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Article

Acyclic Complexes and Graded Algebras

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Abstract: We already know that the noncommutative $\mathbb{N}$-graded Noetherian algebras resemble commutative local Noetherian rings in many respects. We also know that commutative rings have the important property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic, and we want to generalize such properties to noncommutative $\mathbb{N}$-graded Noetherian algebra. By generalizing the conclusions about commutative rings and combining what we already know about noncommutative graded algebras, we identify a class of noncommutative graded algebras with the property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic. We also discuss how the relationship between AS–Gorenstein algebras and AS–Cohen–Macaulay algebras admits a balanced dualizing complex. We show that AS–Gorenstein algebras and AS–Cohen–Macaulay algebras with a balanced dualizing complex belong to this algebra.

Keywords: AS–Gorenstein algebra; AS–Cohen–Macaulay algebra; acyclic complex; totally acyclic complex; balanced dualizing complex

MSC: 16E05; 16E35; 18A50; 18G05; 18G20; 18G35

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1. Introduction

Recent studies have shown that noncommutative $\mathbb{N}$-graded Noetherian algebras are similar to commutative local Noetherian rings in many respects. For example, one has constructed a theory of the G-dimension and local cohomology in noncommutative $\mathbb{N}$-graded algebra; see [1,2]. People have also extended the (commutative) Gorenstein ring and G-class concepts to noncommutative concepts and proved the Auslander–Buchsbaum formula and the Auslander–Bridger formula in noncommutative graded situations; see [1,3,4]. In [5], the authors give the classes of rings that satisfy the property of every minimal acyclic complex of finitely generated free modules is totally acyclic. It is natural to look for the classes of noncommutative $\mathbb{N}$-graded algebras with the property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic.

The classes of commutative rings have the property that every minimal acyclic complex of finitely generated free R-modules is totally acyclic, as studied by Hughes, Meri Jorgensen, David Ega, and Liana [5]. In [5], the authors relied on the G-dimension to demonstrate that every minimal acyclic complex of finitely generated free R-modules is totally acyclic; the Auslander–Buchsbaum formula, the Auslander–Bridger formula, and the depth lemma of the R-module are also used in the proof. The noncommutative versions of these contents have been presented by Jørgense [3], Ueyama, Kenta [1], and Huckaba S, Marley T [6], respectively. In this paper, we will continue to explore the properties of the G-dimension and the G-class of graded modules and combine the G-dimension with the short exact sequence of graded left A-modules to explore the interrelationship of the G-dimension among the three elements in the short exact sequence. I try to find a class of noncommutative graded algebra with the property that every minimal acyclic complex of finitely generated graded free left A-modules is totally acyclic. We will also discuss the relationship between AS–Gorenstein algebras and AS–Cohen–Macaulay algebras with a balanced dualizing complex; the results obtained in this paper will complement the results of [7]. Through the obtained relations, we know that AS–Cohen–Macaulay algebras with a balanced dualizing complex are the ones satisfying the property that every minimal acyclic complex of finitely generated graded free left A-module is totally acyclic. This result complements the conclusions related to the commutative ring in [5] on noncommutative graded algebra. This paper is summarized as follows:

- Section 2. This Section introduces notation and conventions regarding graded algebras and modules and recalls some well-known results.
- Section 3. We introduce the definition of the coproduct and product of graded modules and prove some properties of the functors acting on the direct sum of the graded modules and on the projective graded modules.
- Section 4. We focus on the proof of Theorem 4, which can be seen as the G-class equivalence definition.
- Section 5. We introduce the properties of the G-dimension and G-class in the noncommutative case and combine the G-dimension with the short exact sequence of the graded left A-module to explore the interrelationship of the G-dimension among the three elements in the short exact sequence.
- Section 6. We prove that there is an inequality $\text{G-dim}_A M \leq \text{pd}_A M$, and the equality holds if $\text{pd}_A M < \infty$ for every finitely generated graded left A-module M. We still proved the Auslander–Bridger formula while satisfying the condition $\chi^{\circ}_{\text{depth}_A(A)}({}_A A)$.
- Section 7. In this section, we will find a class of noncommutative graded algebras that satisfies the property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic and that contains the AS–Gorenstein algebra. The following theorem is the main result of this section, and covers the algebraic class we are looking for.

Theorem 1 (Theorem 11)**.** *Let A be a left Noetherian connected graded algebra with $\text{id}_A A = \text{id}_{A^\circ} A = n < \infty$ and satisfying the condition $\chi^{\circ}_{\text{depth}_A(A)}({}_A A)$, then every minimal acyclic complex of finitely generated graded free left A-module is totally acyclic.*

It is easy to know that an AS–Gorenstein algebra has the property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic. But AS–Gorenstein algebras and the property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic are not mutually equivalent. So, the above theorem extends AS–Gorenstein algebras to a larger class of noncommutative graded algebras, such that this class of algebras also satisfies the property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic.

- Section 8. In this section, our main content is to prove two theorems.

Theorem 2 (Theorem 14)**.** *A is an AS–Gorenstein algebra if and only if A is an AS–Cohen–Macaulay algebra with a balanced dualizing complex.*

One should compare Theorem 2 with the results in ([7], Theorem 1.6). We know that this theorem is complementary to the results of the paper [7].

Theorem 3 (Theorem 15)**.** *Let B be an AS–Cohen–Macaulay algebra with a balanced dualizing complex, then B satisfies the property that every minimal acyclic complex of finitely generated graded free left B-modules is totally acyclic.*

According to the relation between an AS–Cohen–Macaulay algebra with a balanced dualizing complex and an AS–Gorenstein algebra given by Theorem 2 and the related results in Section 7, the above theorem shows that an AS–Cohen–Macaulay algebra with a balanced dualizing complex also has the property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic. This theorem complements the conclusions related to the commutative ring in [5] on noncommutative graded algebras.

2. Preliminaries

In this section, let us review some of the basics of coderived functors and of the hyperhomology of the category of graded modules over a graded algebra. We refer to [1,3,8] for more basics about graded algebras.

2.1. Algebras

An $\mathbb{N}$-*graded* k-*algebra* A is a graded k-vector space $A = \oplus_{i \in \mathbb{N}} A_i$ with an associative multiplication, such that unit 1 is in A_0 and the multiplication preserves the grading; thus, $A_i A_j \subseteq A_{i+j}$. $|a|$ denotes the degree of a graded or homogeneous element a. We use A^o to denote the opposite algebra of A.

A graded algebra A is a *connected* graded algebra when $A_0 = k$. In these circumstances, we denote $k = A/m$, $m = \oplus_{i=1}^{\infty} A_i$; it defines an obvious augmentation of A.

A graded algebra A connected over k is called the *AS–Gorenstein* of dimension n, and of Gorenstein parameter l if $\mathrm{id}_A A = \mathrm{id}_{A^o} A = n$ and A satisfies the Gorenstein condition (cf. [4], Definition 1.1), that is,

$$\underline{\mathrm{Ext}}^i_A(k, A) \cong \underline{\mathrm{Ext}}^i_{A^o}(k, A) \cong \begin{cases} k(l) & if\ i = n \\ 0 & otherwise. \end{cases}$$

We call A an *AS–Cohen–Macaulay algebra* when A is a Noetherian, $\mathbb{N}$-graded, connected k-algebra, for which the complex $R\Gamma_{\mathfrak{m}}(A)$ is concentrated in one degree (cf. [7], Definition 1.1).

2.2. Graded Modules

Let A be a graded algebra and M be an A-module. M is called a *left graded module* over A if there exists a family of k-vectorspaces $\{M_n\}_{n \in \mathbf{Z}}$ of M, such that

(1) $M = \bigoplus_{i=-\infty}^{\infty} M_i$; and
(2) $A_n \cdot M_m \subseteq M_{n+m}$ for any $n, m \in \mathbb{Z}$.

Let $M = \bigoplus_{n=-\infty}^{\infty} M_n$ be a graded left A-module and N be a submodule of M. For each $n \in \mathbb{Z}$, let $N_n = N \cap M_n$. If the family of subvectorspaces $\{N_n\}_{n \in \mathbf{Z}}$ makes N a graded left A-module, then N is a *graded submodule* of M.

The n'th *shift* of M is defined by $M(n)_i = M_{n+i}$.

A graded module M is a *left-bounded graded module* if $M_i = 0$ for a sufficiently large negative; *right-boundness graded module* and *boundness graded module* are defined similarly. The graded left A-module M is *a locally finite graded module* if each graded piece M_i is a finitely generated k-vectorspace. The graded left A-module M is called *finitely generated* if there exist $m_1, m_2 \dots m_n$ in M, such that for any m in M, there exists $a_1, a_2 \dots a_n$ in A with $m = a_1 m_1 + a_2 m_2 + \cdots + a_n m_n$. The graded left A-module M is *a free graded module* if it is isomorphic to $\oplus A(n_i)$ for some $n_i \in \mathbb{Z}$.

The graded A-module M is called a *graded Noetherian module* (*graded Artinian module*) if every ascending (descending) chain of graded submodules

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

stops. If ${}_A A$ is a graded Noetherian (graded Artinian) left A-module, then A is a *left graded Noetherian algebra* (*left graded Artinian algebra*). If A is connected and left-Noetherian, then it is a locally finite module over itself.

A finitely generated graded left A-module belongs to the *G-class* $\mathrm{G(A)}$ (cf. [1,9]) if and only if

(1) $\underline{\mathrm{Ext}}^i_A(M, A) = 0$ for $i > 0$;
(2) $\underline{\mathrm{Ext}}^i_{A^o}(\underline{\mathrm{Hom}}_A(M, A), A) = 0$ for $i > 0$; and
(3) The biduality map $\sigma_M : M \to \underline{\mathrm{Hom}}_{A^o}(\underline{\mathrm{Hom}}_A(M, A), A)$ is an isomorphism.

Similarly definable, a finitely generated right A-module M belongs to the G-class; we still call M *belongs to* G-*class*, notated as $M \in \mathrm{G(A^o)}$.

The proofs of $M \in \mathrm{G}(\mathrm{A}^{\mathrm{o}})$ and $M \in \mathrm{G}(\mathrm{A})$ are similar, so we only prove the relevant conclusion of $M \in \mathrm{G}(\mathrm{A})$.

2.3. Homomorphisms

Let A be an $\mathbb{N}$-graded k-algebra and M, N be graded left A-modules. Let $f : M \to N$ be an A-module homomorphism. Then, f is said to be *graded* or *homogeneous* of *degree d* if $f(M_n) \subseteq N_{n+d}$ for all n.

GrMod(A) denoted the category whose objects are graded left-modules over A and whose morphisms are A-linear homomorphisms of degree 0. *grmod(A)* denoted the full subcategory of finitely generated graded modules. We denote $\mathrm{Hom}_{\mathrm{GrMod}(A)}$ for the homomorphism-functor in the category GrMod(A). Define

$$\underline{\mathrm{Hom}}_A(M,N) = \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{\mathrm{GrMod}(A)}(M, N(n)).$$

Let A be a graded algebra; the two graded A-modules M and N are called *isomorphic* as graded modules if there exists a degree 0 isomorphism between them. Likewise, the two graded algebras A and B are said to be *isomorphic* as graded algebra if there exists a degree 0 isomorphism between them.

The two graded left A-modules are *projectively equivalent* if $\exists P$, Q graded projective left A-module with $M \oplus P \cong N \oplus Q$. Notation: $M \approx N$.

2.4. Duality

For a graded left A-module, we are given the following definition:

$$M^* = \underline{\mathrm{Hom}}_A(M,A) \text{ and } M^{**} = \underline{\mathrm{Hom}}_{A^o}(\underline{\mathrm{Hom}}_A(M,A),A)$$

We called the modules M^* and M^{**}, respectively, the *dual* and *bidual* of M. Similarly, if M is a graded right A-module, then $M^* = \underline{\mathrm{Hom}}_{A^o}(M,A)$.

Let $\pi : P_1 \xrightarrow{u} P_0 \xrightarrow{f} M \to 0$ be a sequence of graded module morphisms. π is called a *(finitely generated) projective presentation* if and only if

$$\pi : P_1 \xrightarrow{u} P_0 \xrightarrow{f} M \to 0$$

is an exact sequence, P_1 and P_0 are (finitely generated) projective graded A-modules.

Let M be any finitely generated graded left A-module, and let

$$\pi : P_1 \xrightarrow{u} P_0 \xrightarrow{f} M \to 0$$

be a finitely generated projective presentation of M. The *Auslander dual*, $D(M)$ of M is defined as

$$D(M) = \mathrm{Coker}(u^* : P_0^* \to P_1^*)$$

in other words, by dualizing (π), we obtain an exact sequence

$$\pi^* : 0 \to M^* \xrightarrow{f^*} P_0^* \xrightarrow{u^*} P_1^* \to D(M) \to 0.$$

Note that the Auslander dual of M is not unique, and they are natural isomorphisms to each other in $\underline{\mathrm{grmod}}(A)$.

2.5. Complexes

A *complex* of graded left modules over A is a sequence

$$X = \cdots \to X^{u-1} \xrightarrow{\delta^{u-1}} X^u \xrightarrow{\delta^u} X^{u+1} \to \cdots$$

of morphisms of graded left modules over A such that

$$\delta^u \delta^{u-1} = 0 \text{ for all } u \in \mathbb{Z}.$$

The nth *syzygy module* of X is $\Omega^n X = \mathrm{Coker}\, \delta_X^{n-1}$ for each integer n. The cohomology number m of a complex X is denoted by $h^m(X) = \ker \delta^m / \mathrm{Im}\delta^{m-1}$. *The twisting* of complexes is denoted by [ ] so that $(X[n])^p = X^{n+p}$ and $\delta^p_{x[n]} = (-1)^n \delta_X^{n+p}$, for $n \in \mathbb{Z}$.

The complex X of the finitely generated free graded left A-module is called *acyclic* if $h^m(X) = \ker \delta_X^m / \mathrm{Im}\delta_X^{m-1} = 0, m \in \mathbb{Z}$. We let $(\_\_)^*$ denote the functor $\underline{\mathrm{Hom}}_A(\_\_, A)$. The *dual complex* of X is the complex X^*, which has component $(X^{-n})^*$ in degree n, and differentials $\delta^n_{X^*} = (\delta_X^{-n-1})^* = \underline{\mathrm{Hom}}_A(\delta_X^{-n-1}, A)$. An acyclic complex of finitely generated free graded left A-modules is called *totally acyclic* if $h^m(X^*) = \ker \delta^m_{X^*} / \mathrm{Im}\delta^{m-1}_{X^*} = 0, m \in \mathbb{Z}$. The relevant definitions of the finitely generated free graded right A-modules are similar.

A *morphism* of complexes from a complex X to a complex

$$Y = \dots Y^{u-1} \overset{\epsilon^{u-1}}{\to} Y^u \overset{\epsilon^u}{\to} Y^{u+1} \to \dots$$

is a family $(\gamma^u \in \mathrm{Hom}_{\mathrm{GrMod}(A)}(X^u, Y^u))_{u \in z}$, such that $\gamma^{u+1}\delta^u = \epsilon^u \gamma^u$ for $u \in \mathbb{Z}$.

The induced map $h(f)$ of f on the cohomologies is a morphism of $h(A)$-modules. f is called a *quasi-isomorphism* if $h(f)$ is an isomorphism, denoted $f : X \overset{\simeq}{\to} Y$. X is called *quasitrivial* if $X \simeq 0$, X, Y, and 0 are complexes of graded modules.

A complex $X \in D^-(\mathrm{GrMod}(A))$ consisting of free modules is said to be *finitely generated* if it consists of finitely generated frees.

A complex $X \in D^-(\mathrm{GrMod}(A))$ consisting of free modules is said to be *minimal* if $\mathrm{Im}(\delta_X^{i-1}) \subseteq mX^i$ for each i.

Let A be a Noetherian connected graded k-algebra. A complex $R^{\cdot} \in \mathrm{D^b}(\mathrm{GrMod}A^e)$ is a *dualizing complex* if it meets the following conditions (cf. [8], Definitions 3.3 and 4.1):

(1) $\mathrm{id}_A(R^{\cdot}) < \infty$ and $\mathrm{id}_{A^o}(R^{\cdot}) < \infty$;
(2) $\mathrm{res}_A(R^{\cdot}) \in \mathrm{D^b_{fg}}(\mathrm{GrMod}A)$ and $\mathrm{res}_{A^o}(R^{\cdot}) \in \mathrm{D^b_{fg}}(\mathrm{GrMod}A^o)$;
(3) The natural morphisms $A \to R\underline{\mathrm{Hom}}_A(R^{\cdot}, R^{\cdot})$ and $A \to R\underline{\mathrm{Hom}}_{A^o}(R^{\cdot}, R^{\cdot})$ are isomorphisms in $\mathrm{D^b}(\mathrm{GrMod}A^e)$.

A dualizing complex $R^{\cdot}$ over A is said to be *balanced* if there are isomorphisms

$$R\Gamma_{\mathfrak{m}}(R^{\cdot}) \cong R\Gamma_{\mathfrak{m}^o}(R^{\cdot}) \cong A'$$

in $\mathrm{D^b}(\mathrm{GrMod}A^e)$.

2.6. Truncation

For a complex of graded left A-modules and each integer n, *left brutal truncation* of complex X is denoted by $X_{\leq n}$, which has components

$$X^i_{\leq n} = \begin{cases} X^i, & i \leq n \\ 0, & i > n \end{cases}$$

and differentials

$$\delta^i_{\leq n} = \begin{cases} \delta^i, & i \leq n-1 \\ 0, & i \geq n \end{cases}$$

that is,

$$\cdots \overset{\delta^{n-3}}{\longrightarrow} X^{n-2} \overset{\delta^{n-2}}{\longrightarrow} X^{n-1} \overset{\delta^{n-1}}{\longrightarrow} X^n \longrightarrow 0 \longrightarrow 0 \cdots .$$

Define the *right brutal truncation* $X_{\geq n}$ as the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^n \overset{\delta^n}{\longrightarrow} X^{n+1} \overset{\delta^{n+1}}{\longrightarrow} X^{n+2} \overset{\delta^{n+2}}{\longrightarrow} \cdots .$$

Define the *left truncation* $\sigma_{\leq n}X$ as the complex

$$\cdots \xrightarrow{\delta^{n-3}} X^{n-2} \xrightarrow{\delta^{n-2}} X^{n-1} \longrightarrow \operatorname{Ker}\delta^{n} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots .$$

Define the *right truncation* $\sigma_{\geq n}X$ as the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{Im}\delta^{n-1} \longrightarrow X^{n} \xrightarrow{d^{n}} X^{n+1} \xrightarrow{d^{n+1}} \cdots .$$

Note that the cohomology of the left truncation $\sigma_{\leq n}X$ and the right truncation $\sigma_{\geq n}X$ at n is still $h^n(X)$.

2.7. Resolutions

Let A be an $\mathbb{N}$-graded k-algebra; a *G-resolution* of a finitely generated graded left A-module M is a sequence of modules in $\mathrm{G(A)}$,

$$\cdots G^{-l} \xrightarrow{d^{-l}} G^{-l+1} \xrightarrow{d^{-l+1}} \cdots \to G^{-1} \xrightarrow{d^{-1}} G^{0} \xrightarrow{d^{0}} 0$$

which is exact at G^{-l} for $l > 0$ and has $G^0/\mathrm{Im}(G^{-1} \to G^0) \cong M$. That is, there is an exact sequence,

$$\cdots G^{-l} \xrightarrow{d^{-l}} G^{-l+1} \xrightarrow{d^{-l+1}} \cdots \to G^{-1} \xrightarrow{d^{-1}} G^{0} \xrightarrow{d^{0}} M \to 0.$$

The resolution is said to be of *length n* if $G^{-n} \neq 0$ and $G^{-l} = 0$ for $l > n$.

A finitely generated graded left A-module M is called having a *finite G-dimension*, denoted $\text{G-dim}_A M \leq \infty$, if M has a G-resolution of finite length. We set $\text{G-dim}_A 0 = -\infty$. For $M \neq 0$ and $n \in \mathbb{N}_0$, M has the *G-dimension at most n*, denoted $\text{G-dim}_A M \leq n$, if and only if M has a G-resolution of length n. If M has no G-resolution of finite length, then M has an *infinite G-dimension*, denoted $\text{G-dim}_A M = \infty$.

A finitely generated graded left A-module M is called having a *G-dimension g*, denoted $\text{G-dim}_A M = g$, if g is the smallest integer, such that there exists an exact sequence $0 \to G^g \to G^{g-1} \to \cdots \to G^1 \to G^0 \to M \to 0$, with each G^i belonging to the G-class. Thus, if the finitely generated graded left A-module M belongs to the G-class, then $\text{G-dim} M = 0$.

A *projective resolution* of a finitely generated graded left A-module M is a sequence of projective graded left A-modules,

$$\cdots P^{-l} \xrightarrow{d^{-l}} P^{-l+1} \xrightarrow{d^{-l+1}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{d^{0}} 0$$

which is exact at P^{-l} for $l > 0$ and has $P^0/\mathrm{Im}(P^{-1} \to P^0) \cong M$. That is, there is an exact sequence,

$$\cdots P^{-l} \xrightarrow{d^{-l}} P^{-l+1} \xrightarrow{d^{-l+1}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{d^{0}} M \to 0.$$

The resolution is said to be of *length n* if $P^{-n} \neq 0$ and $P^{-l} = 0$ for $l > n$.

For a finitely generated graded left A-module M, take an exact sequence $0 \to \Omega_A^n(M) \to P^{-n+1} \xrightarrow{d^{-n+1}} \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} M \to 0$, where P^{-i} is a finitely generated graded projective left A-module$(0 \leq i \leq -n+1)$.

For each $n > 0$, the module $\Omega_A^n(M)$ is called the *nth syzygy* module of M. Note that syzygy modules of M are not uniquely determined, while they are natural isomorphisms to each other in $\underline{\mathrm{grmod}}(A)$.

A finitely generated graded left A-module M is said to have *a finite projective dimension*, denoted $\mathrm{pd}_A M \leq \infty$, if M has a projective resolution of finite length. We set $\mathrm{pd}_A 0 = -\infty$. For $M \neq 0$, we define the projective dimension of M as follows: For $n \in \mathbb{N}_0$, M has *projective dimension at most n* and denoted $\mathrm{pd}_A M \leq n$ if and only if M has a projective resolution of length n. If M has no projective resolution of finite length, then M has *infinite projective dimension*, denoted $\mathrm{pd}_A M = \infty$.

A finitely generated graded left A-module M is called have *projective dimension g*, denoted $\mathrm{pd}_A M = g$, if g is the smallest integer, such that there exists an exact sequence $0 \to P^{-g} \to P^{-g+1} \to \cdots \to P^{-1} \to P^0 \to M \to 0$, where each P_i is a projective graded left A-module. Thus, if finitely generated graded left A-module M is a projective module, then $\mathrm{pd}M = 0$.

Let A be an $\mathbb{N}$-graded k-algebra, an *injective resolution* of a finitely generated graded left A-module M is a sequence of injective modules,

$$0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots I^g \xrightarrow{d^g} \cdots$$

which is exact at I^g for $g > 0$ and has $\ker d^0 \cong M$. That is, there is an exact sequence,

$$M \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots I^g \xrightarrow{d^g} \cdots .$$

The resolution is said to be of *length n* if $I^n \neq 0$ and $I^l = 0$ for $l > n$.

A finitely generated graded left A-module M is called having a *finite injective dimension*, denoted $\mathrm{id}_A M \leq \infty$, if M has an injective resolution of finite length. We set $\mathrm{id}_A 0 = -\infty$. For $M \neq 0$ and $n \in \mathbb{N}_0$, M has an *injective dimension at most n*, denoted $\mathrm{id}_A M \leq n$, if and only if M has an injective resolution of length n. If M has no injective resolution of finite length, then M has an *infinite injective dimension*, denoted $\mathrm{id}_A M = \infty$.

A finitely generated graded left A-module M is called having *injective dimension g*, denoted $\mathrm{id}_A M = g$, if g is the smallest integer, such that there exists an exact sequence $M \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \ldots I^g \xrightarrow{d^g} 0$, with each I^i being a graded injective left A-module.

2.8. Category of Graded A-Modules

The category of graded left A-modules is denoted by $\mathrm{GrMod}(A)$, whose morphisms are graded module morphisms. We denote by $\mathrm{grmod}(A)$ the category of finitely generated graded left A-modules whose morphisms are graded module morphisms.

We denote by $\underline{\mathrm{grmod}}(A)$ the stable category of $\mathrm{grmod}(A)$ modulo graded projective modules: the objects are the same as $\mathrm{grmod}(A)$, while the morphism space between two objects M and N is, by definition, the quotient k-vectorspace $\underline{\mathrm{Hom}}_A(M,N)/\underline{\mathrm{P}}_A(M,N)$, where $\underline{\mathrm{P}}_A(M,N)$ is the k-subvectorpace of $\underline{\mathrm{Hom}}_A(M,N)$ consisting of morphisms factoring through projective modules. The stable category $\underline{\mathrm{grmod}}(A)$ is additive, and projective modules are zero objects. Moreover, two modules M and N become isomorphic in $\underline{\mathrm{grmod}}(A)$, denoted by $M \approx N$, if and only if there exist graded projective left A-modules P and Q such that $M \oplus P \cong N \oplus Q$;

The derived category of graded A-modules is denoted by $\mathrm{D}(\mathrm{GrMod}(A))$, which is constructed from the category $\mathrm{GrMod}(A)$ by inverting quasi-isomorphisms. The derived categories will be equipped with superscripts and subscripts when we only consider certain types of complexes: superscripts and subscripts when we only consider certain types of complexes: superscripts "$+$", "$-$", and "b", respectively, decorate the signs for categories of right- and left-bounded complexes, respectively, while subscripts "fg" and "lf", respectively, decorate the signs for categories of complexes of finitely generated respectively locally finite modules. These decorations can be combined arbitrarily; for example, $D^{-}_{fg}(\mathrm{GrMod(A)})$ is the derived category whose objects are left-bounded complexes of finitely generated graded left A-modules.

The right derived functor of Hom is denoted $R\mathrm{Hom}$, and the left derived functor of $\otimes$ is denoted ${}^{L}\otimes$. They can be computed via projective, injective, and flat resolutions of the graded modules. For any $M, N \in \mathrm{D}(\mathrm{GrMod}(A))$ and $T \in \mathrm{D}(\mathrm{GrMod}(A^o))$, let $F \xrightarrow{\simeq} M$,

$N \xrightarrow{\simeq} I$, and $P \xrightarrow{\simeq} T$ be the projective resolution of M, injective resolution of N, and flat resolution of T, respectively. Then,

$$R\mathrm{Hom}_A(M,N) = \mathrm{Hom}_A(F,N) \text{ or } R\mathrm{Hom}_A(M,N) = \mathrm{Hom}_A(M,I)$$

$$T\,{}^L\!\otimes_A M = P \otimes_A M$$

and

$$\underline{\mathrm{Ext}}^i_A(X,Y) = h^i R\underline{\mathrm{Hom}}_A(X,Y), \quad \mathrm{Tor}^A_i(X,Y) = h^{-i}\Big(X\ {}^L\!\otimes_A Y\Big).$$

We say that $\chi^\circ_i(M)$ holds for an A-module M if $\underline{\mathrm{Ext}}^j(A_0, M)$ is bounded for all $j \leq i$. If $\chi^\circ_i(M)$ holds for every finite A-module M, we say that χ°_i holds for the graded algebra A, and if χ°_i holds for every i, we say that χ° holds for A (cf. [10], Definition 3.2).

In the rest of this paper, A will always be a connected and left-Noetherian graded algebra over a field k, All modules mentioned in the rest of this paper are finitely generated graded left A-modules. We denote its maximal graded ideal $\oplus_{i=1}^{\infty} A_i$ by m_A or m. Obviously, the residue field k has a graded A-module structure via the canonical augmentation map $\varepsilon : A \to k$. We abbreviate the property that every minimal acyclic complex of finitely generated graded free modules is totally acyclic as ac = tac.

3. Properties of Finitely Generated Graded Left A-Module

First, we introduce the definition of a coproduct and product of a graded module. Then, we prove some properties of the functors acting on the direct sum of the graded modules and on the graded projective modules.

If M_i, $i \in \mathbb{Z}$ are graded left A-modules, then the category $\mathrm{GrMod}(A)$ has a small *coproduct* and *product*, given by

$$\coprod_i M_i = \bigoplus_n \bigoplus_i M_{i,n}$$

$$\prod_i M_i = \bigoplus_n \prod_i M_{i,n}.$$

If M' and M'' are graded left A-modules, then the formula

$$(M' \oplus M'')_i = M'_i \oplus M''_i$$

defines a graded module $M' \oplus M''$, the *direct sum* of M' and M''. The maps

$$\begin{array}{rcl} M' \xrightarrow{\iota'} & M' \oplus M'' \xrightarrow{\pi'} M' \\ m' \longmapsto & (m', 0) \\ & (m', m'') \longmapsto m' \end{array}$$

$$\begin{array}{rcl} M'' \xrightarrow{\iota''} & M' \oplus M'' \xrightarrow{\pi''} M'' \\ m' \longmapsto & (m', 0) \\ & (m', m'') \longmapsto m' \end{array}$$

are morphisms of graded modules, which produce a natural isomorphism of A-module

$$\begin{array}{rcl} \underline{\mathrm{Hom}}_A(M' \oplus M'', N) & \xrightarrow{\cong} & \underline{\mathrm{Hom}}_A(M', N) \times \underline{\mathrm{Hom}}_A(M'', N) \\ \beta & \longmapsto & (\beta\iota', \beta\iota'') \\ \beta'\pi' + \beta''\pi'' & \longleftarrow & (\beta', \beta'') \end{array}$$

Thus $\underline{\mathrm{Hom}}_A(N \oplus M, X) \cong \underline{\mathrm{Hom}}_A(N, X) \oplus \underline{\mathrm{Hom}}_A(M, X)$.

In the following, we prove the properties of graded projective modules in the noncommutative graded algebra, which are obvious in the commutative algebra but differ in some way from the commutative in the noncommutative. For example, comparing Corollary 1 with ([11], Corollary 1.15) reveals the difference.

Lemma 1. *If P is a finitely generated graded projective left A-module, then $P^* = \underline{\mathrm{Hom}}_A(P, A)$ is a graded projective right A-module.*

Proof. If P is a finitely generated graded projective left A-module, we can find a free graded left A-module, such that $F = P \oplus Q \cong A^n$. Since $\underline{\mathrm{Hom}}_A(F, A) = \underline{\mathrm{Hom}}_A(P \oplus Q, A)$, then $P^* \oplus Q^* \cong \underline{\mathrm{Hom}}_A(A^n, A) = A^n$; thus, $P^* = \underline{\mathrm{Hom}}_A(P, A)$ is a graded projective right A-module. □

Proposition 1. *For every graded projective left A-module P, the canonical map $\sigma_P : P \to P^{**}$ is injective, in which $P^{**} = \underline{\mathrm{Hom}}_{A^o}(\underline{\mathrm{Hom}}_A(P, A), A)$.*

Proof. Let $\{x_i\}, \{f_i\}$ be a dual basis $\{x_i\}, \{f_i\}$ for P. Suppose that $x \neq 0$ and $x \mapsto (f \mapsto f(x)) = 0$, Then, $f(x) = 0$ for all $f \in P^*$, and in particular $f_i(x) = 0$, for all i. But $x = \sum_i f_i(x)x_i = 0$. So, we obtain a contradiction, and we conclude that $\sigma_P : P \to P^{**}$ is injective. □

Corollary 1. *For every finitely generated graded projective A-module P, the canonical map $\sigma_P : P \to P^{**}$ is an isomorphism, in which $P^{**} = \underline{\mathrm{Hom}}_{A^o}(\underline{\mathrm{Hom}}_A(P, A), A)$.*

Proof. By choosing a dual basis $x_1, \cdots, x_n \in P$ and $f_1, \cdots f_n \in P^*$ for P, we prove that $f_1, \cdots f_n \in P^* = \underline{\mathrm{Hom}}_A(P, A)$ and $\widehat{x}_1 \cdots \widehat{x}_n \in P^{**} = \underline{\mathrm{Hom}}_{A^o}(\underline{\mathrm{Hom}}_A(P, A), A)$ form a pair of dual bases of P^*. By the definition of the dual basis for P, we obtain $x = \sum_{i=i}^n f_i(x)x_i$ for all $x \in P$; applying an arbitrary of $f \in P^*$ to both sides of this equality yields $f(x) = f(\sum_{i=1}^n f_i(x)x_i) = \sum_{i=1}^n f_i(x)f(x_i) = \sum_{i=1}^n f_i(x)\widehat{x}_i(f)$. Hence, $f = \sum_{i=1}^n f_i\widehat{x}_i(f)$.

We prove that $\widehat{x}_1 \cdots \widehat{x}_n \in P^{**} = \underline{\mathrm{Hom}}_{A^o}(\underline{\mathrm{Hom}}_A(P, A), A)$ and $\widetilde{f}_1, \cdots \widetilde{f}_n \in P^{***} = \underline{\mathrm{Hom}}_A(P^{**}, A)$ form a pair of the dual bases of P^{**}. By the definition of dual basis for P^*, we obtain $f = \sum_{i=i}^n f_i\widehat{x}_i(f)$ for all $f_i \in P^*$, applying an arbitrary of $\widehat{x} \in P^{**}$ to both sides of this equality yields $\widehat{x}(f) = \widehat{x}(\sum_{i=i}^n f_i\widehat{x}_i(f)) = \sum_{i=1}^n \widehat{x}(f_i)\widehat{x}_i(f) = \sum_{i=1}^n \widetilde{f}_i(\widehat{x})\widehat{x}_i(f)$. Hence $\widehat{x} = \sum_{i=1}^n \widetilde{f}_i(\widehat{x})\widehat{x}_i$. Since each $\widehat{x}_i$ has preimage x_i, then $P \to P^{**} = \underline{\mathrm{Hom}}_{A^o}(\underline{\mathrm{Hom}}_A(P, A), A)$ is a surjective A-linear map. We also showed injectivity in Proposition 1. Thus, this map is an isomorphism. □

Obviously, each finitely generated graded projective A-module belongs to the G-class.

Lemma 2. *Let A be an $\mathbb{N}$-graded k-algebra. The functor $\underline{\mathrm{Hom}}_A(\_\_, A)$ is left exact.*

Proof. For each exact sequence of graded left A-modules: $0 \to M \xrightarrow{\eta} N \xrightarrow{\pi} L \to 0$, we want to prove the complex $0 \to \underline{\mathrm{Hom}}_A(L, A) \xrightarrow{\underline{\mathrm{Hom}}_A(\pi, A)} \underline{\mathrm{Hom}}_A(N, A) \xrightarrow{\underline{\mathrm{Hom}}_A(\eta, A)} \underline{\mathrm{Hom}}_A(N, A)$.

(i) $\ker \underline{\mathrm{Hom}}_A(\pi, A) = \{0\}$.

For each $f \in \underline{\mathrm{Hom}}_A(L, A)$, if $\underline{\mathrm{Hom}}_A(\pi, A)(f) = f\pi = 0$, since π is surjective, thus f=0, that is, $\ker \underline{\mathrm{Hom}}_A(\pi, A) = \{0\}$.

(ii) $\mathrm{Im}\underline{\mathrm{Hom}}_A(\pi, A) = \ker \underline{\mathrm{Hom}}_A(\eta, A)$.

First, since $\underline{\mathrm{Hom}}_A(\eta, A)\underline{\mathrm{Hom}}_A(\pi, A) = \underline{\mathrm{Hom}}_A(\pi\eta, A) = \underline{\mathrm{Hom}}_A(0, A) = 0$, thus $\mathrm{Im}\underline{\mathrm{Hom}}_A(\pi, A) \subseteq \ker \underline{\mathrm{Hom}}_A(\eta, A)$. Second, if $\tau \in \underline{\mathrm{Hom}}_A(N, A)$ and $\underline{\mathrm{Hom}}_A(\eta, A)\tau = 0$. If $c \in L$, then $\pi(b) = c$ for some $b \in N$, because π is surjective. Define $\sigma : L \to A$ by $\sigma(c) = m$ if $\tau(b) = m, m \in A$. Note that σ is well defined: If $\sigma(c') = m'$, $m' \in A$ and $c' \in L$, then $\pi(b') = c', \tau(b') = m'$ for some $b' \in N$ by the definition of σ. If $\pi(b') + \pi(b) = c' + c = \pi(b' + b)$ and $\tau(b') + \tau(b) = m' + m = \tau(b' + b)$, then $\sigma(c' + c) = m' + m = \sigma(c') + \sigma(c)$. If $\pi(ab) = a\pi(b) = ac$ and $\tau(ab) = a\tau(b) = am$, then

$\sigma(ac) = am = a\sigma(c)$. By the definition of σ, then $\sigma \in \underline{\mathrm{Hom}}_A(L, A)$ and $\underline{\mathrm{Hom}}_A(\pi, A)\sigma = \tau$. Thus, $\ker \underline{\mathrm{Hom}}_A(\eta, A) \subseteq \mathrm{Im}\underline{\mathrm{Hom}}_A(\pi, A)$. $\square$

Proposition 2. *For each graded left A-module M, N. If the complex $N \to P^{-i+1} \to P^{-i+2} \cdots \to P^{-1} \to P^0 \to M$ is an exact sequence, and P^{-j} are projective graded left A-modules, $0 \le j \le i-1$, then $\underline{\mathrm{Ext}}_A^{n+i}(M, A) = \underline{\mathrm{Ext}}_A^n(N, A)$, $\forall n \in \mathbb{Z}$.*

Proof. If a complex $\cdots Q^{-1} \to Q^0 \to N$ is a projective resolution of N, Q^{-j} is a graded projective A-module, $\forall j \in \mathbb{Z}$. There is an exact sequence, $\cdots Q^{-1} \to Q^0 \to P^{-i+1} \to P^{-i+2} \cdots \to P^{-1} \to P^0 \to M$. Let $Q^{-j} = P^{-i-j}$; then, $\cdots P^{-i-1} \to P^{-i} \to P^{-i+1} \to P^{-i+2} \cdots \to P^{-1} \to P^0 \to M$ is a projective resolution of M, then $\underline{\mathrm{Hom}}_A(M, A) \to \underline{\mathrm{Hom}}_A(P^0, A) \to \underline{\mathrm{Hom}}_A(P^{-1}, A) \to \cdots \to \underline{\mathrm{Hom}}_A(P^{-i}, A) \to \underline{\mathrm{Hom}}_A(P^{-i-1}, A) \to \cdots = \underline{\mathrm{Hom}}_A(M, A) \to \underline{\mathrm{Hom}}_A(P^0, A) \to \underline{\mathrm{Hom}}_A(P^{-1}, A) \to \cdots \to \underline{\mathrm{Hom}}_A(P^{-i+1}, A) \to \underline{\mathrm{Hom}}_A(Q^0, A) \to \cdots$ and $\underline{\mathrm{Hom}}_A(N, A) \to \underline{\mathrm{Hom}}_A(Q^0, A) \to \underline{\mathrm{Hom}}_A(Q^{-1}, A) \to \cdots$. Thus $\mathrm{Ext}_A^{i+n}(M, A) = h^{i+n}R\underline{\mathrm{Hom}}_A(M, A) = h^n R\underline{\mathrm{Hom}}_A(N, A) = \mathrm{Ext}_A^n(N, A)$. $\square$

4. Properties of G-Class

In this section, we prove the equivalence definition of the G-class by a particular exact sequence.

The following lemma is a necessary condition for our proof of the equivalence definition of G-class; this is a noncommutative version of ([12], Proposition 5).

Lemma 3. *Let $\sigma_M : M \to M^{**}$ be the natural map with* Kernel K_M *and* Cokernel C_M. *Then, we have natural isomorphisms*

$$K_M \cong \underline{\mathrm{Ext}}_{A^o}^1(D(M), A) \ \ and \ \ C_M \cong \underline{\mathrm{Ext}}_{A^o}^2(D(M), A).$$

Proof. Consider the finitely generated projective presentation $\pi : P_1 \xrightarrow{u} P_0 \xrightarrow{f} M \to 0$ of M. Dualizing π, we have an exact sequence $\pi^* : 0 \to M^* \xrightarrow{f^*} P_0^* \xrightarrow{u^*} P_1^* \to D(M) \to 0$. Split π^* into short exact sequence $\pi_0^* : 0 \to M^* \xrightarrow{f^*} P_0^* \xrightarrow{\beta_0} N \to 0$ and $\pi_1^* : 0 \to N \xrightarrow{\beta_1} P_1^* \to D(M) \to 0$, where $N = \mathrm{Coker}(f^*)$ and $\beta_1\beta_0 = u^*$. Dualizing π_0^* and π_1^*, we obtain an exact sequence,

$$\pi_0^{**}: \quad 0 \longrightarrow N^* \xrightarrow{\beta_0^*} P_0^{**} \xrightarrow{f^{**}} M^{**} \longrightarrow \underline{\mathrm{Ext}}_{A^o}^1(N, A) \longrightarrow 0$$

and

$$\pi_1^{**}: \quad 0 \longrightarrow D(M)^* \longrightarrow P_1^{**} \xrightarrow{\beta_1^*} N^* \longrightarrow \underline{\mathrm{Ext}}_{A^o}^1(D(M), A) \longrightarrow 0.$$

Consider the commutative diagram with the exact row

$$\begin{tikzcd} & P_1 \arrow[r, "u"] \arrow[d, "\beta_1^*\sigma_{P_1}"'] & P_0 \arrow[r, "f"] \arrow[d, "\sigma_{P_0}"'] & M \arrow[r] \arrow[d, "\sigma_M"'] & 0 \\ 0 \arrow[r] & N^* \arrow[r, "\beta_0^*"] & P_0^{**} \arrow[r, "f^{**}"] & M^{**} & \end{tikzcd}$$

Since σ_{P_1} is an isomorphism, we obtain $\mathrm{Coker}(\beta_1^*\sigma_{P_1}) = \mathrm{Coker}(\beta_1^*)$; according to π_1^{**}, we obtain $\mathrm{Coker}(\beta_1^*) \cong \underline{\mathrm{Ext}}_{A^o}^1(D(M), A)$, so we have $\mathrm{Coker}(\beta_1^*\sigma_{P_1}) = \mathrm{Coker}(\beta_1^*) \cong \underline{\mathrm{Ext}}_{A^{op}}^1(D(M), A)$. According to the snake lemma, we obtain $\mathrm{Ker}\beta_1^*\sigma_{P_1} \to \mathrm{Ker}\sigma_{P_0} \to \mathrm{Ker}\sigma_M \to \mathrm{Coker}\beta_1^*\sigma_{P_1} \to \mathrm{Coker}\sigma_{P_0} \to \mathrm{Coker}\sigma_M$, since σ_{P_0} is isomorphism, we obtain $K_M = \mathrm{Ker}(\sigma_M) \cong \mathrm{Coker}(\beta_1^*\sigma_{P_1})$, thus $K_M \cong \underline{\mathrm{Ext}}_{A^o}^1(D(M), A)$.

Since f is surjective and σ_{P_0} is an isomorphism, we have $\mathrm{Im}(f^{**}) = \mathrm{Im}\sigma_M$. According to π_0^{**}, we obtain $C_M = \mathrm{Coker}(\sigma_M) = \mathrm{Coker}(f^{**}) \cong \underline{\mathrm{Ext}}_{A^o}^1(N, A)$. According to the short exact sequence, π_1^*, P_1^* is a graded projective right A-module and exact sequence theorem in cohomology; we obtain $\underline{\mathrm{Ext}}^1_{A^o}(N,A) \cong \underline{\mathrm{Ext}}^2_{A^o}(D(M),A)$. So, $C_M \cong \underline{\mathrm{Ext}}^2_{A^o}(D(M),A)$. □

Lemma 4. *For each graded left A-module, we have a exact sequence,*

$$0 \to \underline{\mathrm{Ext}}^1_{A^o}(D(M),A) \to M \to M^{**} \to \underline{\mathrm{Ext}}^2_{A^o}(D(M),A) \to 0$$

Proof. According to Lemma 3, we can obtain two natural isomorphisms $f : K_M \to \underline{\mathrm{Ext}}^1_{A^o}(D(M),A)$ and $g : C_M \to \underline{\mathrm{Ext}}^2_{A^o}(D(M),A)$. We also have the exact sequence $0 \to K_M \xrightarrow{u} M \xrightarrow{\sigma_M} M^{**} \xrightarrow{v} C_M \to 0$; thus, we obtain a exact sequence $0 \to \underline{\mathrm{Ext}}^1_{A^o}(D(M),A) \xrightarrow{fu} M \xrightarrow{\sigma_M} M^{**} \xrightarrow{vg} \underline{\mathrm{Ext}}^2_{A^o}(D(M),A) \to 0$. □

The ideas of the next theorem come from ([13], Proposition 3.8). The following theorem can be used as the equivalent definition of the G-class.

Theorem 4. *Let M be a finitely generated graded left A-module. The following are equivalent:*

(a) G-*dim*$M = 0$;

(b) *M belongs to the* G*-class;*

(c) $\underline{\mathrm{Ext}}^i_A(M,A) = 0$ *and* $\underline{\mathrm{Ext}}^i_{A^o}(D(M),A) = 0, \forall i > 0$.

Proof. $(a) \Rightarrow (b)$ If G-dim$M$=0, then there is a exact sequence $0 \to M^0 \to M \to 0$, where M^0 belongs to the G-class. Thus $M^0 \cong M$; that is, M belongs to the G-class.

$(b) \Rightarrow (a)$ Because M belongs to the G-class, $0 \to M \xrightarrow{id_M} M \to 0$ is a G-resolution of M; that is, G-dim$M$ = 0.

$(b) \Leftrightarrow (c)$ From the exact sequence

$$0 \to \underline{\mathrm{Ext}}^1_{A^o}(D(M),A) \xrightarrow{u} M \xrightarrow{\sigma_M} M^{**} \xrightarrow{v} \underline{\mathrm{Ext}}^2_{A^o}(D(M),A) \to 0,$$

we see that the biduality map $\delta_M : M \to M^{**} = \underline{\mathrm{Hom}}_{A^o}(\underline{\mathrm{Hom}}_A(M,A),A)$ is an isomorphism if and only if $\underline{\mathrm{Ext}}^1_{A^o}(D(M),A) = 0$ and $\underline{\mathrm{Ext}}^2_{A^o}(D(M),A) = 0$. From the exact sequence

$$\pi^* : 0 \to M^* \to P_0^* \to P_1^* \to D(M) \to 0$$

in which $P_1 \to P_0 \to M \to 0$ is a projective presentation of M, we see that $\underline{\mathrm{Ext}}^i_{A^o}(D(M),A) = 0$ if and only if $\underline{\mathrm{Ext}}^{i-2}_{A^o}(M^*,A) = 0$, for each $i > 2$. Thus, (b) and (c) are equivalent. □

5. G-Dimension of Graded Left A-Module

This Section complements the properties of the G-dimension and G-class in the non-commutative case of [1] by the properties of G-dimension in the commutative case and combines the G-dimension with the short exact sequence of the graded left A-module to explore the interrelationship of the G-dimension among the three elements in the short exact sequence.

Proposition 3. *If $M \approx N$, then* G-*dim*M = G-*dim*N*, where M and N are finitely generated graded left A-modules.*

Proof. The following proves that G-dim$M$ and G-dim$N$ are infinite or finite at the same time. When G-dim$M$ and G-dim$N$ are finite, then G-dim$M$ = G-dim$N$. Suppose G-dim$N$ is infinite and G-dim$M < t < \infty$; let us prove the contradiction: Since G-dim$M < t < \infty$, then we have a G-resolution $0 \to G^{-t} \xrightarrow{d^{-t}} G^{-t+1} \xrightarrow{d^{-t+1}} \cdots \to G^{-1} \xrightarrow{d^{-1}} G^0 \xrightarrow{d^0} M \to 0$ of M. Since $M \approx N$, then $M \oplus P \overset{f}{\cong} N \oplus Q$. Since $0 \to G^{-t} \xrightarrow{d^{-t}} G^{-t+1} \xrightarrow{d^{-t+1}} \cdots \to G^{-1} \xrightarrow{d^{-1}\oplus 0} G^0 \oplus P \xrightarrow{d^0 \oplus id_P} M \oplus P \to 0$ is a G-resolution of $M \oplus P$, then $0 \to G^{-t} \xrightarrow{d^{-t}} G^{-t+1} \xrightarrow{d^{-t+1}} \cdots \to$

$G^{-1} \overset{d^{-1}\oplus 0}{\rightarrow} G^0 \oplus P \overset{(d^0 \oplus id_P)f}{\rightarrow} N \oplus Q \rightarrow 0$ is a G-resolution of $N \oplus Q$. Thus, $0 \rightarrow G^{-t} \overset{d^{-t}}{\rightarrow} G^{-t+1} \overset{d^{-t+1}}{\rightarrow} \cdots \rightarrow G^{-2} \overset{d^{-2}}{\rightarrow} G^{-1} \oplus Q \overset{d^{-1}\oplus 0 \oplus id_Q}{\rightarrow} G^0 \oplus P \oplus Q \overset{(d^0 \oplus id_P)f(id_N \oplus 0)}{\rightarrow} N \rightarrow 0$ is a G-resolution of N, so we can obtain G-dim$N \leq \infty$, contradictory to assumption; thus, G-dim$M$ and G-dim$N$ are infinite or finite at the same time. From the above proof, it is clear that each G-resolution of M is also G-resolution of N. Similarly, we can prove that the G-resolution of N is also the G-resolution of M; thus, when G-dim$M$ and G-dim$N$ are finite, G-dim$M$ = G-dim$N$. In summary, G-dim$M$ = G-dim$N$. □

By the result in [13], we can obtain the following long exact sequence of graded left A-modules and obtain the relationship between the G-dimensions of the three elements in the short exact sequence according to the long exact sequence.

Lemma 5. *Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of finitely generated graded left A-modules; then, there exists an exact sequence, $0 \rightarrow M_3^* \rightarrow M_2^* \rightarrow M_1^* \rightarrow D_3 \rightarrow D_2 \rightarrow D_1 \rightarrow 0$, where $D_i \approx D(M_i)$ for $i = 1, 2, 3$, such that there is an exact sequence, $0 \rightarrow D_1^* \rightarrow D_2^* \rightarrow D_3^* \rightarrow 0$.*

Proof. We can find finitely generated graded projective left A-modules P_i, Q_i $(i = 1, 2, 3)$ and maps so that the following diagram is exact and commutative:

$$\begin{array}{ccccccc}
0 \rightarrow & Q_1 \rightarrow & Q_2 \rightarrow & Q_3 \rightarrow 0 \\
& \downarrow & \downarrow & \downarrow \\
0 \rightarrow & P_1 \rightarrow & P_2 \rightarrow & P_3 \rightarrow 0 \\
& \downarrow & \downarrow & \downarrow \\
0 \rightarrow & M_1 \rightarrow & M_2 \rightarrow & M_3 \rightarrow 0 \\
& \downarrow & \downarrow & \downarrow \\
& 0 & 0 & 0
\end{array}$$

We note that the top rows split. If we dualize the first diagram, we obtain the exact commutative diagram:

$$\begin{array}{cccc}
& 0 & 0 & 0 \\
& \downarrow & \downarrow & \downarrow \\
& M_3^* \rightarrow & M_2^* \rightarrow & M_1^* \\
& \downarrow & \downarrow & \downarrow \\
0 \rightarrow & P_3^* \rightarrow & P_2^* \rightarrow & P_1^* \rightarrow 0 \\
& \downarrow & \downarrow & \downarrow \\
0 \rightarrow & Q_3^* \rightarrow & Q_2^* \rightarrow & Q_1^* \rightarrow 0 \\
& \downarrow & \downarrow & \downarrow \\
& D_3 \rightarrow & D_2 \rightarrow & D_1 \\
& \downarrow & \downarrow & \downarrow \\
& 0 & 0 & 0
\end{array}$$

We have an exact sequence $0 \rightarrow M_3^* \rightarrow M_2^* \rightarrow M_1^* \rightarrow D_3 \rightarrow D_2 \rightarrow D_1 \rightarrow 0$ by snake lemma. Since each finitely generated graded projective A-module P is isomorphic to P^{**} and the first diagram is an exact commutative diagram, we can dualize once more and obtain the following exact and commutative diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & D_1^* & \to & D_2^* & \to & D_3^* & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & Q_1 & \to & Q_2 & \to & Q_3 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & P_1 & \to & P_2 & \to & P_3 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & M_1 & \to & M_2 & \to & M_3 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

Applying the snake lemma diagram again, we obtain an exact sequence $0 \to D_1^* \to D_2^* \to D_3^* \to M_1 \to M_2 \to M_3 \to 0$. Furthermore, since $M_1 \to M_2$ is a monomorphism, we conclude that $0 \to D_1^* \to D_2^* \to D_3^* \to 0$ is exact. □

Lemma 6. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be exact with M_i finitely generated. Suppose that* G-*dim* $M_3 = 0$*; then,* G-*dim* $M_2 = 0$ *if and only if* G-*dim*$M_1 = 0$.

Proof. Since G-dim$M_3 = 0$, we know $\underline{\mathrm{Ext}}^i_A(M_3, A) = 0$ and $\underline{\mathrm{Ext}}^i_{A^o}(D(M_3), A) = 0$ for all $i > 0$ by Lemma 4. In particular, $\underline{\mathrm{Ext}}^1_A(M_3, A) = 0$. Dualizing our short exact sequence then gives the exact sequence $0 \to M_3^* \to M_2^* \to M_1^* \to 0$. Now, from Lemma 5, there exists an exact sequence $0 \to M_3^* \to M_2^* \to M_1^* \to D_3 \to D_2 \to D_1 \to 0$, where $D_i \approx D(M_i)$ for $i = 1, 2, 3$. Since $M_2^* \to M_1^*$ is surjective, we deduce the exactness of $0 \to D_3 \to D_2 \to D_1 \to 0$. We now have two long exact sequences obtained from $0 \to M_1 \to M_2 \to M_3 \to 0$ and $0 \to D_3 \to D_2 \to D_1 \to 0$, respectively,

$$\cdots \to \underline{\mathrm{Ext}}^i_A(M_3, A) \to \underline{\mathrm{Ext}}^i_A(M_2, A) \to \underline{\mathrm{Ext}}^i_A(M_1, A) \to \underline{\mathrm{Ext}}^{i+1}_A(M_3, A) \to \cdots$$

and

$$0 \to D_1^* \to D_2^* \to D_3^* \to \underline{\mathrm{Ext}}^1_{A^o}(D_1, A) \to \underline{\mathrm{Ext}}^1_{A^o}(D_2, A) \to \underline{\mathrm{Ext}}^1_{A^o}(D_3, A) \to \cdots$$
$$\to \underline{\mathrm{Ext}}^{i-1}_{A^o}(D_3, A) \to \underline{\mathrm{Ext}}^i_{A^o}(D_1, A) \to \underline{\mathrm{Ext}}^i_{A^o}(D_2, A) \to \underline{\mathrm{Ext}}^i_{A^o}(D_3, A) \to \cdots.$$

Since $\underline{\mathrm{Ext}}^i_A(M_3, A) = 0$ for all $i > 0$, we deduce from the first of these sequences that $\underline{\mathrm{Ext}}^i_A(M_1, A) \cong \underline{\mathrm{Ext}}^i_A(M_2, A)$, $\forall i > 0$. By Lemma 5, we have that $D_2^* \to D_3^*$ is surjective. Furthermore, since G-dim$M_3 = 0$, then $\underline{\mathrm{Ext}}^i_{A^o}(D_3, A) = 0$, $\forall i > 0$. We deduce from the second of these sequences that $\underline{\mathrm{Ext}}^i_{A^o}(D_1, A) \cong \underline{\mathrm{Ext}}^i_{A^o}(D_2, A)$, $\forall i > 0$. In summary, $\forall i > 0$,

$$\underline{\mathrm{Ext}}^i_A(M_1, A) \cong \underline{\mathrm{Ext}}^i_A(M_2, A),$$
$$\underline{\mathrm{Ext}}^i_{A^o}(D_1, A) \cong \underline{\mathrm{Ext}}^i_{A^o}(D_2, A).$$

By the characterization of G-dim$= 0$ contained in Theorem 4, we see that G-dim$M_1 = 0$ if and only if G-dim$M_2 = 0$. □

$N \in$ G(A) also has the same properties as the projective module. The conclusions are as follows.

Lemma 7. *Let $0 \to X \to N \to M \to 0$ be an exact sequence of finitely generated graded left A-modules. If $N \in$ G(A), then there are isomorphisms*

$$\underline{\mathrm{Ext}}^{m+1}_A(M, A) \cong \underline{\mathrm{Ext}}^m_A(X, A),$$

for $m > 0$.

Proof. Dualizing the short exact sequence $0 \to X \to N \to M \to 0$, we have a long exact sequence $0 \to M^* \to N^* \to X^* \to \underline{\mathrm{Ext}}_A^1(M,A) \to \cdots \to \underline{\mathrm{Ext}}_A^m(M,A) \to \underline{\mathrm{Ext}}_A^m(N,A) \to \underline{\mathrm{Ext}}_A^m(X,A) \to \cdots$. Since $N \in \mathrm{G(A)}$, then $\underline{\mathrm{Ext}}_A^m(N,A) = 0$ for $m > 0$. By the long exact sequence, we can easily obtain $\underline{\mathrm{Ext}}_A^{m+1}(M,A) \cong \underline{\mathrm{Ext}}_A^m(X,A)$ for $m > 0$. □

Theorem 5. *Let $0 \to X \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ be an exact sequence of finitely generated graded left A-modules. If $G_i \in \mathrm{G(A)}$, $0 \le i \le n-1$, then there are isomorphisms*

$$\underline{\mathrm{Ext}}_A^{m+n}(M,A) \cong \underline{\mathrm{Ext}}_A^m(X,A),$$

for $m > 0$.

Proof. We set $X_0 = M$, $X_1 = \mathrm{Ker}(G_0 \to M)$, $X = X_n$ and $X_i = \mathrm{Ker}(G_{i-1} \to G_{i-2})$ for $2 \le i \le n-1$. For each $0 \le j \le n$, we have a short exact sequence $0 \to X_j \to G_{j-1} \to X_{j-1} \to 0$. Applying Lemma 7, we obtain isomorphisms $\underline{\mathrm{Ext}}_A^{m+1}(X_{j-1},A) \cong \underline{\mathrm{Ext}}_A^m(X_j,A)$ for $m > 0$, which piece together to give isomorphisms $\underline{\mathrm{Ext}}_A^{m+n}(X,A) \cong \underline{\mathrm{Ext}}_A^m(M,A)$ for $m > 0$. □

Proposition 4. *Let M be a finitely generated graded left A-module, and M belong to the* G*-class; then, M^* also belongs to the* G*-class.*

Proof. Since M belongs to the G-class, then M has the following properties:

(1) $\underline{\mathrm{Ext}}_A^i(M,A) = 0$ for $i > 0$;
(2) $\underline{\mathrm{Ext}}_{A^{op}}^i(\underline{\mathrm{Hom}}_A(M,A),A) = 0$ for $i > 0$; and
(3) the biduality map $\delta_M : M \to \underline{\mathrm{Hom}}_{A^{op}}(\underline{\mathrm{Hom}}_A(M,A),A) = 0$ is an isomorphism.

Since M and M^{**} are isomorphic, $\underline{\mathrm{Ext}}_A^i(M,A) = \underline{\mathrm{Ext}}_A^i((M^*)^*,A) = 0$ for $i > 0$ and $M^* \cong (M^*)^{**}$; thus, M^* belong to the G-class. □

By the result of [9], we have the following characterization.

Lemma 8. *Let A be an $\mathbb{N}$-graded k-algebra and M be a finitely generated graded left A-module of finite* G*-dimension. If $\underline{\mathrm{Ext}}_A^m(M,A) = 0$ for all $m > 0$, then $M \in \mathrm{G(A)}$.*

Proof. First, let G-dim$M \le 1$, then we obtain an exact sequence, $0 \to G^{-1} \to G^0 \to M \to 0$, where the modules G^{-1} and G^0 belong to $\mathrm{G(A)}$. As $\underline{\mathrm{Ext}}_A^1(M,A) = 0$, this sequence dualizes to give a short exact sequence, $0 \to M^* \to G^{0*} \to G^{-1*} \to 0$. Since $G^{0*}, G^{-1*} \in \mathrm{G(A^o)}$, $M^* \in \mathrm{G(A^o)}$, in particular $\underline{\mathrm{Ext}}_A^m(M^*,A) = 0$ for $m > 0$. Since $G^{0**}, G^{-1**} \in \mathrm{G(A)}$, $\underline{\mathrm{Ext}}_A^1(G^{-1**},A) = 0$. Dualizing once more, we have the second row in the short exact ladder

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G^{-1} & \longrightarrow & G^0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_{G^{-1}}} & & \downarrow{\scriptstyle \delta_{G^0}} & & \downarrow{\scriptstyle \delta_M} & & \\
0 & \longrightarrow & G^{-1**} & \longrightarrow & G^{0**} & \longrightarrow & M^{**} & \longrightarrow & 0
\end{array}$$

By five lemma, we know that δ_M is an isomorphism; thus, $M \in \mathrm{G(A)}$.
Now, let $n > 1$ and G-dim$M \le n-1$ imply $M \in \mathrm{G(A)}$.
If G-dim$M \le n$, then M has a G-resolution of length n

$$0 \to G^{-n} \to G^{-n+1} \to \cdots \to G^{-1} \to G^0 \to M \to 0.$$

Let $K = \mathrm{Ker}(G^{-n+2} \to G^{-n+3})$, then there is a exact sequence $0 \to G^{-n} \to G^{-n+1} \to K$. By the definition of the G-dimension, we know G-dim$(K) \le 1$. since $\underline{\mathrm{Ext}}_A^m(K,A) = \underline{\mathrm{Ext}}_A^{m+n-1}(M,A) = 0$ for $m > 0$, So $K \in \mathrm{G(A)}$ by the above. Now, the exact sequence

$$0 \to K \to G^{-n+2} \to \cdots \to G^0 \to M \to 0$$

shows that G-dim$M \leq n-1$. By the induction hypothesis, $M \in \mathrm{G}(\mathrm{A})$. $\square$

The G-dimension, projective dimension, and injective dimension have similar conclusions about cohomology. From [9], we can obtain the following conclusions about G-dimension.

Theorem 6. *Let M be a finitely generated graded left A-module and $n \in \mathbb{N}$. The following are equivalent:*

(a) G-*dim*$M \leq n$.

(b) G-*dim*$M < \infty$ *and* $\underline{\mathrm{Ext}}_A^m(M, A) = 0$, *for* $m > n$.

Proof. $(a) \Rightarrow (b)$

If G-dim$M \leq n$, then M has a G-resolution of length n

$$0 \to G^{-n} \to \cdots \to G^{-1} \to G^0 \to M \to 0.$$

From Theorems 4 and 5, $\underline{\mathrm{Ext}}_A^{m+n}(M, A) \cong \underline{\mathrm{Ext}}_A^m(G^{-n}, A) = 0$, for $m > 0$, that is, $\underline{\mathrm{Ext}}_A^m(M, A) = 0$, for $m > n$.

$(b) \Rightarrow (a)$

We assume that M has a G-resolution of finite length p:

$$0 \to G^{-p} \to \cdots \to G^{-1} \to G^0 \to M \to 0.$$

If $p \leq n$, there is nothing to prove. So, we assume that $p > n$. Defining $K = \mathrm{Ker}(G^{-n+1} \to G^{-n+2})$, we obtain an exact sequence,

$$0 \to K \to G^{-n+1} \to \cdots \to G^{-1} \to G^0 \to M \to 0.$$

Since $0 \to G^{-p} \to \cdots \to G^{-n-1} \to K$ is an exact sequence, K has finite G-dimension at most $p-n$. Since $\underline{\mathrm{Ext}}_A^m(M, A) = 0$, for $m > n$, then $\underline{\mathrm{Ext}}_A^{m+n}(M, A) \cong \underline{\mathrm{Ext}}_A^m(K, A) = 0$, for $m > 0$. From Lemma 8, $K \in \mathrm{G}(\mathrm{A})$. So, M has a G-resolution of length n. $\square$

Notation. From Theorem 6, we can immediately conclude that $G\text{-}dim_A(M)= \sup\{m \in \mathbb{N} | \underline{\mathrm{Ext}}_A^m(M, A) \neq 0\}$ for modules of the finite G-dimension.

Theorem 7. *Let A be an $\mathbb{N}$-graded k-algebra. If $X \in D^-(\mathrm{GrMod}(A)$ and n is an integer, the following conditions are equivalent:*

(a) *We have* $\mathrm{pd}_A(M) \leq n$;

(b) *For any* $X \in \mathrm{GrMod}(A)$ *and any* $m > n$ *we have* $\underline{\mathrm{Ext}}_A^m(M, X) = 0$;

(c) *For any* $X \in \mathrm{grmod}(A)$ *and any* $m > n$ *we have* $\underline{\mathrm{Ext}}_A^m(M, X) = 0$.

Proof. See [3]. $\square$

Corollary 2. *If there exist two exact sequences, as follows:*

$$0 \to K \to G^{-n+1} \to \cdots \to G^{-1} \to G^0 \to M \to 0,$$

$$0 \to N^{-n} \to N^{-n+1} \to \cdots \to N^{-1} \to N^0 \to M \to 0.$$

where N^{-j},$(j = 1, \ldots, n)$ *and* G^{-i},$(i = 1, \ldots, n-1)$ *belong to the G-class, then K belongs to the G-class.*

That is, G-*dim*$M \leq n$, *and there is an exact sequence* $0 \to K \to G^{-n+1} \to \cdots \to G^{-1} \to G^0 \to M \to 0$; *then, K belongs to the* G-*class.*

Proof. Since $0 \to N^{-n} \to N^{-n+1} \to \cdots \to N^{-1} \to N^0 \to M \to 0$ is the G-resolution of M, then $G\text{-dim}M \leq n$. From Theorem 6, we can obtain $\underline{\text{Ext}}^m_A(M,A) = 0$, for $m > n$. From the observation $\underline{\text{Ext}}^{m+n}_A(M,A) \cong \underline{\text{Ext}}^m_A(K,A) = 0$, for $m > 0$. Then, $K \in \text{G(A)}$, from Theorem 5. □

Analogously to the commutative case ([13], Corollary 3.16), we have the following characteristics in the noncommutative case.

Theorem 8. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of finitely generated graded left A-modules; if two modules have a finite* G*-dimension, then so does the third.*

Proof. First, assume that $\text{G-dim}M_3 = n < \infty$; we prove that $\text{G-dim}M_1 \leq t$ if and only if $\text{G-dim}M_2 \leq t$, where $n < t < \infty$.

Consider the exact and commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M_1 \arrow[u] \arrow[r] & M_2 \arrow[u] \arrow[r] & M_3 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & P^0 \arrow[u] \arrow[r] & P^0 \oplus Q^0 \arrow[u] \arrow[r] & Q^0 \arrow[u] \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
0 \arrow[r] & P^{-t+1} \arrow[u] \arrow[r] & P^{-t+1} \oplus Q^{-t+1} \arrow[u] \arrow[r] & Q^{-t+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & K_1 \arrow[u] \arrow[r] & K_2 \arrow[u] \arrow[r] & K_3 \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where P_i and Q_i are graded projective left A-modules ($0 \leq i \leq t-1$), then $K_i \approx \Omega^t M_i$, we can obtain $\text{G-dim}(K_i) = \text{G-dim}\Omega^t_A(M_i)$ for $(i = 1,2,3)$ by Proposition 3. Since $n \leq t$, we have $\text{G-dim}\Omega^t M_3 = \text{G-dim}K_3 = 0$ by Corollary 2.

Since $0 \to K_1 \to K_2 \to K_3 \to 0$ are exact, we have $\text{G-dim}K_2 = 0$ if and only if $\text{G-dim}K_1 = 0$ by Lemma 6. By definition of the G-dimension, this is equivalent to saying $\text{G-dim}M_2 \leq t$ if and only if $\text{G-dim}M_1 \leq t$.

Second, assume that $\text{G-dim}M_1 \leq n$ and $\text{G-dim}M_1 \leq n$; we prove that $\text{G-dim}M_3 \leq t+1$, where $n < t < \infty$.

Consider the exact and commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & M_3 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^0 \arrow[r] \arrow[u] & P^0 \oplus Q^0 \arrow[r] \arrow[u] & Q^0 \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
0 \arrow[r] & P^{-t+1} \arrow[r] \arrow[u] & P^{-t+1} \oplus Q^{-t+1} \arrow[r] \arrow[u] & Q^{-t+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & K_1 \arrow[r] \arrow[u] & K_2 \arrow[r] \arrow[u] & K_3 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where the P_i and Q_i are graded projective left A-modules ($0 \leq i \leq t-1$), then $K_i \approx \Omega^t(M_i)$, we can obtain $\text{G-dim}(K_i) = \text{G-dim}\Omega^t_A(M_i)$ for $(i = 1,2,3)$. Since $\text{G-dim}M_1 \leq n$ and $\text{G-dim}M_2 \leq n$, $\text{G-dim}(K_i) = \text{G-dim}\Omega^t_A(M_i) = 0$,$(i = 1,2)$ by Corollary 2. From Theorem 4, we can obtain K_1, K_2 belonging to G-class; we have an exact sequence $0 \to K_1 \to K_2 \to Q^{-t+1} \to \cdots \to Q^0 \to M_3 \to 0$. Thus, $\text{G-dim}M_3 \leq t+1$. $\square$

6. The Auslander–Bridger Theorem

The generalized Auslander–Buchsbaum formula and the generalized Auslander–Bridger formula have already been proved by Jørgensen in ([8], Theorem 3.2) and by ([1], Theorem 4.3), respectively. From the existence theorem due to Van den Bergh ([14], Theorem 6.3), we obtain that if A admits a balanced dualizing complex, then A satisfies the χ°-condition on both sides; thus, the conditions assumed in ([1], Theorem 4.3) are stronger. Rogalski and Sierra [15] found that there is a Noetherian connected graded algebra that does not satisfy the Auslander–Buchsbaum formula; that is, the χ°-condition is in some sense necessary for Jørgensen's results. However, in the Auslander–Bridger formula, we only use the conclusion of the A-module, so we use the relationship between the projective dimension and the G-dimension of A-module to give a simple proof and only assume A satisfies the χ°-condition.

Definition 1 ([2])**.** *Let A be a connected graded algebra. For each $X \in D(\mathrm{GrMod}A)$, we define* the Bass-numbers

$$\mu^i(X) = \dim_k \underline{\mathrm{Ext}}^i_A(k, X),$$

the depth *of X*

$$\mathrm{depth}_A(X) = \inf R\underline{\mathrm{Hom}}_A(k, X) = \inf\left\{ i \mid \mu^i(X) \neq 0 \right\},$$

and the k-injective dimension *of X*

$$\mathrm{k.\,id}_A(X) = \sup R\underline{\mathrm{Hom}}_A(k, X) = \sup\left\{ i \mid \mu^i(X) \neq 0 \right\}.$$

Theorem 9 (The Auslander–Buchsbaum theorem)**.** *Let A be an $\mathbb{N}$-graded k-algebra which is left-Noetherian and connected and satisfies the condition $\chi^\circ_{\mathrm{depth}_A(A)}({}_A A)$ (this χ°-condition is vacuous if A has infinite depth as a left-module over itself).*

Given $X \in D^b_{fg}(\mathrm{GrMod(A)})$ *with* $\mathrm{pd}_A(X) < \infty$, *we have*

$$\mathrm{pd}_A(X) + \mathrm{depth}_A(X) = \mathrm{depth}_A(A).$$

Proof. See ([3], Theorem 3.2). □

Depending on the commutative case [9], we can obtain the relevant inequality on the noncommutative graded algebra.

Lemma 9. *For every finitely generated graded left* A*-module* M*, there is an inequality*

$$\mathrm{G{-}dim}_A M \leq \mathrm{pd}_A M,$$

and equality holds if $\mathrm{pd}_A M < \infty$.

Proof. If M has infinite projective dimension, then the inequality certainly holds. Assume that M has a finite projective dimension, say p. Since M is a finitely generated graded left A-module, it has a resolution of minimal length by the finitely generated graded projective left A-module $0 \to P^{-p} \to \cdots \to P^{-1} \to P^0 \to M$. In particular, this resolution is a G-resolution, so $\mathrm{G\text{-}dim}_A M \leq p$. By Theorem 7, $\underline{\mathrm{Ext}}^{p+1}_A(M, \_\_) = 0$ and a finitely generated graded left A-module T exists, such that $\underline{\mathrm{Ext}}^p_A(M, T) \neq 0$. Applying the functor $\underline{\mathrm{Hom}}_A(M, \_\_)$ to the short exact sequence $0 \to K \to \oplus^n_{i=1} A \to T \to 0$, we have an exact sequence

$$\cdots \to \underline{\mathrm{Ext}}^p_A(M, K) \to \underline{\mathrm{Ext}}^p_A(M, \oplus^n_{i=1} A) \to \underline{\mathrm{Ext}}^p_A(M, T) \to 0.$$

Show that $\underline{\mathrm{Ext}}^p_A(M, \oplus^n_{i=1} A) \neq 0$, and therefore, $\oplus^n_{i=1} \underline{\mathrm{Ext}}^p_A(M, A) = \underline{\mathrm{Ext}}^p_A(M, \oplus^n_{i=1} A) \neq 0$, that is, $\underline{\mathrm{Ext}}^p_A(M, A) \neq 0$; thus, $\mathrm{G\text{-}dim}_A M = p$. □

Theorem 10 (The Auslander–Bridge theorem)**.** *Let* A *be an* $\mathbb{N}$*-graded* k*-algebra that is left-Noetherian and connected and satisfies the condition* $\chi^\circ_{\mathrm{depth}_A(A)}({}_A A)$ *(this* χ°*-condition is vacuous if* A *has infinite depth as a left-module over itself).*

Given $X \in \mathrm{grmod}(A)$ *with* $\mathrm{pd}_A(X) < \infty$, *we have*

$$\mathrm{G{-}dim}_A X + \mathrm{depth}_A(X) = \mathrm{depth}_A(A).$$

Proof. According to Lemma 9 and Theorem 10, we can immediately obtain the Auslander–Bridger theorem. □

7. Complexes of Graded Modules

In this section, we identify a class of noncommutative graded algebra that satisfies ac = tac. We also give an example belonging to this algebra. We default A to be a connection left-Noetherian graded algebra and satisfy the condition $\chi^\circ_{\mathrm{depth}_A(A)}({}_A A)$; the following lemma is similar to ([5], Lemma 1.3).

Lemma 10. *Let* X *be an acyclic complex of finitely generated graded free left* A*-modules. The following conditions are equivalent:*

(a) X *is totally acyclic;*
(b) $\Omega^i X$ *belongs to* G*-class for all* $i \in \mathbb{Z}$*;*
(c) $\mathrm{G\text{-}dim}_A(\Omega^i X) < \infty$ *for some* $i \in \mathrm{Z}$*;*

Proof. Let X be an acyclic complex of finitely generated graded free left A-modules, that is,

$$X = \cdots \to X^{u-1} \xrightarrow{\delta^{u-1}} X^u \xrightarrow{\delta^u} X^{u+1} \to \cdots,$$

where X^i is finitely generated graded free left A-module for $i \in \mathbb{Z}$.

$(a) \Rightarrow (b)$ Fix $n \in \mathbb{Z}$ and set $G = \Omega^n X = \mathrm{Coker}\delta^{n-1}$. As $X_{\leq n}[n]$ is a projective resolution of G, we have $\underline{\mathrm{Ext}}^i_A(G, A) \cong h^{i-n}(X^*) = 0$ for $i > 0$ and $G^* \cong \mathrm{Ker}(\delta^{n-1})^* = \mathrm{Ker}(\delta^{-n}_{X^*})$. As X^* is exact, $\mathrm{Ker}\delta^{-u}_{X^*} = \Omega^{-n-1}X^*$ and $X^*_{\leq -n-1}[-n-1]$ is a projective resolution of G^*. Since $X^{**} = X$, we obtain $\underline{\mathrm{Ext}}^i_A(G^*, A) \cong h^{i+n+1}(X^{**}) = 0$ for $i > 0$ and $G^{**} \cong \mathrm{Ker}(\delta^{X^*}_{-n-2})^* = \mathrm{Ker}\delta^{n+1}_{X^{**}} = \mathrm{Ker}\delta^{n+1}_X = G$.

$(b) \Rightarrow (a)$ For each $n \in \mathbb{Z}$, the complex $X_{\leq n-1}[n-1]$ is a projective resolution of $\Omega^{n-1}X$, so $h^n(\underline{\mathrm{Hom}}_A(X, A)) \cong \underline{\mathrm{Ext}}^1_A(\Omega^{-n+1}X, A) = 0$. Since this holds for all $n \in \mathbb{Z}$, the conclusion holds.

$(b) \Rightarrow (c)$ From Theorem 4, the conclusion clearly holds.

$(c) \Rightarrow (b)$ From Lemma 8, we know that in a short sequence, if two modules have a finite G-dimension, then so does the third. Recursive use of the short exact sequence $0 \to \Omega^{i-1}X \to X^i \to \Omega^i X \to 0$ gives then $\text{G-dim}_A(\Omega^i X) < \infty$ for all i. For any i, we can obtain an exact sequence, $0 \to \Omega^{i-1}X \to X^i \to \Omega^i X \to 0$. Since X^i is a finitely generated graded free left A-module, X^i belongs to the G-class, that is, $\text{G-dim}X^i = \mathrm{depth}A - \mathrm{depth}X^i = 0$. From the depth lemma cf. [7], we can obtain $\mathrm{depth}\Omega^{i-1}X \geq \inf\{\mathrm{depth}X^i, \mathrm{depth}\Omega^i X + 1\}$.

(i) Assume $\mathrm{depth}\Omega^{i-1}X \geq \mathrm{depth}X^i$.

From the Auslander–Bridger theorem, we can obtain $\text{G-dim}\Omega^{i-1}X = \mathrm{depth}A - \mathrm{depth}\Omega^{i-1}X = \mathrm{depth}X^i - \mathrm{depth}\Omega^{i-1}X \leq 0$; thus, $\text{G-dim}\Omega^{i-1}X = 0$.

(ii) Assume $\mathrm{depth}\Omega^{i-1}X \geq \mathrm{depth}\Omega^i X + 1$.

From the Auslander–Bridger theorem, we can obtain $\text{G-dim}\Omega^{i-1}X = \mathrm{depth}A - \mathrm{depth}\Omega^{i-1}X = \mathrm{depth}X^i - \mathrm{depth}\Omega^{i-1}X \leq \mathrm{depth}A - \mathrm{depth}\Omega^i X - 1$.

Next, we consider the exact sequence $0 \to \Omega^i X \to X^{i+1} \to \Omega^{i+1}X \to 0$.

Assume $\mathrm{depth}\Omega^i X \geq \mathrm{depth}X^{i+1} = \mathrm{depth}A$, then $\text{G-dim}\Omega^{i-1}X \leq -1$. Assumptions do not hold.

Assume $\mathrm{depth}\Omega^i X \geq \mathrm{depth}\Omega^{i+1}X + 1$, then $\text{G-dim}\Omega^{i-1}X \leq \mathrm{depth}A - \mathrm{depth}\Omega^{i+1}X - 2$, and so on, we can obtain $\text{G-dim}\Omega^{i-1}X \leq 0$. Therefore, assumption (ii) is not valid.

Summarizing the above, $\text{G-dim}\Omega^{i-1}X = 0$. That is, conclusion (b) holds. □

In the following theorem, we will give a class of noncommutative graded algebra that satisfies ac = tac.

Theorem 11. *Let A be a left Noetherian connected graded algebra with $\mathrm{id}_A A = \mathrm{id}_{A^\circ} A = n < \infty$ and satisfying the condition $\chi^\circ_{\mathrm{depth}_A(A)}({}_A A)$, then A satisfies* ac = tac.

Proof. First, let us suppose A is a left Noetherian connected graded algebra with $\mathrm{id}_A A = \mathrm{id}_{A^o} A = n < \infty$. If M is a finitely generated graded left A-module, we show that $\text{G-dim}M \leq n$.

Consider $\Omega^n M$. Since $\mathrm{id}_A A = n$, then we have $\underline{\mathrm{Ext}}^i_A(\Omega^n M, A) \cong \underline{\mathrm{Ext}}^{n+i}_A(M, A) = 0$.

Now, consider an exact sequence $P^{-n-1} \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to \Omega^n M$, where P^{-i} are finitely generated projective graded left A-modules. Since $\underline{\mathrm{Ext}}^i_A(\Omega^n M, A) = 0$ for all $i > 0$ and the functor $\underline{\mathrm{Hom}}_A(\_\_, A)$ is left exact. We have, upon dualizing, the following exact sequence:

$$0 \to (\Omega^n M)^* \to (P^0)^* \to (P^{-1})^* \to \cdots \to (P^{-n})^* \to (P^{-n-1})^*$$

Let $C = \mathrm{Coker}((P^{-n})^* \to (P^{-n-1})^*)$. Since $D(\Omega^n M) \approx \mathrm{Coker}((\mathrm{P}^0)^* \to (\mathrm{P}^{-1})^*)$, then we have an exact sequence,

$$0 \to D(\Omega^n M) \to (P^{-2})^* \to (P^{-3})^* \to \cdots \to (P^{-n})^* \to (P^{-n-1})^* \to C \to 0.$$

Therefore, $D(\Omega^n M) \simeq \Omega^n C$. Since $\mathrm{id}_{A^o} A = n$, we have $\underline{\mathrm{Ext}}^i_{A^o}(D(\Omega^n M), A) = \underline{\mathrm{Ext}}^{n+i}_{A^o}(C, A) = 0$ for all $i > 0$. From Theorem 4, we obtain that G-dim$\Omega^n M = 0$. So, G-dim$M \le n < \infty$. For each minimal acyclic complex of finitely generated graded free left A-modules,

$$L : \cdots \to L^{n-1} \to L^n \to L^{n+1} \to \cdots .$$

Since L^i, $i \in \mathbb{Z}$ are finitely generated graded free left A-modules, then $\Omega^i L$ is a finitely generated graded left A-module, so we can obtain G-dim$\Omega^i L \le n < \infty$. From Lemma 10, we obtain that L is totally acyclic. □

Example 1. *When A is an AS–Gorenstein algebra, A belongs to the algebra in Theorem 11, so the AS–Gorenstein algebra also satisfies* ac $=$ tac.

8. AS–Cohen–Macaulay Algebra

In this section, we will discuss the relationship between AS–Gorenstein algebra and AS–Cohen–Macaulay algebra with a balanced dualizing complex. The results obtained in this paper will be complementary to the results of [7]. Through the obtained relations, we know that AS–Cohen–Macaulay algebra with a balanced dualizing complex satisfies ac $=$ tac; this result complements the conclusions related to the commutative ring in [5] on the noncommutative graded algebra.

Throughout this section, we will suppose that B is an AS–Cohen–Macaulay algebra with a balanced dualizing complex. According to [7], we know that it is a graded factor of an AS–Gorenstein algebra. We will suppose that A is an AS–Gorenstein algebra, that $\mathfrak{a}$ is a graded ideal in A, and that $B = A/\mathfrak{a}$. Note that by ([14], Theorem 6.3), this entails that B satisfies χ° and $\mathrm{lcd}(B)$ is finite. We will denote $\mathfrak{m} = A_{\ge 1}$ and $\mathfrak{n} = B_{\ge 1} = \mathfrak{m}/\mathfrak{a}$.

Lemma 11. *If $X, Y \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(B))$, then for each i, we have*

$$\underline{\mathrm{Ext}}^i_B(X, Y) \cong \underline{\mathrm{Ext}}^i_{B^0}\big(R\Gamma_{\mathfrak{n}}(Y)', R\Gamma_{\mathfrak{n}}(X)'\big).$$

Proof. see ([2], Lemma 4.2). □

Proposition 5. *For $X \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(B))$, we have*

$$\mathrm{depth}_B(X) = \inf\Big\{i | R^i\Gamma_{\mathfrak{n}}(X) \neq 0\Big\};$$

$$\mathrm{k.id}_B(X) = \sup\Big\{i | R^i\Gamma_{\mathfrak{n}}(X) \neq 0\Big\};$$

Proof. Considering a minimal injective resolution $X \xrightarrow{\simeq} I$, we will write $\mathfrak{n} = B_{\ge 1}$. Because of the χ°-condition, each I^i only contains a finite number of direct summands isomorphic B', so $(R\Gamma_{\mathfrak{n}}(X)' = (R\Gamma_{\mathfrak{n}}(I)' = F$ is a complex of finitely generated free modules. Moreover, the minimality of I implies the minimality of F. According to Lemma 11, $\underline{\mathrm{Ext}}^i_B(k, X) \cong \underline{\mathrm{Ext}}^i_{B^o}(R\Gamma_{\mathfrak{n}}(X)', R\Gamma_{\mathfrak{n}}(k)') \cong \underline{\mathrm{Ext}}^i_{B^o}(F, k) \cong \underline{\mathrm{Hom}}_{B^o}(F_{-i}, k)$.

$\mathrm{depth}_B(X) = \inf R\underline{\mathrm{Hom}}_B(k, X) = \inf\{i | \underline{\mathrm{Ext}}^i_B(k, X) \neq 0\}$
$= \inf\{i | \underline{\mathrm{Hom}}_{B^o}(F_{-i}, k) \neq 0\} = \sup\{i | \underline{\mathrm{Hom}}_{B^o}(F_i, k) \neq 0\} = \inf\{i \mid R^i\Gamma_{\mathfrak{n}}(X) \neq 0\}$,

$\mathrm{k.id}_B(X) = \sup R\underline{\mathrm{Hom}}_B(k, X) = \sup\{i | \underline{\mathrm{Ext}}^i_B(k, X) \neq 0\}$
$= \sup\{i | \underline{\mathrm{Hom}}_{B^o}(F_{-i}, k) \neq 0\} = \inf\{i | \underline{\mathrm{Hom}}_{B^o}(F_i, k) \neq 0\} = \sup\{i \mid R^i\Gamma_{\mathfrak{n}}(X) \neq 0\}$. □

Theorem 12. *If $X \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(B))$, then*

$$\mathrm{id}_B(X) = \mathrm{k.id}_B(X).$$

Proof. see ([2], Theorem 4.5). □

Theorem 13. *Let C be a Noetherian connected graded k-algebra with a balanced dualizing complex $R^{\cdot}$. Then, the following are equivalent:*

(1) *C is AS–Gorenstein;*
(2) $\mathrm{id}_C\, C < \infty$;
(3) $\mathrm{pd}_C\, R^{\cdot} < \infty$;
(4) *For any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(C))$, $\mathrm{pd}_C\, X < \infty$ if and only if $\mathrm{id}_C\, X < \infty$.*

Proof. see ([16], Theorem 3). □

The following Theorem is complementary to the results of [7].

Theorem 14. *A is an AS–Gorenstein algebra if and only if A is an AS–Cohen–Macaulay algebra with a balanced dualizing complex.*

Proof. Assume that A is an AS–Gorenstein algebra. From ([2], Theorem 1.2), we know that A has a balanced dualizing complex $A_\alpha(-l)[d]$ for some graded automorphism α of A, some integer l, and $d = \mathrm{id}_{A^o}(A) = \mathrm{id}_A(A)$. So, $A_\alpha(-l)$ is a balanced dualizing module, and A is an AS–Cohen–Macaulay algebra with a balanced dualizing complex.

Next, assume that A is an AS–Cohen–Macaulay algebra with a balanced dualizing complex. According to [7], we know that it is a graded factor of an AS–Gorenstein algebra. Because A is an AS–Cohen–Macaulay algebra, $R\Gamma_{\mathfrak{n}}(A)$ is thus concentrated in one degree. Because $A \in D^{\mathrm{b}}_{\mathrm{fg}}(\mathrm{GrMod}(A))$, we know that $\mathrm{depth}_B(X) = \inf\{i | R^i\Gamma_{\mathfrak{n}}(X) \neq 0\} = \sup\{i | R^i\Gamma_{\mathfrak{n}}(X) \neq 0\} = \mathrm{k}.\,\mathrm{id}_B(X) = n < \infty$ from Lemma 5. From Theorem 12, $\mathrm{id}_A(A) = \mathrm{k} \cdot \mathrm{id}_A(A) = n < \infty$. From Theorem 13, we know that A is an AS–Gorenstein algebra. □

The following theorem complements the conclusions related to the commutative ring in [5] on noncommutative graded algebra.

Theorem 15. *Let B be an AS–Cohen–Macaulay algebra with a balanced dualizing complex; then, B satisfies* $\mathrm{ac} = \mathrm{tac}$.

Proof. From Theorem 14, it follows that when B is an AS–Cohen–Macaulay algebra with a balanced dualizing complex, B is also an AS–Gorenstein algebra. Thus, the AS–Cohen–Macaulay algebra with a balanced dualizing complex satisfies $\mathrm{ac} = \mathrm{tac}$. □

9. Conclusions

In this paper, the author identified a class of noncommutative graded algebra that satisfies $\mathrm{ac} = \mathrm{tac}$. The author also discussed the relationship between AS–Gorenstein algebra and AS–Cohen–Macaulay algebra with a balanced dualizing complex and showed that AS–Gorenstein algebra and AS–Cohen–Macaulay algebra with a balanced dualizing complex belong to this algebra. Unfortunately, we could not give a specific example for the above result. We did not find all the noncommutative graded algebraic classes satisfying $\mathrm{ac} = \mathrm{tac}$, and we did not find conditions such that AS–Cohen–Macaulay algebra is equivalent to $\mathrm{ac} = \mathrm{tac}$. Our next goal is to find all the noncommutative graded algebraic classes that satisfy $\mathrm{ac} = \mathrm{tac}$, and to try to find conditions such that AS–Cohen–Macaulay algebra is equivalent to $\mathrm{ac} = \mathrm{tac}$. It is also our next task to give concrete examples of the relevant results.

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