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Some Double q-Series by Telescoping

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Abstract: By means of the telescoping method, we derived two general double series formulas that encapsulate the Riemann zeta values $\zeta(s)$, the Catalan constant G, $\log(2)$, π and several other significant mathematical constants.

1. Introduction

Recently, Chu [1], using the telescoping method, obtained the following double series expressions of $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and the Catalan constant $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$:

$$\frac{\zeta(2)}{\lambda^2} = \sum_{i,j=1}^{\infty} \frac{(\lambda i - 1)!(j-1)!}{(\lambda i + j)!},$$

$$8G = \sum_{i,j=1}^{\infty} \frac{(i-1)!(j-1)!}{\left(\frac{1}{2}\right)_{i+j}},$$

where $\lambda \in \mathbb{N}$ and the rising factorial $(x)_n = \Gamma(x+n)/\Gamma(x)$. With the works mentioned above as a source of inspiration, we derived two general double series formulas that encapsulate the Riemann zeta values $\zeta(s)$, the Catalan constant G, $\log(2)$, π and several other significant mathematical constants. We highlight some identities as examples (see Equations (20), (22), (24) and (25) in Section 3.2).

$$\frac{\log(2)}{2} = \sum_{i,j=1}^{\infty} \frac{(2i-2)!(2j-2)!(i+j-1)!}{(i-1)!(j-1)!(2i+2j-1)!},$$

$$4\log(2) - \zeta(2) = \sum_{i,j=1}^{\infty} \frac{4(2i-2)!(2j-2)!(i+j-1)!}{i!(j-1)!(2i+2j-1)!},$$

$$G = \sum_{i,j=1}^{\infty} \frac{\binom{\lambda i+j}{j}}{\binom{2\lambda i+2j}{2j}} \frac{2\lambda i(-1)^{i-1}}{(2i-1)^2(2j-1)},$$

$$\frac{\sqrt{3}}{18}\pi = \sum_{i,j} \frac{(i+j)!i!(2j)!}{(2i+2j)!j!(2j-1)}.$$

Throughout this paper, we assume that $0 < q < 1$. For complex numbers, x, α defined the q-shifted factorial by [2]

$$(x;q)_{\infty} := \prod_{i=0}^{\infty}(1 - xq^i), \quad (x;q)_{\alpha} := \frac{(x;q)_{\infty}}{(xq^{\alpha};q)_{\infty}},$$

where the principal value of q^{α} is taken. For $z \in \mathbb{C}$, we introduce the notation

$$[z]_q := \frac{1 - q^z}{1 - q}.$$

Jackson defined the q-Gamma function $\Gamma_q(x)$ by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}.$$

Thus, we can write

$$(1-q)^{x-1}\Gamma_q(x) = (q;q)_{x-1}.$$

The q-Gamma function satisfies the fundamental functional relation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x). \tag{1}$$

The Gaussian polynomial is the q-analogue of the binomial coefficient. It is defined by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q := \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha-\beta+1)} = \frac{(q^{\alpha-\beta+1};q)_\beta}{(q;q)_\beta}. \tag{2}$$

It is known that q-analogue is a powerful tool that generalizes mathematical expressions by replacing variables with q-deformed ones. One important example of q-analogue is the Gaussian polynomial, which is a q-analogue of the binomial coefficient. The q-binomial coefficient has found applications in many areas, including q-series, combinatorics and algebraic geometry. In algebraic geometry, there exists a close relationship between q-binomial coefficients and Grassmannians.

There have been many recent developments in the study of q-analogue. For example, q-deformed conformal field theory has been studied in [3,4], q-hypergeometric series have been studied in [5] and q-analogue of the Riemann zeta function has been studied in [6,7]. Other recent works on q-analogue can be found in [2,8].

In this paper, we begin with the base of the method of telescoping sums to provide the following two main q-analogue formulas:

Theorem 1. *Let $f_q(z)$ be a q-function of z and m, n, k be any positive integers. Then,*

$$\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{f_q(i)q^j[ni]_q}{[j]_q \begin{bmatrix} ni+j \\ j \end{bmatrix}_q} = \sum_{i=1}^{m} f_q(i)\left(1 - \frac{(q;q)_k}{(q^{ni+1};q)_k}\right), \tag{3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{\begin{bmatrix} ni+j \\ j \end{bmatrix}_{q^2}}{\begin{bmatrix} 2ni+2j \\ 2j \end{bmatrix}_q} \cdot \frac{[2ni]_q f_q(i) q^{2j-1}}{[2j-1]_q} = \sum_{i=1}^{m} f_q(i)\left(1 - \frac{(q;q^2)_k}{(q^{2ni+1};q^2)_k}\right). \tag{4}$$

Let $f(i) = \lim_{q\to 1^-} f_q(i)$ exist and the series $\sum f(i)$ converge. In the above formulas, as k and m tend towards infinity and q approaches 1^-, the resulting equations can be written as follows (see Corollary 2 in Section 2.2 and Corollary 3 in Section 3.2).

$$\sum_{i,j=1}^{\infty} \frac{nif(i)}{j\binom{ni+j}{j}} = \sum_{i=1}^{\infty} f(i),$$

$$\sum_{i,j=1}^{\infty} \frac{\binom{ni+j}{j}}{\binom{2ni+2j}{2j}} \cdot \frac{2ni\,f(i)}{2j-1} = \sum_{i=1}^{\infty} f(i).$$

Hence, we successfully reduce double series to single series, enabling us to quickly obtain results when computing such double series. Consequently, we can derive numerous

elegant representations of classical constants in the form of double series. We list three formulas as examples (see Equations (10) and (11) in Section 2.3 and Equation (27) in Section 3.2).

$$\frac{7}{4}\zeta(3) = \sum_{i,j=1}^{\infty} \frac{O_i}{ij\binom{i+j}{j}},$$

$$7\zeta(3) - 2\pi G = \sum_{i,j=1}^{\infty} \frac{O_{2i}}{ij\binom{i+j}{j}},$$

$$\frac{\pi^2}{48} = \sum_{i,j=1}^{\infty} \frac{\binom{2i+j}{j}}{\binom{4i+2j}{2j}} \cdot \frac{O_i}{(2i+1)(2j-1)},$$

where $O_n = \sum_{k=1}^{n} \frac{1}{2k-1}$.

2. The First q-Formula in Theorem 1

Firstly, we utilize the q-Gamma function to create our lemma that we will employ, primarily relying on the telescoping method.

2.1. Basic Lemma

For any two complex numbers x, y, we define a sequence $Q_j(x, y)$ by

$$Q_j(x,y) := \frac{\Gamma_q(x+j)}{\Gamma_q(y+j)}. \tag{5}$$

This sequence clearly converges and we write the limit as Q_∞, i.e.,

$$Q_\infty := \lim_{j\to\infty} Q_j(x,y) = (1-q)^{y-x}.$$

It is a basic calculation that we have

$$\frac{Q_m}{Q_0} = \frac{(q^x;q)_m}{(q^y;q)_m}, \quad \text{and} \quad \frac{Q_\infty}{Q_0} = \lim_{m\to\infty} \frac{Q_m}{Q_0} = (q^x;q)_{y-x}. \tag{6}$$

Lemma 1. *For any two distinct complex numbers x, y, and a positive integer k, we have*

$$\sum_{j=1}^{k} q^{x+j-1} \frac{\Gamma_q(x+j-1)}{\Gamma_q(y+j)} = \frac{\Gamma_q(x)}{[y-x]_q \Gamma_q(y)} \left(1 - \frac{(q^x;q)_k}{(q^y;q)_k}\right).$$

Proof. Since

$$\begin{aligned}
Q_{j-1}(x,y) - Q_j(x,y) &= \frac{\Gamma_q(x+j-1)}{\Gamma_q(y+j-1)} - \frac{\Gamma_q(x+j)}{\Gamma_q(y+j)} \\
&= \frac{\Gamma_q(x+j-1)}{\Gamma_q(y+j)} \left([y+j-1]_q - [x+j-1]_q\right) \\
&= \frac{\Gamma_q(x+j-1)}{\Gamma_q(y+j)} q^{x+j-1} [y-x]_q,
\end{aligned}$$

we have

$$\sum_{j=1}^{k} q^{x+j-1} \frac{\Gamma_q(x+j-1)}{\Gamma_q(y+j)} = \frac{1}{[y-x]_q} \sum_{j=1}^{k} Q_{j-1}(x,y) - Q_j(x,y)$$

$$= \frac{Q_0 - Q_k}{[y-x]_q} = \frac{Q_0}{[y-x]_q} \left(1 - \frac{Q_k}{Q_0} \right).$$

By using Equations (5) and (6), we can establish the formula we want to prove. $\square$

When k tends to infinity, by imposing the condition $\Re(y) > \Re(x)$ on the summation mentioned above, we arrive at the subsequent outcome.

Corollary 1. *For any two complex numbers* x, y *with* $\Re(y) > \Re(x)$, *we have*

$$\sum_{j=1}^{\infty} q^{x+j-1} \frac{\Gamma_q(x+j-1)}{\Gamma_q(y+j)} = \frac{\Gamma_q(x)}{[y-x]_q \Gamma_q(y)} \left(1 - (q^x; q)_{y-x} \right).$$

2.2. The Proof of the First q-Formula

Moving forward, we will utilize Lemma 1 to deduce our first double q-summation formula, Equation (3).

We rewrite the following sum as

$$\sum_{i=1}^{m} \sum_{j=1}^{k} \frac{f_q(i) q^j [ni]_q}{[j]_q \begin{bmatrix} ni+j \\ j \end{bmatrix}_q} = \sum_{i=1}^{m} f_q(i) [ni]_q \sum_{j=1}^{k} \frac{q^j}{[j]_p \begin{bmatrix} ni+j \\ j \end{bmatrix}_q}.$$

The inner sum can be simplified

$$\sum_{j=1}^{k} \frac{q^j}{[j]_p \begin{bmatrix} ni+j \\ j \end{bmatrix}_q} = \sum_{j=1}^{k} \frac{q^j \Gamma_q(ni+1) \Gamma_q(j+1)}{[j]_p \Gamma_q(ni+j+1)}$$

$$= \sum_{j=1}^{k} \frac{q^j \Gamma_q(ni+1) \Gamma_q(j)}{\Gamma_q(ni+j+1)}.$$

We apply $x = 1$ and $y = ni + 1$ in Lemma 1 and we have

$$\sum_{j=1}^{k} \frac{q^j \Gamma_q(j)}{\Gamma_q(ni+j+1)} = \frac{1}{[ni]_q \Gamma_q(ni+1)} \left(1 - \frac{(q;q)_k}{(q^{ni+1};q)_k} \right).$$

Therefore, the inner sum becomes

$$\frac{1}{[ni]_q} \left(1 - \frac{(q;q)_k}{(q^{ni+1};q)_k} \right).$$

We substitute this result into our original double sums, and then we obtain our desired result.

If we let $k \to \infty$ and $m \to \infty$ in Equation (3), then we have

$$\sum_{i,j=1}^{\infty} \frac{f_q(i) q^j [ni]_q}{[j]_q \begin{bmatrix} ni+j \\ j \end{bmatrix}_q} = \sum_{i=1}^{\infty} f_q(i) (1 - (q;q)_{ni}), \tag{7}$$

if both the series on the left and right sides of the equation converge.

Corollary 2. *Let $f(i) = \lim_{q \to 1^-} f_q(i)$ exist and the series $\sum f(i)$ converge. Then, for any positive integer n, we have*

$$\sum_{i,j=1}^{\infty} \frac{nif(i)}{j\binom{ni+j}{j}} = \sum_{i=1}^{\infty} f(i). \tag{8}$$

2.3. Examples of the First Formula

We provide some applications. Let $f(i) = 1/i^2$ and $n = \lambda$ in Equation (8). Then,

$$\zeta(2) = \sum_{i=1}^{\infty} \frac{1}{i^2} = \sum_{i,j=1}^{\infty} \frac{\lambda}{ij\binom{\lambda i+j}{j}}.$$

This equation appeared in [1], Theorem 4. Based on some well-known results ([9], Equations (15) and (19)):

$$\sum_{k=1}^{\infty} \frac{\psi(\frac{1}{2} \pm k) - \psi(\frac{1}{2})}{k^2} = \frac{7}{2}\zeta(3), \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\left(\psi(\frac{1}{2} \pm k) - \psi(\frac{1}{2})\right) = 2\pi G - \frac{7}{2}\zeta(3).$$

We know that $\psi(\frac{1}{2} \pm k) - \psi(\frac{1}{2}) = 2O_k$. Thus, we have

$$\sum_{i=1}^{\infty} \frac{O_i}{i^2} = \frac{7}{4}\zeta(3), \quad \sum_{i=1}^{\infty} \frac{O_{2i}}{i^2} = 7\zeta(3) - 2\pi G.$$

We list three more identities in the following.

$$2\zeta(3) = \sum_{i=1}^{\infty} \frac{H_i}{i^2} = \sum_{i,j=1}^{\infty} \frac{nH_i}{ij\binom{ni+j}{j}}, \tag{9}$$

$$\frac{7}{4}\zeta(3) = \sum_{i=1}^{\infty} \frac{O_i}{i^2} = \sum_{i,j=1}^{\infty} \frac{nO_i}{ij\binom{ni+j}{j}}, \tag{10}$$

$$7\zeta(3) - 2\pi G = \sum_{i=1}^{\infty} \frac{O_{2i}}{i^2} = \sum_{i,j=1}^{\infty} \frac{nO_{2i}}{ij\binom{ni+j}{j}}, \tag{11}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the harmonic number. Let $n = 1$ in Equation (9); this provides the identity appearing in [1], Corollary 7:

$$2\zeta(3) = \sum_{i,j=1}^{\infty} \frac{(i-1)!(j-1)!H_i}{(i+j)!}.$$

Aside from the mentioned applications, we can also consider the case of finite summations. Simply by taking the q parameter close to 1^- in Equation (3), we arrive at the following formula.

$$\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{nif(i)}{j\binom{ni+j}{j}} = \sum_{i=1}^{m} f(i)\left(1 - \frac{1}{\binom{ni+k}{k}}\right). \tag{12}$$

Since ([10], Equation (5.9))

$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n},$$

we let $f(i) = \binom{k+i}{i}$ and $n = 1$ in Equation (12). We obtain

$$\sum_{i=1}^{m}\sum_{j=1}^{k} \frac{i\binom{k+i}{i}}{j\binom{i+j}{j}} = \sum_{i=1}^{m}\left(\binom{k+i}{i} - 1\right) = \binom{k+1+m}{m} - 1 - m. \tag{13}$$

The following result is easily obtained from [11], Proposition 1:

$$\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{k+n}{k}} = \sum_{k=1}^{\infty} \frac{1}{(k+n)^2}.$$

Taking $m \to \infty$ with $n = 1$ in Equation (12) and $f(i) = 1/i^2$, we have

$$\sum_{i=1}^{\infty} \sum_{j=1}^{k} \frac{1}{ij\binom{i+j}{j}} = \sum_{i=1}^{\infty} \left(\frac{1}{i^2} - \frac{1}{i^2 \binom{i+k}{k}} \right)$$

$$= \sum_{i=1}^{\infty} \frac{1}{i^2} - \frac{1}{(i+k)^2} \quad = \quad H_k^{(2)}, \tag{14}$$

where $H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2}$.

3. The Second q-Formula in Theorem 1

3.1. The Proof of the Second Formula

Let A be the left-hand side of Equation (4)

$$A := \sum_{i=1}^{m} \sum_{j=1}^{k} \frac{\begin{bmatrix} ni+j \\ j \end{bmatrix}_{q^2}}{\begin{bmatrix} 2ni+2j \\ 2j \end{bmatrix}_q} \cdot \frac{[2ni]_q f_q(i) q^{2j-1}}{[2j-1]_q}.$$

Rewrite A by using Equation (2):

$$A = \sum_{i=1}^{m} \sum_{j=1}^{k} \frac{\Gamma_{q^2}(ni+j+1)\Gamma_q(2j+1)\Gamma_q(2ni+1)[2ni]_q q^{2j-1} h_q(i)}{\Gamma_q(2ni+2j+1)\Gamma_{q^2}(j+1)\Gamma_{q^2}(ni+1)[2j-1]_q}.$$

By using Equation (1), we simplify the factor

$$\frac{\Gamma_{q^2}(ni+j+1)\Gamma_q(2j+1)\Gamma_q(2ni+1)}{\Gamma_q(2ni+2j+1)\Gamma_{q^2}(j+1)\Gamma_{q^2}(ni+1)}$$

as the following

$$\frac{\Gamma_{q^2}(ni+j)\Gamma_q(2j)\Gamma_q(2ni)}{\Gamma_q(2ni+2j)\Gamma_{q^2}(j)\Gamma_{q^2}(ni)} \cdot \frac{[ni+j]_{q^2}[2j]_q[2ni]_q}{[2ni+2j]_q[j]_{q^2}[ni]_{q^2}} = \frac{\Gamma_{q^2}(ni+j)\Gamma_q(2j)\Gamma_q(2ni)}{\Gamma_q(2ni+2j)\Gamma_{q^2}(j)\Gamma_{q^2}(ni)} \cdot [2]_q.$$

Therefore, A becomes

$$A = \sum_{i=1}^{m} \sum_{j=1}^{k} \frac{\Gamma_{q^2}(ni+j)\Gamma_q(2j)\Gamma_q(2ni)[2ni]_q q^{2j-1}(q+1)h_q(i)}{\Gamma_q(2ni+2j)\Gamma_{q^2}(j)\Gamma_{q^2}(ni)[2j-1]_q}$$

$$= \sum_{i=1}^{m} \frac{\Gamma_q(2ni)[2ni]_q(1+q)h_q(i)}{\Gamma_{q^2}(ni)} \sum_{j=1}^{k} \frac{\Gamma_{q^2}(ni+j)\Gamma_q(2j-1)q^{2j-1}}{\Gamma_q(2ni+2j)\Gamma_{q^2}(j)}.$$

Let B_k be the above inner sum, that is,

$$B_k := \sum_{j=1}^{k} \frac{\Gamma_{q^2}(ni+j)\Gamma_q(2j-1)q^{2j-1}}{\Gamma_q(2ni+2j)\Gamma_{q^2}(j)}.$$

A q-analogue of Legendre duplication formula for the Gamma function [2] is

$$\Gamma_q(2x)\Gamma_{q^2}\left(\frac{1}{2}\right) = \Gamma_{q^2}(x)\Gamma_{q^2}\left(x+\frac{1}{2}\right)(1+q)^{2x-1}. \tag{15}$$

We set $x = j - \frac{1}{2}$ and $x = ni + j$, respectively, and then we obtain

$$\frac{\Gamma_q(2j-1)}{\Gamma_{q^2}(j)} = \frac{\Gamma_{q^2}\left(j-\frac{1}{2}\right)}{\Gamma_{q^2}\left(\frac{1}{2}\right)}(1+q)^{2j-2} \tag{16}$$

and $\quad \dfrac{\Gamma_{q^2}(ni+j)}{\Gamma_q(2ni+2j)} = \dfrac{\Gamma_{q^2}\left(\frac{1}{2}\right)}{\Gamma_{q^2}\left(ni+j+\frac{1}{2}\right)}(1+q)^{1-2ni-2j}. \tag{17}$

Substituting Equations (16) and (17) into B_k, we have

$$B_k = \sum_{j=1}^{k} \frac{\Gamma_{q^2}\left(j-\frac{1}{2}\right)q^{2j-1}}{\Gamma_{q^2}\left(ni+j+\frac{1}{2}\right)(1+q)^{2ni+1}}.$$

We let $x = \frac{1}{2}$, $y = ni + \frac{1}{2}$ and replace q with q^2 in Lemma 1; we can rewrite B_k as follows.

$$B_k = \frac{\Gamma_{q^2}\left(\frac{1}{2}\right)}{[ni]_{q^2}\Gamma_{q^2}\left(ni+\frac{1}{2}\right)(1+q)^{2ni+1}}\left(1 - \frac{(q;q^2)_k}{(q^{2ni+1};q^2)_k}\right).$$

Thus,

$$A = \sum_{i=1}^{m} \frac{\Gamma_q(2ni)[2ni]_q h_q(i)}{\Gamma_{q^2}(ni)(1+q)^{2ni}} \frac{\Gamma_{q^2}\left(\frac{1}{2}\right)}{[ni]_{q^2}\Gamma_{q^2}\left(ni+\frac{1}{2}\right)}\left(1 - \frac{(q;q^2)_k}{(q^{2ni+1};q^2)_k}\right).$$

We use Equation (15) again with $x = ni$; therefore,

$$\frac{\Gamma_q(2ni)\Gamma_{q^2}\left(\frac{1}{2}\right)}{\Gamma_{q^2}(ni)\Gamma_{q^2}\left(ni+\frac{1}{2}\right)} = (1+q)^{2ni-1}.$$

Substituting this result into the representation of A, we have

$$A = \sum_{i=1}^{m} \frac{[2ni]_q h_q(i)}{(1+q)[ni]_{q^2}}\left(1 - \frac{(q;q^2)_k}{(q^{2ni+1};q^2)_k}\right) = \sum_{i=1}^{m} h_q(i)\left(1 - \frac{(q;q^2)_k}{(q^{2ni+1};q^2)_k}\right).$$

Hence, we have obtained the expected equation and conclude the proof.

If we let $k \to \infty$ and $m \to \infty$ in Equation (4), then we have

$$\sum_{i,j=1}^{\infty} \frac{\left[\begin{array}{c} ni+j \\ j \end{array}\right]_{q^2}}{\left[\begin{array}{c} 2ni+2j \\ 2j \end{array}\right]_q} \frac{[2ni]_q h_q(i) q^{2j-1}}{[2j-1]_q} = \sum_{i=1}^{\infty} h_q(i)\left(1 - (q;q^2)_{ni}\right) \tag{18}$$

if both the series on the left and right sides of the equation converge.

Corollary 3. *Let $h(i) = \lim_{q \to 1^-} h_q(i)$ exist and the series $\sum h(i)$ converge. Then, for any positive integer n, we have*

$$\sum_{i,j=1}^{\infty} \frac{\binom{ni+j}{j}}{\binom{2ni+2j}{2j}} \cdot \frac{2ni\, h(i)}{2j-1} = \sum_{i=1}^{\infty} h(i). \tag{19}$$

3.2. Examples of the Second Formula

We provide some applications. Let $n = 1$ and $h(i) = \frac{1}{(2i)(2i-1)}$ in Equation (19) and we obtain a symmetric double series:

$$\frac{\log(2)}{2} = \sum_{i=1}^{\infty} \frac{1}{(4i)(2i-1)} = \sum_{i,j=1}^{\infty} \frac{(2i-2)!(2j-2)!(i+j-1)!}{(i-1)!(j-1)!(2i+2j-1)!}. \tag{20}$$

Moreover, let $n = 1$ and $h(i) = \frac{1}{i^s(2i-1)}$ in Equation (19), where $s \in \mathbb{N}$. Since the partial fraction decomposition is

$$\frac{1}{i^s(2i-1)} = \frac{2^{s-1}}{i(2i-1)} - \sum_{k=2}^{s} \frac{2^{s-k}}{i^k},$$

we have

$$2^s \log(2) - \sum_{k=2}^{s} 2^{s-k} \zeta(k) = \sum_{i,j=1}^{\infty} \frac{\binom{i+j}{j}}{\binom{2i+2j}{2j}} \frac{2\, i^{1-s}}{(2i-1)(2j-1)}. \tag{21}$$

The following are the formulas with $s = 2, 3$:

$$4\log(2) - \zeta(2) = \sum_{i,j=1}^{\infty} \frac{4(2i-2)!(2j-2)!(i+j-1)!}{i!(j-1)!(2i+2j-1)!}, \tag{22}$$

$$8\log(2) - 2\zeta(2) - \zeta(3) = \sum_{i,j=1}^{\infty} \frac{4(2i-2)!(2j-2)!(i+j-1)!}{i \cdot i!(j-1)!(2i+2j-1)!}. \tag{23}$$

Building upon the known series expansions for G and π, we can formulate a double series expansion that specifically represents these constants. For example, we use the power seies expansion

$$\frac{2x \arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)! n} x^{2n}.$$

Setting $x = 1/2$ in the above series, we have

$$\frac{\pi}{3\sqrt{3}} = \sum_{n=1}^{\infty} \frac{n! n!}{n(2n)!}.$$

Thus, if we let $h(i) = \frac{i! i!}{i(2i)!}$ and $n = 1$ in Equation (19), then

$$\frac{\sqrt{3}\pi}{18} = \sum_{i,j=1}^{\infty} \frac{(i+j)! i!(2j)!}{(2i+2j)! j!(2j-1)}. \tag{24}$$

Also, if we let $h(i) = \frac{(-1)^{i-1}}{(2i-1)^2}$ in Equation (19), we obtain a double series representation for the Catalan constant G:

$$G = \sum_{i,j=1}^{\infty} \frac{\binom{ni+j}{j}}{\binom{2ni+2j}{2j}} \frac{2(-1)^{i-1}ni}{(2i-1)^2(2j-1)}. \tag{25}$$

Aliev and Dil [12] proved that

$$\frac{\zeta(2)}{4} = \sum_{k=1}^{\infty} \frac{O_k}{2k(2k+1)}.$$

Using Equation (19) with $h(i) = \frac{O_i}{2i(2i+1)}$, we have

$$\frac{\pi^2}{24} = \sum_{i,j=1}^{\infty} \frac{\binom{ni+j}{j}}{\binom{2ni+2j}{2j}} \cdot \frac{nO_i}{(2i+1)(2j-1)}. \tag{26}$$

Indeed, we can derive

$$\sum_{n=1}^{\infty} \frac{O_n}{(2n)(2n-1)} = \frac{\pi^2}{12}$$

using the method outlined in [13]. Moreover, by substituting $h(i) = \frac{O_i}{(2i)(2i-1)}$ and $n = 1$ into Equation (19), we obtain the following equation, which exhibits enhanced symmetry:

$$\frac{\pi^2}{12} = \sum_{i,j=1}^{\infty} \frac{\binom{i+j}{j}}{\binom{2i+2j}{2j}} \cdot \frac{O_i}{(2i-1)(2j-1)}. \tag{27}$$

Lastly, we would like to emphasize that, by taking the q parameter towards 1^- in Equation (4), we obtain a concise summation formula that can be applied

$$\sum_{i=1}^{m} \sum_{j=1}^{k} \frac{\binom{ni+j}{j}}{\binom{2ni+2j}{2j}} \cdot \frac{2nih(i)}{2j-1} = \sum_{i=1}^{m} h(i)\left(1 - \frac{1}{\binom{2ni+k}{k}}\right). \tag{28}$$

4. Conclusions

In this paper, our focus is on demonstrating the effective application of the "telescoping method" in handling summation expressions of q-series (ref. Equations (3) and (4)). Specifically, we are primarily concerned with the summation of finite series (ref. Equations (12), (13) and (28)) or infinite series (ref. Equations (8) and (19)) that involve coefficients represented by binomial coefficients.

Utilizing the telescoping method, we derived two general double series formulas that encompass notable mathematical constants, including the Riemann zeta values $\zeta(s)$ (ref. Equations (9), (10) and (21)–(23)), the Catalan constant G (ref. Equations (11) and (25)), $\log(2)$ (ref. Equations (20) and (21)), π (ref. Equations (11) and (24)) and various other significant mathematical constants (ref. Equations (26) and (27)).

Interestingly, there are still many intriguing double series worth exploring, such as the work by Aliev and Dil [12]:

$$\sum_{n,m \geq 1} \frac{H_{n+m}}{nm(n+m)} = 6\zeta(4),$$

or the double series for π established by Wei [8], initially conjectured by Guo and Lian [14]:

$$\frac{\pi}{12} = \sum_{k=1}^{\infty} (6k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3 4^k} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} - \frac{1}{16j^2}.$$

These equations, among others, present intriguing topics worthy of investigation. Of course, since they were derived using different approaches, it would be fascinating to obtain similar equations using the telescoping method.

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