ \ b = b,\ \ \lambda_0 = 2\,\frac{q_1 q_2}{q_1 q_3 + 1},\ \ w_1 = 1/2\,\frac{q_1 q_3 + 2}{\sqrt{q_1}(-1 + q_4)}, \tag{120}$$

$$\lambda_2 = -\frac{q_2 q_1{}^2(b q_1 q_3 - q_2 q_4 + b + q_2)}{(q_1{}^2 q_3{}^2 + 2\,q_1 q_3 + 1)\mu_2(-1 + q_4)},\ \ w_2 = w_3 = 0,$$

$$\psi(\eta) = 2\,\frac{q_1 q_2}{q_1 q_3 + 1} - \frac{q_2 q_1{}^2(b q_1 q_3 - q_2 q_4 + b + q_2)}{(q_1{}^2 q_3{}^2 + 2\,q_1 q_3 + 1)\mu_2(-1 + q_4)}\tan^2\left(\frac{\phi(\eta)}{2}\right) + \mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right),\ \ \eta = -\frac{-1 + q_4}{q_1 q_3 + 2}x + bt. \tag{121}$$

As a consequence (Product 9), the kink soliton solution is presented by

$$\psi(\eta) = 2\,\frac{q_1 q_2}{q_1 q_3 + 1} - \frac{q_2 q_1{}^2(b q_1 q_3 - q_2 q_4 + b + q_2)}{(q_1{}^2 q_3{}^2 + 2\,q_1 q_3 + 1)\mu_2(-1 + q_4)}\tan^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1} + 1}, -\frac{e^{2\,\eta\,w_1} - 1}{e^{2\,\eta\,w_1} + 1}\right)\right) \tag{122}$$

$$+\mu_2\cot^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1} + 1}, -\frac{e^{2\,\eta\,w_1} - 1}{e^{2\,\eta\,w_1} + 1}\right)\right),\ \ \eta = -\frac{-1 + q_4}{q_1 q_3 + 2}x + bt.$$

4.1.27. Set XXVII

$$a = 1/3 - 1/3\,q_4,\ \ b = b,\ \ \lambda_0 = 1/2\,\frac{3\,b q_1 q_3 + 2\,q_2 q_4 - 3\,b - 2\,q_2}{q_3(-1 + q_4)},\ \ w_1 = 3/2\,\frac{\sqrt{q_3}}{-1 + q_4}, \tag{123}$$

$$\lambda_2 = -1/16\,\frac{(3\,b q_1 q_3 + 2\,q_2 q_4 - 3\,b - 2\,q_2)(b q_1 q_3 - 2\,q_2 q_4 + 3\,b + 2\,q_2)}{q_3{}^2\mu_2(q_4{}^2 - 2\,q_4 + 1)},\ \ \lambda_1 = \mu_1 = w_2 = w_3 = 0,$$

$$\psi(\eta) = \frac{1}{2}\frac{3b q_1 q_3 + 2q_2 q_4 - 3b - 2q_2}{q_3(-1 + q_4)} - \frac{1}{16}\frac{(3b q_1 q_3 + 2q_2 q_4 - 3b - 2q_2)(b q_1 q_3 - 2\,q_2 q_4 + 3b + 2\,q_2)}{q_3{}^2\mu_2(q_4{}^2 - 2q_4 + 1)}\tan^2\left(\frac{\phi(\eta)}{2}\right) \tag{124}$$

$$+\mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right),\ \ \eta = (1/3 - 1/3\,q_4)x + bt,$$

$$\phi(\eta) = \arctan\left(2\,\frac{e^{3/2\,\frac{\sqrt{q_3}}{-1+q_4}\eta}}{e^{3\frac{\sqrt{q_3}}{-1+q_4}\eta} + 1}, -\frac{e^{2\,\eta\,w_1} - 1}{e^{3\frac{\sqrt{q_3}}{-1+q_4}\eta} + 1}\right).$$

As a consequence (Product 9), the kink soliton solution is shown by

$$\psi(\eta) = 2\,\frac{q_1q_2}{q_1q_3+1} - \frac{q_2{q_1}^2(bq_1q_3 - q_2q_4 + b + q_2)}{({q_1}^2{q_3}^2 + 2\,q_1q_3 + 1)\mu_2(-1+q_4)}\tan^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1}, -\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right) \tag{125}$$
$$+\mu_2\cot^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1}, -\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right),\;\; \eta = -\frac{-1+q_4}{q_1q_3+2}x + bt.$$

4.1.28. Set XXVIII

$$a = 1/4 - 1/4\,q_4,\;\; b = 1/2\,\frac{q_2(-1+q_4)}{q_1q_3+1},\;\; \lambda_0 = 2\,\frac{q_1q_2}{q_1q_3+1},\;\; \lambda_1 = 1/4\,\frac{q_1q_2s_2}{(q_1q_3+1)q_3}\sqrt{-2\,\frac{s_1}{q_3}}, \tag{126}$$
$$\mu_1 = \frac{q_1q_2}{q_1q_3+1}\sqrt{-1/2\,\frac{s_1}{q_3}},\;\; w_2 = \frac{\sqrt{{w_3}^2{q_4}^2 - 2\,{w_3}^2q_4 + {w_3}^2 - 4\,q_3}}{-1+q_4},\;\; \lambda_2 = \mu_2 = w_1 = 0,$$
$$s_1 = {w_3}^2{q_4}^2 + \sqrt{{w_3}^2{q_4}^2 - 2\,{w_3}^2q_4 + {w_3}^2 - 4\,q_3}\,w_3(-1+q_4) - 2\,{w_3}^2q_4 + {w_3}^2 - 2\,q_3,$$
$$s_2 = -{w_3}^2{q_4}^2 + \sqrt{{w_3}^2{q_4}^2 - 2\,{w_3}^2q_4 + {w_3}^2 - 4\,q_3}\,w_3(-1+q_4) + 2\,{w_3}^2q_4 - {w_3}^2 + 2\,q_3,$$

$$\psi(\eta) = 2\,\frac{q_1q_2}{q_1q_3+1} + 1/4\,\frac{q_1q_2s_2}{(q_1q_3+1)q_3}\sqrt{-2\,\frac{s_1}{q_3}}\tan\left(\frac{\phi(\eta)}{2}\right) + \frac{q_1q_2}{q_1q_3+1}\sqrt{-1/2\,\frac{s_1}{q_3}}\cot\left(\frac{\phi(\eta)}{2}\right), \tag{127}$$
$$\eta = (1/4 - 1/4\,q_4)x + 1/2\,\frac{q_2(-1+q_4)}{q_1q_3+1}t.$$

As a consequence (Product 2), the soliton solution is discussed by

$$\psi(\eta) = 2\,\frac{q_1q_2}{q_1q_3+1} + 1/4\,\frac{q_1q_2s_2}{(q_1q_3+1)q_3}\sqrt{-2\,\frac{s_1}{q_3}}\,\frac{\sqrt{w_2^2 - w_3^2}}{w_2 - w_3}\tanh\left(\frac{\sqrt{w_2^2 - w_3^2}}{2}\eta\right) \tag{128}$$
$$+\frac{q_1q_2}{q_1q_3+1}\sqrt{-1/2\,\frac{s_1}{q_3}}\,\frac{w_2 - w_3}{\sqrt{w_2^2 - w_3^2}}\coth\left(\frac{\sqrt{w_2^2 - w_3^2}}{2}\eta\right),\;\; \eta = (1/4 - 1/4\,q_4)x + 1/2\,\frac{q_2(-1+q_4)}{q_1q_3+1}t.$$

4.1.29. Set XXIX

$$b = \frac{-6(-1+q_4)q_2}{4\,q_1(-1+q_4)^2({w_2}^2 - {w_3}^2) - 9},\;\; a = \frac{1-q_4}{3}, \tag{129}$$
$$\lambda_0 = \frac{108q_1q_2({w_2}^2 - {w_3}^2)(-1+q_4)^2}{\left(-4\,(-1+q_4)^2({w_2}^2 - {w_3}^2) + 9\,q_3\right)\left(4\,q_1(-1+q_4)^2({w_2}^2 - {w_3}^2) - 9\right)},$$
$$\lambda_2 = \frac{-54q_2q_1(w_2 - w_3)^2(-1+q_4)^2}{\left(-4\,(-1+q_4)^2({w_2}^2 - {w_3}^2) + 9\,q_3\right)\left(4\,q_1(-1+q_4)^2({w_2}^2 - {w_3}^2) - 9\right)},\;\; \lambda_1 = \mu_1 = w_1 = 0,$$
$$\mu_2 = \frac{-54q_1q_2(w_2 + w_3)^2(-1+q_4)^2}{\left(-4\,(-1+q_4)^2({w_2}^2 - {w_3}^2) + 9\,q_3\right)\left(4\,q_1(-1+q_4)^2({w_2}^2 - {w_3}^2) - 9\right)},$$

$$\psi(\eta) = \frac{108q_1q_2({w_2}^2 - {w_3}^2)(-1+q_4)^2}{\left(-4\,(-1+q_4)^2({w_2}^2 - {w_3}^2) + 9\,q_3\right)\left(4\,q_1(-1+q_4)^2({w_2}^2 - {w_3}^2) - 9\right)} \tag{130}$$
$$-\frac{54\,q_2q_1(w_2 - w_3)^2(-1+q_4)^2}{\left(-4\,(-1+q_4)^2({w_2}^2 - {w_3}^2) + 9\,q_3\right)\left(4\,q_1(-1+q_4)^2({w_2}^2 - {w_3}^2) - 9\right)}\tan^2\left(\frac{\phi(\eta)}{2}\right)$$

$$-\frac{54\,q_1q_2(w_2+w_3)^2(-1+q_4)^2}{\left(-4\,(-1+q_4)^2({w_2}^2-{w_3}^2)+9\,q_3\right)\left(4\,q_1(-1+q_4)^2({w_2}^2-{w_3}^2)-9\right)}\cot^2\left(\frac{\phi(\eta)}{2}\right),$$

$$\eta=\frac{1-q_4}{3}x+1/2\,\frac{q_2(-1+q_4)}{q_1q_3+1}t,\;\;\tan(\phi(\eta))=\frac{\sqrt{w_2^2-w_3^2}}{w_2-w_3}\tanh\left(\frac{\sqrt{w_2^2-w_3^2}}{2}\eta\right).$$

4.1.30. Set XXX

$$a=-\frac{-1+q_4}{q_1q_3+2},\;\;b=\frac{q_2(-1+q_4)}{q_1q_3+1},\;\;\lambda_0=1/4\,\frac{{\lambda_1}^2q_1q_3+8\,\lambda_2q_2q_1+{\lambda_1}^2}{(q_1q_3+1)\lambda_2},\;\;w_1=1/2\,\frac{q_1q_3+2}{(-1+q_4)\sqrt{q_1}},\tag{131}$$

$$w_2=1/4\,\frac{\lambda_1(q_1q_3+2)\sqrt{q_1}}{q_1(-1+q_4)\lambda_2},\;\;\mu_1=\mu_2=0,\;\;w_3=1/4\,\frac{\lambda_1(q_1q_3+2)\sqrt{q_1}}{q_1(-1+q_4)\lambda_2},$$

$$\psi(\eta)=1/4\,\frac{{\lambda_1}^2q_1q_3+8\,\lambda_2q_2q_1+{\lambda_1}^2}{(q_1q_3+1)\lambda_2}+\lambda_1\tan\left(\frac{\phi(\eta)}{2}\right)+\lambda_2\tan^2\left(\frac{\phi(\eta)}{2}\right),\tag{132}$$

$$\eta=-\frac{-1+q_4}{q_1q_3+2}x+\frac{q_2(-1+q_4)}{q_1q_3+1}t,\;\;\phi(\eta)=2\arctan\left(\frac{\mathrm{e}^{\eta\,w_1}-w_3}{w_1}\right).$$

4.1.31. Set XXXI

$$a=-\frac{-1+q_4}{q_1q_3+2},\;\;b=\frac{q_2(-1+q_4)}{q_1q_3+1},\;\;\lambda_0=1/4\,\frac{{\mu_1}^2q_1q_3+8\,\mu_2q_2q_1+{\mu_1}^2}{(q_1q_3+1)\mu_2},\;\;w_1=\frac{q_1q_3+2}{2(-1+q_4)\sqrt{q_1}},\tag{133}$$

$$w_2=-1/4\,\frac{\mu_1(q_1q_3+2)\sqrt{q_1}}{q_1\mu_2(-1+q_4)},\;\;\lambda_1=\lambda_2=0,\;\;w_3=1/4\,\frac{\mu_1(q_1q_3+2)\sqrt{q_1}}{q_1\mu_2(-1+q_4)},$$

$$\psi(\eta)=1/4\,\frac{{\mu_1}^2q_1q_3+8\,\mu_2q_2q_1+{\mu_1}^2}{(q_1q_3+1)\mu_2}+\lambda_1\tan\left(\frac{\phi(\eta)}{2}\right)+\lambda_2\tan^2\left(\frac{\phi(\eta)}{2}\right),\tag{134}$$

$$\eta=-\frac{-1+q_4}{q_1q_3+2}x+\frac{q_2(-1+q_4)}{q_1q_3+1}t,\;\;\;\phi(\eta)=2\arctan\left[\frac{\frac{q_1q_3+2}{2(-1+q_4)\sqrt{q_1}}e^{\frac{q_1q_3+2}{2(-1+q_4)\sqrt{q_1}}\eta}}{1-w_3e^{\frac{q_1q_3+2}{2(-1+q_4)\sqrt{q_1}}\eta}}\right].$$

4.1.32. Set XXXII

$$a=2/3-2/3\,q_4,\;\;b=8/3\,\frac{q_2(-1+q_4)}{q_1q_3+2},\;\;\lambda_0=-6\,\mu_2,\;\;\lambda_1=-4\,\sqrt{-\frac{{\mu_2}^2q_3q_1+\mu_2q_2q_1+2\,{\mu_2}^2}{q_1q_3+2}},\tag{135}$$

$$\lambda_2=\mu_2,\;\;w_2=w_3=0,\;\;\mu_1=4\,\sqrt{-\frac{{\mu_2}^2q_3q_1+\mu_2q_2q_1+2\,{\mu_2}^2}{q_1q_3+2}},\;\;w_1=\frac{3}{4}\,\frac{\sqrt{-2\,q_3}}{-1+q_4},$$

$$\psi(\eta)=-6\,\mu_2-4\,\sqrt{-\frac{{\mu_2}^2q_3q_1+\mu_2q_2q_1+2\,{\mu_2}^2}{q_1q_3+2}}\tan\left(\frac{\phi(\eta)}{2}\right)+\mu_2\tan^2\left(\frac{\phi(\eta)}{2}\right)\tag{136}$$

$$+4\,\sqrt{-\frac{{\mu_2}^2q_3q_1+\mu_2q_2q_1+2\,{\mu_2}^2}{q_1q_3+2}}\cot\left(\frac{\phi(\eta)}{2}\right)+\mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right),\;\;\eta=2/3-2/3\,q_4x+8/3\,\frac{q_2(-1+q_4)}{q_1q_3+2}t.$$

As a consequence (Product 9), the kink soliton solution is presented by

$$\psi_1(x,t)=-6\,\mu_2-4\,\sqrt{-\frac{{\mu_2}^2q_3q_1+\mu_2q_2q_1+2\,{\mu_2}^2}{q_1q_3+2}}\tan\left(\arctan\left(2\,\frac{\mathrm{e}^{\eta\,w_1}}{\mathrm{e}^{2\,\eta\,w_1}+1},-\frac{\mathrm{e}^{2\,\eta\,w_1}-1}{\mathrm{e}^{2\,\eta\,w_1}+1}\right)\right)\tag{137}$$

$$+\mu_2\tan^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1},-\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right)$$
$$+4\,\sqrt{-\frac{{\mu_2}^2q_3q_1+\mu_2q_2q_1+2\,{\mu_2}^2}{q_1q_3+2}}\cot\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1},-\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right)$$
$$+\mu_2\cot^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1},-\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right),\ w_1=\frac{3}{4}\,\frac{\sqrt{-2\,q_3}}{-1+q_4},\ \eta=2/3-2/3\,q_4x+8/3\,\frac{q_2(-1+q_4)}{q_1q_3+2}t.$$

4.1.33. Set XXXIII

$$\begin{cases} a=\frac{2-2q_4}{3},\ \ b=\frac{8}{3}\frac{q_2(-1+q_4)}{q_1q_3+2},\ \ \lambda_0=3/8\,\frac{{\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2}{\mu_2(q_1q_3+2)},\ \ \lambda_1=\frac{1}{16}\frac{\mu_1\left({\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2\right)}{{\mu_2}^2(q_1q_3+2)},\\ \lambda_2=\frac{\left({\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2\right)^2}{256\,(q_1q_3+2)^2{\mu_2}^3},\ \ w_2=w_3=0,\ \ w_1=3/2\,\frac{\sqrt{-q_3}}{\sqrt{2}(-1+q_4)},\\ \psi(\eta)=3/8\,\frac{{\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2}{\mu_2(q_1q_3+2)}+1/16\,\frac{\mu_1\left({\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2\right)}{{\mu_2}^2(q_1q_3+2)}\tan\left(\frac{\phi(\eta)}{2}\right)+\\ \frac{\left({\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2\right)^2}{256\,(q_1q_3+2)^2{\mu_2}^3}\tan^2\left(\frac{\phi(\eta)}{2}\right)+\mu_1\cot\left(\frac{\phi(\eta)}{2}\right)+\mu_2\cot^2\left(\frac{\phi(\eta)}{2}\right),\\ \eta=\frac{2-2q_4}{3}x+\frac{8}{3}\frac{q_2(-1+q_4)}{q_1q_3+2}t. \end{cases} \tag{138}$$

As a consequence (Product 9), the kink soliton solution is shown by

$$\begin{cases} \psi_1(x,t)=\frac{3}{8}\frac{{\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2}{\mu_2(q_1q_3+2)}+\frac{\mu_1\left({\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2\right)}{16{\mu_2}^2(q_1q_3+2)}\tan\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1},-\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right)\\ +\frac{\left({\mu_1}^2q_1q_3+16\,\mu_2q_1q_2+2\,{\mu_1}^2\right)^2}{256\,(q_1q_3+2)^2{\mu_2}^3}\tan^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1},-\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right)\\ +\mu_1\cot\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1},-\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right)\\ +\mu_2\cot^2\left(\arctan\left(2\,\frac{e^{\eta\,w_1}}{e^{2\,\eta\,w_1}+1},-\frac{e^{2\,\eta\,w_1}-1}{e^{2\,\eta\,w_1}+1}\right)\right),\\ w_1=\frac{3}{4}\,\frac{\sqrt{-2\,q_3}}{-1+q_4},\ \eta=(2/3-2/3\,q_4)x+8/3\,\frac{q_2(-1+q_4)}{q_1q_3+2}t. \end{cases} \tag{139}$$

5. He's Variational Direct Technique

The fundamental steps will be considered:

Step 1. Let a nonlinear PDE be stated in the general form

$$\mathcal{W}(x,t,\psi,\psi_x,\psi_t,\psi_{xx},\psi_{tt},\dots)=0. \tag{140}$$

We commence the below transformations:

$$\psi(x,t)=\mathfrak{z}(\xi),\quad \xi=ax+bt, \tag{141}$$

where a and b are values that will be determined.

Step 2. Engaging the reduction (141), one becomes

$$\mathfrak{H}(\mathfrak{z},\mathfrak{z}',\mathfrak{z}'',\mathfrak{z}''',\mathfrak{z}'''',\dots)=0, \tag{142}$$

where $\mathfrak{z}'=\frac{d\mathfrak{z}}{d\xi},\mathfrak{z}''=\frac{d^2\mathfrak{z}}{d\xi^2},\mathfrak{z}'''=\frac{d^3\mathfrak{z}}{d\xi^3},\mathfrak{z}''''=\frac{d^4\mathfrak{z}}{d\xi^4}$ and so on.

Step 3. By utilizing the variational technique to Equation (142), the following functional can be presented as

$$J(\mathfrak{z})=\int\mathfrak{L}(\mathfrak{z},\mathfrak{z}',\mathfrak{z}'',\mathfrak{z}''',\mathfrak{z}'''',\dots)\,d\xi. \tag{143}$$

Step 4. The solution of Equation (142) can be reached in the following different forms:

Set I: $\mathfrak{z}(\xi)=\delta\,sec(\mu\,\xi)$.

$$\text{Set II: } \mathfrak{Z}(\xi) = \delta\,\frac{\cosh(\mu\,\xi)}{1+\cosh(\mu\,\xi)}.$$

$$\text{Set III: } \mathfrak{Z}(\xi) = \delta\,sech^2(\mu\,\xi).$$

$$\text{Set IV: } \mathfrak{Z}(\xi) = \delta\,\cos(\mu\,\xi).$$

$$\text{Set V: } \mathfrak{Z}(\xi) = \delta\,\frac{sec(\mu\,\xi)}{1+sec(\mu\,\xi)}.$$

Step 5. Employing the Ritz-like technique with the stationary requirement, we reach:

$$\frac{\partial J(\mathfrak{Z})}{\partial \delta} = 0, \qquad \frac{\partial J(\mathfrak{Z})}{\partial \mu} = 0. \tag{144}$$

5.1. *Performance of He's VDT for a GP Equation*

Substituting Equation (141) into Equation (1), the following ODE can be received:

$$-\frac{a\mathfrak{Z}(\xi)^2 q_3}{2} + 2q_2 a\mathfrak{Z}(\xi) + b\mathfrak{Z}(\xi) - \frac{a^3\left(\frac{d}{d\xi}\mathfrak{Z}(\xi)\right)^2 (q_4-1)}{2} - (\mathfrak{Z}(\xi)a + bq_1)\left(\frac{d^2}{d\xi^2}\mathfrak{Z}(\xi)\right)a^2 = 0, \tag{145}$$

where a, b are free values. The functional form is read as

$$J(\mathfrak{Z}) = \int\int\Bigg\{-\frac{a\mathfrak{Z}(\xi)^2 q_3}{2} + 2q_2 a\mathfrak{Z}(\xi) + b\mathfrak{Z}(\xi) - \frac{a^3\left(\frac{d}{d\xi}\mathfrak{Z}(\xi)\right)^2 (q_4-1)}{2} - (\mathfrak{Z}(\xi)a + bq_1)\left(\frac{d^2}{d\xi^2}\mathfrak{Z}(\xi)\right)a^2\Bigg\}\frac{d}{d\xi}\mathfrak{Z}(\xi)d\xi\,d\xi. \tag{146}$$

5.1.1. Periodic Wave Solution

Collection 1. *The first set of solutions for Equation (145) is considered as*

$$\mathfrak{Z}(\xi) = \delta\,sec(\mu\,\xi). \tag{147}$$

Putting Equation (147) into Equation (146) results in

$$J(\delta,\mu) = \int_0^{T/8} (\bullet)d\xi = -\frac{2\left(20a^3\delta\,\mu^2 q_4 + 236a^3\delta\,\mu^2 + 168a^2 b\,\mu^2 q_1 + 49\delta a q_3 - 210q_2 a - 105b\right)\delta^2}{315\mu}. \tag{148}$$

Using the following derivatives, we conclude:

$$\frac{\partial J}{\partial \delta} = 0, \qquad \frac{\partial J}{\partial \mu} = 0. \tag{149}$$

So we have:

$$\frac{\partial J}{\partial \delta} = -\frac{2\delta\left(20a^3\delta\,\mu^2 q_4 + 236a^3\delta\,\mu^2 + 112a^2 b\,\mu^2 q_1 + 49\delta a q_3 - 140q_2 a - 70b\right)}{105\mu} = 0, \tag{150}$$

$$\frac{\partial J}{\partial \mu} = -\frac{2\delta^2\left(20a^3\delta\,\mu^2 q_4 + 236a^3\delta\,\mu^2 + 168a^2 b\,\mu^2 q_1 - 49\delta a q_3 + 210q_2 a + 105b\right)}{315\mu^2} = 0, \tag{151}$$

which leads to:

$$\mu = \frac{\sqrt{2}\sqrt{B_3\left(-B_2 + \sqrt{-4B_1B_3 + B_2^2}\right)}}{2aB_3}, \quad \delta = -\frac{14\left(8b\,a^2\mu^2 q_1 - 10q_2 a - 5b\right)}{a(20a^2\mu^2 q_4 + 236a^2\mu^2 + 49q_3)}, \tag{152}$$

$$B_1 = 490aq_2q_3 + 245bq_3, \quad B_2 = 1000aq_2q_4 + 1960bq_1q_3 + 11800q_2a + 500bq_4 + 5900b,$$
$$B_3 = 160bq_1q_4 + 1888bq_1.$$

The final solution is mentioned as

$$\psi(x,t) = -\frac{14\left(8b\,a^2\mu^2q_1 - 10q_2a - 5b\right)}{a(20a^2\mu^2q_4 + 236a^2\mu^2 + 49q_3)} sec\left(\frac{\sqrt{2}\sqrt{B_3\left(-B_2 + \sqrt{-4B_1B_3 + B_2^2}\right)}}{2aB_3}(ax+bt)\right), \tag{153}$$

provided that $B_2^2 - 4B_1B_3 > 0$ *and* $B_3(-B_2 + \sqrt{B_2^2 - 4B_1B_3}) > 0$.

5.1.2. Soliton Solution

Collection 2. *The second set of solutions for Equation (145) is considered as*

$$\mathfrak{z}(\xi) = \delta\,\frac{cosh^2(\mu\,\xi)}{1 + cosh^2(\mu\,\xi)}. \tag{154}$$

Plugging Equation (154) into Equation (146), one becomes

$$J(\delta,\mu) = \int_0^\infty (\bullet)d\xi = \frac{\delta^2\left(5\delta\,a^3\mu^2q_4 - 25\delta\,a^3\mu^2 - 21b\,a^2\mu^2q_1 + 224a\delta q_3 - 1050q_2a - 525b\right)}{630\mu}. \tag{155}$$

With the following derivatives:

$$\frac{\partial J}{\partial \delta} = \frac{\delta\left(5\delta\,a^3\mu^2q_4 - 25\delta\,a^3\mu^2 - 21b\,a^2\mu^2q_1 + 224a\delta q_3 - 1050q_2a - 525b\right)}{315\mu} + \frac{\delta^2\left(5a^3\mu^2q_4 - 25a^3\mu^2 + 224aq_3\right)}{630\mu}, \tag{156}$$
$$\frac{\partial J}{\partial \mu} = \frac{\delta^2\left(10a^3\delta\mu q_4 - 50a^3\delta\mu - 42a^2b\mu q_1\right)}{630\mu} - \frac{\delta^2\left(5\delta\,a^3\mu^2q_4 - 25\delta\,a^3\mu^2 - 21b\,a^2\mu^2q_1 + 224a\delta q_3 - 1050q_2a - 525b\right)}{630\mu^2}.$$

From which we have

$$\mu = \frac{\sqrt{2}\sqrt{B_3\left(-B_2 + \sqrt{-4B_1B_3 + B_2^2}\right)}}{2aB_3}, \quad \delta = \frac{14b\,a^2\mu^2q_1 + 700q_2a + 350b}{a(5a^2\mu^2q_4 - 25a^2\mu^2 + 224q_3)}, \tag{157}$$

$$B_1 = -2240aq_2q_3 - 1120bq_3, \quad B_2 = -250aq_2q_4 + 224bq_1q_3 + 1250q_2a - 125bq_4 + 625b, \quad B_3 = bq_1q_4 - 5bq_1.$$

The last solution of Equation (1) is mentioned as

$$\psi(x,t) = \frac{14b\,a^2\mu^2q_1 + 700q_2a + 350b}{a(5a^2\mu^2q_4 - 25a^2\mu^2 + 224q_3)}\,\frac{\cosh\left(\frac{\sqrt{2}\sqrt{B_3\left(-B_2+\sqrt{-4B_1B_3+B_2^2}\right)}}{2aB_3}(ax+bt)\right)}{1+\cosh\left(\frac{\sqrt{2}\sqrt{B_3\left(-B_2+\sqrt{-4B_1B_3+B_2^2}\right)}}{2aB_3}(ax+bt)\right)}, \tag{158}$$

provided that $B_3 \neq 0$, $B_2^2 - 4B_1B_3 > 0$ *and* $B_3(-B_2 + \sqrt{B_2^2 - 4B_1B_3}) > 0$.

5.1.3. Bright Soliton Solution

Collection 3. *The third set of solutions of Equation (145) is considered as*

$$\mathfrak{z}(\xi) = \delta\,sech^2(\mu\,\xi). \tag{159}$$

Plugging Equation (159) into Equation (146), one becomes

$$J(\delta,\mu) = \int_0^{\infty}(\bullet)d\xi = -\frac{8\delta^2\left(\delta\,\mu^2\left(q_4 - \frac{4}{5}\right)a^3 + \frac{21b\,a^2\mu^2 q_1}{10} + \left(\frac{7\delta q_3}{10} - \frac{21 q_2}{4}\right)a - \frac{21b}{8}\right)}{63\mu}. \tag{160}$$

With the following derivatives:

$$\frac{\partial J}{\partial \delta} = -\frac{16\delta\left(\delta\,\mu^2\left(q_4 - \frac{4}{5}\right)a^3 + \frac{21b\,a^2\mu^2 q_1}{10} + \left(\frac{7\delta q_3}{10} - \frac{21 q_2}{4}\right)a - \frac{21b}{8}\right)}{63\mu} - \frac{8\delta^2\left(\mu^2\left(q_4 - \frac{4}{5}\right)a^3 + \frac{7aq_3}{10}\right)}{63\mu}, \tag{161}$$

$$\frac{\partial J}{\partial \mu} = \frac{8\delta^2\left(\delta\,\mu^2\left(q_4 - \frac{4}{5}\right)a^3 + \frac{21b\,a^2\mu^2 q_1}{10} + \left(\frac{7\delta q_3}{10} - \frac{21 q_2}{4}\right)a - \frac{21b}{8}\right)}{63\mu^2} - \frac{8\delta^2\left(2\delta\mu\left(q_4 - \frac{4}{5}\right)a^3 + \frac{21a^2 b\mu q_1}{5}\right)}{63\mu}.$$

From which, we have

$$\mu = \frac{\sqrt{2}\sqrt{B_3\left(-B_2 + \sqrt{-4B_1B_3 + B_2^2}\right)}}{2aB_3}, \quad \delta = -\frac{7(4b\,a^2\mu^2 q_1 - 10q_2 a - 5b)}{2a(10a^2\mu^2 q_4 - 8a^2\mu^2 + 7q_3)}, \tag{162}$$

$$B_1 = 70aq_2q_3 + 35bq_3, \quad B_2 = 500aq_2q_4 + 140bq_1q_3 - 400q_2a + 250bq_4 - 200b, \quad B_3 = 40bq_1q_4 - 32bq_1.$$

The last solution of Equation (1) is mentioned as

$$\psi(x,t) = -\frac{7(4b\,a^2\mu^2 q_1 - 10q_2 a - 5b)}{2a(10a^2\mu^2 q_4 - 8a^2\mu^2 + 7q_3)} sech^2\left(\frac{\sqrt{2}\sqrt{B_3\left(-B_2 + \sqrt{-4B_1B_3 + B_2^2}\right)}}{2aB_3}(ax + bt)\right), \tag{163}$$

provided that $B_3 \neq 0$, $B_2^2 - 4B_1B_3 > 0$ *and* $B_3(-B_2 + \sqrt{B_2^2 - 4B_1B_3}) > 0$.

5.1.4. Periodic Wave form Solution

Collection 4. *The fourth set of solutions of Equation (145) is considered as*

$$\mathfrak{Z}(\xi) = \delta\cos(\mu\,\xi). \tag{164}$$

Plugging Equation (164) into Equation (146), one becomes

$$J(\delta,\mu) = \int_0^{T/4}(\bullet)d\xi = \frac{\delta^2\left(-7\delta\left(\mu^2\left(q_4 - \frac{11}{7}\right)a^2 + \frac{2q_3}{7}\right)a + \frac{9\pi\left(b\,a^2\mu^2 q_1 + 2q_2 a + b\right)}{4}\right)}{18\mu}. \tag{165}$$

With the following derivatives:

$$\frac{\partial J}{\partial \delta} = \frac{\delta\left(-7\delta\left(\mu^2\left(q_4 - \frac{11}{7}\right)a^2 + \frac{2q_3}{7}\right)a + \frac{9\pi\left(b\,a^2\mu^2 q_1 + 2q_2 a + b\right)}{4}\right)}{9\mu} - \frac{7\delta^2\left(\mu^2\left(q_4 - \frac{11}{7}\right)a^2 + \frac{2q_3}{7}\right)a}{18\mu}, \tag{166}$$

$$\frac{\partial J}{\partial \mu} = -\frac{\delta^2\left(-7\delta\left(\mu^2\left(q_4 - \frac{11}{7}\right)a^2 + \frac{2q_3}{7}\right)a + \frac{9\pi\left(b\,a^2\mu^2 q_1 + 2q_2 a + b\right)}{4}\right)}{18\mu^2} + \frac{\delta^2\left(-14\delta\mu\left(q_4 - \frac{11}{7}\right)a^3 + \frac{9\pi\,a^2 b\mu q_1}{2}\right)}{18\mu}.$$

From which, we have

$$\mu = \frac{\sqrt{2}\sqrt{bq_1(7q_4 - 11)(B_1 + \sqrt{B_2})}}{2bq_1(7q_4 - 11)a}, \quad \delta = \frac{3\pi(b\,a^2\mu^2 q_1 + 2q_2 a + b)}{2a(7a^2\mu^2 q_4 - 11a^2\mu^2 + 2q_3)}, \tag{167}$$

$$B_1 = 70aq_2q_4 - 10bq_1q_3 - 110q_2a + 35bq_4 - 55b,$$

$$B_2 = 100a^2q_2^2(7q_4-11)^2 - 4abq_2(7q_4-11)(46q_1q_3-175q_4+275) + \Big(100q_3^2q_1^2 - 644q_1q_3q_4 + 1012q_1q_3 + 1225q_4^2 - 3850q_4\Big)b^2.$$

The last solution of Equation (1) is mentioned as

$$\psi(x,t) = \frac{3\pi(b\,a^2\mu^2q_1+2q_2a+b)}{2a(7a^2\mu^2q_4-11a^2\mu^2+2q_3)}\cos\left(\frac{\sqrt{2}\sqrt{bq_1(7q_4-11)(B_1+\sqrt{B_2})}}{2bq_1(7q_4-11)a}(ax+bt)\right), \tag{168}$$

provided that $B_2>0$ *and* $q_1(7q_4-11)(B_1+\sqrt{B_2})>0.$

5.1.5. Another form of Periodic Wave Solution

Collection 5. *The fifth set of solutions of Equation (145) is considered as*

$$\mathfrak{z}(\xi) = \delta\,\frac{sec(\mu\,\xi)}{1+sec(\mu\,\xi)}. \tag{169}$$

Plugging Equation (169) into Equation (146), one becomes

$$J(\delta,\mu) = \int_0^{T/4}(\bullet)d\xi = \tag{170}$$

$$\frac{\delta^2(a^2\mu^2(105\pi K_1-20a\delta q_4-236a\delta-336bq_1)-14a(-15\pi\delta q_3+90\pi q_2+14\delta q_3)-630\pi b+1680q_2a+840b)}{2520\mu}.$$

With the following derivatives:

$$\frac{\partial J}{\partial\delta} = \frac{\delta(a^2\mu^2(105\pi K_1-20a\delta q_4-236a\delta-336bq_1)-14a(90\pi q_2-15\pi\delta q_3+14\delta q_3)-630\pi b+1680q_2a+840b)}{1260\mu} \tag{171}$$

$$+\frac{\delta^2(a^2\mu^2(105\pi(aq_4+4a)-20aq_4-236a)-14a(-15\pi q_3+14q_3))}{2520\mu},$$

$$\frac{\partial J}{\partial\mu} = -\frac{\delta^2(a^2\mu^2(105\pi K_1-20a\delta q_4-236a\delta-336bq_1)-14a(90\pi q_2-15\pi\delta q_3+14\delta q_3)-630\pi b+1680q_2a+840b)}{2520\mu^2}$$

$$+\frac{\delta^2a^2(105\pi(a\delta q_4+4a\delta+6bq_1)-20a\delta q_4-236a\delta-336bq_1)}{1260},\quad K_1=(a\delta q_4+4a\delta+6bq_1).$$

From which, we have

$$\mu = \frac{\sqrt{2}\sqrt{B_3\left(-B_2+\sqrt{-4B_3B_1+B_2^2}\right)}}{2aB_3},\quad \delta = -\frac{7(4b\,a^2\mu^2q_1-10q_2a-5b)}{2a(10a^2\mu^2q_4-8a^2\mu^2+7q_3)}, \tag{172}$$

$$B_1 = 6300\pi^2aq_2q_3+3150\pi^2bq_3-14280\pi aq_2q_3-7140\pi bq_3+7840aq_2q_3+3920bq_3,$$

$$B_2 = 15750\pi^2aq_2q_4+15750\pi^2bq_1q_3+63000\pi^2aq_2+7875\pi^2bq_4-24000\pi aq_2q_4-23100\pi bq_1q_3+31500\pi^2b- 119400\pi aq_2-12000\pi bq_4+4000q_2aq_4+7840bq_1q_3-59700\pi b+47200q_2a+2000bq_4+23600b,$$

$$B_3 = 1575\pi^2bq_1q_4+6300\pi^2bq_1-1140\pi bq_1q_4-6900\pi bq_1+160bq_1q_4+1888bq_1.$$

The last solution of Equation (1) is mentioned as

$$\psi(x,t) = -\frac{7\left(4b\,a^2\mu^2 q_1 - 10q_2 a - 5b\right)}{2a(10a^2\mu^2 q_4 - 8a^2\mu^2 + 7q_3)} \frac{sec\left(\frac{\sqrt{2}\sqrt{B_3\left(-B_2+\sqrt{-4B_3B_1+B_2^2}\right)}}{2aB_3}(ax+bt)\right)}{1+sec\left(\frac{\sqrt{2}\sqrt{B_3\left(-B_2+\sqrt{-4B_3B_1+B_2^2}\right)}}{2aB_3}(ax+bt)\right)}, \tag{173}$$

provided that $B_3 \neq 0$, $B_2 > 0$ *and* $q_1(7q_4 - 11)\left(B_1 + \sqrt{B_2}\right) > 0$.

Remark 1. *We will utilize one test to verify the obtained solutions in Section 7. We obtained the values* $q_1 = 0.1, q_2 = 0.2, q_3 = 0.1, q_4 = 0.1, a = -3, b = 1$*, then the manner of acting of the periodic wave solution in Equation (153) is designed in Figure 5 by the 3D and 2D graphs. The manner of acting of the soliton solution in Equation (158) is designed in Figure 6 in the form of 3D and 2D graphs. The bright soliton manner of acting of Equation (163) is investigated in Figure 7. Moreover, the periodic wave behavior of Equation (168), and also the periodic manner of acting of Equation (173), have been designed, respectively, in Figures 8 and 9 via the 3D and 2D plots. Obviously, the Figures 5–9 are "the periodic wave", "soliton", "bright soliton", and "periodic" wave solutions.*

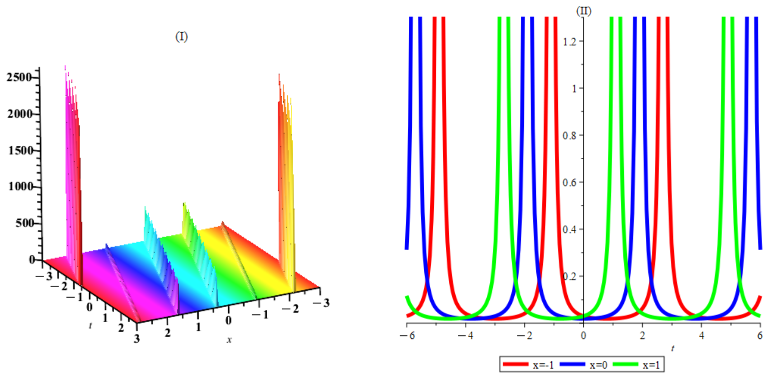

Figure 5. Plots of periodic wave solution (153) ψ for the parameter values $q_1 = 0.1, q_2 = 0.2,$ $q_3 = 0.1, q_4 = 0.1, a = -3, b = 1$.

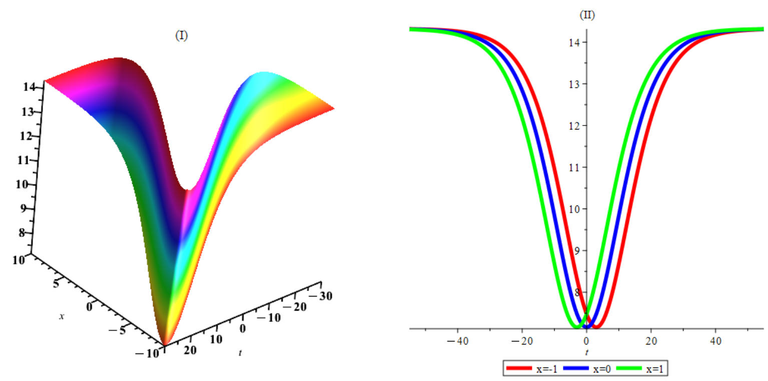

Figure 6. Plots of soliton solution (158) ψ for the parameter values $q_1 = 0.1, q_2 = 0.2, q_3 = 0.1,$ $q_4 = 0.1, a = 3, b = 1$.

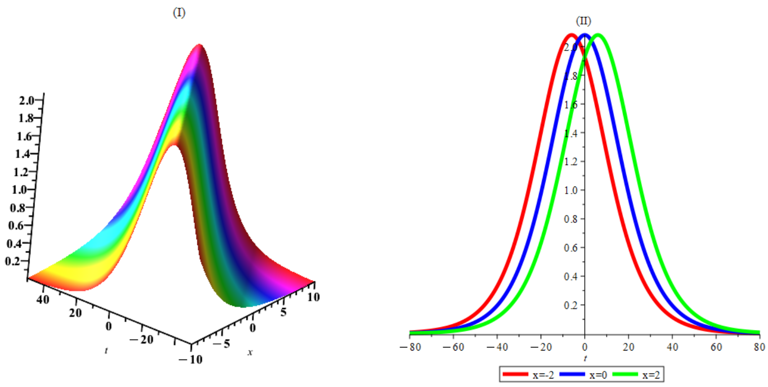

Figure 7. Plots of bright soliton solution (163) ψ for the parameter values $q_1 = 0.1, q_2 = 0.2, q_3 = 0.1, q_4 = 0.1, a = -3, b = 1$.

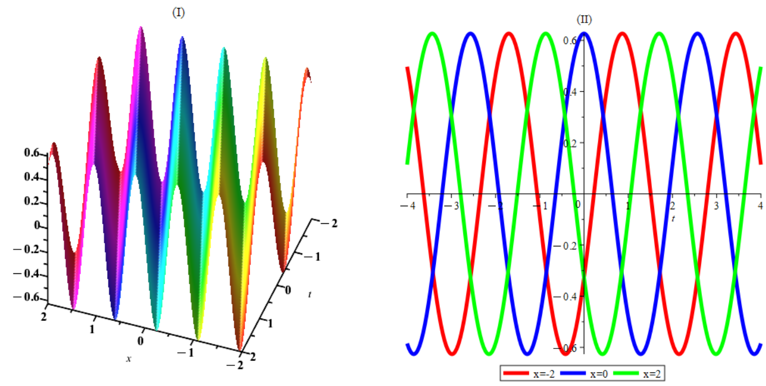

Figure 8. Plots of periodic wave solution (168) ψ for the parameter values $q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 2, a = 3, b = 1$.

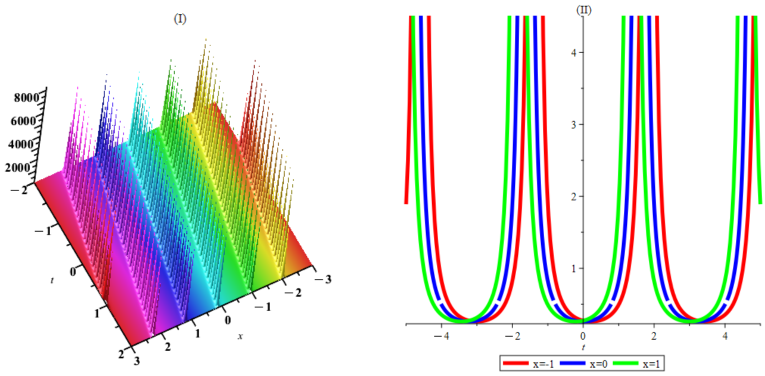

Figure 9. Plots of periodic wave solution (173) ψ for the parameter values $q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 2, a = -3, b = 1$.

6. Interpretation and Discussion

In this part, the graphical interpretation of the developed results are discussed. Using the computer software Maple 22, the standard $\tan(\phi/2)$-expansion technique, the Paul–Painlevé approach, and He's variational direct technique are implemented to establish the analytical solitary wave solutions of the nonlinear GP model. A few of the new kind of solutions, which have not been added to the literature previously, are successfully developed in this article. In Figures 1–4, we find three-dimensional surface plot, contour, density and 2D-parametric for different values of arbitrary constants. The importance of this study lies in the fact that it can serve as a base for the experimental work that we want to undertake on the plasma physics and crystal lattice theory. A comparison of our recently published design solutions reveals that some of the results are novel. Some of the results are similar to recently published results by choosing specific values q_1, q_2, q_3, and q_4 [1–4].

7. Conclusions

The standard $\tan(\phi/2)$-expansion technique, the Paul–Painlevé approach and He's variational direct technique have been successfully applied to construct some novel exact solitary wave solutions of the nonlinear GP Model. The proposed methods are most efficient, simple, direct and more capable to implement, and also these were quite convenient to develop traveling wave solutions of most of the NLPDEs. Some 3D-surface figures, density graphs and contour graphs to show some physical meaning to a few of the gained solutions to some specific values for the arbitrary parameters were also shown. We observed that the new extracted solutions included some remarkable kinds of soliton solutions such as bright–dark solitons, the singular periodic shape solution, bright soliton, and singular soliton solutions. The results will help in solving and understanding the mechanism of suggested problem, which is one of cardinal focuses. The methods used in this study were not only very effective, but also really well-suited to the purpose of looking for closed form soliton solutions to the problems discussed in this study. Our study employed advanced analytical and numerical techniques to obtain numerical solutions to the problem, which were validated through graphical representations and revalidation using the Maple 18 software. Our results were found to have negligible errors, indicating their high reliability. This study provides new insights into the behavior of incompressible fluids and bio-elastomers and has significant implications for future research in this area.

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