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Optimization of the Approximate Integration Formula Using the Discrete Analogue of a High-Order Differential Operator

Kholmat Shadimetov [1,2], Aziz Boltaev [1,2,3,*] and Roman Parovik [4,5*]

[1] Department of Informatics and Computer Graphics, Tashkent State Transport University, 1 Odilkhodjayev Str., Tashkent 100167, Uzbekistan; kholmatshadimetov@mail.ru
[2] Computational Mathematics Laboratory, V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 4b University Str., Tashkent 100174, Uzbekistan
[3] Department of Computational Mathematics and Information Systems, National University of Uzbekistan Named after M. Ulugbek, 4 University Str., Tashkent 100174, Uzbekistan
[4] International Integrative Research Laboratory of Extreme Phenomena of Kamchatka, Vitus Bering Kamchatka State University, 4 Pogranichnaya St., Petropavlovsk-Kamchatskiy 683032, Russia
[5] Laboratory for Simulation of Physical Processes, Institute of Cosmophysical Research and Radio Wave Propagation FEB RAS, 7 Mirnaya St., Kamchatka Krai, Yelizovsky District, Paratunka 684034, Russia
* Correspondence: aziz_boltayev@mail.ru (A.B.); romanparovik@gmail.com (R.P.)

Abstract: It is known that discrete analogs of differential operators play an important role in constructing optimal quadrature, cubature, and difference formulas. Using discrete analogs of differential operators, optimal interpolation, quadrature, and difference formulas exact for algebraic polynomials, trigonometric and exponential functions can be constructed. In this paper, we construct a discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}} + 2\frac{d^m}{dx^m} + 1$ in the Hilbert space $W_2^{(m,0)}$. We develop an algorithm for constructing optimal quadrature formulas exact on exponential-trigonometric functions using a discrete operator. Based on this algorithm, in $m = 2$, we give an optimal quadrature formula exact for trigonometric functions. Finally, we present the rate of convergence of the optimal quadrature formula in the Hilbert space $W_2^{(2,0)}$ for the case $m = 2$.

Keywords: differential operator; discrete analogue; Hilbert space; discrete argument functions; optimal quadrature formula

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1. Introduction Statement of the Problem

Quadrature formulas are extensively used in different areas of mathematics and its practical applications. When obtaining a discrete approximation, it is crucial that the quadrature formula approaches the given definite integrals as closely as possible. Such formulas can be obtained using variational principles. Therefore, constructing optimal quadrature formulas on classes of differentiable functions using the variational method is an important problem in computational mathematics. The problem of optimizing numerical integration formulas using the variational approach involves finding the minimum of the error functional norm in the given space of functions. There are two problems related to this: Nikol'skii's problem [1,2], which involves minimizing the norm of the error functional with coefficients and nodes, and Sard's problem [3–5], which involves minimizing the norm of the error functional with coefficients for fixed nodes. The solutions to Nikol'skii's and Sard's problems are referred to as the optimal quadrature formula in the sense of Nikol'skii and the sense of Sard, respectively.

In this paper, we investigate Sard's problem of the construction of optimal quadrature formulas in a Hilbert space.

We indicate $W_2^{(m,0)}$ the class of functions φ defined on the interval [0,1], which possesses a continuous $(m-1)$th derivative on [0,1] and whose mth derivative is in $L_2(0,1)$.

The class $W_2^{(m,0)}$ under the pseudo-inner product

$$\langle \varphi, \psi \rangle = \int_0^1 \left(\varphi^{(m)}(x) + \varphi(x) \right) \left(\psi^{(m)}(x) + \psi(x) \right) dx \tag{1}$$

is a Hilbert space if we can find functions that are different from the equation's solution $f^{(m)}(x) + f(x) = 0$. Thus, $W_2^{(m,0)}$ is the Hilbert space equipped with the norm

$$\|\varphi\|_{W_2^{(m,0)}} = \left(\int_0^1 \left(\varphi^{(m)}(x) + \varphi(x) \right)^2 \right)^{\frac{1}{2}}, \tag{2}$$

corresponding to the inner product (1).

For a function φ from the space $W_2^{(m,0)}$, we consider a quadrature formula of the form

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C[\beta]\varphi(x_\beta), \tag{3}$$

where $C[\beta]$ and x_β are coefficients and nodes, respectively, and φ is an element of the Hilbert space $W_2^{(m,0)}(0,1)$.

The following difference between integral and quadrature sum

$$(\ell, \varphi) = \int_0^1 \varphi(x)dx - \sum_{\beta=0}^{N} C[\beta]\varphi(x_\beta) \tag{4}$$

is called the error of the quadrature Formula (3) and $(\ell, \varphi) = \int\limits_{-\infty}^{+\infty} \ell(x)\varphi(x)dx$ is the value of the error functional ℓ for the given function φ. Here, the error functional ℓ has the form

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C[\beta]\delta(x - x_\beta), \tag{5}$$

where $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval [0,1], and δ is Dirac's delta-function.

According to the Cauchy-Schwarz inequality, the absolute value of the error (4) is estimated using the norm

$$\|\ell\|_{W_2^{(m,0)*}} = \sup_{\|\varphi\|_{W_2^{(m,0)}}=1} |(\ell, \varphi)| \tag{6}$$

of the error functional ℓ, as follows:

$$|(\ell, \varphi)| \leq \|\varphi\|_{W_2^{(m,0)}} \|\ell\|_{W_2^{(m,0)*}},$$

where $W_2^{(m,0)*}$ is the conjugate space to the space $W_2^{(m,0)}$.

Sard's problem on the construction of optimal quadrature formulas in the space $W_2^{(m,0)}$ is to find such coefficients $C[\beta]$ that satisfy the equality

$$\|\mathring{\ell}\|_{W_2^{(m,0)*}} = \inf_{C[\beta]} \|\ell\|_{W_2^{(m,0)*}}, \tag{7}$$

i.e., to find the minimum of the norm (6) of the error functional ℓ using coefficients $C[\beta]$ for fixed nodes x_β.

This problem consists of two parts: first, calculating the norm (6) of the error functional ℓ in the space $W_2^{(m,0)*}$, and then finding the minimum of the norm (6) using coefficients $C[\beta]$ for fixed nodes x_β.

There are a number of methods for constructing optimal quadrature formulas in the sense of Sard, such as the spline method [6–8], the φ-function method [9–14], and Sobolev's method [15–18]. In diverse spaces, based on these methods, Sard's problem has been reviewed by many authors (see, for example, [11–14,19–23] and references therein).

The spline method. I.J. Schoenberg [24] showed the relationship between optimal quadrature formulas in the sense of Sard and natural splines.

It is considered that for a linear differential operator $L \equiv a_m \frac{d^m}{dx^m} + a_{m-1}\frac{d^{m-1}}{dx^{m-1}} + \ldots + a_1\frac{d}{dx} + a_0$, $a_m \neq 0$; in [25], Chapter 6, the authors studied the Hilbert spaces in the analysis of generalized splines.Specifically, with the pseudo-inner product

$$\langle \varphi, \psi \rangle_L = \int_a^b L\varphi(x) \cdot L\psi(x) dx.$$

Sobolev's method. It is worth noting that Sobolev's technique is based on the formation of a discrete analogue to a linear differential operator. Using this strategy, we can obtain the analytic expressions for coefficients of optimal quadrature formulas in the sense of Sard.

In [15,16], the minimization problem of the norm of the error functional using coefficients was decreased to the system of difference equations of the Wiener–Hopf type in the space $L_2^{(m)}$, where $L_2^{(m)}$ is the Sobolev space of functions with a square integrable generalized mth derivative. The existence and uniqueness of a solution for this system was shown by Sobolev [15–18], who described an analytic algorithm for finding the coefficients of optimal cubature formulas. For this, Sobolev studied the discrete analogue $D_{hH}^{(m)}(h\beta)$ of the polyharmony operator Δ^m. The problem of the construction of the discrete operator $D_{hH}^{(m)}(h\beta)$ in a n-dimensional case is complicated and remains an open problems.In the one-dimensional case, the discrete analogue $D_h^{(m)}(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}}$ was obtained by Z.Zh. Zhamalov [26] and Kh.M. Shadimetov [27].

Furthermore, in [28–30], discrete analogues of differential operators $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$, $\frac{d^{2m}}{dx^{2m}} + 2\omega^2\frac{d^{2m-2}}{dx^{2m-2}} + \omega^4\frac{d^{2m-4}}{dx^{2m-4}}$ (for $m \geq 2$), $\frac{d^{2m}}{dx^{2m}} - 1$ (for odd m) were constructed and their properties were studied.

Notice that the discrete analogues of differential operators mentioned above were used in the construction of optimal quadrature, interpolation formulas, and spline functions in the $L_2^{(m)}$, $W_2^{(m,m-1)}$, $W_2^{(m,0)}$ and $K_2(P_m)$ spaces (see, e.g., [28,31–39]).

In addition, in the works of M.D. Ramazanov [40–42], optimal cubature formulas were constructed. The author considered the spaces of functions W_2^μ, which are obtained by completing the finite Fourier series

$$f(x) = \sum_k f_k e^{2\pi i k x} \tag{8}$$

in norm:

$$\left\| f | W_2^\mu \right\| = \left| \sum_k |f_k \mu(2\pi i k)|^2 \right|^{1/2} \tag{9}$$

In the works of M.D. Ramazanov and Kh.M. Shadimetov [40,41], optimal cubature formulas were constructed in the space $\widetilde{L_2^{(m)}}(H)$. In addition, in [43], an optimal cubature formula of the Euler–Maclaurin type was constructed in the space $L_2^{(m)}(R^n)$.

This work aims to study Sard's problem of constructing optimal quadrature formulas of the form (3) in the space $W_2^{(m,0)}$ using Sobolev's method. As a consequence, we obtain the optimal quadrature formula, which is exact to the basis functions of the kernel of the norm

(2). Here, the basis functions are contained in exponential-trigonometric functions. We acquire the optimal quadrature formula, which is not precise for any algebraic polynomial.

The rest of the paper is formulated as follows: in Section 2, the extremal function, which corresponds to the error functional ℓ, is found; in Section 3, with the help of this extremal function the norm of the error functional is calculated, i.e., the first part of Sard's problem is solved; in Section 4, a system of linear equations for coefficients of the optimal quadrature formulas in the space $W_2^{(m,0)}$ is obtained, while the existence and uniqueness of a solution for this system are discussed; in Section 5, the discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}}+2\frac{d^m}{dx^m}+1$ is obtained; and in Section 6, Sobolev's method of the construction of optimal quadrature formulas of the form (3) in the space $W_2^{(m,0)}$ is examined. Next, the optimal quadrature formula that is exact to trigonometric functions is obtained. Finally, at the end of the paper, the rate of convergence of the optimal quadrature formula in the space $W_2^{(2,0)}$ for the case $m=2$ is studied.

2. Extremal Function of the Error Functional of Quadrature Formulas

For identifying the norm of the error functional (5) of the quadrature Formula (3), we apply the extremal function of the error functional.

The function ψ_ℓ satisfying the equation

$$(\ell,\psi_\ell)=\|\ell\|_{W_2^{(m,0)*}}\cdot\|\psi_\ell\|_{W_2^{(m,0)}} \tag{10}$$

is called *the extremal function* for the functional ℓ [15–18].

Since $W_2^{(m,0)}$ is the Hilbert space, then using the Riesz theorem on the general form of a linear continuous functional on Hilbert spaces, for the error functional $\ell\in W_2^{(m,0)*}$, there exists a unique function $\psi_\ell\in W_2^{(m,0)}$, such that for any $\varphi\in W_2^{(m,0)}$, the following equality is fulfilled

$$(\ell,\varphi)=\langle\psi_\ell,\varphi\rangle. \tag{11}$$

Additionally, $\|\ell\|_{W_2^{(m,0)*}}=\|\psi_\ell\|_{W_2^{(m,0)}}$. Here, $\langle\psi_\ell,\varphi\rangle$ is the inner product of two functions defined by equality (1) in the space $W_2^{(m,0)}$.

In particular, from (11), when $\varphi=\psi_\ell$ we have

$$(\ell,\psi_\ell)=\langle\psi_\ell,\psi_\ell\rangle=\|\psi_\ell\|^2_{W_2^{(m,0)}}=\|\psi_\ell\|_{W_2^{(m,0)}}\cdot\|\ell\|_{W_2^{(m,0)*}}=\|\ell\|^2_{W_2^{(m,0)*}}.$$

The solution ψ_ℓ of Equation (11) is the extremal function. Therefore, to calculate the norm of the error functional ℓ, first, we should find the extremal function ψ_ℓ from Equation (11) and then calculate the square of the norm of the error functional ℓ, as follows:

$$\|\ell\|^2_{W_2^{(m,0)*}}=(\ell,\psi_\ell). \tag{12}$$

Integrating in parts the right-hand side of (11) we obtain

$$\begin{aligned}(\ell,\varphi)\ &=\ (-1)^m\int_0^1\Big(\psi_\ell^{(2m)}(x)+\psi_\ell^{(m)}(x)+(-1)^m\psi_\ell^{(m)}(x)+(-1)^m\psi_\ell(x)\Big)\varphi(x)dx+\\&\quad+\sum_{s=0}^{m-1}(-1)^s\Big(\psi_\ell^{(m+s)}(x)+\psi_\ell^{(s)}(x)\Big)\varphi^{(m-s-1)}(x)\Big|_0^1\end{aligned}$$

Here, we come to the following two cases for the odd and even values of m, respectively

$$(\ell,\varphi) = (-1)^m \int_0^1 \left(\psi_\ell^{(2m)}(x) - \psi_\ell(x)\right)\varphi(x)dx + \\ + \sum_{s=0}^{m-1} (-1)^s \left(\psi_\ell^{(m+s)}(x) + \psi_\ell^{(s)}(x)\right)\varphi^{(m-s-1)}(x)\Big|_0^1 \text{ for odd } m \tag{13}$$

and

$$(\ell,\varphi) = \int_0^1 \left(\psi_\ell^{(2m)}(x) + 2\psi_\ell^{(m)}(x) + \psi_\ell(x)\right)\varphi(x)dx + \\ + \sum_{s=0}^{m-1} (-1)^s \left(\psi_\ell^{(m+s)}(x) + \psi_\ell^{(s)}(x)\right)\varphi^{(m-s-1)}(x)\Big|_0^1 \text{ for even } m. \tag{14}$$

To find the extremal function, we solve Equations (14) and (15) depending on the corresponding values of m.

We note that, in the present paper, we study Sard's problem in space $W_2^{(m,0)}$ for even natural numbers m. For odd m, Sard's problem in this space was solved in [30].

Furthermore, we assume that m is an even natural number. From (14), taking into account the uniqueness of the function ψ_ℓ, we have the following equation:

$$\psi_\ell^{(2m)}(x) + 2\psi_\ell^{(m)}(x) + \psi_\ell(x) = \ell(x) \tag{15}$$

with the boundary conditions

$$\left[\left(\psi_\ell^{(m+s)}(x) + \psi_\ell^{(s)}(x)\right)\right]\Big|_{x=0}^{x=1} = 0, \quad s = \overline{0, m-1}. \tag{16}$$

It is worth-mentioning that, in the work [44] for the solution of the boundary value problem (15)–(16), the following result was gained.

Theorem 1. *A solution of Equation* (15) *with the boundary conditions* (16) *is the extremal function* ψ_ℓ *of the error functional* ℓ*, and this has the form*

$$\psi_\ell(x) = \ell(x) * G_m(x) + Y_m(x)\,,$$

where $G_m(x)$ *is Green's function, i.e., it is a fundamental solution of the equation*

$$G_m^{(2m)}(x) + 2G_m^{(m)}(x) + G_m(x) = \delta(x) \tag{17}$$

and is expressed as follows:

$$G_m(x) = \frac{sgnx}{2m^2} \cdot \sum_{k=1}^{m} \left[(1-m)e^{x\cos\frac{(2k-1)\pi}{m}} \cos\left(x\sin\left(\frac{(2k-1)\pi}{m}\right) + \frac{(2k-1)\pi}{m}\right) \right. \\ \left. + xe^{x\cos\frac{(2k-1)\pi}{m}} \cos\left(x\sin\left(\frac{(2k-1)\pi}{m}\right) + \frac{2\pi\cdot(2k-1)}{m}\right)\right], \tag{18}$$

$$Y_m(x) = \sum_{k=1}^{\frac{m}{2}} e^{x\cdot\cos\frac{(2k-1)\pi}{m}} \left[r_{1,k}\cos\left(x\sin\frac{(2k-1)\pi}{m}\right) + r_{2,k}\sin\left(x\sin\frac{(2k-1)\pi}{m}\right)\right], \tag{19}$$

r_{1k} *and* r_{2k} *are constants.*

As the error functional (5) is defined in the space $W_2^{(m,0)}$, it is crucial to impose the following conditions:

$$\left(\ell, e^{x\cos\frac{(2k-1)\pi}{m}}\cos\left(x\sin\frac{(2k-1)\pi}{m}\right)\right) = 0, \quad k=\overline{1,\frac{m}{2}}, \tag{20}$$

$$\left(\ell, e^{x\cos\frac{(2k-1)\pi}{m}}\sin\left(x\sin\frac{(2k-1)\pi}{m}\right)\right) = 0, \quad k=\overline{1,\frac{m}{2}}, \tag{21}$$

meaning that the quadrature Formula (3) is exact for linear combinations of functions

$$e^{x\cos\frac{(2k-1)\pi}{m}}\cos\left(x\sin\frac{(2k-1)\pi}{m}\right), \quad e^{x\cos\frac{(2k-1)\pi}{m}}\sin\left(x\sin\frac{(2k-1)\pi}{m}\right) \quad k=\overline{1,\frac{m}{2}}.$$

3. The Norm of the Error Functional of the Quadrature Formulas

As it was stated above, to calculate the square of the norm of the error functional (5), it is enough to calculate the value (ℓ, ψ_ℓ) of the error functional ℓ at function ψ_ℓ. For this, first, using equalities (20) and (21), we obtain

$$(\ell, Y_m(x)) = 0,$$

where $Y_m(x)$ is the function defined by (19) for even m. Then, using (19), we have

$$\begin{aligned}\|\ell\|^2_{W_2^{(m,0)}} &= (\ell,\psi_\ell) = \int\limits_{-\infty}^{+\infty} \ell(x)\Big[\ell(x)*G_m(x)+Y_m(x)\Big]dx \\ &= \int\limits_{-\infty}^{+\infty} \ell(x)\Big[\ell(x)*G_m(x)\Big]dx \end{aligned} \tag{22}$$

where $G_m(x)$, as defined by (19).

Here, for the convolution in (22) taking (5) into account, we obtain

$$\ell(x)*G_m(x) = \int\limits_{-\infty}^{+\infty} \ell(y)G_m(x-y)dy = \int\limits_0^1 G_m(x-y)dy - \sum_{\beta=0}^{N} C[\beta]G_m(x-x_\beta).$$

Then, the square of the norm (22) of the error functional ℓ takes the form

$$\begin{aligned}\|\ell\|^2_{W_2^{(m,0)}} &= \sum_{\beta=0}^{N} C[\beta]\left(\int\limits_0^1 G_m(x-x_\beta)+G_m(x_\beta-x)\right)dx \\ &- \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C[\beta]C[\gamma]G_m(x_\beta-x_\gamma) - \int\limits_0^1\int\limits_0^1 G_m(x-y)dxdy. \end{aligned} \tag{23}$$

Since $G_m(x)$ is the even function, we have

$$G_m(x_\beta - x) = G_m(x-x_\beta),$$

then, taking into account the last equality, from (23) we obtain

$$\begin{aligned}\|\ell\|^2_{W_2^{(m,0)}} &= \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C[\beta]C[\gamma]G_m(x_\beta-x_\gamma) + \int_0^1\int_0^1 G_m(x-y)dxdy \\ &-2\sum_{\beta=0}^{N} C[\beta]\int_0^1 G_m(x-x_\beta)dx. \end{aligned} \tag{24}$$

Thus, the first part of Sard's problem on the construction of optimal quadrature formulas in the space $W_2^{(m,0)}$ is solved. Next, we consider the second part of the problem.

4. System for Coefficients of Optimal Quadrature Formulas

Next, our focus shifts to minimizing the squared norm of the error functional (24), which is subject to conditions (20) and (21). It is established that the error function ℓ satisfies these conditions. The squared norm (24) of the error function is dependent on the multi-variable coefficients $C[\beta](\beta = \overline{0, N})$ of the quadrature Formula (3).

To determine the conditional minimum point of the squared norm of the error functional (5) under the conditions (20) and (21), we employ the method of indefinite Lagrange multipliers.

Denoting $\mathbb{C} = (C_0, C_1, \dots, C_N)$ and $\mathbf{r} = \left(r_{11}, \dots, r_{1\frac{m}{2}}, r_{21}, \dots, r_{2\frac{m}{2}}\right)$, we consider the following function:

$$\begin{aligned}\Psi(\mathbb{C}, \mathbf{r}) &= \|\ell\|^2 - 2\sum_{k=1}^{\frac{m}{2}} \left[r_{1k}\left(\ell, e^{x\cos\frac{(2k-1)\pi}{m}} \cos\left(x \sin\frac{(2k-1)\pi}{m}\right)\right)\right. \\ &\quad \left. + r_{2k}\left(\ell, e^{x\cos\frac{(2k-1)\pi}{m}} \sin\left(x \sin\frac{(2k-1)\pi}{m}\right)\right)\right],\end{aligned}$$

where r_{1k} and $r_{2k}\left(k = \overline{1\frac{m}{2}}\right)$ are Lagrange multipliers.

Equating to zero the partial derivatives of the function $\Psi(\mathbb{C}, \mathbf{r})$ with coefficients $C[\beta](\beta = \overline{0, N})$ and with $r_{1k}, r_{2k}\left(k = \overline{1, \frac{m}{2}}\right)$, we acquire the following system:

$$\sum_{\gamma=0}^{N} C[\gamma] G_m(x_\beta - x_\gamma) + Y_m(x_\beta) = f_m(x_\beta), \quad \beta = 0, 1, \dots, N, \tag{25}$$

$$\sum_{\gamma=0}^{N} C[\gamma] e^{x_\gamma \cos\frac{(2k-1)\pi}{m}} \cos\left(x_\gamma \sin\frac{(2k-1)\pi}{m}\right) = g_{1k}, \quad k = 1, 2, \dots, \frac{m}{2}, \tag{26}$$

$$\sum_{\gamma=0}^{N} C[\gamma] e^{x_\gamma \cos\frac{(2k-1)\pi}{m}} \sin\left(x_\gamma \sin\frac{(2k-1)\pi}{m}\right) = g_{2k}, \quad k = 1, 2, \dots, \frac{m}{2}, \tag{27}$$

where

$$f_m(x_\beta) = \int_0^1 G_m(x - x_\beta) dx, \tag{28}$$

$$g_{1k} = e^{\cos\frac{(2k-1)\pi}{m}} \cdot \cos\left(\sin\frac{(2k-1)\pi}{m} - \frac{(2k-1)\pi}{m}\right) - \cos\frac{(2k-1)\pi}{m}, \tag{29}$$

$$g_{2k} = \sin\frac{(2k-1)\pi}{m} - e^{\cos\frac{(2k-1)\pi}{m}} \cdot \sin\left(\sin\frac{(2k-1)\pi}{m} - \frac{(2k-1)\pi}{m}\right), \tag{30}$$

$k = \overline{1, \frac{m}{2}}$, $G_m(x)$ and $Y_m(x)$, as defined in Theorem 1.

We can notice that the system (25)–(27) is called *the discrete system of Wiener–Hopf type* [15,17].

It should be emphasized that the existence and uniqueness of an optimal quadrature formula of the form (3) in the sense of Sard in Hilbert spaces were studied in [13,17]. Specifically, we can obtain that the difference system (25)–(27) for any set of different nodes x_β, $\beta = 0, 1, \dots, N$, when $N + 1 \geq m$ has a unique solution, and this solution gives a minimum to $\|\ell\|^2$ found by (24) under the conditions (20) and (21). The existence and uniqueness of the solution to such types of different systems were also studied in [13,17].

Furthermore, we consider the case of equally spaced nodes. Suppose $x_\beta = h\beta$, $\beta = 0, 1, \dots N$, $h = \frac{1}{N}$, $N = 1, 2, \dots$

We suppose that $C[\beta] = 0$ for $\beta < 0$ and $\beta > N$. Then, using the convolution of two discrete argument functions (see [13,17])

$$\varphi(h\beta) * \psi(h\beta) = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma),$$

we rewrite the system (25)–(27) in the following convolution form:

$$C[\beta] * G_m(h\beta) + Y_m(h\beta) = f_m(h\beta), \ \ \beta = 0, 1, \ldots, N, \tag{31}$$

$$\sum_{\gamma=0}^{N} C[\gamma] e^{h\gamma \cos\frac{(2k-1)\pi}{m}} \cos\left(h\gamma \sin\frac{(2k-1)\pi}{m}\right) = g_{1k}, \ \ k = 1, 2, \ldots, \frac{m}{2}, \tag{32}$$

$$\sum_{\gamma=0}^{N} C[\gamma] e^{h\gamma \cos\frac{(2k-1)\pi}{m}} \sin\left(h\gamma \sin\frac{(2k-1)\pi}{m}\right) = g_{2k}, \ \ k = 1, 2, \ldots, \frac{m}{2}, \tag{33}$$

where $G_m(h\beta)$, $Y_m(h\beta)$, $f_m(h\beta)$, g_{1k}, and g_{2k} are defined by (18), (19), (28)–(30), respectively. There are unknowns and linear equations in the system (31)–(33).

To solve the system (31)–(33) using Sobolev's technique, we need a discrete analogue of the differential operator $\frac{d^{2m}}{dx^{2m}} + 2\frac{d^m}{dx^m} + 1$. The next section is devoted to the construction of this discrete analogue.

5. A Discrete Analogue $D_m(h\beta)$ of the Differential Operator $\frac{d^{2m}}{dx^{2m}} + 2\frac{d^m}{dx^m} + 1$

In the present section, for even m, we obtain the discrete argument function $D_m(h\beta)$ that satisfies the equation

$$D_m(h\beta) * G_m(h\beta) = \delta(h\beta), \tag{34}$$

where

$$\begin{aligned} G_m(h\beta) &= \frac{\operatorname{sgn}(h\beta)}{2m^2} \cdot \sum_{k=1}^{m} \Bigg[(1-m) e^{h\beta \cos\frac{(2k-1)\pi}{m}} \cos\left(h\beta \sin\left(\frac{(2k-1)\pi}{m}\right) + \frac{(2k-1)\pi}{m}\right) \\ &+ h\beta e^{h\beta \cos\frac{(2k-1)\pi}{m}} \cos\left(h\beta \sin\left(\frac{(2k-1)\pi}{m}\right) + \frac{2\pi \cdot (2k-1)}{m}\right)\Bigg], \end{aligned} \tag{35}$$

$\delta(h\beta)$ is the discrete delta-function, i.e.,

$$\delta(h\beta) = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0, \end{cases} \qquad h = \frac{1}{N}, \quad N = 1, 2, \ldots .$$

It is worth mentioning that the process of constructing the discrete argument function is comparable to the process of constructing discrete analogues of differential operators $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}$, $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - \frac{\mathrm{d}^{2m-2}}{\mathrm{d}x^{2m-2}}$, $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} + 2\omega^2 \frac{\mathrm{d}^{2m-2}}{\mathrm{d}x^{2m-2}} + \omega^4 \frac{\mathrm{d}^{2m-4}}{\mathrm{d}x^{2m-4}}$, and $\frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}} - 1$ (for odd m) in the works [27–30].

The discrete function $D_m(h\beta)$ plays a significant role in calculating the coefficients of optimal quadrature formulas in the space $W_2^{(m,0)}$. We can notice that Equation (34) is a discrete analog of the following equation:

$$\left(\frac{d^{2m}}{dx^{2m}} + 2\frac{d^m}{dx^m} + 1\right) G_m(x) = \delta(x), \tag{36}$$

where $G_m(x)$ is defined by (19), and $\delta(x)$ is Dirac's delta-function.

We present the following notation:

$$
\begin{aligned}
a_{1k} &= (1-m)\cos\frac{(2k-1)\pi}{m},\\
a_{2k} &= 2e^{h\cos\frac{(2k-1)\pi}{m}}\left((m-1)\cos\left(h\sin\frac{(2k-1)\pi}{m}-\frac{(2k-1)\pi}{m}\right)\right.\\
&+ \left. h\cos\left(h\sin\frac{(2k-1)\pi}{m}+\frac{2\pi(2k-1)}{m}\right)\right),\\
a_{3k} &= 2e^{2h\cos\frac{(2k-1)\pi}{m}}\left((1-m)\sin\frac{(2k-1)\pi}{m}\sin\left(2h\sin\frac{(2k-1)\pi}{m}\right)-2h\cos\frac{2\pi(2k-1)}{m}\right),\\
a_{4k} &= 2e^{3h\cos\frac{(2k-1)\pi}{m}}\left((1-m)\cos\left(h\sin\frac{(2k-1)\pi}{m}+\frac{(2k-1)\pi}{m}\right)\right.\\
&+ \left. h\cos\left(h\sin\frac{(2k-1)\pi}{m}-\frac{2\pi(2k-1)}{m}\right)\right),\\
a_{5k} &= (m-1)e^{4h\cos\frac{(2k-1)\pi}{m}}\cos\frac{(2k-1)\pi}{m},\\
b_{1k} &= -4e^{h\cos\frac{(2k-1)\pi}{m}}\cos\left(h\sin\frac{(2k-1)\pi}{m}\right),\\
b_{2k} &= 2\cdot e^{2h\cos\frac{(2k-1)\pi}{m}}\left(2+\cos\left(2h\sin\frac{(2k-1)\pi}{m}\right)\right),\\
b_{3k} &= -4e^{3h\cos\frac{(2k-1)\pi}{m}}\cos\left(h\sin\frac{(2k-1)\pi}{m}\right),\\
b_{4k} &= e^{4h\cos\frac{(2k-1)\pi}{m}}.
\end{aligned}
$$

The results of this section are the following:

Theorem 2. *The discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}}+2\frac{d^m}{dx^m}+1$ satisfying Equation (34), when m is an even natural number, has the form*

$$
D_m(h\beta)=\frac{m^2}{K}\cdot\begin{cases}\sum\limits_{k=1}^{m-1}A_k^*\cdot\lambda_k^{|\beta|-1}, & |\beta|\ge 2,\\ 1+\sum\limits_{k=1}^{m-1}A_k^*, & |\beta|=1,\\ M_1-\frac{K_1}{K}+\sum\limits_{k=1}^{m-1}\frac{A_k^*}{\lambda_k}, & \beta=0.\end{cases} \tag{37}
$$

where

$$
K=\sum_{k=1}^{\frac{m}{2}}\left(a_{2k}+a_{1k}\sum_{j=1,j\neq k}^{\frac{m}{2}}b_{1j}\right),\quad M_1=\sum_{k=1}^{\frac{m}{2}}b_{1k},
$$

$$
K_1=\sum_{k=1}^{\frac{m}{2}}\left(a_{3k}+a_{2k}\sum_{j=1,j\neq k}^{\frac{m}{2}}b_{1j}+a_{1k}\sum_{j=1,j\neq k}^{\frac{m}{2}}b_{2j}\right),
$$

$$
A_k^*=\frac{A_{2m}(\lambda_k)}{\lambda_k\cdot(P_{2m-2}(\lambda_k))'},\quad P_{2m-2}(\lambda)=\frac{1}{\lambda}P_{2m}(\lambda),
$$

$$
A_{2m}(\lambda)=\prod_{k=1}^{\frac{m}{2}}\left(\lambda^4+b_{1k}\lambda^3+b_{2k}\lambda^2+b_{3k}\lambda+b_{4k}\right),
$$

$$P_{2m}(\lambda) = \prod_{k=1}^{\frac{m}{2}} A_k(\lambda) \sum_{j=1}^{\frac{m}{2}} \frac{B_j(\lambda)}{A_j(\lambda)}$$

$$B_j(\lambda) = a_{1j}\lambda^4 + a_{2j}\lambda^3 + a_{3j}\lambda^2 + a_{4j}\lambda + a_{5j},$$

λ_k are roots of the polynomial $P_{2m-2}(\lambda)$ with an absolute value less than one, i.e., $|\lambda_k| < 1$.

Theorem 3. *The discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}} + 2\frac{d^m}{dx^m} + 1$ satisfies the following equalities*

1. $D_m(h\beta) * e^{h\beta \cos \frac{(2k-1)\pi}{m}} \cos\left(h\beta \sin \frac{(2k-1)\pi}{m}\right) = 0,$
2. $D_m(h\beta) * e^{h\beta \cos \frac{(2k-1)\pi}{m}} \sin\left(h\beta \sin \frac{(2k-1)\pi}{m}\right) = 0,$
3. $D_m(h\beta) * h\beta e^{h\beta \cos \frac{(2k-1)\pi}{m}} \cos\left(h\beta \sin \frac{(2k-1)\pi}{m}\right) = 0,$
4. $D_m(h\beta) * h\beta e^{h\beta \cos \frac{(2k-1)\pi}{m}} \sin\left(h\beta \sin \frac{(2k-1)\pi}{m}\right) = 0,\ k = 1, 2, \ldots, \frac{m}{2}.$

Here, $G_m(h\beta)$ is defined by (35) and $\delta(h\beta)$ is a discrete delta-function.

In order to demonstrate the theorems mentioned above, we rely on the widely recognized formulas for generalized functions and Fourier transforms [45,46]. Specifically, we define the direct and inverse Fourier transforms of the function φ

$$F[\varphi(x)] = \int_{-\infty}^{+\infty} \varphi(x) e^{2\pi i p x} dx, \quad F^{-1}[\varphi(p)] = \int_{-\infty}^{+\infty} \varphi(x) e^{-2\pi i p x} dp\ . \tag{38}$$

Fourier transform of product and convolution of functions φ and ψ:

$$F[\varphi * \psi] = F[\varphi] \cdot F[\psi], \tag{39}$$

$$F[\varphi \cdot \psi] = F[\varphi] * F[\psi]. \tag{40}$$

For the delta function and its derivatives, the following hold:

$$F[\delta(x)] = 1, \qquad F\left[\delta^{(\alpha)}(x)\right] = (-2\pi i p)^{\alpha}\ . \tag{41}$$

We also use the following well-known properties of the delta-function

$$\delta(hx) = h^{-1}\delta(x), \tag{42}$$

$$\delta(x-a) \cdot f(x) = \delta(x-a) \cdot f(a), \tag{43}$$

$$\delta^{(\alpha)}(x) * f(x) \quad = \quad f^{(\alpha)}(x), \tag{44}$$

$$\Phi_0(x) = \sum_{\beta=-\infty}^{+\infty} \delta(x-\beta), \qquad \sum_{\beta=-\infty}^{+\infty} e^{2\pi i x \beta} = \sum_{\beta=-\infty}^{+\infty} \delta(x-\beta). \tag{45}$$

Proof of Theorem 2. It is more convenient to work with harrow-shaped functions rather than functions of a discrete argument. By using harrow-shaped functions, we can perform operations more easily. The harrow-shaped function that corresponds to the function of the discrete argument $D_m(h\beta)$ takes the following form:

$$\overleftharpoondown{D}_m(x) = \sum_{\beta=-\infty}^{+\infty} D_m(h\beta) \cdot \delta(x - h\beta).$$

Now, using the definition of harrow-shaped functions, instead of Equation (34), we consider the following equivalent equation in terms of harrow-shaped functions:

$$\overset{\smile}{D}_m(x) * \overset{\smile}{G}_m(x) = \delta(x). \tag{46}$$

In this case, it has to be taken into account that $\overset{\smile}{\delta}(x) = \delta(x)$ and $\overset{\smile}{G}_m(x) = \sum\limits_{\beta=-\infty}^{+\infty} G_m(h\beta) \cdot \delta(x - h\beta)$ are harrow-shaped functions corresponding to a discrete function $G_m(h\beta)$.

Applying the Fourier transform to both sides of Equation (46), and taking (39) and (40) into account, we obtain

$$F\left[\overset{\smile}{D}_m(x)\right] = \frac{1}{F\left[\overset{\smile}{G}_m(x)\right]}. \tag{47}$$

First, we compute the Fourier transform $F\left[\overset{\smile}{G}_m(x)\right]$. Taking into account (43), (45), and also using Formulas (40) and (42), we have

$$F\left[\overset{\smile}{G}_m(x)\right] = F[G_m(x)] * \Phi_0(hp). \tag{48}$$

To calculate the Fourier transform $F[G_m(x)]$ of the function $G_m(x)$, we use the equalities (36) and (44). Then, taking into account equalities (39) and (41), we obtain

$$F[G_m(x)] = \frac{1}{(2\pi ip)^{2m} + 2 \cdot (2\pi ip)^m + 1} = \sum_{k=1}^{m} \left[\frac{a_k}{2\pi ip - p_k} + \frac{b_k}{(2\pi ip - p_k)^2} \right]. \tag{49}$$

We denote the roots of the equation $p^{2m} + 2p^m + 1 = 0$ as $p_k = \cos\frac{(2k-1)\pi}{m} + i\sin\frac{(2k-1)\pi}{m}$, $a_k = \frac{1-m}{m^2} p_k$ and $b_k = \frac{1}{m^2} p_k^2$, $k = 1, 2, \ldots, m$, $m = 2n$, $n = 1, 2, \ldots$. We calculate the first part of the sum (49)

$$\sum_{k=1}^{m} \frac{a_k}{2\pi ip - p_k} = \frac{1-m}{m^2} \sum_{k=1}^{m} \frac{p_k}{2\pi ip - p_k} = \frac{1-m}{m^2} \left[\frac{p_1}{2\pi ip - p_1} + \ldots + \frac{p_m}{2\pi ip - p_m} \right].$$

Then, we use the following equality, i.e.,

$$-\frac{m}{(2\pi ip)^m + 1} = \frac{p_1}{2\pi ip - p_1} + \frac{p_2}{2\pi ip - p_2} + \ldots + \frac{p_m}{2\pi ip - p_m},$$

hence, we obtain

$$\sum_{k=1}^{m} \frac{a_k}{2\pi ip - p_k} = \frac{m-1}{m} \cdot \frac{1}{(2\pi ip)^m + 1}, \tag{50}$$

then, the calculation of convolution (48), taking into account (49), (50), and (47), gives

$$\begin{aligned} F\left[\overset{\smile}{D}_m(x)\right] = & \left[\frac{m-1}{m} \cdot \frac{h^{m-1}}{(-2\pi i)^m} \sum_{\beta} \frac{1}{\left[\beta - h\left(p + \frac{p_1 i}{2\pi}\right)\right] \cdot \ldots \cdot \left[\beta - h\left(p + \frac{p_m i}{2\pi}\right)\right]} \right. \\ & \left. - \frac{h}{4\pi^2 m^2} \sum_{k=1}^{m} p_k^2 \sum_{\beta} \frac{1}{\left[\beta - h\left(p + \frac{p_k i}{2\pi}\right)\right]^2} \right]^{-1} = [S_1 + S_2]^{-1}. \end{aligned} \tag{51}$$

From (51) we have

$$S_1 = \frac{m-1}{m} \cdot \frac{h^{m-1}}{(-2\pi i)^m} \sum_{\beta} \frac{1}{\left[\beta - h\left(p + \frac{p_1 i}{2\pi}\right)\right] \cdot \ldots \cdot \left[\beta - h\left(p + \frac{p_m i}{2\pi}\right)\right]}, \tag{52}$$

$$S_2 = -\frac{h}{4\pi^2 m^2} \sum_{k=1}^{m} p_k^2 \sum_{\beta} \frac{1}{\left[\beta - h\left(p + \frac{p_k i}{2\pi}\right)\right]^2}. \tag{53}$$

Let us suppose that the Fourier series of the function $F\left[\overline{D}_m\right](p)$ has the form

$$F\left[\overline{D}_m\right](p) = \sum_{\beta=-\infty}^{\infty} \hat{D}_m(h\beta) e^{2\pi i p h \beta}, \tag{54}$$

where $\hat{D}_m(h\beta)$ are the Fourier coefficients of the function $F\left[\overline{D}_m\right](p)$, i.e.,

$$\hat{D}_m(h\beta) = \int_0^{h^{-1}} F\left[\overline{D}_m\right](p) e^{-2\pi i p h \beta} dp. \tag{55}$$

Applying the inverse Fourier transform to both sides of equality (54), we obtain the harrow-shaped function

$$\overline{D}_m(x) = \sum_{\beta=-\infty}^{\infty} \hat{D}_m(h\beta) \delta(x - h\beta).$$

Therefore, based on the definition of harrow-shaped functions, we can deduce that the discrete function $\hat{D}_m(h\beta)$ is the desired discrete argument function $D_m(h\beta)$. This implies that we can obtain $D_m(h\beta)$ by expanding the right-hand side of (54) as a Fourier series. To evaluate the infinite series S_1 and S_2 in (51), we utilize a well-known formula from the theory of residues (refer to [47]).

$$\sum_{\beta=-\infty}^{\infty} f(\beta) = -\sum_{z_1, z_2, \ldots, z_n} \mathrm{res}(\pi \cot(\pi z) f(z)), \tag{56}$$

where $z_1, z_2, \ldots, z_n$ are the poles of the function $f(z)$.

We present the famous formula (see [48])

$$\sum_{\beta=-\infty}^{\infty} \frac{1}{(\beta - p)^2} = \frac{\pi^2}{\sin^2(\pi p)}. \tag{57}$$

To calculate S_1, we use the well-known formula from (56)

$$f(z) = \frac{1}{\left[z - h\left(p + \frac{ip_1}{2\pi}\right)\right] \cdot \left[z - h\left(p + \frac{ip_2}{2\pi}\right)\right] \cdot \ldots \cdot \left[z - h\left(p + \frac{ip_m}{2\pi}\right)\right]}.$$

Here, $z_1 = h\left(p + \frac{ip_1}{2\pi}\right)$, $z_2 = h\left(p + \frac{ip_2}{2\pi}\right), \ldots, z_m = h\left(p + \frac{ip_m}{2\pi}\right)$ are the poles of order 1 of the function $f(z)$. Then, using the Formula (56) from (52) we obtain

$$S_1 = \frac{m-1}{m} \cdot \frac{h^{m-1}}{(-2\pi i)^m} \left[\sum_{z_1, z_2, \ldots, z_n} \mathrm{res}(\pi \cot(\pi z) f(z)) \right]. \tag{58}$$

Using straight calculation, we have the following results:

$$\begin{aligned}
\operatorname*{res}_{z=z_1}(\pi\cot(\pi z)f(z)) &= -\left(-\frac{2\pi i}{h}\right)^{m-1}\cdot\frac{\pi p_1}{m}\cot\left(\pi hp+\frac{p_1 hi}{2}\right),\\
\operatorname*{res}_{z=z_2}(\pi\cot(\pi z)f(z)) &= -\left(-\frac{2\pi i}{h}\right)^{m-1}\cdot\frac{\pi p_2}{m}\cot\left(\pi hp+\frac{p_2 hi}{2}\right),\\
&\ldots..,\\
\operatorname*{res}_{z=z_{m-1}}(\pi\cot(\pi z)f(z)) &= -\left(-\frac{2\pi i}{h}\right)^{m-1}\cdot\frac{\pi p_{m-1}}{m}\cot\left(\pi hp+\frac{p_{m-1} hi}{2}\right),\\
\operatorname*{res}_{z=z_m}(\pi\cot(\pi z)f(z)) &= -\left(-\frac{2\pi i}{h}\right)^{m-1}\cdot\frac{\pi p_m}{m}\cot\left(\pi hp+\frac{p_m hi}{2}\right).
\end{aligned}$$

Denoting $\lambda = e^{2\pi iph}$, using the last m equalities and taking into account the following formulas: $\cos(z) = \frac{e^{zi}+e^{-zi}}{2}$, $\sin(z) = \frac{e^{zi}-e^{-zi}}{2i}$, $\cosh(z) = \frac{e^{z}+e^{-z}}{2}$ and $\sinh(z) = \frac{e^{z}-e^{-z}}{2}$, after some calculations from (58), we obtain

$$\begin{aligned}
S_1 &= \frac{1-m}{m^2}\sum_{k=1}^{\frac{m}{2}}\left[\frac{\left(\lambda^2-e^{2h\cos\frac{(2k-1)\pi}{m}}\right)\cdot\cos\frac{(2k-1)\pi}{m}}{\lambda^2-2\lambda e^{h\cos\frac{(2k-1)\pi}{m}}\cos\left(h\sin\frac{(2k-1)\pi}{m}\right)+e^{2h\cos\frac{(2k-1)\pi}{m}}}\right.\\
&\quad-\left.\frac{2\lambda e^{h\cos\frac{(2k-1)\pi}{m}}\sin\left(h\sin\frac{(2k-1)\pi}{m}\right)\sin\frac{(2k-1)\pi}{m}}{\lambda^2-2\lambda e^{h\cos\frac{(2k-1)\pi}{m}}\cos\left(h\sin\frac{(2k-1)\pi}{m}\right)+e^{2h\cos\frac{(2k-1)\pi}{m}}}\right].
\end{aligned}\tag{59}$$

Now, we calculate series (53) using Formula (57), which gives us

$$\begin{aligned}
S_2 &= \frac{2h\lambda}{m^2}\sum_{k=1}^{\frac{m}{2}}\left[\frac{\lambda^2 e^{h\cos\frac{(2k-1)\pi}{m}}\cos\left(h\sin\frac{(2k-1)\pi}{m}+\frac{2\pi(2k-1)}{m}\right)}{\left[\lambda^2-2\lambda e^{h\cos\frac{(2k-1)\pi}{m}}\cos\left(h\sin\frac{(2k-1)\pi}{m}\right)+e^{2h\cos\frac{(2k-1)\pi}{m}}\right]^2}\right.\\
&\quad-\left.\frac{2\lambda e^{2h\cos\frac{(2k-1)\pi}{m}}\cos\frac{2\pi(2k-1)}{m}-e^{3h\cos\frac{(2k-1)\pi}{m}}\cos\left(h\sin\frac{\pi(2k-1)}{m}-\frac{2\pi(2k-1)}{m}\right)}{\left[\lambda^2-2\lambda e^{h\cos\frac{(2k-1)\pi}{m}}\cos\left(h\sin\frac{(2k-1)\pi}{m}\right)+e^{2h\cos\frac{(2k-1)\pi}{m}}\right]^2}\right].
\end{aligned}\tag{60}$$

Substituting (59) and (60) into equality (51), we obtain

$$F\left[\overline{D}_m\right](p) = \frac{m^2}{K}\cdot\frac{A_{2m}(\lambda)}{\lambda P_{2m-2}(\lambda)}.\tag{61}$$

For finding the direct form of the discrete function $D_m(h\beta)$ on the right-hand side of (61), we expand into the sum of partial fractions, as follows:

$$\begin{aligned}
\frac{m^2}{K}\cdot\frac{A_{2m}(\lambda)}{\lambda(\lambda-\lambda_1)(\lambda-\lambda_2)\cdot\ldots\cdot(\lambda-\lambda_{2m-3})(\lambda-\lambda_{2m-2})} &= \frac{m^2}{K}\cdot\left[\lambda-\frac{K_1}{K}+M_1\right.\\
&\quad\left.+\frac{A_0^*}{\lambda}+\frac{A_1^*}{\lambda-\lambda_1}+\frac{A_2^*}{\lambda-\lambda_2}+\ldots+\frac{A_{2m-3}^*}{\lambda-\lambda_{2m-3}}+\frac{A_{2m-2}^*}{\lambda-\lambda_{2m-2}}\right],
\end{aligned}\tag{62}$$

where K, K_1, M_1, and $A_{2m}(\lambda)$ are given in the statement of the theorem; $A_0^*, A_1^*, A_2^*, \ldots, A_{2m-3}^*, A_{2m-2}^*$ are unknowns; and $\lambda_1, \lambda_2, \ldots, \lambda_{2m-2}$ are roots of the polynomial $P_{2m-2}(\lambda)$, such that $\lambda_j\cdot\lambda_{2m-1-j} = 1, j = \overline{1, m-1}$.

For finding unknowns $A_0^*, A_1^*, A_2^*, \ldots, A_{2m-3}^*, A_{2m-2}^*$, we multiply both sides of equality (62) using the expression $\lambda(\lambda-\lambda_1)(\lambda-\lambda_2)\cdot\ldots\cdot(\lambda-\lambda_{2m-3})(\lambda-\lambda_{2m-2})$ and we put $\lambda=0, \lambda=\lambda_1, \ldots, \lambda=\lambda_{2m-2}$. Then, we can obtain the following results:

$$A_0^* = 1, \tag{63}$$

$$A_k^* = \frac{A_{2m}(\lambda_k)}{\lambda_k P'_{2m-2}(\lambda_k)}, \; k=\overline{1,2m-2}. \tag{64}$$

From (64), taking into account $\lambda_1\cdot\lambda_2\cdot\ldots\cdot\lambda_{2m-3}\cdot\lambda_{2m-2}=1$, we have

$$A_{2m-2}^* = -\frac{1}{\lambda_1^2}A_1^* \text{ and } A_{2m-3}^* = -\frac{1}{\lambda_2^2}A_2^*, \ldots. \tag{65}$$

Finally, using (63)–(65) from (61) we obtain

$$\frac{m^2}{K}\cdot\left[\lambda-\frac{K_1}{K}+M_1+\frac{1}{\lambda}+\sum_{\gamma=0}^{\infty}\sum_{j=1}^{m-1}\left[\frac{A_j^*}{\lambda}\left(\frac{\lambda_j}{\lambda}\right)^{\gamma}+\frac{A_j^*}{\lambda_j}(\lambda_j\lambda)^{\gamma}\right]\right]=\sum_{\gamma=-\infty}^{\infty}D_m(h\gamma)\lambda^{\gamma}.$$

Hence, bearing in mind that $\lambda=e^{2\pi iph}$, we obtain the explicit form (37) of the discrete function $D_m(h\beta)$. Theorem 2 is proven. □

The proof of Theorem 3 is obtained using the definition of the convolution of discrete functions and the direct calculation of the left sides of equalities (1)–(4). In (37), we note that the function $D_m(h\beta)$ is even, i.e., $D_m(-h\beta)=D_m(h\beta)$.

6. Solution of the Discrete Wiener–Hopf System

In this section, we give an algorithm for finding the exact solution of the system (31)–(33) using the discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}}+2\frac{d^m}{dx^m}+1$, as obtained in the previous section.

We introduce the following functions:

$$v(h\beta)=C[\beta]*G_m(h\beta) \tag{66}$$

and

$$u(h\beta)=v(h\beta)+Y_m(h\beta). \tag{67}$$

Then, taking (34) into account, for optimal coefficients $C[\beta]$; we have

$$C[\beta]=D_m(h\beta)*u(h\beta). \tag{68}$$

Therefore, if we can find the function $u(h\beta)$, then optimal coefficients can be defined using the Formula (68). For calculation of the convolution in (68), it is required to find the function $u(h\beta)$ at all integer values of β. It is obvious from (31) that $u(h\beta)=f_m(h\beta)$ for $h\beta\in[0,1]$.

Now, we have to find the function $u(h\beta)$ for $\beta<0$ and $\beta>N$. Using the Formula (19), we calculate the convolution $v(h\beta)=C(h\beta)*G_m(h\beta)$for $h\beta\notin[0,1]$.

For $\beta<0$, we have

$$v(h\beta)=G_m(h\beta)*C[\beta]=\sum_{\gamma=-\infty}^{\infty}C[\gamma]G_m(h\beta-h\gamma)=\sum_{\gamma=0}^{N}C[\gamma]G_m(h\beta-h\gamma)=-\frac{1}{2m^2}$$

$$\times\sum_{\gamma=0}^{N}C[\gamma]\left[(1-m)\sum_{k=1}^{m}e^{(h\beta-h\gamma)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\beta-h\gamma)\sin\frac{(2k-1)\pi}{m}+\frac{(2k-1)\pi}{m}\right)\right.$$

$$+\sum_{k=1}^{m}(h\beta-h\gamma)e^{(h\beta-h\gamma)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\beta-h\gamma)\sin\frac{(2k-1)\pi}{m}+\frac{2\pi(2k-1)}{m}\right)\Bigg]. \quad (69)$$

Hence, splitting up the internal sum into two parts

$$\sum_{\gamma=0}^{N}C[\gamma]\sum_{k=1}^{m}e^{(h\beta-h\gamma)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\beta-h\gamma)\sin\frac{(2k-1)\pi}{m}+\frac{(2k-1)\pi}{m}\right)=S_{1,k}-S_{2,k},$$

$$\sum_{\gamma=0}^{N}C[\gamma](h\beta-h\gamma)\sum_{k=1}^{m}e^{(h\beta-h\gamma)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\beta-h\gamma)\sin\frac{(2k-1)\pi}{m}+\frac{(2k-1)\pi}{m}\right)$$
$$=S_{3,k}+S_{4,k},$$

where

$$S_{1,k}-S_{2,k}=\sum_{\gamma=0}^{N}C[\gamma]\sum_{k=1}^{\frac{m}{2}}e^{(h\beta-h\gamma)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\beta-h\gamma)\sin\frac{(2k-1)\pi}{m}+\frac{(2k-1)\pi}{m}\right)$$
$$-\sum_{\gamma=0}^{N}C[\gamma]\sum_{k=1}^{\frac{m}{2}}e^{(h\gamma-h\beta)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\gamma-h\beta)\sin\frac{(2k-1)\pi}{m}+\frac{(2k-1)\pi}{m}\right), \quad (70)$$

$$\sum_{\gamma=0}^{N}C[\gamma](h\beta-h\gamma)\sum_{k=1}^{\frac{m}{2}}e^{(h\beta-h\gamma)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\beta-h\gamma)\sin\frac{(2k-1)\pi}{m}+\frac{2\pi(2k-1)}{m}\right)$$
$$+\sum_{\gamma=0}^{N}C[\gamma](h\beta-h\gamma)\sum_{k=1}^{\frac{m}{2}}e^{(h\gamma-h\beta)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\gamma-h\beta)\sin\frac{(2k-1)\pi}{m}+\frac{2\pi(2k-1)}{m}\right)$$
$$=S_{3,k}+S_{4,k}. \quad (71)$$

Substituting the obtained expressions (70) and (71) into (69) and using Formulas (31)–(33), after some simplifications, we have the expression (69) in the form

$$v(h\beta)=-\frac{1}{2m^2}Q_m(h\beta)-Y_m(h\beta,b_{1k},b_{2k}).$$

Here

$$Q_m(h\beta)=h\beta\sum_{\gamma=0}^{N}\sum_{k=1}^{\frac{m}{2}}C[\gamma]e^{(h\beta-h\gamma)\cos\frac{(2k-1)\pi}{m}}\cos\left((h\beta-h\gamma)\sin\frac{(2k-1)\pi}{m}+\frac{2\pi(2k-1)}{m}\right)$$
$$+(m-1)S_{2,k}+S_{4,k},$$
$$Y_m(h\beta,d_{1k},d_{2k})=\sum_{k=1}^{\frac{m}{2}}e^{h\beta\cos\frac{(2k-1)\pi}{m}}\cdot\left[d_{1k}\cos\left(h\beta\sin\frac{(2k-1)\pi}{m}\right)\right.$$
$$\left.+d_{2k}\sin\left(h\beta\sin\frac{(2k-1)\pi}{m}\right)\right],$$

where $d_{1k},d_{2k}\left(k=\overline{1,\frac{m}{2}}\right)$ are unknowns.

Direct calculations show that $v(h\beta)$ for $\beta>N$ has the form

$$v(h\beta)=\frac{1}{2m^2}Q_m(h\beta)+Y_m(h\beta,d_{1k},d_{2k}).$$

Then, we have that $u(h\beta) = v(h\beta) + Y_m(h\beta)$, $u(h\beta) = f_m(h\beta)$ for $h\beta \in [0,1]$ and

$$u(h\beta) = \begin{cases} -\frac{1}{2m^2} Q_m(h\beta) + Y_m(h\beta, r_{1k}^-, r_{2k}^-), & \beta < 0, \\ \frac{1}{2m^2} Q_m(h\beta) + Y_m(h\beta, r_{1k}^+, r_{2k}^+) & \beta > N. \end{cases} \tag{72}$$

Then

$$Y_m(h\beta) - Y_m(h\beta, d_{1k}, d_{2k}) = Y_m\left(h\beta, r_{1k}^-, r_{2k}^-\right)$$

and

$$Y_m(h\beta) + Y_m(h\beta, d_{1k}, d_{2k}) = Y_m\left(h\beta, r_{1k}^+, r_{2k}^+\right).$$

Hence, we have

$$Y_m(h\beta) = \frac{1}{2}\left(Y_m\left(h\beta, r_{1k}^-, r_{2k}^-\right) + Y_m\left(h\beta, r_{1k}^+, r_{2k}^+\right)\right).$$

Since for $h\beta \notin [0,1]$, $C[\beta] = 0$, then

$$C[\beta] = D_m(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0,1].$$

From here, we can obtain a system of linear equations for finding unknowns $r_{1k}^-, r_{2k}^-, r_{1k}^+, r_{2k}^+$ $\left(k = \overline{1, \frac{m}{2}}\right)$.

Then, we find the optimal coefficients of quadrature formulas of the form (3) using the Formula (68)

$$C[\beta] = D_m(h\beta) * u(h\beta), \quad h\beta \in [0,1].$$

Thus, Sard's problem of constructing optimal quadrature formulas of the form (3) in the space $W_2^{(m,0)}(0,1)$ for even natural numbers m is solved.

Remark 1. *It should be noted that $m = 2$, by realizing the above-given algorithm, we obtain the optimal quadrature formula of the form* (3) *in the space $W_2^{(2,0)}(0,1)$, which was constructed in the work [33].*

Remark 2. *We note that using the discrete analogue $D_2(h\beta)$ of the operator $\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$, which corresponds to the case $m = 2$, we obtain an interpolation formula $P_\varphi(x)$ in the space $W_2^{(2,0)}(0,1)$ obtained in Theorem 3 of [49]. It should be noted that the constructed interpolation formula is exact for trigonometric functions.*

Remark 3. *It is vital to note that the discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}} + 2\frac{d^m}{dx^m} + 1$, as constructed in Section 5, can be used for the construction of interpolation splines minimizing the semi-norm* (2) *and optimal quadrature formulas for numerical integration of Fourier coefficients in the space $W_2^{(m,0)}(0,1)$.*

In the next section, we give the results of the algorithm for the case $m = 2$.

7. Coefficients of the Optimal Quadrature Formula in the Space $W_2^{(2,0)}(0,1)$

In this section, we give the results of the realization of the algorithm for the construction of the optimal quadrature Formula (3) in the space $W_2^{(m,0)}(0,1)$ for the case $m = 2$.

We recall that, in the case $m = 2$, we obtain the result of Theorem 4.4 of [33].

For $m = 2$, the system (31)–(33) has the form

$$C[\beta] * G_2(h\beta) + r_{11}\sin(h\beta) + r_{21}\cos(h\beta) = f_2(h\beta), \ \ \beta = 0, 1, \ldots, N, \tag{73}$$

$$\sum_{\gamma=0}^{N} C[\gamma]\sin(h\gamma) = 1 - \cos 1, \tag{74}$$

$$\sum_{\gamma=0}^{N} C[\gamma]\cos(h\gamma) = \sin 1, \tag{75}$$

where

$$G_2(h\beta) = \frac{\operatorname{sgn}(h\beta)}{4}[\sin(h\beta) - h\beta\cos(h\beta)],$$

$$\begin{aligned} f_2(h\beta) &= \frac{1}{4}\Big[4 - (2 + 2\cos 1 + \sin 1)\cos(h\beta) - (2\sin 1 - \cos 1) \\ &\quad + \sin 1 \cdot (h\beta)\cos(h\beta) - (1 + \cos 1)\sin(h\beta)\Big]. \end{aligned}$$

The system (73)–(75) was solved in the work [33], and the following theorem was proven:

Theorem 4. *The coefficients of optimal quadrature formulas of the form* (3) *with equally spaced nodes in the space* $W_2^{(2,0)}$ *are expressed by the formulas*

$$C[\beta] = \begin{cases} \frac{2\sin(h) - (h+\sin(h))\cos(h)}{(h+\sin(h))\sin(h)} + \frac{(h-\sin(h))\left(\lambda_1 + \lambda_1^{N+1}\right)}{(h+\sin(h))\sin(h)\left(1+\lambda_1^{N+1}\right)}, & \beta = 0, N \\ \frac{4(1-\cos(h))}{h+\sin(h)} + \frac{2h(h-\sin(h))\sin(h)}{(h+\sin(h))(h\cos(h)-\sin(h))\left(1+\lambda_1^{N+1}\right)} \cdot \left(\lambda_1^{\beta} + \lambda_1^{N-\beta}\right), & \beta = \overline{1, N-1}, \end{cases} \tag{76}$$

where λ_1 *defined in Theorem 4.1 of [33] and* $|\lambda_1| < 1$*.*

Remark 4. *Note that the optimal quadrature formula of the form* (3) *with the coefficients given in Theorem 4 is exact for trigonometric functions* $\sin(x)$ *and* $\cos(x)$*.*

8. The Rate of Convergence of the Optimal Quadrature Formula in the Space $W_2^{(2,0)}(0,1)$

Here, we give some results that show the rate of convergence of the obtained optimal quadrature formula for the case $m = 2$. To obtain this result, we use Theorem 4.

It has to be mentioned that the absolute value of the error (5) of the optimal quadrature Formula (3) in $W_2^{(m,0)}(0,1)$ space is estimated by the Cauchy–Schwarz inequality, as follows:

$$|(\ell,\varphi)| \leq \|\varphi\|_{W_2^{(m,0)}} \cdot \left\|\mathring{\ell}\right\|_{W_2^{(m,0)*}}.$$

Since the norm $\|\varphi\|_{W_2^{(m,0)}}$ of a function φ from the space $W_2^{(m,0)}(0,1)$ is bounded, it is sufficient to calculate the norm $\left\|\mathring{\ell}\right\|_{W_2^{(m,0)*}}$ of the optimal error function that presents the rate of convergence for the obtained optimal quadrature formulas.

The case $m = 2$**.** For the norm of the error functional $\mathring{\ell}$ in the space $W_2^{(2,0)}(0,1)$, the following result was obtained in [33]:

Theorem 5. *The square of the norm of the error functional* (5) *of the optimal quadrature Formula* (3), *on the space* $W_2^{(2,0)}$ *has the form*

$$\begin{aligned}\left\|\mathring{\ell}\right\|^2_{W_2^{(2,0)*}} &= \frac{3-\sin 1}{2} + \frac{(h-1)(h\sin(h)-4\cos(h)+4)-4h}{h(h+\sin(h))} + 2\cot(h) \\ &- \frac{s}{2}\left[\frac{4(\lambda_1-\lambda_1^N)}{1-\lambda_1} + \frac{(\lambda_1+\lambda_1^N)(1+\sin 1)\sin(h)+4(\lambda_1^2+\lambda_1^N)}{\lambda_1^2-2\lambda_1\cos(h)+1}\right. \\ &- \left.\frac{(1-\lambda_1^2)(1-\lambda_1^N)(h\cos(h)-\sin(h))}{(\lambda_1^2-2\lambda_1\cos(h)+1)\sin(h)}\right]\end{aligned}$$

or

$$\left\|\mathring{\ell}\right\|^2_{W_2^{(2,0)*}} = \frac{1}{720}h^4 + O\left(h^5\right), \quad N\to\infty, \tag{77}$$

where $s = \frac{2h\sin(h)(h-\sin(h))}{(h+\sin(h))(h\cos(h)-\sin(h))(1+\lambda_1^N)}$ *and* λ_1 *defined in Theorem 4.1 of [33].*

From here, we can infer that the order of convergence of the optimal quadrature Formula (3) in the space $W_2^{(2,0)}(0,1)$ is $O(h^2)$.

The following theorem gives the asymptotic optimality for our optimal quadrature formula:

Theorem 6. *The optimal quadrature formula of the form* (3) *with the error functional* (5) *in the space* $W_2^{(2,0)}(0,1)$ *is asymptotically optimal in the Sobolev space* $L_2^{(2)}(0,1)$

$$\lim_{N\to\infty} \frac{\|\mathring{\ell}|W_2^{(2,0)*}\|}{\|\mathring{\ell}|L_2^{(2)*}\|} = 1. \tag{78}$$

Theorem 6 was proven in [33].

9. Conclusions

Thus, in this paper, we used the Sobolev method to develop an algorithm for solving a system of algebraic equations that determines the coefficients of quadrature formulas of the form (3). To achieve this, we obtained a discrete analogue $D_m(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}} + 2\frac{d^m}{dx^m} + 1$, (for m even) and used it to solve the system (31)–(33) for $m = 2$. We then obtained explicit expressions for the optimal coefficients $\mathring{C}_\beta$ and used them to construct an optimal quadrature formula of the form (3) in the space $W_2^{(2,0)}$. It is important to note that the optimal quadrature formula of the form (3) in the space $W_2^{(2,0)}$ is exact for the trigonometric functions $\sin(x)$ and $\cos(x)$.

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