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Superconvergent Nyström Method Based on Spline Quasi-Interpolants for Nonlinear Urysohn Integral Equations

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Abstract: Integral equations play an important role for their applications in practical engineering and applied science, and nonlinear Urysohn integral equations can be applied when solving many problems in physics, potential theory and electrostatics, engineering, and economics. The aim of this paper is the use of spline quasi-interpolating operators in the space of splines of degree d and of class C^{d-1} on uniform partitions of a bounded interval for the numerical solution of Urysohn integral equations, by using a superconvergent Nyström method. Firstly, we generate the approximate solution and we obtain outcomes pertaining to the convergence orders. Additionally, we examine the iterative version of the method. In particular, we prove that the convergence order is $(2d+2)$ if d is odd and $(2d+3)$ if d is even. In case of even degrees, we show that the convergence order of the iterated solution increases to $(2d+4)$. Finally, we conduct numerical tests that validate the theoretical findings.

Keywords: Urysohn integral equation; quasi-interpolating spline; Nyström method; superconvergence

MSC: 65R20; 65D07; 65J15

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1. Introduction

In this paper, we consider nonlinear Urysohn integral equations of the form:

$$u - K(u) = f, \tag{1}$$

where K is the Urysohn integral operator:

$$K(u)(s) := \int_0^1 k(s,t,u(t))dt, \qquad s \in I := [0,1], \qquad u \in X := C(I). \tag{2}$$

The kernel $k(s,t,u)$ is a real valued function defined on $I \times I \times \mathbb{R}$ and we assume that, for $f \in C(I)$, Equation (1) has a unique solution $\bar{u}$. Defining the operator T by $T(u) = f + K(u)$, Equation (1) can be written as $u = T(u)$.

Moreover, for the study proposed in the paper, we need the following assumptions. Let a and b be two real numbers, such that:

$$[\min_{s\in I} \bar{u}(s), \max_{s\in I} \bar{u}(s)] \subset [a,b].$$

Define $\Omega := I \times I \times [a,b]$. Let $\alpha \geq 1$ and assume that $k, \frac{\partial k}{\partial u} \in C^\alpha(\Omega)$, $f \in C^\alpha(I)$. Therefore, K is a compact operator from $C(I)$ to $C^\alpha(I)$ and $\bar{u} \in C^\alpha(I)$, and K is Fréchet differentiable and its derivative is given by:

$$(K'(u)q)(s) = \int_0^1 \frac{\partial k}{\partial u}(s,t,u(t))q(t)dt$$

and K' is Lipschitz continuous in a neighbourhood of $\bar{u}$.

As noticed in [1,2] (see also [3]), integral equations of various kinds play an important role for their applications in practical engineering and applied science, and Urysohn integral equations can be applied in solving many problems in physics, potential theory and electrostatics, engineering, and economics. Furthermore, numerous problems addressed in the theory of partial differential equations (PDEs) result in Urysohn integral equations (see, e.g., [4]).

These equations usually cannot be solved explicitly and they are required to be approximate numerically. Commonly used methods to solve numerically Equation (1) are projection methods [4,5], such as Galerkin, collocation methods, and, recently, a modified projection one [6]. They are based on a sequence of orthogonal or interpolatory projectors onto finite dimensional subspaces X_n approximating X, and X_n is usually the space of piecewise polynomials of degree d at its most continuous (see, e.g., [5–8] and references therein). Other well-known methods are Nyström and degenerate kernel ones (see, e.g., [4,9,10] and references therein).

The solutions reconstructed in the above mentioned spaces are, at most, continuous and do not preserve the smoothness of the exact solution. In order to avoid this drawback, an alternative recently proposed in the literature is the reconstruction of the approximate solution by methods based on quasi-interpolating (QI) operators in the space of splines of degree d and smoothness C^{d-1} (see, e.g., [11–13] and also the recent papers [14,15] in the context of PDEs). Moreover, in case of piecewise polynomials, the dimension of the space of approximants is related to the product between the number of subintervals n, and the degree d with smooth splines instead is related to the sum between n and d. Therefore, using smooth spaces we have also an advantage from the computational point of view, for increasing values of n.

Therefore, in this paper, we use spline QI operators that are not necessarily projectors to solve Equation (1) by the superconvergent Nyström method introduced in [10]. In addition to preserving the smoothness of the exact solution, this method is more accurate than the classical Nyström and more economical than Kulkarni's scheme (for a detailed comparison on the computational cost in the linear case, see Remark 4.1 in [11]). Moreover, the use of QI operators that are not projectors allows a reduction of the computational costs in terms of the number of evaluation points in the operator construction with respect to the construction of QI projectors.

This paper's structure is as follows: Section 2 presents a review of definitions and properties of spline QI operators of general degree d and class C^{d-1}. Then, in Section 3, we consider integral Equation (1) and we describe the superconvergent Nyström method, constructing the approximate solution and studying the convergence order and providing its iterated version. We prove that the convergence order is $(2d+2)$ if d is odd and $(2d+3)$ if d is even. In case of even degrees, we show that the convergence order of the iterated solution increase to $(2d+4)$. Finally, in Section 4, we provide numerical tests confirming the theoretical results.

2. Spline Quasi-Interpolants

For a fixed $d \geq 1$, we define $\mathcal{S}_d^{d-1}(I, \Theta_n)$ as the space of splines of degree d and of class C^{d-1} on the uniform partition $\Theta_n := \{t_i = ih,\ h = \frac{1}{n},\ i = 0 \dots n\}$. Adding multiple knots at the endpoints, i.e., adding $t_{-d} = \ldots = t_{-1} = t_0$, $t_n = t_{n+1} = \ldots = t_{n+d}$, we consider the basis formed by the $n+d$ normalized B-splines $\{B_j,\ j = 1, 2, \ldots, n+d\}$, with support $\Sigma_j := \operatorname{supp}(B_j) = [t_{j-d-1}, t_j]$.

We consider discrete spline QI operators of the form:

$$Q_n^d : C(I) \to \mathcal{S}_d^{d-1}(I, \Theta_n), \qquad Q_n^d u := \sum_{j=1}^{n+d} \lambda_j(u) B_j, \tag{3}$$

where $\lambda_j(u)$ are linear combinations of values of u at some points in a neighbourhood of Σ_j. i.e.,

$$\lambda_j(u) := \sum_i \alpha_{ij} u(\xi_i). \tag{4}$$

The quasi-interpolation nodes $\mathcal{E}_n := \{\xi_i\}_{i=0}^N$ are defined in this way:

$$\xi_i = \begin{cases} s_i = \frac{t_{i-1}+t_i}{2}, & N = n+1, \quad \text{for } d \text{ even}, \\ t_i, & N = n, \quad \text{for } d \text{ odd}. \end{cases}$$

The coefficients α_{ij} are chosen such that $Q_n^d u = u$, for all $u \in \mathbb{P}_d$, where $\mathbb{P}_d$ denotes the space of polynomials of degree d. The QI Q_n^d can be written in quasi-Lagrange form:

$$Q_n^d u = \sum_{j=0}^{N} u(\xi_j) L_j,$$

where the quasi-Lagrange functions L_j are linear combinations of a finite number of B-splines, according to Equation (4).

It is well known (see, e.g., [16]) that the operators Q_n^d are uniformly bounded and for smooth functions $u \in C^{d+1}(I)$, there exists a positive constant C_1 independent of h, such that:

$$\|Q_n^d u - u\|_\infty \le C_1 \|u^{(d+1)}\|_\infty h^{d+1}. \tag{5}$$

In the case of an even degree, Q_n^d presents an interesting property. More precisely, we have the following theorem (see [12] for the proof):

Theorem 1. *Let Q_n^d be the discrete QI operator of degree d, d even, defined on $\mathcal{S}_d^{d-1}(I, \Theta_n)$ by Equation (3). Then, there exists a positive constant C_2 independent of h, such that:*

$$\left| \int_0^1 g(t)(Q_n^d u(t) - u(t))dt \right| \le C_2 h^{d+2} \|u^{(d+2)}\|_\infty,$$

for any function $u \in C^{d+2}(I)$ and any weight function g with $\|g'\|_1$ bounded.

Examples of spline QI operators of this kind can be found in [16,17]. In particular, in Section 4, we will use the operators Q_n^2 and Q_n^3, defined on $\mathcal{S}_2^1(I, \Theta_n)$ and $\mathcal{S}_3^2(I, \Theta_n)$, respectively (see [17] for their definition).

3. Superconvergent Nyström Method Based on Spline Quasi-Interpolants

In this section, we propose a method for the solution of Equation (1) based on the Nyström operator:

$$(K_n^N(u))(s) := \int_0^1 Q_n^d k(s, t, u(t))dt = \sum_{j=0}^{N} w_j k(s, \xi_j, u(\xi_j)),$$

where $w_j := \displaystyle\int_0^1 L_j(t)dt$. The operator K in Equation (2) is then approximated by:

$$K_n := Q_n^d K + K_n^N - Q_n^d K_n^N, \tag{6}$$

and the approximate solution satisfies:

$$u_n - K_n(u_n) = f. \tag{7}$$

Defining the operator T_n by $T_n(u_n) = f + K_n(u_n)$, Equation (7) can be written as $u_n = T_n(u_n)$.

3.1. Construction of the Approximate Solution

From Equations (6) and (7), we have:

$$\begin{aligned} u_n(s) = \;& f(s) + Q_n^d\left(\int_0^1 k(s,t,u_n(t))dt\right) + \sum_{j=0}^{N} w_j k(s,\xi_j,u_n(\xi_j)) \\ & - Q_n^d\left(\sum_{j=0}^{N} w_j k(s,\xi_j,u_n(\xi_j))\right) \\ = \;& f(s) + \sum_{j=0}^{N}\left(\int_0^1 k(\xi_j,t,u_n(t))dt\right)L_j(s) + \sum_{j=0}^{N} w_j k(s,\xi_j,u_n(\xi_j)) \\ & - \sum_{j=0}^{N}\left(\sum_{i=0}^{N} w_i k(\xi_j,\xi_i,u_n(\xi_i))\right)L_j(s). \end{aligned}$$

Denoting by $X_j := u_n(\xi_j)$ and $Y_j := \int_0^1 k(\xi_j,t,u_n(t))dt - \sum_{i=0}^{N} w_i k(\xi_j,\xi_i,u_n(\xi_i))$, then the approximate solution u_n can be written as:

$$u_n(s) = f(s) + \sum_{j=0}^{N} w_j k(s,\xi_j,X_j) + \sum_{j=0}^{N} Y_j\, L_j(s).$$

Fixing $s = \xi_\ell$, we obtain the $2(N+1)$ unknowns X_ℓ, Y_ℓ, $\ell = 0,\dots,N$ by solving:

$$\begin{cases} X_\ell = f(\xi_\ell) + \sum_{j=0}^{N} w_j k(\xi_\ell,\xi_j,X_j) + \sum_{j=0}^{N} Y_j\, L_j(\xi_\ell) \\ Y_\ell = \int_0^1 k\left(\xi_\ell,t,f(t) + \sum_{i=0}^{N} w_i k(t,\xi_i,X_i) + \sum_{i=0}^{N} Y_i\, L_i(t)\right)dt - \sum_{i=0}^{N} w_i k(\xi_\ell,\xi_i,X_i) \end{cases}.$$

3.2. Convergence of the Method

Within this section, we delve into the examination of the local existence and uniqueness of the approximate solution u_n and analyze the convergence of the method.

Theorem 2. *Let $\bar{u}$ be the unique solution of Equation (1) and assume that 1 is not an eigenvalue of $K'(\bar{u})$. Let $Q_n^d : C(I) \to \mathcal{S}_d^{d-1}(I,\Theta_n)$ be a spline QI operator of kind Equation (3) and let k, $\frac{\partial k}{\partial u} \in C^{d+1}(\Omega)$, $f \in C^{d+1}(I)$. Then, there exists a neighbourhood of $\bar{u}$, $\Lambda(\bar{u},\delta)$, which contains, for all n large enough, a unique solution u_n of Equation (7), and we have:*

$$c_1\alpha_n \le \|\bar{u} - u_n\|_\infty \le c_2\alpha_n, \tag{8}$$

where c_1 and c_2 are positive constants and:

$$\alpha_n = \left\|(I - T_n'(\bar{u}))^{-1}(T(\bar{u}) - T_n(\bar{u}))\right\|_\infty \to 0, \quad n \to \infty.$$

Moreover, it holds:

$$\|\bar{u} - u_n\|_\infty = \begin{cases} O(h^{2d+2}), & \text{if } d \text{ is odd} \\ O(h^{2d+3}), & \text{if } d \text{ is even} \end{cases}.$$

Proof. From the stated hypothesis and from Equation (5), we can use Lemmas 1 and 2 in [9] and deduce that $(I - T_n'(\bar{u}))$ is invertible for n large enough, such that:

$$\|(I - T_n'(\bar{u}))^{-1}\| \le \Gamma_1,$$

and that $K_n'(\bar{u})$ is Lipschitz continuous in a neighbourhood of $\bar{u}$, with constant Γ_2. Therefore, for $\|\bar{u} - u\|_\infty \le \delta$:

$$\|T_n'(\bar{u}) - T_n'(u)\|_\infty = \|K_n'(\bar{u}) - K_n'(u)\|_\infty \le \Gamma_2\delta,$$

and

$$\sup_{\|\bar{u}-u\|_\infty \le \delta} \left\| (I - T_n'(\bar{u}))^{-1}(T_n'(\bar{u}) - T_n'(u)) \right\|_\infty \le \Gamma_1\Gamma_2\delta = \Gamma.$$

We can choose δ small enough that $\Gamma \le 1$. Moreover,

$$\begin{aligned} \alpha_n &= \left\|(I - T_n'(\bar{u}))^{-1}(T(\bar{u}) - T_n(\bar{u}))\right\|_\infty \le \Gamma_1 \|T(\bar{u}) - T_n(\bar{u})\|_\infty \\ &= \Gamma_1 \|K(\bar{u}) - K_n(\bar{u})\|_\infty = \Gamma_1 \left\|(I - Q_n^d)(K(\bar{u}) - K_n^N(\bar{u}))\right\|_\infty \\ &\le \Gamma_1 \left(1 + \left\|Q_n^d\right\|_\infty\right) \max_{s\in I} \int_0^1 \left|(I - Q_n^d)k(s,t,u(t))\right| dt, \end{aligned}$$

then $\alpha_n \to 0$, when $n \to 0$ and we can choose n large enough, such that $\alpha_n \le \delta(1-\Gamma)$. Therefore, we can apply Theorem 2 of [18] to conclude that the approximate solution is unique and Equation (8) holds. Now,

$$\|\bar{u} - u_n\|_\infty \le c_2\alpha_n \le c_2\Gamma_1 \left\|(I - Q_n^d)(K(\bar{u}) - K_n^N(\bar{u}))\right\|_\infty.$$

Since

$$\left[K(\bar{u}) - K_n^N(\bar{u})\right]^{(d+1)}(s) = \int_0^1 (I - Q_n^d)\frac{\partial^{(d+1)}k}{\partial s^{(d+1)}}(s,t,u(t))dt,$$

from Equation (5),

$$\left|\left[K(\bar{u}) - K_n^N(\bar{u})\right]^{(d+1)}(s)\right| \le C_1 \left\|\frac{\partial^{(d+1)}k}{\partial s^{(d+1)}}\right\|_\infty h^{d+1}$$

and, applying again Equation (5),

$$\begin{aligned} \left\|(I - Q_n^d)(K(\bar{u}) - K_n^N(\bar{u}))\right\|_\infty &\le C_1 \left\|\left[K(\bar{u}) - K_n^N(\bar{u})\right]^{(d+1)}(s)\right\|_\infty h^{d+1} \\ &\le C_1^2 \left\|\frac{\partial^{(d+1)}k}{\partial s^{(d+1)}}\right\|_\infty h^{2(d+1)}. \end{aligned}$$

If d is even, it is known that the error in the Nyström method is of order $d+2$. Therefore, in this case, the error in the superconvergent Nyström method is $O(h^{2d+3})$, which gives the thesis. □

3.3. Iterated Version

The iterated solution is defined by:

$$\tilde{u}_n = K(u_n) + f, \qquad (9)$$

with u_n given by Equation (7). In this case, we show that the convergence order is improved in case of degrees d even.

Theorem 3. *Let $\bar{u}$ be the unique solution of (1) and assume that 1 is not an eigenvalue of $K'(\bar{u})$. Let $Q_n^d : C(I) \to \mathcal{S}_d^{d-1}(I, \Theta_n)$ be a spline QI operator of kind (3) and let k, $\frac{\partial k}{\partial u} \in C^{d+2}(\Omega)$. Then, for all n large enough, the iterated solution defined by Equation (9) satisfies:*

$$\|\bar{u} - \tilde{u}_n\|_\infty = \begin{cases} O(h^{2d+2}), & \text{if } d \text{ is odd} \\ O(h^{2d+4}), & \text{if } d \text{ is even} \end{cases}.$$

Proof. From Equations (6) and (9), we have:

$$\begin{aligned} &\bar{u} - \tilde{u}_n = K(\bar{u}) - K(u_n) \\ &\tilde{u}_n - u_n = K(u_n) - K_n(u_n) = (I - Q_n^d)\big(K(u_n) - K_n^N(u_n)\big) \end{aligned}$$

and we consider the following identity:

$$
\begin{aligned}
(I-K'(\bar{u}))(\bar{u}-\tilde{u}_n) = \; & \left[I-K'(\bar{u})(I-Q_n^d)\right][K(\bar{u})-K(u_n)-K'(\bar{u})(\bar{u}-u_n)] \\
& +K'(\bar{u})(I-Q_n^d)(K-K_n^N)(\bar{u}) \\
& +K'(\bar{u})(I-Q_n^d)\left[K_n^N(\bar{u})-K_n^N(u_n)-K_n'(\bar{u})(\bar{u}-u_n)\right] \\
& -K'(\bar{u})(I-Q_n^d)(K'(\bar{u})-K_n'(\bar{u}))(\bar{u}-u_n).
\end{aligned}
$$

Multiplying by $(I-K'(\bar{u}))^{-1}$, we obtain:

$$
\begin{aligned}
\bar{u}-\tilde{u}_n = \; & \left[I+(I-K'(\bar{u}))^{-1}K'(\bar{u})Q_n^d\right][K(\bar{u})-K(u_n)-K'(\bar{u})(\bar{u}-u_n)] \\
& +(I-K'(\bar{u}))^{-1}K'(\bar{u})(I-Q_n^d)(K-K_n^N)(\bar{u}) \\
& +(I-K'(\bar{u}))^{-1}K'(\bar{u})(I-Q_n^d)\left[K_n^N(\bar{u})-K_n^N(u_n)-K_n'(\bar{u})(\bar{u}-u_n)\right] \\
& -(I-K'(\bar{u}))^{-1}K'(\bar{u})(I-Q_n^d)(K'(\bar{u})-K_n'(\bar{u}))(\bar{u}-u_n)
\end{aligned}
$$

and, therefore:

$$
\begin{aligned}
\|\bar{u}-\tilde{u}_n\|_\infty \le \; & \left\|I+(I-K'(\bar{u}))^{-1}K'(\bar{u})Q_n^d\right\|_\infty \|K(\bar{u})-K(u_n)-K'(\bar{u})(\bar{u}-u_n)\|_\infty \\
& +\left\|(I-K'(\bar{u}))^{-1}\right\|_\infty \left\|K'(\bar{u})(I-Q_n^d)(K-K_n^N)(\bar{u})\right\|_\infty \\
& +\left\|(I-K'(\bar{u}))^{-1}K'(\bar{u})(I-Q_n^d)\right\|_\infty \|K_n^N(\bar{u})-K_n^N(u_n)-K_n'(\bar{u})(\bar{u}-u_n)\|_\infty \\
& +\left\|(I-K'(\bar{u}))^{-1}\right\|_\infty \left\|K'(\bar{u})(I-Q_n^d)(K'(\bar{u})-K_n'(\bar{u}))\right\|_\infty \|\bar{u}-u_n\|_\infty \\
= \; & (a)+(b)+(c)+(d).
\end{aligned}
$$

By using the generalized Taylor's theorem (see e.g., [19]), it can be shown that:

$$
\begin{aligned}
(a) \le C_2\|\bar{u}-u_n\|_\infty^2 &= \begin{cases} O(h^{2(2d+2)}), & \text{if } d \text{ is odd} \\ O(h^{2(2d+3)}), & \text{if } d \text{ is even} \end{cases}, \\
(c) \le C_3\|\bar{u}-u_n\|_\infty^2 &= \begin{cases} O(h^{2(2d+2)}), & \text{if } d \text{ is odd} \\ O(h^{2(2d+3)}), & \text{if } d \text{ is even} \end{cases},
\end{aligned}
$$

for suitable constants C_2 and C_3. Moreover, by applying Theorem 1, Equation (5) and taking into account that the error in the Nyström method is of order $d+2$, if d is even, we obtain:

$$
\begin{aligned}
(b) &= \begin{cases} O(h^{2d+2}), & \text{if } d \text{ is odd} \\ O(h^{2d+4}), & \text{if } d \text{ is even} \end{cases}, \\
(d) &= \begin{cases} O(h^{3d+3}), & \text{if } d \text{ is odd} \\ O(h^{3d+4}), & \text{if } d \text{ is even} \end{cases},
\end{aligned}
$$

and the thesis follows. □

4. Numerical Results

This section includes the presentation of various test equations of type Equation (1), which are also explored in [6,9,10,13]. We solve these equations by the proposed method and its iterated version, based on the spline QI operators Q_n^2 and Q_n^3. We run the numerical alghoritms in Matlab on a PC with 11th Gen Intel(R) Core(TM) i7-1165G7 @ 2.80 GHz, 16 GB RAM.

The most expensive steps in the construction of the approximate solution and its iterated version are integral evaluation and the solution of nonlinear systems: the integrals appearing in the construction of the approximate solution are evaluated numerically by employing a conventional composite Gauss–Legendre quadrature formula known for its high accuracy, and the nonlinear systems are solved by using the Matlab function `fsolve`.

To illustrate the theoretical results of the previous sections, we compute the maximum absolute errors for increasing values of n:

$$
E_\infty := \max_{v\in G} |\bar{u}(v)-u_n(v)|, \qquad \tilde{E}_\infty := \max_{v\in G} |\bar{u}(v)-\tilde{u}_n(v)|,
$$

where G is a set of 1500 equally spaced points in I. Additionally, we calculate the corresponding numerical convergence orders, denoted as O_∞ and $\tilde{O}_\infty$, by taking the logarithm to the base 2 of the ratio between two consecutive errors. Observe that when we do not have results of errors for certain values of n in Tables 1–4, it is because the double float precision of Matlab implementation was reached.

The numerical tests confirm the theoretical properties proved in Sections 3.2 and 3.3. Moreover, we can compare the obtained results with others proposed in the literature [6,9,10,13]. In [9,10], a superconvergent Nyström method based on piecewise polynomials is proposed, and a Kulkarni technique based on piecewise polynomials and on spline QI projectors is considered in [6,13], respectively. We can state that the results are comparable with those obtained [13], whose numerical tests are based on splines of the same degree.

Finally, we remark that methods based on spline QI operators provide smooth approximate solutions of class C^1 if based on Q_n^2 and of class C^2 if based on Q_n^3.

Test 1

Consider the following Hammerstein integral operator K:

$$K(u)(s) = \int_0^1 p(s)q(t)u^2(t)dt, \qquad s \in I,$$

where $p(s) = \cos(11\pi s)$, $q(t) = \sin(11\pi t)$.

Then, K is compact and the integral equation $u - K(u) = f$ has a unique solution for $f \in C(I)$.

We choose $f(s) = \left(1 - \dfrac{2}{33\pi}\right)\cos(11\pi s)$, so that the exact solution is $\bar{u}(s) = \cos(11\pi s)$.

The results obtained, as presented in Table 1, corroborate the theoretical findings outlined in Theorems 2 and 3.

Table 1. Maximum absolute errors for Test 1.

n	E_∞	O_∞	$\tilde{E}_\infty$	$\tilde{O}_\infty$	n	E_∞	O_∞	$\tilde{E}_\infty$	$\tilde{O}_\infty$
Methods Based on Q_n^2					**Methods Based on Q_n^3**				
40	2.84 (−07)		9.10 (−09)		40	3.31 (−05)		3.04 (−07)	
80	3.99 (−08)	2.8	1.04 (−09)	3.1	80	4.60 (−07)	6.2	6.79 (−09)	5.5
160	2.98 (−10)	7.1	5.44 (−12)	7.6	160	5.86 (−10)	9.6	1.49 (−11)	8.8
320	2.27 (−12)	7.0	2.42 (−14)	7.8	320	6.93 (−12)	6.4	1.97 (−13)	6.2

Test 2

Consider the following Hammerstein integral equation:

$$u(s) + \int_0^1 e^{s-2t}u^3(t)dt = e^{s+1}, \qquad s \in I,$$

with the exact solution $\bar{u}(s) = e^s$.

We obtain the results reported in Table 2 that confirm the theoretical ones stated in Theorems 2 and 3.

Test 3

Consider the following Urysohn integral equation:

$$u(s) - \int_0^1 \frac{dt}{s + t + u(t)} = f(s), \qquad s \in I.$$

We choose the function f in a way that $\bar{u}(t) = \dfrac{1}{t+c}$, $c > 0$ becomes a solution. For our analysis, we take two cases: $c = 1$ and $c = 0.1$. It should be noted that the exact solution is ill behaved when $c = 0.1$.

Table 2. Maximum absolute errors for Test 2.

n	E_∞	O_∞	$\tilde{E}_\infty$	$\tilde{O}_\infty$	n	E_∞	O_∞	$\tilde{E}_\infty$	$\tilde{O}_\infty$
Methods Based on Q_n^2					**Methods Based on Q_n^3**				
8	1.15 (−10)		2.54 (−12)		8	1.67 (−10)		3.96 (−11)	
16	1.09 (−12)	6.8	9.77 (−15)	8.0	16	1.27 (−12)	7.0	4.03 (−13)	6.6
32	1.07 (−14)	6.7	-	-	32	8.88 (−15)	7.2	1.20 (−14)	-
64	-	-	-	-	64	-	-	-	-
128	-	-	-	-	128	-	-	-	-

In this context, we first focus on the case where $c = 1$, and the obtained results are shown in Table 3.

Table 3. Maximum absolute errors for Test 3, with $c = 1$.

n	E_∞	O_∞	$\tilde{E}_\infty$	$\tilde{O}_\infty$	n	E_∞	O_∞	$\tilde{E}_\infty$	$\tilde{O}_\infty$
Methods Based on Q_n^2					**Methods Based on Q_n^3**				
8	3.97 (−09)		2.71 (−11)		8	2.25 (−08)		5.72 (−10)	
16	3.61 (−11)	6.8	1.57 (−13)	7.4	16	1.95 (−10)	6.9	1.02 (−11)	5.8
32	2.93 (−13)	6.9	1.78 (−15)	-	32	9.60 (−13)	7.7	6.39 (−14)	7.3
64	5.77 (−15)	-	-	-	64	3.33 (−15)	6.2	1.89 (−15)	-
128	-	-	-	-	128	-	-	-	-

When we take $c = 0.1$ and apply the same procedures, we obtain the results presented in Table 4.

Table 4. Maximum absolute errors for Test 3, with $c = 0.1$.

n	E_∞	O_∞	$\tilde{E}_\infty$	$\tilde{O}_\infty$	n	E_∞	O_∞	$\tilde{E}_\infty$	$\tilde{O}_\infty$
Methods Based on Q_n^2					**Methods Based on Q_n^3**				
8	2.55 (−10)		4.69 (−12)		8	4.34 (−10)		2.33 (−11)	
16	2.28 (−12)	6.8	2.66 (−14)	7.5	16	5.69 (−12)	6.3	1.71 (−13)	7.1
32	1.60 (−14)	7.2	1.78 (−15)	-	32	1.42 (−14)	8.6	1.78 (−15)	-
64	-	-	-	-	64	-	-	-	
128	-	-	-	-	128	-	-	-	

Additionally, Test 3 provides further confirmation of the theoretical results mentioned in Theorems 2 and 3.

5. Conclusions

In this paper, we have proposed a superconvergent Nyström method based on spline quasi-interpolating operators in the space of splines of degree d and of class C^{d-1} on uniform partitions of a bounded interval for the numerical solution of nonlinear Urysohn integral equations, getting results related to the convergence orders and also analyzing the iterated version. In particular, we have proved that the convergence order is $(2d+2)$ if d is odd, $(2d+3)$ if d is even and, in case of even degrees, the convergence order of the iterated solution increases to $(2d + 4)$. Finally, we have provided numerical tests confirming the theoretical results.

We recall that the proposed method outperforms the classical Nyström approach in accuracy and is also more cost-effective than Kulkarni's scheme, as evidenced by a detailed comparison of their computational costs in the linear case (refer to Remark 4.1 in [11]). Moreover, it preserves the smoothness of the exact solution and allows a reduction of the computational costs in terms of the number of evaluation points in the operator construction with respect to the construction of QI projectors.

The results obtained in this work can be extended to integral equations with not sufficiently smooth kernels (e.g., Green's-function-type kernels) or weakly singular kernels (e.g., logarithmic kernels).

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