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Distributional Chaos and Sensitivity for a Class of Cyclic Permutation Maps

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Abstract: Several chaotic properties of cyclic permutation maps are considered. Cyclic permutation maps refer to p -dimensional dynamical systems of the form $\varphi(b_1, b_2, \dots, b_p) = (u_p(b_p), u_1(b_1), \dots, u_{p-1}(b_{p-1}))$, where $b_j \in H_j$ ($j \in \{1, 2, \dots, p\}$), $p \geq 2$ is an integer, and H_j ($j \in \{1, 2, \dots, p\}$) are compact subintervals of the real line $\mathbb{R} = (-\infty, +\infty)$. $u_j : H_j \rightarrow H_{j+1}$ ($j = 1, 2, \dots, p-1$) and $u_p : H_p \rightarrow H_1$ are continuous maps. Necessary and sufficient conditions for a class of cyclic permutation maps to have Li–Yorke chaos, distributional chaos in a sequence, distributional chaos, or Li–Yorke sensitivity are given. These results extend the existing ones.

Keywords: cyclic permutation map; sensitivity; chaos

MSC: 54H20; 18F60; 91B55; 26A18



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1. Introduction

It is well known that Li and Yorke [1] first proposed the mathematical definition of chaos. Since then, many different rigorous concepts of chaos have been given to depict different kinds of unpredictability in the evolution of a system. A very important generalization is distributional chaos, which was defined by Schweizer and Smítal [2], mainly because it is equivalent to positive topological entropy and some other notions of chaos for some spaces (see [2,3]). It is noted that this equivalence does not transfer to higher dimensions. For example, positive topological entropy does not imply distributional chaos for triangular maps of the unit square [4] (the same happens when the dimension is zero [5]). In [6], Wang et al. defined distributional chaos in a sequence and showed that it is equivalent to Li–Yorke chaos for continuous interval maps.

During the last few years, many researchers have paid attention to the chaotic behavior of Cournot maps [7–13]. Let $u : C \rightarrow B$ and $v : B \rightarrow C$ be continuous maps of the compact subintervals B and C of $\mathbb{R}$, and let $\varphi : B \times C \rightarrow B \times C$ be a continuous map defined as $\varphi(b, c) = (u(c), v(b))$ for any $(b, c) \in B \times C$. This aims to provide a mathematical description of competition in a duopolistic market, which is called Cournot duopoly ([7]). So, φ is called a Cournot map. u and v are called reaction functions; that is, u and v give laws to organize the production of some firms which are competitors in a market. According to [7,11], it is also called Markov perfect equilibria (MPE henceforth) processes. Only one player moves at each discrete time: that is, the two players move alternatively, each choosing the best reply to the previous action of the other player. Markov perfect equilibria occurs when the phase point (c_t, b_t) in the space $B \times C$ belongs alternatively

to the graphs of the reaction curves $c = v(b)$ and $b = u(c)$. This condition is satisfied when the initial condition (b_0, c_0) belongs to a reaction curve (i.e., $c_0 = v(b_0)$ (player 1 moves first) or $b_0 = u(c_0)$ (player 2 moves first)). Let $W_1 = \{(u(c), c) : c \in C\}$ and $W_2 = \{(b, v(b)) : b \in B\}$. The set $W_{12} = W_1 \cup W_2$ represents the union of the graphs of the two reaction function and is φ -invariant, i.e., $\varphi(W_{12}) \subseteq W_{12}$. Cáovas and Marín called the set W_{12} a MPE set for φ ([8]). Moreover, they considered some kinds of chaos of a Cournot map and showed that for Devaney chaos, RT chaos, topological chaos, and Li–Yorke chaos, ‘ φ is chaotic’ is not equivalent to so is $\varphi|_{W_{12}}$. Chaotic properties of a Cournot map have been studied in the literature [14–17] also. In 2013, Lu and Zhu studied the dynamical properties of the maps $\varphi|_{W_{12}}$, $\varphi^2|_{W_1}$ and $\varphi^2|_{W_2}$ [12]. In 2016, Bas and Lopez [18] introduced the concept of a cyclically permuted direct product map and discussed the topics of transitivity and mixing by studying the relationship between the dynamics of G and that of the compositions $g_{\alpha(k)} \circ \dots \circ g_{\alpha^p(k)}$, where $p \geq 2$ is an integer, $k \in \{1, \dots, p\}$, $G(h_1, h_2, \dots, h_p) = (g_{\alpha(1)}(h_{\alpha(1)}), g_{\alpha(2)}(h_{\alpha(2)}), \dots, g_{\alpha(p)}(h_{\alpha(p)}))$ is said to be a cyclically permuted direct product map. For any $(h_1, h_2, \dots, h_p) \in H_1 \times H_2 \times \dots \times H_p$ ($H_1, H_2, \dots, H_p$ are topological spaces), every map $g_{\alpha(k)} : H_{\alpha(k)} \rightarrow H_k$, $k \in \{1, 2, \dots, p\}$ is continuous and α is a cyclic permutation of $1, 2, \dots, p$. The authors in [18] also obtained several results on transitivity for cyclically permuted direct product maps of the Cartesian product J^p , where $J = [0, 1]$. In particular, for any integer $p > 2$, the transitivity of G is equivalent to the total transitivity. And then, they extended several well-known properties of transitivity from interval maps to cyclically permuted direct product maps [19]. These maps appear associated with a certain economical model: the so-called Cournot duopoly ([8–10,20], etc.). One can also find them in age-structured population models, as in [21], where it is called the Leslie model and analyzed. In [22], we considered and obtained several chaotic properties of a cyclic permutation map. Necessary and sufficient conditions for a cyclic permutation map to be LY-chaotic, h-chaotic, RT-chaotic, or D-chaotic were established. Moreover, we showed that the topological entropy of such a cyclic permutation map is the same as the topological entropy of each of the coordinates maps and that it is sensitive if and only if so is at least one of the coordinates maps.

Obviously, up to now, research on the relationship between the chaotic properties of cyclic permutation maps on high-dimensional spaces and their chaotic properties limited to subsets of the spaces has been very rare. Regarding the traditional Cournot map, scholars have only studied some of its chaotic properties in two-dimensional space and obtained its chaotic properties limited to MPE sets. In order to study cyclic permutation maps on higher-dimensional spaces, we first generalize the definition of a Cournot map. Let $\varphi(b_1, b_2, \dots, b_p) = (u_p(b_p), u_1(b_1), \dots, u_{p-1}(b_{p-1}))$, where $b_j \in H_j$ for any $j \in \{1, 2, \dots, p\}$, $p \geq 2$ is an integer. $u_j : H_j \rightarrow H_{j+1}$ ($j = 1, 2, \dots, p-1$) and $u_p : H_p \rightarrow H_1$ are continuous. H_j ($j \in \{1, 2, \dots, s\}$) represents compact subintervals of the real line $\mathbb{R} = (-\infty, +\infty)$. Such mappings are called generalized Cournot maps on p -dimensional spaces.

Motivated by [8,12], this paper discusses the dynamical properties of the above p -dimensional dynamical system. Particularly, the necessary and sufficient conditions for a generalized Cournot map $\varphi(b_1, b_2, \dots, b_p) = (v_p(b_p), v_1(b_1), \dots, v_{p-1}(b_{p-1}))$ to be Li–Yorke chaotic, distributional chaotic (in a sequence), Li–Yorke sensitive, or sensitive dependence on the initial conditions (briefly, sensitive) are obtained. Our results extend the existing ones on low-dimensional dynamical systems.

In Section 2, we reviewed some notations and concepts, and we defined generalized Cournot maps (which are all cyclic permutation maps) as well as some closed subsets covered in this paper. In Section 3, we investigated the relationship between some chaotic properties of generalized Cournot maps and the corresponding chaotic properties restricted to some closed subsets. The results obtained are generalizations or improvements of some known results in the corresponding literature. In Section 4, we briefly summarized the results of this research and proposed future research directions.

2. Preliminaries

In this paper, let U be a metric space with metric d .

Definition 1 ([1]). Let u be a continuous map over a metric space (U, d) . For any $a_1, a_2 \in U$ and any $\lambda > 0$, (a_1, a_2) is said to be a Li–Yorke pair of modulus λ if

$$\liminf_{m \rightarrow +\infty} d(u^m(a_1), u^m(a_2)) = 0$$

and

$$\limsup_{m \rightarrow +\infty} d(u^m(a_1), u^m(a_2)) \geq \lambda.$$

A subset $C \subset U$ is said to be a Li–Yorke chaotic set if for any two points $a_1, a_2 \in C : a_1 \neq a_2$, (a_1, a_2) is a Li–Yorke pair. A dynamic system (U, u) or a map $u : U \rightarrow U$ is called to be Li–Yorke chaotic if U contains an uncountable Li–Yorke chaotic set. The set of Li–Yorke pairs of modulus λ is written by $LY(u, \lambda)$, and the set of Li–Yorke pairs is written by $LY(u)$.

Let $\{s_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers; for any two different points $a, b \in U$ and any $h > 0$, we set

$$F_{ab}^*(h, \{s_i\}_{i=1}^{\infty}, u) = \limsup_{m \rightarrow +\infty} \frac{1}{m} \sum_{i=1}^m \chi_{[0, h)}(d(u^{s_i}(a), u^{s_i}(b)))$$

and

$$F_{ab}(h, \{s_i\}_{i=1}^{\infty}, u) = \liminf_{m \rightarrow +\infty} \frac{1}{m} \sum_{i=1}^m \chi_{[0, h)}(d(u^{s_i}(a), u^{s_i}(b))),$$

where $\chi_{[0, h)}(y)$ denotes the characteristic function of the set $[0, h)$.

Definition 2 ([6]). Let $\{s_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers and u be a continuous map on a metric space (U, d) . A subset $U_0 \subset U$ is a distributional chaotic set in the sequence $\{s_i\}_{i=1}^{\infty}$ if for any two different points $a, b \in U_0$ satisfying that $F_{ab}^*(h, \{s_i\}_{i=1}^{\infty}, u) = 1$ for any $h > 0$, and that $F_{ab}(\eta, \{s_i\}_{i=1}^{\infty}, u) = 0$ for some $\eta > 0$. A dynamic system (U, u) or a map $u : U \rightarrow U$ is called to be distributional chaotic in a sequence if it has an uncountable distributional chaotic set in a sequence. If a dynamic system (U, u) or a map $u : U \rightarrow U$ is called to be distributional chaotic in the sequence $\{1, 2, \dots\}$, then it is called to be distributional chaotic.

Definition 3 ([8]). A dynamic system (U, u) or a map $u : U \rightarrow U$ is called to be transitive if for any nonempty open sets B and C of X , there is an integer $m > 0$ such that $u^m(B) \cap C \neq \emptyset$.

Definition 4 ([8]). A dynamic system (U, u) or a map $u : U \rightarrow U$ is called to be sensitive if there is an $\varepsilon > 0$ such that for any nonempty open set B of U , there are points $x, y \in B$ with $d(u^m(x), u^m(y)) > \varepsilon$ for some integer $m > 0$, where ε is said to be a sensitive constant of u .

Definition 5 ([8]). A dynamic system (U, u) or a map $u : U \rightarrow U$ is called to be chaotic in the sense of Devaney (or D-chaotic, for short) if u is transitive and sensitive with $\overline{\text{Per}(u)} = U$, where $\overline{B}$ denotes the closure of the set B .

Definition 6 ([23]). Let u be a continuous map over a metric space (U, d) . A dynamic system (U, u) or a map $u : U \rightarrow U$ is called to be Li–Yorke sensitive if there is $\lambda > 0$ such that for any $a \in U$ and any $\eta > 0$, there exists some $b \in U$ with $d(a, b) < \eta$ and $LY(u, \lambda)$.

Let H_i be a compact interval of the real line $\mathbb{R} = (-\infty, +\infty)$ for any $i \in \{1, 2, \dots, p\}$ and $H_1 \times H_2 \times \dots \times H_p$ endowed with the product metric ρ , which is given by

$$\rho((a_1, a_2, \dots, a_p), (b_1, b_2, \dots, b_p)) = \max_{1 \leq i \leq p} \{|b_i - a_i|\}$$

for any

$$(a_1, a_2, \dots, a_p), (b_1, b_2, \dots, b_p) \in H_1 \times H_2 \times \dots \times H_p.$$

For a generalized Cournot map

$$\varphi(b_1, b_2, \dots, b_p) = (u_p(b_s), u_1(b_1), \dots, u_{p-1}(b_{p-1}))$$

where

$$(b_1, b_2, \dots, b_p) \in H_1 \times H_2 \times \dots \times H_p,$$

let

$$W_p = \{(u_p(b_p), u_1 \circ u_p(b_p), u_2 \circ u_1 \circ u_p(b_p), \dots, u_{p-2} \circ \dots \circ u_1 \circ u_p(b_p), b_p) : b_p \in H_p\},$$

$$W_{p-1} = \{(u_p(a_p), u_1 \circ u_p(a_p), u_2 \circ u_1 \circ u_p(a_p), \dots, u_{p-3} \circ \dots \circ u_1 \circ u_p(a_p), b_{p-1}, a_p) : b_{p-1} \in H_{p-1}\}, \dots,$$

and

$$W_1 = \{(b_1, u_1(b_1), \dots, u_{p-1} \circ \dots \circ u_1(b_1)) : b_1 \in H_1\}$$

where $a_p = u_{p-1}(b_{p-1})$. Let $W_{12\dots p} = W_1 \cup W_2 \cup \dots \cup W_p$ be the MPE-set for φ .

3. Main Results

In this paper, $\mathcal{P}$ -chaotic denotes one of the five properties: Li-Yorke chaos, distributional chaos in a sequence, Li-Yorke sensitivity, sensitivity, or distributional chaos. Meanwhile, $\mathcal{P}_1$ denotes one of three properties: Li-Yorke chaos, distributional chaos in a sequence, or distributional chaos.

Lemma 1 ([12]). A dynamical system (U, u) is $\mathcal{P}$ -chaotic if and only if so is (U, u^m) for any integer $m > 0$.

Lemma 2 ([12]). Assume that (U, d_1) and (V, d_2) are two compact metric spaces. $u : V \rightarrow U$ and $v : U \rightarrow V$ are continuous maps. Then, $u \circ v$ is $\mathcal{P}_1$ -chaotic if and only if so is $v \circ u$.

Lemma 3. For a cyclic permutation map

$$\varphi(b_1, b_2, \dots, b_p) = (u_p(b_p), u_1(b_1), \dots, u_{p-1}(b_{p-1})),$$

$u_i (i > 0)$ is $\mathcal{P}_1$ -chaotic if and only if so is u_k for some integer $k > 0 : k \neq i$.

Proof. By using the mathematical inductive method, the definitions, and Lemma 2, one can easily prove Lemma 3. $\square$

Lemma 4 ([12]). Assume that $J_1 \subset (-\infty, +\infty)$ and $J_2 \subset (-\infty, +\infty)$ are two compact subintervals. $u_1 : J_1 \rightarrow J_1$ and $u_2 : J_2 \rightarrow J_2$ are continuous. Then, $u_1 \times u_2$ is $\mathcal{P}$ -chaotic if and only if either u_1 or u_2 is $\mathcal{P}$ -chaotic.

Lemma 5. Assume that $J_i \subset (-\infty, +\infty)$ is a compact subinterval for any $i \in \{1, 2, \dots, p\}$. $u_i : J_i \rightarrow J_i$ is continuous for any $i \in \{1, 2, \dots, p\}$. Then, $u_1 \times u_2 \times \dots \times u_p$ is $\mathcal{P}$ -chaotic if and only if u_j is $\mathcal{P}$ -chaotic for some $j \in \{1, 2, \dots, p\}$.

Proof. It is easy to prove this lemma by using Proposition 2 of Reference [3] in [12], Theorem 3 of Reference [13] in [12], Theorem 3.5 of [15], and mathematical induction. $\square$

Remark 1. Lemma 5 is a generalization of Theorem 3.1 in [12].

Theorem 1. For a cyclic permutation map

$$\varphi(b_1, b_2, \dots, b_p) = (u_s(b_p), u_1(b_1), \dots, u_{p-1}(b_{p-1})),$$

the following conditions are equivalent.

- (1) φ is $\mathcal{P}_1$ -chaotic;
- (2) $\varphi|_{W_{12\dots p}}$ is $\mathcal{P}_1$ -chaotic;
- (3) $\varphi^p|_{W_j}$ is $\mathcal{P}_1$ -chaotic for some $j \in \{1, 2, \dots, p\}$;
- (4) $\varphi^p|_{W_j}$ is $\mathcal{P}_1$ -chaotic for any $j \in \{1, 2, \dots, p\}$.

Proof. It is clear that

$$\varphi^p = \varphi_1 \times \varphi_2 \times \dots \times \varphi_p.$$

Since the proof is similar for any integer $p \geq 3$, to be succinct, we only consider $p = 3$.

(i) Li–Yorke chaos.

By the definition, one can easily show that (4) $\Rightarrow$ (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) $\Rightarrow$ (1). So, it is enough to prove that (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4). Since φ is Li–Yorke chaotic, then φ_i is Li–Yorke chaotic for any $i \in \{1, 2, \dots, p\}$. Assume that D is an uncountable Li–Yorke chaotic set of φ_3 and that $\mathcal{B} = \{(u_3(b_3), u_1 \circ u_3(b_3), b_3) : b_3 \in D\}$. It is obvious that $\mathcal{B} \subset W_3$ and $\mathcal{B}$ is uncountable.

Now, we show that $\mathcal{B}$ is a Li–Yorke chaotic set of $\varphi^3|_{W_3}$. Let

$$(u_3(a_3), u_1 \circ u_3(a_3), a_3), (u_3(b_3), u_1 \circ u_3(b_3), b_3)) \in \mathcal{B}$$

with $a_3 \neq b_3$. Then, one has that

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \rho(\varphi^{3m}(u_3(a_3), u_1 \circ u_3(a_3), a_3), \varphi^{3m}(u_3(b_3), u_1 \circ u_3(b_3), b_3)) \\ & \geq \limsup_{m \rightarrow \infty} |\varphi_3^m(a_3) - \varphi_3^m(b_3)| > 0. \end{aligned}$$

Since

$$\liminf_{m \rightarrow \infty} |\varphi_3^m(a_3) - \varphi_3^m(b_3)| = 0,$$

by the definition, there is an increasing positive integer sequence $\{m_j\}_{j=1}^\infty$ satisfying

$$\lim_{j \rightarrow \infty} |\varphi_3^{m_j}(a_3) - \varphi_3^{m_j}(b_3)| = 0.$$

Since u_3 and $u_1 \circ u_3$ are uniformly continuous, one has that

$$\lim_{j \rightarrow \infty} |\varphi_1^{m_j}(u_3(a_3)) - \varphi_1^{m_j}(u_3(b_3))| = 0$$

and

$$\lim_{j \rightarrow \infty} |\varphi_2^{m_j}(u_1 \circ u_3(a_3)) - \varphi_2^{m_j}(u_1 \circ u_3(b_3))| = 0.$$

Consequently,

$$\begin{aligned} & \liminf_{m \rightarrow \infty} \rho(\varphi^{3m}(u_3(a_3), u_1 \circ u_3(a_3), a_3), \varphi^{3m}(u_3(b_3), u_1 \circ u_3(b_3), b_3)) \\ & \leq \liminf_{j \rightarrow \infty} \rho(\varphi^{3m_j}(u_3(a_3), u_1 \circ u_3(a_3), a_3), \varphi^{3m_j}(u_3(b_3), u_1 \circ u_3(b_3), b_3)) \\ & \leq \lim_{j \rightarrow \infty} (|\varphi_1(u_3(a_3)) - \varphi_1(u_3(b_3))| + |\varphi_2(u_1 \circ u_3(a_3)) - \varphi_2(u_1 \circ u_3(b_3))| \\ & \quad + |\varphi_3(a_3) - \varphi_3(b_3)|) \\ & = 0. \end{aligned}$$

This means that $\varphi^3|_{W_3}$ is Li-Yorke chaotic.

Similarly, one can easily prove that $\varphi^3|_{W_1}$ and $\varphi^3|_{W_2}$ are Li-Yorke chaotic.

(ii) Distributional chaos in a sequence.

By the definition, one can easily prove that (4) $\Rightarrow$ (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Therefore, it is enough to show that (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4). Since φ is distributional chaotic in a sequence, by Lemmas 1 and 5, φ_i is distributional chaotic in a sequence for any $i \in \{1, 2, 3\}$. Assume that D is an uncountable distributional chaotic set of φ_3 in a sequence $\{t_i\}_{i=1}^\infty$ and that $\mathcal{B} = \{(u_3(b_3), u_1 \circ u_3(b_3), b_3) : b_3 \in D\}$. It is obvious that $\mathcal{B} \subset W_3$ and $\mathcal{B}$ is uncountable.

The following shows that $\mathcal{B}$ is a distributional chaotic set of $\varphi^3|_{W_3}$ in the sequence $\{t_i\}_{i=1}^\infty$. Let

$$a = (u_3(a_3), u_1 \circ u_3(a_3), a_3), b = (u_3(b_3), u_1 \circ u_3(b_3), b_3) \in \mathcal{B} : a_3 \neq b_3, a, b \in \mathcal{B}.$$

By hypothesis and the definition, there is some $\varepsilon > 0$ such that

$$F_{a,b}(\varepsilon, \{t_i\}_{i=1}^\infty, \varphi^3) = 0.$$

It is clear that

$$\rho(\varphi^{3t_i}(a), \varphi^{3t_i}(b)) \geq |\varphi_3^{t_i}(a_3) - \varphi_3^{t_i}(b_3)|$$

for any integer $i > 0$. This implies that

$$\begin{aligned} & F_{a,b}(\varepsilon, \{t_i\}_{i=1}^\infty, \varphi^3) \\ &= \liminf_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \chi_{[0,\varepsilon)} \rho(\varphi^{3t_i}(a), \varphi^{3t_i}(b)) \\ &\leq \liminf_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \chi_{[0,\varepsilon)} |\varphi_3^{t_i}(a_3) - \varphi_3^{t_i}(b_3)| \\ &= F_{a,b}(\varepsilon, \{t_i\}_{i=1}^\infty, \varphi_3) \\ &= 0. \end{aligned}$$

Since u_1 and u_3 are uniformly continuous, for any $s > 0$, there is some $s' : 0 < s' < s$ satisfying that $|u_3(a_3) - u_3(b_3)| < s$ and $|u_1 \circ u_3(a_3) - u_1 \circ u_3(b_3)| < s$ for any $a_3, b_3 \in H_3 : |a_3 - b_3| < s'$. Therefore,

$$\begin{aligned} F_{a,b}^*(s, \{t_i\}_{i=1}^\infty, \varphi^3) &= \limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \chi_{[0,s)} \rho(\varphi^{3t_i}(a), \varphi^{3t_i}(b)) \\ &= \limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \chi_{[0,s)} \max\{|\varphi_1^{t_i}(u_3(a_3)) - \varphi_1^{t_i}(u_3(b_3))|, \\ &\quad |\varphi_2^{t_i}(u_1 \circ u_3(a_3)) - \varphi_2^{t_i}(u_1 \circ u_3(b_3))|\} \\ &\geq \limsup_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \chi_{[0,s')} |\varphi_3^{t_i}(a_3) - \varphi_3^{t_i}(b_3)| \\ &= F_{a,b}^*(s', \{t_i\}_{i=1}^\infty, \varphi_3) = 1. \end{aligned}$$

Consequently, $\varphi^3|_{W_3}$ is distributional chaotic. Similarly, one can easily show that $\varphi^3|_{W_1}$ and $\varphi^3|_{W_2}$ are distributional chaotic in the sequence $\{t_i\}_{i=1}^\infty$.

(iii) Distributional chaos

The proof is similar to that of (ii). $\square$

Remark 2. Theorem 1 is a generalization of Theorem 3.2 in [12].

Theorem 2. For a cyclic permutation map

$$\varphi(b_1, b_2, \dots, b_p) = (u_s(b_p), u_1(b_1), \dots, u_{p-1}(b_{p-1})),$$

the following conditions are equivalent.

- (1) φ is Li–Yorke sensitive;
- (2) $\varphi^p|_{W_i}$ is Li–Yorke sensitive for some $i \in \{1, 2, \dots, p\}$.

Proof. Assume that φ is Li–Yorke sensitive. By Lemma 1, φ^3 is Li–Yorke sensitive. By the definition and the above argument, there is some $\varepsilon > 0$ satisfying that, for any $a = (u_3(a_3), u_1 \circ u_3(a_3), a_3) \in W_3 \subset H_1 \times H_2 \times H_3$ and any $\delta > 0$, there is a $b = (b_1, b_2, b_3) \in H_1 \times H_2 \times H_3 : \rho(a, b) < \delta$ and $(a, b) \in \text{LY}(\varphi^3, \varepsilon)$. By the Li–Yorke sensitivity of $\varphi^3 = \varphi_1 \times \varphi_2 \times \varphi_3$ and Lemma 5, one can deduce that φ_1 , φ_2 , or φ_3 is Li–Yorke sensitive. Without loss of generality, one can assume that φ_3 is Li–Yorke sensitive. By the definition, there is some $\varepsilon > 0$ satisfying that for any $a_3 \in H_3$ and any $\delta > 0$, there is a $a_3(\delta) \in H_3 : |a_3 - a_3(\delta)| < \delta$ and $(a_3, a_3(\delta)) \in \text{LY}(\varphi_3, \varepsilon)$. Since u_1 and u_3 are uniformly continuous, for the above $\delta > 0$ and any $h_3, h'_3 \in H_3 : h_3 \neq h'_3$, there exists some $\delta' > 0 : \delta' < \delta$ satisfying that,

$$|h_3 - h'_3| < \delta'$$

implies

$$|u_3(h_3) - u_3(h'_3)| < \delta$$

and

$$|u_1 \circ u_3(h_3) - u_1 \circ u_3(h'_3)| < \delta.$$

Therefore,

$$\rho((u_3(a_3), u_1 \circ u_3(a_3), a_3), (u_3(a_3(\delta')), u_1 \circ u_3(a_3(\delta')), a_3(\delta'))) < \delta$$

and

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \rho(\varphi^{3k}((u_3(a_3), u_1 \circ u_3(a_3), a_3), (u_3(a_3(\delta')), u_1 \circ u_3(a_3(\delta')), a_3(\delta')))) \\ & \geq \limsup_{k \rightarrow \infty} |\varphi_3^k(a_3) - \varphi_3^k(a_3(\delta'))| > \varepsilon. \end{aligned}$$

Similarly, it is easy to verify that

$$\liminf_{k \rightarrow \infty} \rho(\varphi^{3k}((u_3(a_3), u_1 \circ u_3(a_3), a_3), (u_3(a_3(\delta')), u_1 \circ u_3(a_3(\delta')), a_3(\delta')))) = 0.$$

This shows that $((u_3(a_3), u_1 \circ u_3(a_3), a_3), (u_3(a_3(\delta')), u_1 \circ u_3(a_3(\delta')), a_3(\delta')))) \in \text{LY}(\varphi^3, \varepsilon)$. By the arbitrariness of δ and $a_3 \in H_3$, φ^3 is Li–Yorke sensitive.

On the contrary, assume that φ^3 is Li–Yorke sensitive. Since u_1 and u_3 are uniformly continuous, one can easily prove that for any $\varepsilon > 0$ and any $((u_3(a_3), u_1 \circ u_3(a_3), a_3), (u_3(a'_3), u_1 \circ u_3(a'_3), a'_3)) \in \text{LY}(\varphi^3|_{W_3}, \varepsilon)$, there is some $\varepsilon' > 0$ such that $(a_3, a'_3) \in \text{LY}(\varphi_3, \varepsilon')$. By the above argument, the definition, and the Li–Yorke sensitivity of $\varphi^3|_{W_3}$, φ_3 is Li–Yorke sensitive. By Lemma 1 and Lemma 5, φ is Li–Yorke sensitive. $\square$

Remark 3. Theorem 2 is a generalization of Theorem 3.3 in [12].

Theorem 3. For a cyclic permutation map

$$\varphi(b_1, b_2, \dots, b_p) = (u_s(b_p), u_1(b_1), \dots, u_{p-1}(b_{p-1})),$$

the following statements are equivalent.

- (1) φ is sensitive;
- (2) $\varphi^p|_{W_i}$ is sensitive for some $i \in \{1, 2, \dots, p\}$.

Proof. Assume that φ is sensitive. By Theorem 31 in [23], $\varphi^3 = \varphi_1 \times \varphi_2 \times \varphi_3$ is sensitive. Then, by Theorem 3.1 in [24], one can see that φ_1 , φ_2 , or φ_3 is sensitive. Without loss of generality, assume that φ_3 is sensitive. By the definition, there exists a sensitivity constant $\delta > 0$ of φ_3 such that for any point $h \in H_3$ and any $\varepsilon > 0$, there is a $h_\varepsilon \in H_3 : |h - h_\varepsilon| < \varepsilon$ and

$$|\varphi_3^k(h) - \varphi_3^k(h_\varepsilon)| > \delta$$

for some integer $k > 0$. Since u_1 and u_3 are uniformly continuous, there is a $\varepsilon' \in (0, \varepsilon)$ such that

$$|u_1 \circ u_3(h_3) - u_1 \circ u_3(h'_3)| < \varepsilon$$

for any $h_1, h'_1 \in H_1 : |h_3 - h'_3| < \varepsilon'$, and $|u_3(h_3) - u_3(h'_3)| < \varepsilon$ for any $h_3, h'_3 \in H_3 : |h_3 - h'_3| < \varepsilon'$.

Clearly,

$$\rho(\varphi^{3k}(u_3(h), u_1 \circ u_3(h), h), \varphi^{3k}(u_3(h_\varepsilon), u_1 \circ u_3(h_\varepsilon), h_\varepsilon)) \geq |\varphi_3^k(h) - \varphi_3^k(h_\varepsilon)| > \delta.$$

This shows that $\varphi^3|_{W_3}$ is sensitive.

On the contrary, assume that $\varphi^3|_{W_3}$ is sensitive with a sensitivity constant $\delta > 0$. By the uniform continuity of u_1 and u_3 , for the above $\delta > 0$ and any $h_3, h'_3 \in H_3 : h_3 \neq h'_3$, there exist some $\delta' > 0 : \delta' < \delta$; satisfying that, $|h_3 - h'_3| \leq \delta'$ implies

$$|u_3(h_3) - u_3(h'_3)| \leq \delta \text{ and } |u_1 \circ u_3(h_3) - u_1 \circ u_3(h'_3)| \leq \delta.$$

Therefore, for any $h_3, h'_3 \in H_3 : h_3 \neq h'_3$ and $|h_3 - h'_3| \leq \delta'$, if

$$\rho(\varphi^{3k}((u_3(h_3), u_1 \circ u_3(h_3), h_3), u_3(h'_3), u_1 \circ u_3(h'_3), h'_3))) > \delta$$

for some integer $k > 0$, then

$$|\varphi_3^k(h_3) - \varphi_3^k(h'_3)| > \delta'.$$

This means that φ_3 is sensitive. By Theorem 3.1 in [24], $\varphi^3 = \varphi_1 \times \varphi_2 \times \varphi_3$, is sensitive. By Theorem 31 in [23], φ is sensitive. $\square$

Remark 4. Theorem 3 is a generalization of Theorem 2.2 in [17].

Remark 5. By using the uniform continuity of the mappings involved in this paper, the results of the above three theorems can be extended to any nonempty invariant compact subset of the space.

4. Conclusions

Some preliminary discussions on the relationship between cyclic permutation maps on high-dimensional spaces and the chaotic properties limited to their spatial subsets are presented. Specifically, for generalized Cournot maps, we have demonstrated that Li–Yorke chaoticity, distributional chaoticity, and the sensitivity of cyclic permutation maps is equivalent to the maps limited in MPE sets. These conclusions extend the ones for Cournot maps in duopoly games. The issue of the relationship between cyclic permutation maps on high-dimensional spaces and their chaotic properties restricted to other subsets is still open and requires further in-depth research in the future.

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