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Fuzzy Differential Subordination and Superordination Results for Fractional Integral Associated with Dziok-Srivastava Operator

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Abstract: Fuzzy set theory, introduced by Zadeh, gives an adaptable and logical solution to the provocation of introducing, evaluating, and opposing numerous sustainability scenarios. The results described in this article use the fuzzy set concept embedded into the theories of differential subordination and superordination from the geometric function theory. In 2011, fuzzy differential subordination was defined as an extension of the classical notion of differential subordination, and in 2017, the dual concept of fuzzy differential superordination appeared. These dual notions are applied in this paper regarding the fractional integral applied to Dziok–Srivastava operator. New fuzzy differential subordinations are proved using known lemmas, and the fuzzy best dominants are established for the obtained fuzzy differential subordinations. Dual results regarding fuzzy differential superordinations are proved for which the fuzzy best subordinates are shown. These are the first results that link the fractional integral applied to Dziok–Srivastava operator to fuzzy theory.

Keywords: analytic function; fuzzy differential subordination; fuzzy differential superordination; fractional integral; Dziok–Srivastava operator

MSC: 30C45; 30A20; 34A40

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Citation: Alb Lupaş, A. Fuzzy Differential Subordination and Superordination Results for Fractional Integral Associated with Dziok-Srivastava Operator. *Mathematics* **2023**, *11*, 3129. https://doi.org/10.3390/math11143129

Academic Editor: Nicoleta Breaz

Received: 22 June 2023
Revised: 7 July 2023
Accepted: 14 July 2023
Published: 15 July 2023

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1. Introduction

The concept of the fuzzy set was defined in 1965 by Zadeh [1] and applications of this concept appeared in [2,3]. It has numerous applications in technology and science. Fuzzy mathematical models are introduced in these studies using fuzzy set theory to estimate the development of the scientific and socio-environmental world. Fuzzy set theory links human expectations for development as stated in language concepts to numerical facts reflected in measurements of sustainability indicators, despite the fact that decision-making regarding sustainable development is subjective. An intuitionistic fuzzy set was applied to define a new extension to the multi-criteria decision-making model for sustainable supplier selection based on sustainable supply chain management practices in [4], based on the fact that selecting a corresponding supplier is the key element of contemporary businesses from a sustainability perspective. One of the generalized forms of orthopairs uses intuitionistic fuzzy sets. Orthopair fuzzy sets were introduced by [5], who studied the basic properties of a generalized frame for orthopair fuzzy sets called "(m, n)-Fuzzy sets". Supply chain sustainability was considered in the fuzzy context for the steel industry in [6], and a model for sustainable energy usage in the textile sector based on intuitionistic fuzzy sets appeared in [7]. The nonlinear integrated fuzzy modeling was used to predict how comfortable an office building can be and how that would affect people's health for optimized sustainability in [8].

The concept of a fuzzy set used in research imposed development in mathematics. The notion of fuzzy differential subordination introduced in 2011 [9] has developed an entire theory since the classical theory of differential subordination [10] was adapted to the

fuzzy theory in 2012 [11]. The fuzzy differential superordination was introduced as the dual notion of fuzzy subordination in 2017 [12]. Numerous results regarding fuzzy differential subordinations and superordinations [13] were established regarding several known operators: Sălăgean and Ruscheweyh operators [14], a linear operator [15], generalized Noor–Sălăgean operator [16] or Wanas operator [17,18].

There is no indication up to this point as to how these notions can be applied to other branches of research or in real life. This is a new line of research that has seen development using the geometric function theory. The link between the fuzzy sets and geometric function theories has been illustrated in [13]. The fractional integral of confluent hypergeometric functions was investigated by applying classical theories of differential subordination and superordination in [19] and the corresponding fuzzy theories in [20,21]. In [22] was defined and investigated using the fuzzy differential subordination theory an operator, used in [23] to obtain results regarding the classical theory of differential subordination. This proved that both approaches give important results and that research from the fuzzy perspective can generate nice consequences when classical theories of differential subordination and superordination are applied to the same subject. Numerous researchers have studied analytic functions using fuzzy concepts, and all aspects of classical concepts from geometric function theory are viewed from the fuzzy perspective. In [24], meromorphic functions were studied in fuzzy conditions, and in [25], fuzzy differential subordinations are obtained for strong Janowski functions. In [26], fuzzy α-convex functions are investigated regarding the quantum calculus aspects in [27] and regarding Hadamard product in [28]. In [29–31] q-analog operators involving analytic functions are studied regarding fuzzy theory. In [32], spiral-like functions were investigated using the fuzzy differential subordination aspects.

The development facilitated by the addition of fractional calculus aspects as well as quantum calculus to geometric function theory was highlighted in Srivastava's recent review paper [33]. The study described in [34] successfully developed a novel complex integrodifferential operator that is connected to both the Mittag–Leffler function and the meromorphic functions in the punctured unit disk. The research reported in [35] offers a set of formulas for specific fractional differintegral operators defined using classical (Riemann–Liouville) fractional integrals. An investigation into some qualitative analysis for a nonlinear Langevin integro-fractional differential equation was the study's goal described in [36]. In article [37], novel connections between the well-known Mittag–Leffler functions of one, two, three, and four parameters are made using fractional calculus. Therefore, this work examines a number of new analytical features by utilizing fractional integration and differentiation for the Mittag–Leffler function created by confluent hypergeometric functions. In the study seen in [38], the well-known tools Mittag–Leffler function and confluent hypergeometric function were used to develop and study a new function called Mittag–Leffler-confluent hypergeometric function. Various analytic implementations of the integral equations were also investigated. The analysis in [39] focused on the study of special functions combined with fractional calculus and aimed to introduce and further explore novel variants of the Gamma and Kummer functions in terms of Mittag–Leffler functions.

In the following, the main concepts used for the study are recalled and basic lemmas used for the proofs of the main results are listed. The next section contains the main results of this investigation, consisting of theorems that concern fuzzy differential subordinations and the dual fuzzy differential superordinations involving fractional integral of extended Dziok–Srivastava operator for which fuzzy best dominants and fuzzy best subordinates are given, respectively. Interesting consequences derive when particular functions with remarkable geometric properties are chosen to act as fuzzy best subordinates and fuzzy best dominants in the proved theorems.

$\mathcal{H}(U)$ represents the family of analytic functions from $U = \{z \in \mathbb{C} : |z| < 1\}$. The dual theories of differential subordination and superordination use specific subclasses of $\mathcal{H}(U)$:

$$\mathcal{A}_n = \{f(z) = z + a_{n+1}z^{n+1} + \dots\} \subset \mathcal{H}(U),$$

denoted by $\mathcal{A}$ for $n = 1$, and

$$\mathcal{H}[a, n] = \{f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\} \subset \mathcal{H}(U),$$

taking $n \in \mathbb{N}$ and $a \in \mathbb{C}$.

Definition 1 ([9]). *The pair $(A, \mathcal{F}_{\mathcal{A}})$ is the fuzzy subset of $\mathcal{X}$, where $\mathcal{F}_{\mathcal{A}} : \mathcal{X} \to [0, 1]$ is the membership function of the fuzzy set $(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$ and the set $\mathcal{A} = \{x \in \mathcal{X} : 0 < \mathcal{F}_{\mathcal{A}}(x) \leq 1\}$ is the support of the fuzzy set $(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$, denoted $\mathcal{A} = \text{supp}(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$.*

The notion of fuzzy differential subordination is defined as follows:

Definition 2 ([9]). *Between the functions $f, g \in \mathcal{H}(D)$ a fuzzy subordination exists when $f(z_0) = g(z_0)$, for $z_0 \in D \subset \mathbb{C}$ a fixed point and $\mathcal{F}_{f(D)} f(z) \leq \mathcal{F}_{g(D)} g(z)$, $z \in D$, and this relation is denoted by $f \prec_{\mathcal{F}} g$.*

Definition 3 ([11], Definition 2.2). *Considering g a univalent function in U and $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$, such that $g(0) = \varphi(a, 0; 0) = a$, when the function p is analytic in U, with $p(0) = a$ and the fuzzy subordination is verified by*

$$\mathcal{F}_{\varphi(\mathbb{C}^3 \times U)} \varphi(p(z), zp'(z), z^2 p''(z); z) \leq \mathcal{F}_{g(U)} g(z), \quad z \in U, \tag{1}$$

then p represents a fuzzy solution of the fuzzy subordination. A fuzzy dominant of the fuzzy subordination is the univalent function q, which verifies $\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{q(U)} q(z)$, $z \in U$, when p satisfying (1). The fuzzy best dominant is a fuzzy dominant $\tilde{q}$ such that $\mathcal{F}_{\tilde{q}(U)} \tilde{q}(z) \leq \mathcal{F}_{q(U)} q(z)$, $z \in U$, for all fuzzy dominants q.

The following lemmas are useful to explore strong differential subordination.

Lemma 1 ([40]). *Considering the function $p \in \mathcal{H}[a, n]$ satisfying the fuzzy differential subordination*

$$\mathcal{F}_{p(U)} \left(p(z, \zeta) + \frac{1}{\eta} z p'_z(z, \zeta) \right) \leq \mathcal{F}_{g(U)} g(z), \quad z \in U,$$

where g is a convex function such that $g(0) = a$ and $\eta \in \mathbb{C}^*$ with $Re\eta \geq 0$,
we obtain the fuzzy differential subordinations

$$\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{q(U)} q(z) \leq \mathcal{F}_{g(U)} g(z), \quad z \in U,$$

and the convex function $q(z) = \frac{\eta}{n z^{\frac{\eta}{n}}} \int_0^z g(t) t^{\frac{\eta}{n} - 1} dt$ is the fuzzy best dominant.

Lemma 2 ([40]). *Considering the holomorphic function*

$$p(z) = q(0) + p_n z^n + p_{n+1} z^{n+1} + \dots,$$

in U satisfying the fuzzy differential subordination

$$\mathcal{F}_{p(U)} \left(p(z) + \eta z p'(z) \right) \leq \mathcal{F}_{g(U)} g(z), \quad z \in U,$$

where q is a convex function and

$$g(z) = q(z) + n\eta z q'(z),$$

for n a positive integer and $\eta > 0$,
we obtain the sharp fuzzy differential subordination

$$\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{q(U)} q(z).$$

The theory of fuzzy differential superordination uses the following notions:

Definition 4 ([12]). *Considering g an analytic function in U and $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$, when p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and it is verified the fuzzy superordination*

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{\varphi(\mathbb{C}^3\times U)}\varphi(p(z), zp'(z), z^2p''(z); z), \quad z \in U, \tag{2}$$

then p represents a fuzzy solution of the fuzzy superordination. A fuzzy subordinate of the fuzzy differential superordination is the analytic function q which verifies $\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z)$, $z \in U$, when p satisfying (2). The fuzzy best subordinate is a fuzzy subordinate $\widetilde{q}$ such that $\mathcal{F}_{q(U)}q \leq \mathcal{F}_{q(U)}\widetilde{q}$ for all fuzzy subordinate q.

Definition 5 ([11]). *Q consists all the injective analytic functions on $\overline{U}\backslash E(f, \zeta)$, satisfying $f'(y) \neq 0$ with $y \in \partial U\backslash E(f)$, and $E(f) = \{y \in \partial U : \lim_{z\to y} f(z) = +\infty\}$. When $f(0) = a$, Q is denoted by $Q(a)$.*

The following lemmas are useful for exploring fuzzy differential superordination.

Lemma 3 ([10], Corollary 2.6 g. 2, p. 66). *Considering the function $p \in \mathcal{H}[a, n] \cap Q$, satisfying the fuzzy differential superordination*

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}\left(p(z) + \frac{1}{\eta}zp'(z)\right), \quad z \in U,$$

and $p(z) + \frac{1}{\eta}zp'(z)$ is univalent in U, where g is a convex function such that $g(0) = a$, and $\eta \in \mathbb{C}^*$ with $Re\ \eta \geq 0$,
we obtain the fuzzy differential superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z),$$

and the convex function $q(z) = \frac{\eta}{nz^{\frac{\eta}{n}}}\int_0^z g(t)t^{\frac{\eta}{n}-1}dt$, $z \in U$, is the fuzzy best subordinate.

Lemma 4 ([10]). *Considering the function $p \in \mathcal{H}[a, n] \cap Q$ satisfying the fuzzy differential superordination*

$$\mathcal{F}_{q(U)}\left(q(z) + \frac{1}{\eta}zq'(z)\right) \leq \mathcal{F}_{p(U)}\left(p(z) + \frac{1}{\eta}zp'(z)\right), \quad z \in U,$$

and $p(z) + \frac{1}{\eta}zp'(z)$ is univalent in U, where q is a convex function and

$$g(z) = q(z) + \frac{1}{\eta}zq'(z),$$

for $\eta \in \mathbb{C}^*$ with $Re\ \eta \geq 0$, we obtain the fuzzy differential superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z), \quad z \in U,$$

and the convex function $q(z) = \frac{\eta}{nz^{\frac{\eta}{n}}}\int_0^z g(t)t^{\frac{\eta}{n}-1}dt$, $z \in U$, is the fuzzy best subordinate.

We recall the definitions of fractional integral, Dziok–Srivastava operator and fractional integral of Dziok–Srivastava operator defined in [41] and studied again in this paper regarding fuzzy theory.

Definition 6 ([42,43]). *For an analytic function f in a simply-connected region of the z-plane, which contains the origin, the fractional integral of order γ ($\gamma > 0$) is given by*

$$D_z^{-\gamma} f(z,\zeta) = \frac{1}{\Gamma(\gamma)} \int_0^z \frac{f(t)}{(z-t)^{1-\gamma}} dt, \tag{3}$$

when $(z-t) > 0$, removing the multiplicity of $(z-t)^{\gamma-1}$ by requiring $\log(z-t)$ to be real.

Definition 7 ([44]). *For $f \in \mathcal{A}$, the Dziok–Srivastava operator is given by*
$H_m^l(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m) : \mathcal{A} \to \mathcal{A}$,

$$H_m^l(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m) f(z) = z + \sum_{j=2}^{+\infty} \frac{(\alpha_1)_{j-1}(\alpha_2)_{j-1} \ldots (\alpha_l)_{j-1}}{(\beta_1)_{j-1}(\beta_2)_{j-1} \ldots (\beta_m)_{j-1} \Gamma(j)} a_j z^j, \tag{4}$$

$\alpha_k \in \mathbb{C}$, $k = 1, 2, \ldots, l$, $\beta_i \in \mathbb{C} \backslash \{0, -1, -2, \ldots\}$, $i = 1, 2, \ldots, m$,
and the Pochhammer symbol $(x)_j$ is defined by

$$(x)_j = \frac{\Gamma(x+j)}{\Gamma(x)} = \begin{cases} 1, \text{ for } j = 0 \text{ and } x \in \mathbb{C} \backslash \{0\}, \\ x(x+1)\ldots(x+j-1), \text{ for } j \in \mathbb{N} \text{ and } x \in \mathbb{C}. \end{cases}$$

For simplicity , we write

$$H_m^l[\alpha_1, \beta_1] f(z) = H_m^l(\alpha_1, \alpha_2, \ldots, \alpha_l; \beta_1, \beta_2, \ldots, \beta_m) f(z). \tag{5}$$

Applying fractional integral to the Dziok–Srivastava operator we obtained the following operator [41]:

Definition 8. *The fractional integral applied to the Dziok–Srivastava operator is defined by*

$$D_z^{-\gamma} H_m^l[\alpha_1, \beta_1] f(z) = \frac{1}{\Gamma(\gamma)} \int_0^z \frac{H_m^l[\alpha_1, \beta_1] f(t)}{(z-t)^{1-\gamma}} dt,$$

which can be written by making a simple calculation as follows:

$$D_z^{-\gamma} H_m^l[\alpha_1, \beta_1] f(z) = \frac{1}{\Gamma(2+\gamma)} z^{1+\gamma} +$$

$$\sum_{j=2}^{+\infty} \frac{j}{(j+\gamma)!} \frac{(\alpha_1)_{j-1}(\alpha_2)_{j-1} \ldots (\alpha_l)_{j-1}}{(\beta_1)_{j-1}(\beta_2)_{j-1} \ldots (\beta_m)_{j-1}(j-1)!} a_j z^{j+\gamma}. \tag{6}$$

considering the function $f(z) = z + \sum_{j=2}^{+\infty} a_j z^j \in \mathcal{A}$.

After a short computation, we obtain the following result

$$z\Big(D_z^{-\gamma} H_m^l[\alpha_1, \beta_1] f(z)\Big)' = \tag{7}$$

$$\alpha_1 D_z^{-\gamma} H_m^l[\alpha_1 + 1, \beta_1] f(z) - [\alpha_1 - (1+\gamma)] D_z^{-\gamma} H_m^l[\alpha_1, \beta_1] f(z).$$

A similar result can be obtained regarding the parameter β_1.

2. Fuzzy Differential Subordination

In this part of the article, we obtain fuzzy differential subordinations regarding the fractional integral of the Dziok–Srivastava operator.

Theorem 1. *Taking the convex function q with the property $q(0) = 0$, we define the function $g(z) = q(z) + \gamma z q'(z)$, $z \in U$, for γ a positive integer.*

If the fuzzy differential subordination

$$\mathcal{F}_{\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f\right)'(U)} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)' \leq \mathcal{F}_{g(U)} g(z), \tag{8}$$

is satisfied for $f \in \mathcal{A}$, then we obtain the following sharp fuzzy differential subordination

$$\mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z} \leq \mathcal{F}_{q(U)} q(z).$$

Proof. Take $p(z) = \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z} \in \mathcal{H}[0,\gamma]$, $z \in U$, then $D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z) = zp(z)$ and differentiating the relation, we obtain $\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)' = p(z) + zp'(z)$. Then fuzzy subordination (8) has the following form

$$\mathcal{F}_{p(U)} \left(p(z) + zp'(z)\right) \leq \mathcal{F}_{q(U)} \left(q(z) + \gamma z q'(z)\right),$$

for which applying Lemma 2, we obtain

$$\mathcal{F}_{p(U)} p(z) \geq \mathcal{F}_{q(U)} q(z), \quad \text{i.e.} \quad \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z,\zeta)}{z} \leq \mathcal{F}_{q(U)} q(z).$$

□

Theorem 2. *If g is a convex function such that $g(0) = 0$, which satisfies the fuzzy differential subordination*

$$\mathcal{F}_{\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f\right)'(U)} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)' \leq \mathcal{F}_{g(U)} g(z), \tag{9}$$

for $f \in \mathcal{A}$, then we obtain the following fuzzy differential subordination

$$\mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z} \leq \mathcal{F}_{q(U)} q(z),$$

and the convex function $q(z) = \frac{1}{z}\int_0^z g(t)dt$ is the fuzzy best dominant.

Proof. Let $p(z) = \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z} \in \mathcal{H}[0,\lambda]$, $z \in U$.

Differentiating relation $D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z) = zp(z)$, yields $\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)' = p(z) + zp'(z)$, and the fuzzy subordination (9) is

$$\mathcal{F}_{p(U)} \left(zp'(z) + p(z)\right) \leq \mathcal{F}_{g(U)} g(z),$$

and applying Lemma 1, we obtain

$$\mathcal{F}_{p(U)} p(z) \leq \mathcal{F}_{q(U)} q(z), \quad \text{i.e.,} \quad \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z} \leq \mathcal{F}_{q(U)} q(z),$$

with $q(z) = \frac{1}{z}\int_0^z g(t)dt$ as the fuzzy best dominant. □

Corollary 1. *Considering the convex function* $g(z) = \frac{1+(2\lambda-1)z}{1+z}$ *for* $0 \leq \lambda < 1$ *which satisfies the fuzzy subordination*

$$\mathcal{F}_{\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f\right)'(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' \leq \mathcal{F}_{g(U)}g(z), \tag{10}$$

for $f \in \mathcal{A}$, *then we obtain the fuzzy subordination*

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z} \leq \mathcal{F}_{q(U)}q(z),$$

and the convex function $q(z) = 2\lambda - 1 + 2(1-\lambda)\frac{\ln(1+z)}{z}$, $z \in U$, *is the fuzzy best dominant.*

Proof. Repeating the steps from the proof of Theorem 2 for $p(z) = \frac{D_z^{-\lambda}H_m^l[\alpha_1,\beta_1]f(z)}{z}$, the fuzzy subordination (10) takes the form

$$\mathcal{F}_{p(U)}\left(p(z) + zp'(z)\right) \leq \mathcal{F}_{g(U)}g(z),$$

for which applying Lemma 1, we obtain $\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{q(U)}q(z)$, i.e.,

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z} \leq \mathcal{F}_{q(U)}q(z)$$

and

$$q(z) = \frac{1}{z}\int_0^z g(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\lambda-1)t}{1+t}dt =$$
$$2\lambda - 1 + 2(1-\lambda)\frac{\ln(z+1)}{z}, \ z \in U.$$

□

Theorem 3. *Considering the convex function* q *with the property* $q(0) = 0$, *we define the function* $g(z) = q(z) + \frac{1}{\mu}zq'(z)$, μ *a positive integer,* $z \in U$.

If the fuzzy subordination is satisfied

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' \leq \mathcal{F}_{g(U)}g(z), \tag{11}$$

for $f \in \mathcal{A}$, *then we obtain the following sharp fuzzy subordination*

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{D_z^{-\lambda}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu} \leq \mathcal{F}_{q(U)}q(z).$$

Proof. Taking $p(z) = \left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu} \in \mathcal{H}[0,\gamma\mu]$, $z \in U$ and applying differentiation, we obtain

$$zp'(z) = \mu\left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' - \mu\left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu}$$
$$= \mu\left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' - \mu p(z),$$

written as $p(z) + \frac{1}{\mu}zp'(z) = \left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'$.

In these conditions, fuzzy subordination (11) takes the form

$$\mathcal{F}_{p(U)}\left(p(z)+\frac{1}{\mu}zp'(z)\right)\leq\mathcal{F}_{q(U)}\left(q(z)+\frac{1}{\mu}zq'(z)\right),$$

and applying Lemma 2, we obtain

$$\mathcal{F}_{p(U)}p(z)\leq\mathcal{F}_{q(U)}q(z),\quad \text{i.e.,}\quad \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{D_z^{-\lambda}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu}\leq\mathcal{F}_{q(U)}q(z).$$

□

Theorem 4. *If the convex function g with $g(0)=0$ verifies the fuzzy subordination*

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'\leq\mathcal{F}_{g(U)}g(z),\quad(12)$$

for $f\in\mathcal{A}$ and μ a positive integer, then we obtain the following fuzzy subordination

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{D_z^{-\lambda}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu}\leq\mathcal{F}_{q(U)}q(z),$$

and the convex function $q(z)=\frac{\mu}{z^{\mu}}\int_0^z g(t)t^{\mu-1}dt$ is the fuzzy best dominant.

Proof. Let $p(z)=\left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu}\in\mathcal{H}[0,\gamma\mu]$, $z\in U$.

Using the computation from the proof of Theorem 3, we have

$$p(z)+\frac{1}{\mu}zp'(z)=\left(\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)',$$

and the fuzzy subordination (12) becomes

$$\mathcal{F}_{p(U)}\left(p(z)+\frac{1}{\mu}zp'(z)\right)\leq\mathcal{F}_{g(U)}g(z)$$

and satisfies the conditions from Lemma 1, so we obtain the fuzzy subordination

$$\mathcal{F}_{p(U)}p(z)\leq\mathcal{F}_{q(U)}q(z),\quad\text{i.e.}\quad\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{D_z^{-\lambda}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu}\leq\mathcal{F}_{q(U)}q(z),$$

with $q(z)=\frac{\mu}{z^{\mu}}\int_0^z g(t)t^{\mu-1}dt$ as the fuzzy best dominant. □

Theorem 5. *Considering the convex function q with the property $q(0)=\frac{1}{1+\gamma}$, we define the function $g(z)=q(z)+zq'(z)$, $z\in U$.*

If the fuzzy subordination

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{2\alpha_1D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)^2}{\left(\alpha_1D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)^2}+\right.$$

$$\left.\frac{\alpha_1^2\left(D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)\right)^2-\alpha_1(\alpha_1+1)D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{\left(\alpha_1D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)^2}\right)$$

$$\leq\mathcal{F}_{g(U)}g(z),\quad(13)$$

holds for $f \in \mathcal{A}$, then we obtain the sharp fuzzy subordination

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'} \leq \mathcal{F}_{q(U)}q(z).$$

Proof. Considering $p(z) = \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'}$ and differentiation yields

$$1 - \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)''}{\left[\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'\right]^2} = p(z) + zp'(z).$$

After a short computation and applying relation (7), we obtain

$$1 - \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)''}{\left[\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'\right]^2} =$$

$$\frac{\alpha_1^2\big(D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)\big)^2-\alpha_1(\alpha_1+1)D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}+$$

$$\frac{2\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]\big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}$$

In these conditions, the fuzzy subordination (13) becomes

$$\mathcal{F}_{p(U)}\big(p(z)+zp'(z)\big) \leq \mathcal{F}_{q(U)}\big(q(z)+zq'(z)\big),$$

and applying Lemma 2, we obtain the sharp fuzzy subordination

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{q(U)}q(z), \quad \text{i.e.} \quad \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'} \leq \mathcal{F}_{q(U)}q(z).$$

□

Theorem 6. *Taking the convex function g such that $g(0) = \frac{1}{1+\gamma}$, which verifies the fuzzy differential subordination*

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{2\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]\big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}+\right.$$

$$\left.\frac{\alpha_1^2\big(D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)\big)^2-\alpha_1(\alpha_1+1)D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)-[\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}\right)$$

$$\leq \mathcal{F}_{g(U)}g(z), \tag{14}$$

for $f \in \mathcal{A}$, then we obtain the fuzzy subordination

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'} \leq \mathcal{F}_{q(U)}q(z),$$

and the convex function $q(z) = \frac{1}{z}\int_0^z g(t)dt$ is the fuzzy best dominant.

Proof. Denote $p(z) = \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)'}$.

Using the computation from the proof of Theorem 5, we have

$$p(z) + zp'(z) = 1 - \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z) \cdot \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)''}{\left[\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)'\right]^2},$$

and the fuzzy subordination (14) takes the form

$$\mathcal{F}_{p(U)}\left(p(z) + zp'(z)\right) \leq \mathcal{F}_{g(U)}g(z),$$

and applying Lemma 1, we obtain the fuzzy subordination

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{q(U)}q(z), \quad \text{i.e.,} \quad \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)'} \leq \mathcal{F}_{q(U)}q(z),$$

and the fuzzy best dominant is the function $q(z) = \frac{1}{z}\int_0^z g(t)dt$. □

Theorem 7. *Taking the convex function q with $q(0) = 0$, we define the function $g(z) = q(z) + \gamma z q'(z)$, with γ a positive integer, $z \in U$.*

If the fuzzy subordination

$$\mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)}\left(\alpha_1(\alpha_1+1)\frac{D_z^{-\gamma} H_m^l[\alpha_1+2,\beta_1]f(z)}{z} + \right.$$

$$\left. \alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z)}{z} + (\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right) \tag{15}$$

$$\leq \mathcal{F}_{g(U)}g(z),$$

is verified for $f \in \mathcal{A}$, then we obtain the sharp fuzzy subordination

$$\mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)' \leq \mathcal{F}_{q(U)}q(z).$$

Proof. Let

$$p(z) = \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)' \in \mathcal{H}[0,\gamma]. \tag{16}$$

Using relation (7), we obtain

$$zp(z) = \alpha_1 D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1 - (1+\gamma)]D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z),$$

and differentiation yields

$$p(z) + zp'(z) = \alpha_1(\alpha_1+1)\frac{D_z^{-\gamma} H_m^l[\alpha_1+2,\beta_1]f(z)}{z} + \alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z)}{z} + (\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}.$$

In these conditions, the fuzzy subordination (16) takes the form

$$\mathcal{F}_{p(U)}\left(p(z) + zp'(z)\right) \leq \mathcal{F}_{q(U)}\left(q(z) + \gamma z q'(z)\right),$$

and applying Lemma 2, we obtain the sharp fuzzy subordination

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{q(U)}q(z), \quad \text{i.e.} \quad \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' \leq \mathcal{F}_{q(U)}q(z).$$

□

Theorem 8. *Taking the convex function g with $g(0) = 0$, which verifies the fuzzy subordination*

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\alpha_1(\alpha_1+1)\frac{D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{z}+\right.$$

$$\left.\alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)}{z}+(\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)$$

$$\leq \mathcal{F}_{g(U)}g(z), \tag{17}$$

for $f \in \mathcal{A}$, then we have the fuzzy subordination

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' \leq \mathcal{F}_{q(U)}q(z),$$

and the convex function $q(z) = \frac{1}{z}\int_0^z g(t)dt$ is the fuzzy best dominant.

Proof. Let $p(z) = \left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' \in \mathcal{H}[0,\gamma], z \in U$.

Using the computation from the proof of Theorem 7, we have

$$zp'(z)+p(z) = \alpha_1(\alpha_1+1)\frac{D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{z}+\alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)}{z}+$$
$$(\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z},$$

and the fuzzy subordination (17) can be written as

$$\mathcal{F}_{p(U)}\left(p(z)+zp'(z)\right) \leq \mathcal{F}_{g(U)}g(z),$$

which satisfies Lemma 1, obtaining

$$\mathcal{F}_{p(U)}p(z) \leq \mathcal{F}_{q(U)}q(z), \quad \text{i.e.} \quad \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' \leq \mathcal{F}_{q(U)}q(z),$$

and the fuzzy best dominant is the function $q(z) = \frac{1}{z}\int_0^z g(t)dt$. □

3. Fuzzy Differential Superordination

In this part of the paper, we obtain fuzzy differential superordinations regarding the fractional integral of the Dziok–Srivastava operator.

Theorem 9. *Taking the convex function q with the property $q(0) = 0$, we define the function $g(z) = q(z) + \gamma zq'(z)$, for γ a positive integer, $z \in U$. Assume that $\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z} \in Q \cap \mathcal{H}[0,\gamma]$ and $\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'$ is univalent and the fuzzy differential superordination is verified*

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)', \tag{18}$$

for $f \in \mathcal{A}$, then we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z},$$

and the convex function $q(z) = \frac{1}{z}\int_0^z g(t)dt$ *is the fuzzy best subordinate.*

Proof. Consider $p(z) = \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z} \in \mathcal{H}[0,\gamma], z \in U$.

Differentiating the relation $D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z) = zp(z)$, we obtain $\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' = p(z) + zp'(z)$.

The fuzzy superordination (18) takes the following form

$$\mathcal{F}_{q(U)}\left(q(z) + \gamma z q'(z)\right) \leq \mathcal{F}_{p(U)}\left(p(z) + zp'(z)\right),$$

and applying Lemma 4, we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z), \quad \text{i.e.,} \quad \mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z},$$

and the fuzzy best subordinate is the function $q(z) = \frac{1}{z}\int_0^z g(t)dt$. □

Theorem 10. *If* g *is a convex function such that* $g(0) = 0$, *assume that* $\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z} \in Q \cap \mathcal{H}[0,\gamma]$ *and* $\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'$ *is univalent and satisfies the fuzzy differential superordination*

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)', \tag{19}$$

for $f \in \mathcal{A}$, *then we obtain the following fuzzy superordination*

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z},$$

and the convex function $q(z) = \frac{1}{z}\int_0^z g(t)dt$ *is the fuzzy best subordinate.*

Proof. Let $p(z) = \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z} \in \mathcal{H}[0,\lambda], z \in U$.

Applying differentiation to the relation $D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z) = zp(z)$, we obtain $\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' = p(z) + zp'(z)$, and the fuzzy superordination (19) takes the form

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}\left(zp'(z) + p(z)\right),$$

for which applying Lemma 3, we obtain

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z), \quad \text{i.e.,} \quad \mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z},$$

and the fuzzy best subordinate is the function $q(z) = \frac{1}{z}\int_0^z g(t)dt$. □

Corollary 2. *Considering the convex function* $g(z) = \frac{1+(2\lambda-1)z}{1+z}$ *for* $0 \leq \lambda < 1$, $f \in \mathcal{A}$, *we assume that* $\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z} \in Q \cap \mathcal{H}[0,\gamma]$, $\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'$ *is univalent, and the fuzzy superordination*

$$\mathcal{F}_{g(u)}g(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)', \tag{20}$$

is verified, then we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z},$$

and the convex function $q(z) = 2\lambda - 1 + 2(1-\lambda)\frac{\ln(1+z)}{z}$, $z \in U$ *is the fuzzy best subordinate.*

Proof. Repeating the steps from the proof of Theorem 10 for $p(z) = \frac{D_z^{-\lambda} H_m^l[\alpha_1,\beta_1]f(z)}{z}$, the fuzzy superordination (20) takes the form

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{p(U)} \left(p(z) + zp'(z)\right).$$

Using Lemma 3 yields $\mathcal{F}_{q(U)} q(z) \leq \mathcal{F}_{p(U)} p(z)$, i.e.,

$$\mathcal{F}_{q(U)} q(z) \leq \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}$$

and the fuzzy best subordinate is the function

$$q(z) = \frac{1}{z}\int_0^z \frac{1+(2\lambda-1)t}{1+t}dt = 2\lambda - 1 + 2(1-\lambda)\frac{\ln(z+1)}{z}, \ z \in U.$$

□

Theorem 11. *Considering the convex function q with the property $q(0) = 0$, we define the function $g(z) = q(z) + \frac{1}{\mu} zq'(z)$, with μ a positive integer, $z \in U$. Assume that for $f \in \mathcal{A}$, $\left(\frac{D_z^{-\lambda} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu} \in Q \cap \mathcal{H}[0,\gamma\mu]$, $\left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)'$ is univalent and the fuzzy superordination*

$$\mathcal{F}_{g(U)} g(z) \leq \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)', \quad (21)$$

is verified, then we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)} q(z) \leq \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \left(\frac{D_z^{-\lambda} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu},$$

and the convex function $q(z) = \frac{\mu}{z^{\mu}} \int_0^z g(t) t^{\mu-1} dt$ is the fuzzy best subordinate.

Proof. Consider $p(z) = \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu} \in \mathcal{H}[0,\gamma\mu]$, $z \in U$.

Differentiation yields

$$zp'(z) = \mu\left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)' - \mu\left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu} =$$
$$\mu\left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)' - \mu p(z);$$

therefore, $p(z) + \frac{1}{\mu} zp'(z) = \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)'$.

In these conditions, the fuzzy superordination (21) can be written as

$$\mathcal{F}_{q(U)}\left(q(z) + \frac{1}{\mu} zq'(z)\right) \leq \mathcal{F}_{p(U)}\left(p(z) + \frac{1}{\mu} zp'(z)\right),$$

and by Lemma 4, we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)} q(z) \leq \mathcal{F}_{p(U)} p(z), \quad \text{i.e.,} \quad \mathcal{F}_{q(U)} q(z) \leq \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu},$$

and the fuzzy best subordinate is the function $q(z) = \frac{\mu}{z^{\mu}} \int_0^z g(t) t^{\mu-1} dt$. □

Theorem 12. *Taking the convex function g with the property $g(0) = 0$, with μ a positive integer, we assume for $f \in \mathcal{A}$ that $\left(\frac{D_z^{-\lambda} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu} \in Q \cap \mathcal{H}[0,\gamma\mu]$, $\left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1}$ $\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)'$ is univalent and the fuzzy superordination is satisfied*

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)', \quad (22)$$

then the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu},$$

holds and the convex function $q(z) = \frac{\mu}{z^{\mu}} \int_0^z g(t)t^{\mu-1}dt$ is the fuzzy best subordinate.

Proof. Let $p(z) = \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu} \in \mathcal{H}[0,\gamma\mu]$, $z \in U$.

Using the computation from the proof of Theorem 11 yields

$$p(z) + \frac{1}{\mu} z p'(z) = \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu-1} \left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)',$$

and the fuzzy superordination (22) is written as

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}\left(p(z) + \frac{1}{\mu} z p'(z)\right).$$

By Lemma 3, we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z), \quad \text{i.e.,} \quad \mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)} \left(\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z}\right)^{\mu},$$

and the fuzzy best subordinate is the function $q(z) = \frac{\mu}{z^{\mu}} \int_0^z g(t)t^{\mu-1}dt$. □

Theorem 13. *Considering the convex function q with the property $q(0) = \frac{1}{1+\gamma}$, we define the function $g(z) = q(z) + zq'(z)$, and assume that $f \in \mathcal{A}$, $\frac{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)'} \in Q \cap \mathcal{H}\left[\frac{1}{1+\gamma}, 1\right]$,*
$\frac{\alpha_1^2\left(D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z)\right)^2 - \alpha_1(\alpha_1+1)D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma} H_m^l[\alpha_1+2,\beta_1]f(z)}{\left(\alpha_1 D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)^2} +$
$\frac{2\alpha_1 D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)^2}{\left(\alpha_1 D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)^2}$ *is univalent and verifies the fuzzy differential superordination*

$$\mathcal{F}_{g(U)}g(z) \leq$$
$$\mathcal{F}_{D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{\alpha_1^2\left(D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z)\right)^2 - \alpha_1(\alpha_1+1)D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma} H_m^l[\alpha_1+2,\beta_1]f(z)}{\left(\alpha_1 D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)^2}\right.$$
$$\left. + \frac{2\alpha_1 D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]\left(D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)^2}{\left(\alpha_1 D_z^{-\gamma} H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma} H_m^l[\alpha_1,\beta_1]f(z)\right)^2}\right), \quad (23)$$

then we obtain the fuzzy differential superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'},$$

and the fuzzy best subordinate is the convex function $q(z) = \frac{1}{z}\int_0^z g(t)dt$.

Proof. Differentiate the relation $p(z) = \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'}$ yields

$$1 - \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)''}{\left[\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'\right]^2} = p(z) + zp'(z).$$

Making a short computation and applying relation (7) we obtain

$$1 - \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)''}{\left[\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'\right]^2} =$$

$$\frac{\alpha_1^2\big(D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)\big)^2 - \alpha_1(\alpha_1+1)D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}$$

$$\frac{2\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]\big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}.$$

In these conditions, the fuzzy superordination takes the form

$$\mathcal{F}_{q(U)}\big(q(z) + zq'(z)\big) \leq \mathcal{F}_{p(U)}\big(p(z) + zp'(z)\big),$$

and applying Lemma 4, we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z), \quad \text{i.e.,} \quad \mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\Big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\Big)'},$$

and the fuzzy best subordinate represents the function $q(z) = \frac{1}{z}\int_0^z g(t)dt$. □

Theorem 14. *Taking the convex function* g *such that* $g(0) = \frac{1}{1+\gamma}$, *we assume that* $f \in \mathcal{A}$, $\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'} \in Q \cap \mathcal{H}\left[\frac{1}{1+\gamma}, 1\right]$, *the function*
$\frac{\alpha_1^2\big(D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)\big)^2 - \alpha_1(\alpha_1+1)D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2} +$
$\frac{2\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]\big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}$ *is univalent and verifies the fuzzy superordination*

$$\mathcal{F}_{g(U)}g(z) \leq$$

$$\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\frac{\alpha_1^2\big(D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)\big)^2 - \alpha_1(\alpha_1+1)D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}\right.$$

$$\left. + \frac{2\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]\big(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}{\big(\alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\big)^2}\right), \tag{24}$$

then we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'},$$

and the fuzzy best subordinate is the convex function $q(z) = \frac{1}{z}\int_0^z g(t)dt$.

Proof. Let $p(z) = \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'}$.

Using the computation from the proof of Theorem 13 yields

$$p(z) + zp'(z) = 1 - \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\cdot\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)''}{\left[\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'\right]^2},$$

and the fuzzy superordination (24) can be written as

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{p(U)}\left(p(z) + zp'(z)\right),$$

and by Lemma 3, we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{p(U)}p(z), \quad \text{i.e.} \quad \mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)} \frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'}$$

and the fuzzy best subordinate is the function $q(z) = \frac{1}{z}\int_0^z g(t)dt$. □

Theorem 15. *Taking the convex function* q *with* $q(0) = 0$, *we define the function* $g(z) = q(z) + \gamma zq'(z)$, *with* γ *a positive integer,* $z \in U$.

Assume that $\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' \in Q \cap \mathcal{H}[0,\gamma]$ *and* $\alpha_1(\alpha_1+1)\frac{D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{z} + \alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)}{z} + (\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}$ *is univalent, for* $f \in \mathcal{A}$ *and verifies the fuzzy superordination*

$$\mathcal{F}_{g(U)}g(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\alpha_1(\alpha_1+1)\frac{D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{z} + \right.$$

$$\left.\alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)}{z} + (\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right), \quad (25)$$

then we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z) \leq \mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)',$$

and the fuzzy best subordinate is the convex function $q(z) = \frac{1}{z}\int_0^z g(t)dt$.

Proof. Let

$$p(z) = \left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)' \in \mathcal{H}[0,\gamma], z \in U. \quad (26)$$

Using relation (7) yields

$$zp(z) = \alpha_1 D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z) - [\alpha_1-(1+\gamma)]D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z),$$

and differentiating it, we obtain

$$p(z)+zp'(z)=\alpha_1(\alpha_1+1)\frac{D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{z}+$$
$$\alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)}{z}+(\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}.$$

In these conditions, the fuzzy superordination (26) has the following form

$$\mathcal{F}_{q(U)}\left(q(z)+\gamma zq'(z)\right)\leq\mathcal{F}_{p(U)}\left(p(z)+zp'(z)\right),$$

and verifies Lemma 4. Therefore we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z)\leq\mathcal{F}_{p(U)}p(z),\quad\text{i.e.,}\quad\mathcal{F}_{q(U)}q(z)\leq\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)',$$

and the fuzzy best subordinate is represented by $q(z)=\frac{1}{z}\int_0^z g(t)dt$. □

Theorem 16. *Taking the convex function g with $g(0)=0$, we assume that $f\in\mathcal{A}$, $(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z))'\in Q\cap\mathcal{H}[0,\gamma]$, the function $\alpha_1(\alpha_1+1)\frac{D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{z}+\alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)}{z}+(\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}$ is univalent and verifies the fuzzy differential superordination*

$$\mathcal{F}_{g(U)}g(z)\leq\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(\alpha_1(\alpha_1+1)\frac{D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{z}+\right.$$

$$\left.\alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)}{z}+(\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z}\right),\qquad(27)$$

then the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z)\leq\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)',$$

holds and the fuzzy best subordinate is the convex function $q(z)=\frac{1}{z}\int_0^z g(t)dt$.

Proof. Let $p(z)=\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)'\in\mathcal{H}[0,\gamma]$, $z\in U$.

Using the computation from the proof of Theorem 15 yields

$$p(z)+zp'(z)=\alpha_1(\alpha_1+1)\frac{D_z^{-\gamma}H_m^l[\alpha_1+2,\beta_1]f(z)}{z}+\alpha_1(1-2\alpha_1+2\gamma)\frac{D_z^{-\gamma}H_m^l[\alpha_1+1,\beta_1]f(z)}{z}+$$
$$(\alpha_1-1-\gamma)^2\frac{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)}{z},$$

and the fuzzy superordination (27) can be written as

$$\mathcal{F}_{g(U)}g(z)\leq\mathcal{F}_{p(U)}\left(p(z)+zp'(z)\right),$$

and applying Lemma 3, we obtain the fuzzy superordination

$$\mathcal{F}_{q(U)}q(z)\leq\mathcal{F}_{p(U)}p(z),\quad\text{i.e.,}\quad\mathcal{F}_{q(U)}q(z)\leq\mathcal{F}_{D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(U)}\left(D_z^{-\gamma}H_m^l[\alpha_1,\beta_1]f(z)\right)',$$

and the fuzzy best subordinate is the function $q(z)=\frac{1}{z}\int_0^z g(t)dt$. □

4. Conclusions

Inspired by the motivating results obtained by incorporating fractional calculus in the studies regarding geometric function theory and fuzzy theory, the theories of fuzzy differential subordination and its dual, fuzzy differential superordination, this study embeds such aspects in trying to revive a study begun in [41] but not further investigated so far. The novelty brought by the outcome of this research resides in the manner in which the definition of the fractional integral of the Dziok–Srivastava operator is employed for obtaining new fuzzy differential subordination results, alongside the dual new fuzzy differential superordinations. Fuzzy best dominants and fuzzy best subordinates are given in each theorem proved, respectively. Significant corollaries are deduced when remarkable functions considering their geometric properties are introduced in the theorems replacing the functions obtained as fuzzy best dominant or fuzzy best subordinate.

The paper proposes a new line of research for fuzzy differential subordination and its dual, fuzzy differential superordination theories incorporating fractional calculus. Starting with the paper [45], other fuzzy subordinations and superordinations could be obtained following the steps in this article.

As future research, the operator $D_z^{-\gamma} H_m^l[\alpha_1, \beta_1] f(z)$ studied in this paper could be adapted to quantum calculus and obtain differential subordinations and superordinations for it by involving q-fractional calculus. In addition, several classes of analytical functions can be defined and studied using the studied operator.

In addition, strong differential subordinations and strong differential superordinations results can be obtained regarding the fractional integral of extended Dziok–Srivastava operator.

Funding: The publication of this research was supported by the University of Oradea.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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