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The Limit Properties of Maxima of Stationary Gaussian Sequences Subject to Random Replacing

Yuwei Li [1,2] and Zhongquan Tan [2,*]

[1] Department of Mathematic, Zhejiang Normal University, Jinhua 321004, China; 18832546276@163.com
[2] College of Data Science, Jiaxing University, Jiaxing 314001, China
* Correspondence: tzq728@163.com

Abstract: In applications, missing data may occur randomly and some relevant datum are often used to replace the missing ones. This article mainly explores the influence of the degree of dependence of stationary Gaussian sequences on the joint asymptotic distribution of the maximum of the Gaussian sequence and its maximum when the sequence is subject to random replacing.

Keywords: extreme value theory; random replacing; asymptotic distribution; stationary gaussian sequences

MSC: 60G70; 60G15

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1. Introduction

Data missing is a common phenomenon in the field of applications. When it occurs, the most common approach is to treat the available sample as a non complete sample with a random sample size. Furthermore, it is necessary to study the properties of incomplete samples with random sample sizes. In the field of extreme value theory, refs. [1,2] first studied the effect of the missing data on extremes of original sequences. Let $\{X_n, n \geq 1\}$ be a sequence of stationary random variables with the marginal distribution function $F(x)$, and suppose that some of the random variables in the sequence are missing randomly. Let ε_k be the indicator of the event that random variable X_k is observed. For the random sequence $\{X_n, n \geq 1\}$, define its random missing sequence as:

$$\widetilde{X}_n(\boldsymbol{\varepsilon}) = \varepsilon_n X_n + (1-\varepsilon_n)x_F, \quad n \geq 1, \tag{1}$$

where $x_F = \inf\{x : F(x) > 0\}$. Suppose that the indicator sequence $\boldsymbol{\varepsilon} = \{\varepsilon_n, n \geq 1\}$ is independent of $\{X_n, n \geq 1\}$, and let $S_n = \sum_{k \leq n} \varepsilon_k$ be the numbers of the observed variables satisfying

$$\frac{S_n}{n} \xrightarrow{P} \lambda, \quad \text{as} \quad n \to \infty, \tag{2}$$

where λ is a random or nonrandom variable.

When $\lambda \in [0,1]$ is a constant, under a global dependent condition $D(u_n, v_n)$ (see [2]) and a well-known local dependent condition $D'(u_n)$ (see e.g., [3]), ref. [2] derived the joint asymptotic distribution of the maximum from a stationary sequence and the maximum from its random missing sequence and proved, for any $x < y \in \mathbb{R}$

$$\lim_{n\to\infty} P\Big(M_n(\widetilde{X}(\boldsymbol{\varepsilon})) \leq \widetilde{a}_n^{-1}x + \widetilde{b}_n, M_n(X) \leq \widetilde{a}_n^{-1}y + \widetilde{b}_n\Big) = G^{\lambda}(x)G^{1-\lambda}(y), \tag{3}$$

with $\widetilde{a}_n > 0$ and $\widetilde{b}_n \in \mathbb{R}$, where G is one of the three types of extreme value distributions (see, e.g., [3]) $M_n(\widetilde{X}(\boldsymbol{\varepsilon})) = \max\{\widetilde{X}_k(\boldsymbol{\varepsilon}), k = 1,2,\ldots,n\}$ and $M_n(X) = \max\{X_k, k = 1,2,\ldots,n\}$.

The result in (3) has been extended to many other cases; we refer to [4,5] for Gaussian cases; ref. [6,7] for the almost sure limit theorem; ref. [8,9] for autoregressive process; ref. [10] for non-stationary random fields; ref. [11] for linear process; and refs. [12,13] for point process.

When $\lambda \in [0,1]$ is a random variable, ref. [14] proved a similar result: for any $x < y \in \mathbb{R}$

$$\lim_{n\to\infty} P\Big(M_n(\widetilde{X}(\varepsilon)) \le \widetilde{a}_n^{-1}x + \widetilde{b}_n, M_n(X) \le \widetilde{a}_n^{-1}y + \widetilde{b}_n\Big) = E[G^{\lambda}(x)G^{1-\lambda}(y)]. \tag{4}$$

Ref. [15] extended the results of (4) to weakly and strongly dependent Gaussian sequences. Let $\{X_n, n \ge 1\}$ be a sequence of stationary Gaussian variables with correlation function $r_n = E(X_1X_{n+1})$. If r_n satisfies

$$\lim_{n\to\infty} r_n \log n = \gamma \in [0,\infty), \tag{5}$$

for any $x < y$, ref. [15] proved that

$$\begin{aligned} &\lim_{n\to\infty} P\Big(M_n(\widetilde{X}(\varepsilon)) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n\Big) \\ &= E\bigg(\int_{-\infty}^{+\infty} \exp(-\lambda g(x,z,\gamma) - (1-\lambda)g(x,z,\gamma))d\Phi(z)\bigg), \end{aligned} \tag{6}$$

where $\Phi(x)$ denotes the distribution function of a standard (mean 0 and variance 1) normal random variable, $g(x,z,\gamma) = e^{-x-\gamma+\sqrt{2\gamma}z}$, and the normalizing constants a_n and b_n are defined as

$$a_n = (2\log n)^{1/2}, \quad b_n = (2\log n)^{1/2} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{1/2}}. \tag{7}$$

If r_n satisfies:

(A1) r_n is convex with $r_n = o(1)$;
(A2) $(r_n \log n)^{-1}$ is monotone with $(r_n \log n)^{-1} = o(1)$,

for any $x, y \in \mathbb{R}$, ref. [15] proved that

$$\begin{aligned} &\lim_{n\to\infty} P(M_n(\widetilde{X}(\varepsilon)) \le (r_n)^{1/2}x + (1-r_n)^{1/2}b_n, M_n(X) \le (r_n)^{1/2}y + (1-r_n)^{1/2}b_n) \\ &= \Phi(\min\{x,y\}). \end{aligned} \tag{8}$$

For more related studies of this situation, we refer to [16–19].

In application, in addition to treating the available samples as incomplete samples with a random sample size, we often use another set of samples to replace the randomly missing samples, to obtain a relatively complete sample. However, this raises the question of to what extent can the random missing samples replace the original samples. To answer this question, we must study the relationship between the original samples and samples subject to random replacement. In the field of extreme value theory, we need to study the asymptotic relationship between the maximum of the original samples and their maximum when the samples are subject to random replacement.

For the random sequence $\{X_n, n \ge 1\}$, define the sequence subject to random replacement as

$$X_n(\varepsilon) = \varepsilon_n X_n + (1-\varepsilon_n)\widehat{X_n}, \tag{9}$$

where the sequence $\{\widehat{X_n}, n \ge 1\}$ is an independent copy of $\{X_n, n \ge 1\}$. When the sequence $\{X_n, n \ge 1\}$ is strongly mixed, ref. [20] proved that the maximum sequences and the maximum when the sequence is subject to random replacement are asymptotically dependent.Under the dependent conditions $D(u_n, v_n)$ and $D'(u_n)$, ref. [21] studied the asymptotic distribution of the maximum from a stationary sequence and its maximum subject to random replacement and proved, for $x, y \in \mathbb{R}$,

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \leq \widetilde{a}_n^{-1}x + \widetilde{b}_n, M_n(X) \leq \widetilde{a}_n^{-1}y + \widetilde{b}_n)$$
$$= G(\min\{x,y\})EG^{1-\lambda}(\max\{x,y\}),$$

where $M_n(X(\boldsymbol{\varepsilon})) = \max\{X_k(\boldsymbol{\varepsilon}), 1 \leq k \leq n\}$.

It is worth noting that, in the study of [21], the random replacement sequence was an independent copy of the original sequence, so they had the same dependent structure. However, in practical applications, we may not know the dependent structure of the original sequence, so it is necessary to explore the impact of the dependent structure of the original sequence itself and the dependent structure of the random replacement sequence itself on their maxima.

The main purpose of this article is to explore the influence of the self-dependent structure of an original sequence and the random replacement sequence on the joint asymptotic distribution between their maxima under a Gaussian scenario. The advantages of choosing a Gaussian sequence scenario are as follows: The dependent structure of Gaussian sequences can be characterized by their correlation coefficient functions; in the field of extreme value theory, the dependence of Gaussian sequences can be characterized by the speed at which their correlation coefficient function converges to 0; the relevant conclusions in the case of Gaussian sequences can be easily generalized, such as in the case of chi square sequences, Gaussian ordered sequences, and so on.

The rest of this paper is organized as follows: The main results of the paper are given in Section 2, and their proofs are collected in Section 3. Some conclusions are presented in Section 4.

2. Main Results

In the following part of this paper, let $\{X_n, n \geq 1\}$ and $\{\widehat{X_n}, n \geq 1\}$ be stationary standard Gaussian sequences, with correlation functions r_n and $\widehat{r}_n$, respectively. Let $\boldsymbol{\varepsilon} = \{\varepsilon_n, n \geq 1\}$ be a sequence of indicators and $S_n = \sum_{k\leq n} \varepsilon_k$. Suppose that (2) holds for some random variable $\lambda \in [0,1]$ a.s. In addition, suppose that $\{X_n, n \geq 1\}$, $\{\widehat{X_n}, n \geq 1\}$, $\{\varepsilon_n, n \geq 1\}$ are independent of each other, and suppose that U and V are independent standard Gaussian random variables, which are independent of λ. Let a_n and b_n be defined as in (7).

Theorem 1. *Suppose that r_n and $\widehat{r}_n$ satisfy $\lim_{n\to\infty} r_n \log n = \gamma_1 \in [0,\infty)$ and $\lim_{n\to\infty} \widehat{r}_n \log n = \gamma_2 \in [0,\infty)$, respectively. For any $x, y \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \leq a_n^{-1}x + b_n, M_n(X) \leq a_n^{-1}y + b_n)$$
$$= E[\exp(-(1-\lambda)g(x,V,\gamma_2))\exp(-\lambda g(\min\{x,y\},U,\gamma_1) - (1-\lambda)g(y,U,\gamma_1))].$$

Corollary 1. *(i). Suppose that r_n and $\widehat{r}_n$ satisfy $\lim_{n\to\infty} r_n \log n = 0$ and $\lim_{n\to\infty} \widehat{r}_n \log n = 0$, respectively. For any $x, y \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \leq a_n^{-1}x + b_n, M_n(X) \leq a_n^{-1}y + b_n)$$
$$= \exp\left(-e^{-\min\{x,y\}}\right) E\exp\left(-(1-\lambda)e^{-\max\{x,y\}}\right),$$

(ii). Suppose that r_n and $\widehat{r}_n$ satisfy $\lim_{n\to\infty} r_n \log n = 0$ and $\lim_{n\to\infty} \widehat{r}_n \log n = \gamma \in (0,\infty)$, respectively. For any $x, y \in \mathbb{R}$, we have

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \leq a_n^{-1}x + b_n, M_n(X) \leq a_n^{-1}x + b_n)$$
$$= E\left[\exp(-(1-\lambda)g(x,U,\gamma))\exp\left(-\lambda e^{-\min\{x,y\}} - (1-\lambda)e^{-y}\right)\right].$$

(iii). Suppose that r_n and $\widehat{r}_n$ satisfy $\lim_{n\to\infty} r_n \log n = \gamma \in (0, \infty)$ and $\lim_{n\to\infty} \widehat{r}_n \log n = 0$, respectively. For any $x, y \in \mathbb{R}$, we have

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}x + b_n)$$
$$= E\big[\exp\big(-(1-\lambda)e^{-x}\big)\exp(-\lambda g(\min\{x,y\}, U, \gamma) - (1-\lambda)g(y, U, \gamma))\big].$$

Remark 1. *The first assertion of Corollary 1 indicates that, when both the original sequence and the random replacement sequence are weakly dependent, the result is consistent with that of [21]. The second and third assertions of Corollary 1 indicate that, when the dependent strength between the original sequence and the random replacement sequence are different, the joint asymptotic distribution of the maximum of the original sequence and the maximum of the sequence subject to random replacement is highly dependent on the strength of dependence.*

Corollary 2. *Under the conditions of Theorem 1, for any $x \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} P(M_n(X) \le a_n^{-1}x + b_n) = E\exp(-g(x, U, \gamma_1)).$$

and

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \le a_n^{-1}x + b_n)$$
$$= E[\exp(-(1-\lambda)g(x, V, \gamma_2))\exp(-\lambda g(x, U, \gamma_1))]$$

Theorem 2. *Suppose both r_n and $\widehat{r}_n$ satisfy the conditions A1 and A2. For any $x, y \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \le (r_n)^{1/2}x + (1-r_n)^{1/2}b_n, M_n(X) \le (r_n)^{1/2}y + (1-r_n)^{1/2}b_n)$$
$$= \Phi(x)\Phi(\min\{x,y\}).$$

Corollary 3. *Under the conditions of Theorem 2, for any $x \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} P(M_n(X) \le a_n^{-1}x + b_n) = \Phi(x).$$

and

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \le a_n^{-1}x + b_n) = \Phi^2(x).$$

Remark 2. *Corollaries 2 and 3 indicate that, when both the original sequence and the random replacing sequence are weakly dependent, the limit distribution of the maximum of the original sequence and the limit distribution of the maximum of the sequence subject to random replacing are consistent. At this point, the sequence subject to random replacement can be used to replace the original sequence. When both the original sequence and the sequence subject to random replacement are strongly dependent, the limit distribution of the maximum of the original sequence and the sequence subject to random replacement is inconsistent. In this case, the sequence subject to random replacement cannot be directly used to replace the original sequence.*

Theorem 3. *(i). Suppose that r_n and $\widehat{r}_n$ satisfy $\lim_{n\to\infty} r_n \log n = \gamma \in (0, \infty)$ and the conditions A1 and A2, respectively. For any $x, y \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n)$$
$$= E[\exp(-\lambda g(\min\{x,y\}, U, \gamma) - (1-\lambda)g(y, U, \gamma))].$$

(ii). Suppose that r_n and $\widehat{r}_n$ satisfy the conditions A1 and A2 and $\lim_{n\to\infty} \widehat{r}_n \log n = \gamma \in (0,\infty)$, respectively. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\lim_{n\to\infty} P(M_n(X(\boldsymbol{\varepsilon})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n)\\
&= E[\exp(-(1-\lambda)g(x,U,\gamma))].
\end{aligned}$$

Remark 3. *Note that, for Gaussian random sequences with correlation functions satisfying the conditions A1 and A2, their maxima have a non-degenerate limit under the normalizing level $(r_n)^{1/2}x + (1-r_n)^{1/2}b_n$ and have a degenerate limit 1 under the normalizing level $a_n^{-1}x + b_n$; for Gaussian random sequences with correlation functions satisfying the condition $\lim_{n\to\infty} r_n \log n = \gamma \in (0,\infty)$, their maxima have a non-degenerate limit under the normalizing level $a_n^{-1}x + b_n$ and have a degenerate limit 0 under the normalizing level $(r_n)^{1/2}x + (1-r_n)^{1/2}b_n$. Thus, in order to obtain the non-degenerate limit, we choose the normalizing level $a_n^{-1}x + b_n$ in Theorem 3.*

3. Proofs

Let $\boldsymbol{\alpha} = \{\alpha_n, n \ge 1\}$ be a sequence of 0 and 1 ($\boldsymbol{\alpha} \in \{0,1\}^{\mathbb{N}}$). For the arbitrary random or nonrandom sequence $\boldsymbol{\beta} = \{\beta_n, n \ge 1\}$ of 0 and 1 and subset $I \subset \mathbb{N}$, put

$$M(X(\boldsymbol{\beta}), I) = \max\{X_i(\boldsymbol{\beta}), i \in I\}, \ \ M(X, I) = \max\{X_i, i \in I\}.$$

For any $I \subset \mathbb{N}$, put $I^1(\boldsymbol{\beta}) = \{i : i \in I, \beta_i = 1\}$ and $I^0(\boldsymbol{\beta}) = \{i : i \in I, \beta_i = 0\}$. Set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. For simplicity, in the following part, denote $u_n(x) = a_n^{-1}x + b_n$ and $\rho_n^{(s)} = \frac{\gamma_s}{\log n}$, $s = 1, 2$.

Lemma 1. *Let $\{X_n^*, n \ge 1\}$ be a standard Gaussian sequence with mutually independent elements, which is independent of $\{\varepsilon_n, n \ge 1\}$. Let $\{\widehat{X_n}^*, n \ge 1\}$ be the independent copy. Define $X_n^*(\boldsymbol{\varepsilon}) = \varepsilon_n X_n^* + (1-\varepsilon_n)\widehat{X_n}^*$. Under the conditions of Theorem 1, we have $n \to \infty$,*

$$\begin{aligned}
&\Bigg| P(M_n(X(\boldsymbol{\alpha})) \le u_n(x), M_n(X) \le u_n(y))\\
&- \int_{-\infty}^{+\infty} P(M(\widehat{X}^*, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le v_n(x,z,\gamma_2)) d\Phi(z)\\
&\quad \times \int_{-\infty}^{+\infty} P(M(X^*, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le v_n(x,z,\gamma_1), M_n(X^*) \le v_n(y,z,\gamma_1)) d\Phi(z) \Bigg| \to 0,
\end{aligned}$$

where $v_n(x,z,\gamma_s) = (1-\rho_n^{(s)})^{-1/2}(u_n(x) - (\rho_n^{(s)})^{1/2}z)$, $s = 1, 2$.

Proof. Note that $\mathbb{N}_n^1(\boldsymbol{\alpha}) = \{i : i \in \mathbb{N}_n, \alpha_i = 1\}$ and $\mathbb{N}_n^0(\boldsymbol{\alpha}) = \{i : i \in \mathbb{N}_n, \alpha_i = 0\}$. Let $\eta_n = (1-\rho_n^{(1)})^{1/2}X_n^* + (\rho_n^{(1)})^{1/2}U$, $\widehat{\eta_n} = (1-\rho_n^{(2)})^{1/2}\widehat{X_n}^* + (\rho_n^{(2)})^{1/2}V$, where U, V are independent standard Gaussian random variables and are independent of $\{X_n^*, n \ge 1\}$ and $\{\widehat{X_n}^*, n \ge 1\}$. Let $\eta_n(\boldsymbol{\varepsilon}) = \varepsilon_n \eta_n + (1-\varepsilon_n)\widehat{\eta_n}$. It is easy to see that both η_n, $\widehat{\eta_n}$ and $\eta_n(\boldsymbol{\alpha})$ are standard Gaussian sequences. Using the normal comparison lemma (see, e.g., [3]),

$$
\begin{aligned}
&\Big| P(M_n(X(\boldsymbol{\alpha})) \le u_n(x), M_n(X) \le u_n(y)) - P(M_n(\eta(\boldsymbol{\alpha})) \le u_n(x), M_n(\eta) \le u_n(y)) \Big| \\
&= \Big| P(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le u_n(x), M(X, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le u_n(x), M_n(X) \le u_n(y)) \\
&\quad - P(M(\widehat{\eta}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le u_n(x), M(\eta, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le u_n(x), M_n(\eta) \le u_n(y)) \Big| \\
&\le \Big| P(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le u_n(x)) - P(M(\widehat{\eta}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le u_n(x)) \Big| \\
&\quad + \Big| P(M(X, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le u_n(x), M_n(X) \le u_n(y)) - P(M(\eta, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le u_n(x), M_n(\eta) \le u_n(y)) \Big| \\
&\le Cn \sum_{k=1}^{n} |\widehat{r}_k - \rho_n^{(2)}| \exp\left(-\frac{u_n^2(x)}{1 + w_k^{(2)}} \right) + Cn \sum_{k=1}^{n} |r_k - \rho_n^{(1)}| \exp\left(-\frac{u_n^2(\min\{x, y\})}{1 + w_k^{(1)}} \right),
\end{aligned}
$$

where $w_k^{(1)} = \max\{|r_k|, \rho_n^{(1)}\}$, $w_k^{(2)} = \max\{|\widehat{r}_k|, \rho_n^{(2)}\}$ and C is a constant. Using Lemma 6.4.1 of [3], we know that the above sums tend to 0, as $n \to \infty$. With the definition of $\eta(\boldsymbol{\alpha})$, we have

$$
\begin{aligned}
&P(M_n(\eta(\boldsymbol{\alpha})) \le u_n(x), M_n(\eta) \le u_n(y)) \\
&= P(M(\widehat{\eta}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le u_n(x), M(\eta, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le u_n(x), M_n(\eta) \le u_n(y)) \\
&= P(M(\widehat{\eta}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le u_n(x)) P(M(\eta, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le u_n(x), M_n(\eta) \le u_n(y)) \\
&= \int_{-\infty}^{+\infty} P((1 - \rho_n^{(2)})^{1/2} M(\widehat{X}^*, \mathbb{N}_n^0(\boldsymbol{\alpha})) + (\rho_n^{(2)})^{1/2} V \le u_n(x) | V = z) d\Phi(z) \\
&\quad \times \int_{-\infty}^{+\infty} P((1 - \rho_n^{(1)})^{1/2} M(X^*, \mathbb{N}_n^1(\boldsymbol{\alpha})) + (\rho_n^{(1)})^{1/2} U \le u_n(x), \\
&\quad (1 - \rho_n^{(1)})^{1/2} M_n(X^*) + (\rho_n^{(1)})^{1/2} U \le u_n(y) | U = z) d\Phi(z) \\
&= \int_{-\infty}^{+\infty} P(M(\widehat{X}^*, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le v_n(x, z, \gamma_2)) d\Phi(z) \\
&\quad \times \int_{-\infty}^{+\infty} P(M(X^*, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le v_n(x, z, \gamma_1), M_n(X^*) \le v_n(y, z, \gamma_1)) d\Phi(z).
\end{aligned}
$$

The proof of Lemma 1 is complete. □

For some fixed k, define $K_s = \{j \in \mathbb{N} : (s-1)t + 1 \le j \le st\}$, $1 \le s \le k$, where $t = \lfloor \frac{n}{k} \rfloor$, and $\lfloor x \rfloor$ denotes the integral part of x.

Lemma 2. *Under the conditions of Theorem 1, for any $x, y \in \mathbb{R}$,*

$$
\begin{aligned}
&\Big| P(M(\widehat{X}^*, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le v_n(x, z_1, \gamma_2)) P(M(X^*, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le v_n(x, z_2, \gamma_1), M(X^*, \mathbb{N}_n) \le v_n(y, z_2, \gamma_1)) \\
&- \prod_{s=1}^{k} P(M(\widehat{X}^*, K_s^0(\boldsymbol{\alpha})) \le v_n(x, z_1, \gamma_2)) P(M(X^*, K_s^1(\boldsymbol{\alpha})) \le v_n(x, z_2, \gamma_1), M(X^*, K_s) \le v_n(y, z_2, \gamma_1)) \Big| \\
&\le 2t\overline{\Phi}(v_n(x, z_1, \gamma_2)) + 4t\overline{\Phi}(v_n(\min\{x, y\}, z_2, \gamma_1)),
\end{aligned}
$$

where $\overline{\Phi}(x) = 1 - \Phi(x)$.

Proof. This proof is the same as that of Lemma 3.2 of [21], so we omit the details. □

Lemma 3. *Under the conditions of Theorem 1, for any* $0 \le r \le 2^k - 1$,

$$\begin{aligned}
&1 - \frac{r}{2^k} t\overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1)) - (1 - \frac{r}{2^k}) t\overline{\Phi}(v_n(y,z,\gamma_1)) \\
&\quad + \left(\frac{\sum_{j\in K_s}\alpha_j}{t} - \frac{r}{2^k}\right) t(\overline{\Phi}(v_n(y,z,\gamma_1)) - \overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1))) \\
&\le P(M(X^*, K_s^1(\boldsymbol{\alpha})) \le v_n(x,z,\gamma_1), M(X^*, K_s) \le v_n(y,z,\gamma_1)) \\
&\le 1 - \frac{r}{2^k} t\overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1)) - (1 - \frac{r}{2^k}) t\overline{\Phi}(v_n(y,z,\gamma_1)) \\
&\quad + \left(\frac{\sum_{j\in K_s}\alpha_j}{t} - \frac{r}{2^k}\right) t(\overline{\Phi}(v_n(y,z,\gamma_1)) - \overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1))) \\
&\quad + 3t^2\overline{\Phi}^2(v_n(\min\{x,y\},z,\gamma_1))
\end{aligned}$$

and

$$\begin{aligned}
&1 - (1 - \frac{r}{2^k}) t\overline{\Phi}(v_n(x,z,\gamma_2)) + \left(\frac{\sum_{j\in K_s}\alpha_j}{t} - \frac{r}{2^k}\right) t\overline{\Phi}(v_n(x,z,\gamma_2)) \\
&\le P(M(\widehat{X}^*, K_s^0(\boldsymbol{\alpha})) \le v_n(x,z,\gamma_2)) \\
&\le 1 - (1 - \frac{r}{2^k}) t\overline{\Phi}(v_n(x,z,\gamma_2)) + \left(\frac{\sum_{j\in K_s}\alpha_j}{t} - \frac{r}{2^k}\right) t\overline{\Phi}(v_n(x,z,\gamma_2)) + t^2\overline{\Phi}^2(v_n(x,z,\gamma_2)).
\end{aligned}$$

Proof. Recall that $K_s^1(\boldsymbol{\alpha}) = \{i : i \in K_s, \alpha_i = 1\}$ and $K_s^0(\boldsymbol{\alpha}) = \{i : i \in K_s, \alpha_i = 0\}$. Noting that $\{X_n^*, n \ge 1\}$ is a Gaussian random sequence with mutually independent elements, we have

$$\begin{aligned}
&P(M(X^*, K_s^1(\boldsymbol{\alpha})) \le v_n(x,z,\gamma_1), M(X^*, K_s) \le v_n(y,z,\gamma_1)) \\
&= P(M(X^*, K_s^1(\boldsymbol{\alpha})) \le v_n(\min\{x,y\},z,\gamma_1), M(X^*, K_s^0(\boldsymbol{\alpha})) \le v_n(y,z,\gamma_1)) \\
&\le 1 - P(M(X^*, K_s^1(\boldsymbol{\alpha})) > v_n(\min\{x,y\},z,\gamma_1)) - P(M(X^*, K_s^0(\boldsymbol{\alpha})) > v_n(y,z,\gamma_1)) \\
&\quad + P(M(X^*, K_s^1(\boldsymbol{\alpha})) > v_n(\min\{x,y\},z,\gamma_1), M(X^*, K_s^0(\boldsymbol{\alpha})) > v_n(y,z,\gamma_1)) \\
&\le 1 - \sharp(K_s^1(\boldsymbol{\alpha}))\overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1)) - \sharp(K_s^0(\boldsymbol{\alpha}))\overline{\Phi}(v_n(y,z,\gamma_1)) \\
&\quad + 3t^2\overline{\Phi}^2(v_n(\min\{x,y\},z,\gamma_1)) \\
&= 1 - \frac{r}{2^k} t\overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1)) - (1 - \frac{r}{2^k}) t\overline{\Phi}(v_n(y,z,\gamma_1)) \\
&\quad + \left(\frac{\sum_{j\in K_s}\alpha_j}{t} - \frac{r}{2^k}\right) t(\overline{\Phi}(v_n(y,z,\gamma_1)) - \overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1))) \\
&\quad + 3t^2\overline{\Phi}^2(v_n(\min\{x,y\},z,\gamma_1)),
\end{aligned}$$

where $\sharp(A)$ denotes the cardinality of the set A. Similarly,

$$\begin{aligned}
&P(M(X^*, K_s^1(\boldsymbol{\alpha})) \le v_n(x,z_2,\gamma_1), M(X^*, K_s) \le v_n(y,z_2,\gamma_1)) \\
&\ge 1 - \frac{r}{2^k} t\overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1)) - (1 - \frac{r}{2^k}) t\overline{\Phi}(v_n(y,z,\gamma_1)) \\
&\quad + \left(\frac{\sum_{j\in K_s}\alpha_j}{t} - \frac{r}{2^k}\right) t(\overline{\Phi}(v_n(y,z,\gamma_1)) - \overline{\Phi}(v_n(\min\{x,y\},z,\gamma_1))),
\end{aligned}$$

which completes the proof of the first result. The proof of the second result is similar, so we omit it. □

Now, for the random variable $\lambda \in (0,1]$ a.s., define

$$B_{r,l} = \left\{ w : \lambda(w) \in \left\{ \begin{array}{ll} \left[0, \frac{1}{2^l}\right], & r = 0 \\ \left(\frac{r}{2^l}, \frac{r+1}{2^l}\right], & 0 < r < 2^l - 1 \end{array} \right. \right\}$$

and

$$\widetilde{B}_{\boldsymbol{\alpha},n} = \{w : \varepsilon_j(w) = \boldsymbol{\alpha}_j, 1 \le j \le n\}.$$

Put

$$B_{r,l,\boldsymbol{\alpha},n} = B_{r,l} \cap \widetilde{B}_{\boldsymbol{\alpha},n}.$$

Proof of Theorem 1. Note that

$$\begin{aligned}
&P(M_n(X(\boldsymbol{\varepsilon})) \le u_n(x), M_n(X) \le u_n(y)) \\
&= \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E\Big(P(M_n(X(\boldsymbol{\alpha})) \le u_n(x), M_n(X) \le u_n(y)) I_{[B_{r,k,\boldsymbol{\alpha},n}]}\Big).
\end{aligned}$$

We will split the proof into six steps. The first step, using Lemma 1, we have $n \to \infty$

$$\begin{aligned}
\Sigma_n^{(1)} := \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E\Big(\Big| &P(M_n(X(\boldsymbol{\alpha})) \le u_n(x), M_n(X) \le u_n(y)) \\
&- E\Big(P(M(\widehat{X}^*, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le v_n(x, U, \gamma_2))|U\Big) \\
&\times E\Big(P(M(X^*, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le v_n(x, U, \gamma_1), M(X^*, \mathbb{N}_n) \le v_n(y, U, \gamma_1))|U\Big)\Big| I_{[B_{r,k,\boldsymbol{\alpha},n}]}\Big) \to 0.
\end{aligned}$$

In the second step, we will prove $n \to \infty$ and $k \to \infty$

$$\begin{aligned}
\Sigma_n^{(2)} := \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E\Big(\Big| &E\Big(P(M(\widehat{X}^*, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le v_n(x, U, \gamma_2))|U\Big) \\
&\times E\Big(P(M(X^*, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le v_n(x, U, \gamma_1), M(X^*, \mathbb{N}_n) \le v_n(y, U, \gamma_1))|U\Big) \\
&- E\Big(\prod_{s=1}^{k} P(M(\widehat{X}^*, K_s^0(\boldsymbol{\alpha})) \le v_n(x, U, \gamma_2))|U\Big) \\
&\times E\Big(\prod_{s=1}^{k} P(M(X^*, K_s^1(\boldsymbol{\alpha})) \le v_n(x, U, \gamma_1), M(X^*, K_s) \le v_n(y, U, \gamma_1))|U\Big)\Big| I_{[B_{r,k,\boldsymbol{\alpha},n}]}\Big) \\
&\to 0.
\end{aligned}$$

Using Lemma 2, we have

$$\begin{aligned}
\Sigma_n^{(2)} &\le 2tE\overline{\Phi}(v_n(x, U, \gamma_2)) + 4tE\overline{\Phi}(v_n(\min\{x, y\}, U, \gamma_1)) \\
&\le \frac{2}{k} nE\overline{\Phi}(v_n(x, U, \gamma_2)) + \frac{4}{k} nE\overline{\Phi}(v_n(\min\{x, y\}, U, \gamma_1)).
\end{aligned}$$

It follows from the proof of Theorem 6.5.1 of [3] that

$$v_n(x, z, \gamma) = u_n(x + \gamma - \sqrt{2\gamma} z) + o(a_n^{-1}).$$

Then, as $n \to \infty$

$$E(n\overline{\Phi}(v_n(x, U, \gamma))) = Eg(x, U, \gamma)(1 + o(1)). \tag{10}$$

Thus, as $n \to \infty$ and $k \to \infty$, $\Sigma_n^{(2)}$ tends to 0.

In the third step, we prove that $n \to \infty$ and $k \to \infty$

$$
\begin{aligned}
\Sigma_n^{(3)} :=& \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E\Bigg(\Bigg|E\Bigg(\prod_{s=1}^{k} P(M(\widehat{X}^*, K_s^0(\boldsymbol{\alpha})) \le v_n(x,U,\gamma_2))|U\Bigg) \\
&\times E\Bigg(\prod_{s=1}^{k} P(M(X^*, K_s^1(\boldsymbol{\alpha})) \le v_n(x,U,\gamma_1), M(X^*,K_s) \le v_n(y,U,\gamma_1))|U\Bigg) \\
-&E\Bigg(\Bigg(1-\frac{(1-\frac{r}{2^k})n\overline{\Phi}(v_n(x,U,\gamma_2))}{k}\Bigg)^k |U\Bigg) \\
&\times E\Bigg(\Bigg(1-\frac{\frac{r}{2^k}\overline{\Phi}(v_n(\min\{x,y\},U,\gamma_1)) + (1-\frac{r}{2^k})n\overline{\Phi}(v_n(y,U,\gamma_1))}{k}\Bigg)^k |U\Bigg)\Bigg| I_{[B_{r,k,\boldsymbol{\alpha},n}]}\Bigg) \\
\to& \ 0.
\end{aligned}
$$

By the following basic inequality

$$
\left|\prod_{s=1}^{k} a_s - \prod_{s=1}^{k} b_s\right| \le \sum_{s=1}^{k} |a_s - b_s|, \ \ a_s, b_s \in (0,1], \tag{11}
$$

we obtain

$$
\Sigma_n^{(3)} \le \Sigma_n^{(31)} + \Sigma_n^{(32)},
$$

where, using Lemma 3, we have

$$
\begin{aligned}
\Sigma_n^{(31)} :=& \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E\Bigg(E\Bigg(\sum_{s=1}^{k} \Bigg| P(M(\widehat{X}^*, K_s^0(\boldsymbol{\alpha})) \le v_n(x,U,\gamma_2)) \\
&- \Bigg(1 - \frac{(1-\frac{r}{2^k})\overline{\Phi}(v_n(x,U,\gamma_2))}{k}\Bigg)\Bigg| |U\Bigg) I_{[B_{r,k,\boldsymbol{\alpha},n})}\Bigg) \\
\le& \sum_{r=0}^{2^k-1} E\Bigg(E\Bigg(\sum_{s=1}^{k} \Bigg|\Bigg(\frac{\sum_{j\in K_s}\varepsilon_j}{t} - \frac{r}{2^k}\Bigg) t\overline{\Phi}(v_n(x,U,\gamma_2)) + t^2\overline{\Phi}^2(v_n(x,U,\gamma_2))\Bigg| |U\Bigg) I_{[B_{r,k}]}\Bigg).
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma_n^{(32)} :=& \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E\Bigg(E\Bigg(\sum_{s=1}^{k} \Bigg| P(M(X^*, K_s^1(\boldsymbol{\alpha})) \le v_n(x,U,\gamma_1), M(X^*,K_s) \le v_n(y,U,\gamma_1)) \\
&- \Bigg(1 - \frac{\frac{r}{2^k}\overline{\Phi}(v_n(\min\{x,y\},U,\gamma_1)) + (1-\frac{r}{2^k})\overline{\Phi}(v_n(y,U,\gamma_1))}{k}\Bigg)\Bigg| |U\Bigg) I_{[B_{r,k,\boldsymbol{\alpha},n}]}\Bigg) \\
\le& \sum_{r=0}^{2^k-1} E\Bigg(E\Bigg(\sum_{s=1}^{k} \Bigg|\Bigg(\frac{\sum_{j\in K_s}\varepsilon_j}{t} - \frac{r}{2^k}\Bigg) t(\overline{\Phi}(v_n(y,U,\gamma_1)) - \overline{\Phi}(v_n(\min\{x,y\},U,\gamma_1))) \\
&+ 3t^2\overline{\Phi}^2(v_n(\min\{x,y\},U,\gamma_1))\Bigg| |U\Bigg) I_{[B_{r,k}]}\Bigg).
\end{aligned}
$$

Taking into account (2), we have $t \to \infty$,

$$
\frac{S_{st}}{st} \xrightarrow{p} \lambda, \ \ \frac{S_{(s-1)t}}{(s-1)t} \xrightarrow{p} \lambda;
$$

furthermore, using dominated convergence theorem, we have as $t \to \infty$

$$
E\left|\frac{S_{st}}{st} - \lambda\right| \to 0, \ \ E\left|\frac{S_{(s-1)t}}{(s-1)t} - \lambda\right| \to 0.
$$

Hence , we obtain $t \to \infty$

$$\begin{aligned}
\sum_{r=0}^{2^k-1} E\left|\frac{\sum_{j\in K_s}\varepsilon_j}{t}-\frac{r}{2^k}\right| I_{[B_{r,k}]} &\le E\left|\frac{\sum_{j\in K_s}\varepsilon_j}{t}-\lambda\right| + \sum_{r=0}^{2^k-1} E\left|\lambda-\frac{r}{2^k}\right| I_{[B_{r,k}]} \\
&\le E\left|\frac{S_{st}-S_{(s-1)t}}{t}-\lambda\right| + \frac{1}{2^k} \\
&= E\left|s\left(\frac{S_{st}}{st}-\lambda\right)+(s-1)\left(\frac{S_{(s-1)t}}{(s-1)t}-\lambda\right)\right| + \frac{1}{2^k} \\
&\le sE\left|\frac{S_{st}}{st}-\lambda\right| + (s-1)E\left|\frac{S_{(s-1)t}}{(s-1)t}-\lambda\right| + \frac{1}{2^k} \\
&= o(1)+\frac{1}{2^k}. \qquad (12)
\end{aligned}$$

Combining (10) with (12) and letting $t \to \infty$, we have

$$\begin{aligned}
\lim_{n\to\infty} \Sigma_n^{(3)} \le\ & \frac{Eg(x,U,\gamma_2)}{2^k} + \frac{(Eg(x,U,\gamma_2))^2}{k} \\
&+ \frac{E(g(y,U,\gamma_1)-g(\min\{x,y\},U,\gamma_1))}{2^k} + \frac{3(Eg(\min\{x,y\},U,\gamma_1))^2}{k}.
\end{aligned}$$

Thus, letting $k \to \infty$, $\Sigma_n^{(3)}$ tends to 0.
In the fourth step, we prove $n \to \infty$ and $k \to \infty$

$$\begin{aligned}
\Sigma_n^{(4)} := \sum_{r=0}^{2^k-1} \sum_{\alpha\in\{0,1\}^n} E\Bigg(\Bigg| & E\left(\left(1-\frac{(1-\frac{r}{2^k})n\overline{\Phi}(v_n(x,U,\gamma_2))}{k}\right)^k |U\right) \\
&\times E\left(\left(1-\frac{\frac{r}{2^k}\overline{\Phi}(v_n(\min\{x,y\},U,\gamma_1))+(1-\frac{r}{2^k})n\overline{\Phi}(v_n(y,U,\gamma_1))}{k}\right)^k |U\right) \\
-E\left(\left(1-\frac{(1-\lambda)n\overline{\Phi}(v_n(x,U,\gamma_2))}{k}\right)^k |U\right)& \\
&\times E\left(\left(1-\frac{\lambda n\overline{\Phi}(v_n(\min\{x,y\},U,\gamma_1))+(1-\lambda)n\overline{\Phi}(v_n(y,U,\gamma_1))}{k}\right)^k |U\right)\Bigg| I_{[B_{r,k,\alpha,n}]}\Bigg) \to 0.
\end{aligned}$$

Using (11) and (12) again, we obtain

$$\Sigma_n^{(4)} \le \frac{Eg(x,U,\gamma_2)}{2^k} + \frac{E(g(y,U,\gamma_1)-g(\min\{x,y\},U,\gamma_1))}{2^k}.$$

Thus, letting $k \to \infty$, we have $\Sigma_n^{(4)}$ tends to 0.

In the fifth step, using (10) again, it is easy to show that $n \to \infty$ and $k \to \infty$

$$\Sigma_n^{(5)} := \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E\Bigg(\Bigg|E\Bigg(\Bigg(1 - \frac{(1-\lambda)n\overline{\Phi}(v_n(x,U,\gamma_2))}{k}\Bigg)^k |U\Bigg)$$
$$\times E\Bigg(\Bigg(1 - \frac{\lambda n\overline{\Phi}(v_n(\min\{x,y\},U,\gamma_1)) + (1-\lambda)n\overline{\Phi}(v_n(y,U,\gamma_1))}{k}\Bigg)^k |U\Bigg)$$
$$-E\Bigg(\Bigg(1 - \frac{(1-\lambda)g(x,U,\gamma_2)}{k}\Bigg)^k |U\Bigg)$$
$$\times E\Bigg(\Bigg(1 - \frac{\lambda g(\min\{x,y\},U,\gamma_1) + (1-\lambda)g(y,U,\gamma_1)}{k}\Bigg)^k |U\Bigg)\Bigg| I_{[B_{r,k,\boldsymbol{\alpha},n}]}\Bigg) \to 0.$$

In the last step, letting $k \to \infty$, we have

$$\Sigma_n^{(6)} := \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} E\Bigg(\Bigg|E\Bigg(\Bigg(1 - \frac{(1-\lambda)g(x,U,\gamma_2)}{k}\Bigg)^k |U\Bigg)$$
$$\times E\Bigg(\Bigg(1 - \frac{\lambda g(\min\{x,y\},U,\gamma_1) + (1-\lambda)g(y,U,\gamma_1)}{k}\Bigg)^k |U\Bigg)$$
$$-E\Big(\exp(-(1-\lambda)g(x,U,\gamma_2))|U\Big)$$
$$\times E\Big(\exp(-\lambda g(\min\{x,y\},U,\gamma_1) + (1-\lambda)g(y,U,\gamma_1))|U\Big)\Bigg| I_{[B_{r,k,\boldsymbol{\alpha},n}]}\Bigg) \to 0.$$

The proof of Theorem is complete. □

Proof of Theorem 2. First, note that

$$\begin{aligned} &P(M_n(X(\boldsymbol{\varepsilon})) \le (r_n)^{1/2}x + (1-r_n)^{1/2}b_n, M_n(X) \le (r_n)^{1/2}y + (1-r_n)^{1/2}b_n) \\ &= \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} P(n,\boldsymbol{\alpha})P(\widetilde{B}_{\boldsymbol{\alpha},n}), \end{aligned} \tag{13}$$

where

$$\begin{aligned} P(n,\boldsymbol{\alpha}) &:= P(M_n(X(\boldsymbol{\alpha})) \le (r_n)^{1/2}x + (1-r_n)^{1/2}b_n, M_n(X) \le (r_n)^{1/2}y + (1-r_n)^{1/2}b_n) \\ &= P(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le (r_n)^{1/2}x + (1-r_n)^{1/2}b_n) \\ &\times P(M(X, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le (r_n)^{1/2}x + (1-r_n)^{1/2}b_n, M_n(X) \le (r_n)^{1/2}y + (1-r_n)^{1/2}b_n). \end{aligned} \tag{14}$$

It follows from (3.5) of [15] that

$$\lim_{n\to\infty} P(M(X, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le (r_n)^{1/2}x + (1-r_n)^{1/2}b_n) = \Phi(x). \tag{15}$$

Since $\{X_n, n \ge 1\}$ and $\{\widehat{X_n}, n \ge 1\}$ have the same distribution function, using a similar proof, we have

$$\lim_{n\to\infty} P(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le (r_n)^{1/2}x + (1-r_n)^{1/2}b_n) = \Phi(x). \tag{16}$$

Hence, combining (8) and (16), we have

$$\lim_{n\to\infty} P(n,\boldsymbol{\alpha}) = \Phi(x)\Phi(\min\{x,y\}).$$

Now, we can finish the proof of Theorem 2 by plugging the last equality into (13) and dominated convergence theorem. □

Proof of Theorem 3. We only give the proof of case (i), since the proof of case (ii) is similar. First, note that

$$\begin{aligned} & P(M_n(X(\boldsymbol{\varepsilon})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n) \\ &= \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} \widetilde{P}(n,\boldsymbol{\alpha}) P(B_{r,k,\boldsymbol{\alpha},n}), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \widetilde{P}(n,\boldsymbol{\alpha}) &:= P(M_n(X(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n) \\ &= P(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n) P(M(X, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n). \end{aligned} \tag{18}$$

Obviously,

$$\begin{aligned} & P(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n) \\ &= P\Big(\widehat{r}_n^{-1/2}(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) - (1-\widehat{r}_n)^{1/2}b_n) \le \widehat{r}_n^{-1/2}(a_n^{-1}x + b_n - (1-\widehat{r}_n)^{1/2}b_n)\Big). \end{aligned}$$

Since the correlation function $\widehat{r}_n$ of $\{\widehat{X_n}, n \ge 1\}$ satisfies the conditions A1 and A2, we have $n \to \infty$

$$\begin{aligned} \widehat{r}_n^{-1/2}(a_n^{-1}x + b_n - (1-\widehat{r}_n)^{1/2}b_n) &= \widehat{r}_n^{-1/2}(a_n^{-1}x + \frac{1}{2}\widehat{r}_n b_n + o(\widehat{r}_n b_n)) \\ &= \frac{1}{2}(2\widehat{r}_n \log n)^{1/2} + o((2\widehat{r}_n \log n)^{1/2}) \\ &\to \infty. \end{aligned}$$

Furthermore, using (16), as $n \to \infty$

$$P(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n) \to 1.$$

Hence, for any $\varepsilon > 0$ and sufficiently large n

$$1 - \varepsilon \le P(M(\widehat{X}, \mathbb{N}_n^0(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n) \le 1 + \varepsilon. \tag{19}$$

Thus, for a sufficiently large n

$$\begin{aligned} & \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} (1-\varepsilon) P(M(X, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n) \\ &\le P(M_n(X(\boldsymbol{\varepsilon})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n) \\ &\le \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} (1+\varepsilon) P(M(X, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n). \end{aligned} \tag{20}$$

Now, using the dominated convergence theorem, in order to finishing the proof, we only need to show $n \to \infty$ and $k \to \infty$

$$\begin{aligned} & \sum_{r=0}^{2^k-1} \sum_{\boldsymbol{\alpha}\in\{0,1\}^n} P(M(X, \mathbb{N}_n^1(\boldsymbol{\alpha})) \le a_n^{-1}x + b_n, M_n(X) \le a_n^{-1}y + b_n) \\ &\to E\bigg(\int_{-\infty}^{+\infty} \exp(-\lambda g(\min\{x,y\}, z, \gamma) - (1-\lambda) g(y, z, \gamma)) d\Phi(z)\bigg). \end{aligned} \tag{21}$$

Noting that the correlation function r_n of $\{X_n, n \ge 1\}$ satisfies $\lim_{n\to\infty} r_n \log n = \gamma \in (0,\infty)$, repeating the proof of Theorem 1, we can prove that (21) holds. □

4. Conclusions

The joint asymptotic distribution of the maximum of stationary Gaussian sequence and the maximum of the sequence subject to random replacing is highly dependent on the dependent structure of the original sequence and the replacing sequence.

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References

1. Hall, A.; Hüsler, J. Extremes of stationary sequences with failures. *Stoch. Model.* **2006**, *22*, 537–557. [CrossRef]
2. Mladenović, P.; Piterbarg, V. On asymptotic distribution of maxima of complete and incomplete samples from stationary sequences. *Stoch. Process. Their Appl.* **2006**, *116*, 1977–1991. [CrossRef]
3. Leadbetter, M.R.; Lindgren, G.; Rootzén, H. *Extremes and Related Properties of Random Sequences and Processes*; Springer: New York, NY, USA, 1983.
4. Peng, Z.; Cao, L.; Nadarajah, S. Asymptotic distributions of maxima of complete and incomplete samples from multivariate stationary Gaussian sequences. *J. Multivar. Anal.* **2010**, *101*, 2641–2647. [CrossRef]
5. Cao, L.; Peng, Z. Asymptotic distributions of maxima of complete and incomplete samples from strongly dependent stationary Gaussian sequences. *Appl. Math. Lett.* **2011**, *24*, 243–247. [CrossRef]
6. Tong, B.; Peng, Z. On almost sure max-limit theorems of complete and incomplete samples from stationary sequences. *Acta Math. Sin.* **2011**, *27*, 1323–1332. [CrossRef]
7. Tan, Z.; Wang, Y. Some asymptotic results on extremes of incomplete samples. *Extremes* **2012**, *15*, 319–332. [CrossRef]
8. Mladenović, P.; Zivadinovic, L. Uniform AR(1) Processes and maxima on partial samples. *Commun. Stat.-Theory Methods* **2015**, *44*, 2546–2563. [CrossRef]
9. Glavaš, L.; Mladenović, P.; Samorodnitsky, G. Extreme values of the uniform order 1 autoregressive processes and missing observations. *Extremes* **2017**, *20*, 671–690. [CrossRef]
10. Panga, Z.; Pereira, L. On the maxima and minima of complete and incomplete samples from nonstationary random fields. *Stat. Probab. Lett.* **2018**, *137*, 124–134. [CrossRef]
11. Glavaš, L.; Mladenović, P. Extreme values of linear processes with heavy-tailed innovations and missing observations. *Extremes* **2020**, *23*, 547–567. [CrossRef]
12. Tong, J.; Peng, Z. Asymptotic distributions of the exceedances point processes for a class of dependent normal sequence. *Acta Math. Appl. Sin.* **2010**, *33*, 1–9.
13. Peng, Z.; Tong, J.; Weng, Z. Exceedances point processes in the plane of stationary Gaussian sequences with data missing. *Stat. Probab. Lett.* **2019**, *149*, 73–79. [CrossRef]
14. Krajka, T. The asymptotic behaviour of maxima of complete and incomplete samples from statioanry sequences. *Stoch. Process. Their Appl.* **2011**, *121*, 1705–1719. [CrossRef]
15. Hashorva, E.; Peng, Z.; Weng, Z. On Piterbarg theorem for maxima of stationary Gaussian sequences. *Lith. Math. J.* **2013**, *53*, 280–292. [CrossRef]
16. Hashorva, E.; Weng, Z. Maxima and minima of complete and incomplete stationary sequences. *Stochastics* **2014**, *86*, 707–720. [CrossRef]
17. Krajka, T.; Rychlik, Z. The limiting behaviour of sums and maximums of iid random variables from the viewpoint of different observers. *Probab. Math. Stat.* **2014**, *34*, 237–252.
18. Zheng, S.; Tan, Z. On the maxima of nonstationary random fields subject to missing observations. *arXiv* **2023**, arXiv:2306.13857.
19. Fang, Y.; Tan, Z. On the extreme order statistics for stationary Gaussian sequences subject to random missing observations. *arXiv* **2023**, arXiv:2306.13861.

20. Robert, C.Y. On asymptotic distribution of maxima of stationary sequences subject to random failure or censoring. *Statist. Probab. Lett.* **2010**, *134*, 134–142. [CrossRef]
21. Li, Y.; Tan, Z. The asymptotic distribution of maxima of stationary random sequences under random replacing. *arXiv* **2023**. [CrossRef]