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On Sub Convexlike Optimization Problems

Renying Zeng

Mathematics Department, Saskatchewan Polytechnic, Saskatoon, SK S7L 4J7, Canada; renying.zeng@saskpolytech.ca

Abstract: In this paper, we show that the sub convexlikeness and subconvexlikeness defined by V. Jeyakumar are equivalent in locally convex topological spaces. We also deal with set-valued vector optimization problems and obtained vector saddle-point theorems and vector Lagrangian theorems.

Keywords: locally convex topological space; subconvexlikeness; sub convexlikeness; vector Lagrangian multiplier theorems; vector saddle-point theorems

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1. Introduction

Generalized convex optimization are very well-studied branches of mathematics. There are many very meaningful and useful definitions of generalized convexities. Let X be a normed space, and X_+ a convex cone of X. K. Fan [1] introduced the definition of X_+-convexlike function. Jeyakumar [2] introduced the definition of X_+-sub convexlike function and defined X_+-subconvexlike function in [3]. There are many research articles discussing subconvexlike optimization problems, e.g., see [4–8]. In this paper, by using partial order relations and the absorbing property of bounded convex sets in locally convex topological spaces [9], we proved that the sub convexlikeness introduced in [2] and subconvexlikeness in [3] are equivalent in locally convex topological spaces (including normed linear spaces).

Most papers in set-valued optimization studied the problem with inequality and abstract constraints. In this paper, we consider the set-valued optimization problem with not only inequality and abstract, but also equality constraints. The explicit statement of the equality constraint is very convenient in various applications. For example, recently, mathematical programs with equilibrium constraints have received considerable attention from the optimization community. The mathematical programs with equilibrium constraints are a class of optimization problem with variational inequality constraints. By representing the variational inequality as a generalized equation, e.g., [10–12], a mathematical program with equilibrium constraints can be reformulated as an optimization problem with an equality constraint. This paper works with a set-valued optimization problem with inequality, equality as well as abstract constraints. By using the separation theorem for convex sets, we extend or modify some results (theorems of alternatives, saddle-points theorems and Lagrangian theorems) in [4,7,8,10,13–15] to vector optimization problems with weakened convexities.

2. Preliminary

Let X be a real topological vector space; a subset X_+ of X is said to be a convex cone if

$$\alpha x^1 + \beta x^2 \in X_+, \forall x^1, x^2 \in X_+,\ \forall \alpha, \beta \geq 0.$$

We denoted by 0_X the zero element in the topological space X and simply by 0 if there is no confusion.

A convex cone X_+ of X is called a pointed cone if $X_+ \cap (-X_+) = \{0\}$.

A real topological vector space X with a pointed cone is said to be an ordered topological liner space. We denote $intX_+$ the topological interior of X_+. The partial order on X is defined by

$$x^1 \prec_{X_+} x^2, \text{ if } x^1 - x^2 \in X_+,$$

$$x^1 \prec\prec_{X_+} x^2, \text{ if } x^1 - x^2 \in int\, X_+.$$

Alternatively, if there is no confusion, they may just be denoted by

$$x^1 \prec x^2, \text{ if } x^1 - x^2 \in X_+,$$

$$x^1 \prec\prec x^2, \text{ if } x^1 - x^2 \in int\, X_+.$$

If $A, B \subseteq X$, we denoted by

$$A \prec_{X_+} B, \text{ if } x \prec_{X_+} y \text{ for } \forall x \in A, \forall y \in B,$$

$$A \prec\prec_{X_+} B, \text{ if } x \prec\prec_{X_+} y \text{ for } \forall x \in A, \forall y \in B.$$

Alternatively,

$$A \prec B, \text{ if } x \prec y \text{ for } \forall x \in A, \forall y \in B,$$

$$A \prec\prec B, \text{ if } x \prec\prec y \text{ for } \forall x \in A, \forall y \in B.$$

A linear functional on X is a continuous linear function from X to R (1-dimensinal Euclidean space). The set X^* of all linear functionals on X is the dual space of X. The subset

$$X_+^* = \{\xi \in X^* : \langle x, \xi \rangle \geq 0, \forall x \in X_+\}.$$

of X^* is said to be the dual cone of the cone X_+, where $\langle x, \xi \rangle = \xi(x)$.

Suppose that X and Y are two real topological vector spaces. Let f: $X \to 2^Y$ be a set-valued function, where 2^Y denotes the power set of Y.

Let D be a nonempty subset of X. We set $f(D) = \cup_{x \in D} f(x)$ and

$$\langle f(x), \eta \rangle = \{\langle y, \eta \rangle : y \in f(x)\},$$

$$\langle f(D), \eta \rangle = \cup_{x \in D} \langle f(x), \eta \rangle.$$

For $x \in D,\ \eta \in Y^*$, we wrote

$$\langle f(x), \eta \rangle \geq 0, \text{ if } \langle y, \eta \rangle \geq 0,\ \forall y \in f(x),$$

$$\langle f(D), \eta \rangle \geq 0, \text{ if } \langle f(x), \eta \rangle \geq 0,\ \forall x \in D.$$

The following Definitions 1 and 2 can be found in [4].

Definition 1 (convex, bounded, and absorbing)**.** *A subset M of X is said to be convex, if $x_1, x_2 \in M$ and $0 < \alpha < 1$ implies $\alpha x_1 + (1 - \alpha)x_2 \in M$; M is said to be balanced if $x \in M$ and $|\alpha| \leq 1$ implies $\alpha x \in M$; M is said to be absorbing if, for any given neighborhood U of 0, there exists a positive scalar β, such that $\beta^{-1}M \subseteq U$, where $\beta^{-1}M = \{x \in X; x = \beta^{-1}v; v \in M\}$.*

Definition 2 (locally convex topological space)**.** *A topological vector space X is called a locally convex topological space if any neighborhood of 0_X contains a convex, balanced and absorbing open set.*

From [9], p. 26 Theorem, p. 33 Definition 1, a normed linear space is a locally convex topological space.

3. The Sub Convexlikeness

This section shows that the definitions of sub convexlikeness and subconvexlikeness provided by Jeyakumar [2,3] are actually one.

A set-valued function f: $X \to 2^Y$ is said to be Y_+-convex on D if $\forall x^1$, $x^2 \in D$, $\forall \alpha \in [0, 1]$; one has

$$\alpha\, f\left(x^1\right) + (1-\alpha)\, f\left(x^2\right) \prec_{Y_+} f\left(\alpha\, x^1 + (1-\alpha)\, x^2\right).$$

The following definition of convexlikeness was introduced by Ky Fan [1].

A set-valued function f: $X \to 2^Y$ is said to be Y_+-convexlike on D if $\forall x^1$, $x^2 \in D$, $\forall \alpha \in [0, 1]$, $\exists x^3 \in D$ such that

$$\alpha\, f\left(x^1\right) + (1-\alpha)\, f\left(x^2\right) \prec_{Y_+} f\left(x^3\right).$$

Jeyakumar [9] introduced the following subconvexlikeness.

Definition 3 (subconvexlike)**.** *Let Y be a topological vector space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in Y. A set-valued map f: $D \to 2^Y$ is said to be Y_+-subconvexlike on D if $\exists \theta \in intY_+$, such that $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$ holds*

$$\varepsilon\theta + \alpha f(x_1) + (1-\alpha) f(x_2) \prec_{Y_+} f(x_3).$$

Lemma 1 is Lemma 2.3 in [14].

Lemma 1. *Let Y be a topological vector space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in Y. A set-valued map f: $D \to 2^Y$ is Y_+-sub-convex-like on D if, and only if, $\forall \theta \in intY_+$, $\forall x_1, x_2 \in D$, $\forall \alpha \in [0,1]$, $\exists x_3 \in D$, such that*

$$\theta + \alpha f(x_1) + (1-\alpha) f(x_2) \prec_{Y_+} f(x_3).$$

A bounded function in a topological space can be defined as following Definition 4 (e.g., see Yosida [9]).

Definition 4 (bounded set-valued map)**.** *A subset M of a real topological vector space Y is said to be a bounded subset if, for any given neighborhood U of 0, there exists a positive scalar β such that $\beta^{-1}M \subseteq U$, where $\beta^{-1}M = \{y \in Y; y = \beta^{-1}v; v \in M\}$. A set-valued map f: $D \to Y$ is said to bounded map if $f(Y)$ is a bounded subset of Y.*

Jeyakumar [2] introduced the following sub convexlikeness.

Definition 5 (sub convexlike)**.** *Let Y be a topological vector space and $D \subseteq X$ be a nonempty set. A set-valued map f: $D \to 2^Y$ is said to be Y_+-sub convexlike on D if $\exists$bounded set-valued map u: $D \to Y$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, $\forall \alpha \in [0, 1]$, $\exists x_3 \in D$, such that*

$$\varepsilon u + \alpha f(x_1) + (1-\alpha) f(x_2) \prec_{Y_+} f(x_3).$$

Lemma 2. *Let Y be a locally convex topological space and $D \subseteq X$ be a nonempty set Y. A set-valued map f: $D \to 2^Y$ is Y_+-sub-convex-like on D if, and only if, $f(D) + intY_+$ is Y_+-convex.*

Theorem 1. *Let Y be a locally convex topological space, $D \subseteq X$ a nonempty set, and Y_+ a convex cone in Y. A set-valued map f: $D \to 2^Y$ is Y_+-sub convexlike on D if, and only if, $f(D) + intY_+$ is Y_+-convex.*

Proof. The necessity.

Suppose that f is Y_+-sub convexlike.

$\forall z_1 = y_1 + y_0^1$, $z_2 = y_2 + y_0^2 \in f(D) + intY_+$, $\exists x_1, x_2 \in D$, such that $y_1 \in f(x_1)$, $y_2 \in f(x_2)$. Let

$$y_0 = \alpha y_0^1 + (1-\alpha)y_0^2,$$

Then, $y_0 \in intY_+$. Therefore, $\exists$ neighborhood U of 0, such that $y_+^0 + U$ is a neighborhood of y_+^0 and

$$y_+^0 + U \subseteq intY_+.$$

By Definition 2, we may assume that U is convex, balanced and absorbing.

From the assumption of sub convexlikeness, i.e., $\exists$ bounded set-valued map u:$x_1, x_2 \in D$, $\varepsilon > 0$, $\alpha \in [0,\ 1]$, $\exists x_3 \in D$, such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \subseteq f(x_3) + Y_+.$$

Therefore,

$$\begin{aligned}&\alpha z_1 + (1-\alpha)z_2\\ &= \alpha y_1 + (1-\alpha)y_2 + \alpha y_+^1 + (1-\alpha)y_+^2\\ &\subseteq f(x_3) - \varepsilon u + Y_+ + y_+^0\end{aligned}$$

Since U is convex, balanced and absorbing, we may take $\varepsilon > 0$ to be small enough, such that

$$-\varepsilon u \subseteq U.$$

Therefore,

$$-\varepsilon u + y_+^0 \subseteq y_+^0 + U \subseteq intY_+.$$

Then,

$$\begin{aligned}&\alpha z_1 + (1-\alpha)z_2\\ &= \alpha y_1 + (1-\alpha)y_2 + \alpha y_+^1 + (1-\alpha)y_+^2\\ &\subseteq f(x_3) + intY_+\\ &\subseteq f(D) + intY_+.\end{aligned}$$

Hence, $f(D) + intY_+$ is a Y_+-convex set.

The sufficiency.

If $f(D) + intY_+$ is Y_+-convex, then, by Lemma 1, f is Y_+-subconvexlike. It is clear that Y_+-subconvexlikeness implies Y_+-sub convexlikeness. □

From Lemma 2 and Theorem 1, we obtained Theorem 2.

Theorem 2. *Let Y be a locally convex topological space, $D \subseteq X$ a nonempty set and Y_+ a convex cone in Y. A set-valued map f: $D \to 2^Y$ is Y_+-subconvexlike on D if, and only if, f is Y_+-sub convexlike on D.*

4. Vector Saddle-Point Theorems

This section presents vector saddle-point theorems for set-valued optimization problems.

A set-valued map f: $D \to 2^Y$ is said to be affine on D if $\forall x_1, x_2 \in D$, $\forall \beta \in R$; therefore,

$$\beta f(x_1) + (1-\beta)f(x_2) = f(\beta x_1 + (1-\beta)x_2.$$

We introduced the notion of sub affinelike functions, as follows.

Definition 6 (sub affinelike)**.** *A set-valued map f: $D \to 2^Y$ is said to be Y_+-sub affinelike on D if $\forall x_1, x_2 \in D$, $\forall \alpha \in (0,1)$, $\exists\, v \in intY_+$, $\exists x_3 \in D$; therefore,*

$$v + \alpha f(x_1) + (1-\alpha) f(x_2) = f(x_3).$$

Theorem 3. *Let X, Y, Z and W be real topological vector spaces, $D \subseteq X$. Y_+, Z_+ and W_+ are pointed convex cones of Y, Z and W, respectively. Assume that the functions $f : D \to Y$, $g : D \to Z$, $h : D \to W$ satisfy:*

(a) *f and g are sub convexlike maps on D, i.e., $\forall u_1 \in intY_+, \forall u_2 \in intZ_+, \forall \alpha \in (0,\ 1), \forall x^1, x^2 \in D$, $\exists x', x'' \in D$, such that*

$$u_1 + \alpha f\left(x^1\right) + (1-\alpha) f\left(x^2\right) \prec f(x'),$$
$$u_2 + \alpha g\left(x^1\right) + (1-\alpha) g\left(x^2\right) \prec g(x'');$$

(b) *h is a sub affinelike map on D, i.e., $\forall \alpha \in (0,1), \forall x^1, x^2 \in D, \exists x''' \in D, v \in intW_+$ such that*

$$v + \alpha h\left(x^1\right) + (1-\alpha) h\left(x^2\right) = h(x''');$$

(c) *$int\, h(D) \neq \varnothing$;*

(i) and (ii) denote the system:

(i) $\exists x \in D,\ s.t.,\ f(x) \prec\prec 0,\ g(x) \prec 0,\ h(x) = 0;$

(ii) $\exists (\xi, \eta, \zeta) \in (Y_+^* \times Z_+^* \times W^*) \backslash \{(0_Y, 0_Z, 0_W)\}$ *such that*

$$\xi(f(x)) + \eta(g(x)) + \varsigma(h(x)) \geq 0,\ \forall x \in D.$$

If (i) has no solutions, then (ii) has solutions.
Moreover, if (ii) has a solution (ξ, η, ς) with $\xi \neq 0_{Y^*}$, then (i) has no solutions.

Proof. $\forall w_1, w_2 \in \bigcup_{t>0} th(D) + intW_+, \forall \alpha \in (0,1), \exists x_1, x_2 \in D, \exists b_1, b_2 \in intW_+, \exists t_1, t_2 > 0$, such that

$$\begin{aligned}
&\alpha w_1 + (1-\alpha) w_2 \\
&= \alpha t_1 h(x_1) + (1-\alpha) t_2 h(x_2) + \alpha b_1 + (1-\alpha) b_2 \\
&= (\alpha t_1 + (1-\alpha) t_2)\left[\tfrac{\alpha t_1}{\alpha t_1 + (1-\alpha) t_2} h(x_1) + \tfrac{(1-\alpha) t_2}{\alpha t_1 + (1-\alpha) t_2} h(x_2)\right] + \alpha b_1 + (1-\alpha) b_2.
\end{aligned}$$

By the assumption (b), $\exists x_3 \in D, \exists v \in intW_+, \forall \varepsilon > 0$, such that

$$\frac{\alpha t_1}{\alpha t_1 + (1-\alpha) t_2} h(x_1) + \frac{(1-\alpha) t_2}{\alpha t_1 + (1-\alpha) t_2} h(x_2) = h(x_3) - \varepsilon v.$$

Since $v \in intZ_+$, $\exists$ neighborhood U of 0 in W for which $V = \alpha b_1 + (1-\alpha) b_2 + U$ is a neighborhood of $\alpha b_1 + (1-\alpha) b_2$.

By Definition 2, we may take $\varepsilon > 0$ to be small enough, such that

$$-\varepsilon(\alpha t_1 + (1-\alpha) t_2) v \subseteq U.$$

Then,

$$\alpha b_1 + (1-\alpha) b_2 - \varepsilon(\alpha t_1 + (1-\alpha) t_2) v \subseteq V \subseteq intW_+.$$

Therefore,

$$\begin{aligned}&\alpha w_1 + (1-\alpha)w_2\\ &\subseteq \alpha t_1 h(x_1) + (1-\alpha)t_2 h(x_2) + \alpha b_1 + (1-\alpha)b_2\\ &= (\alpha t_1 + (1-\alpha)t_2)\left[\tfrac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} h(x_1) + \tfrac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} h(x_2)\right] + \alpha b_1 + (1-\alpha)b_2\\ &= (\alpha t_1 + (1-\alpha)t_2)h(x_3) + \alpha b_1 + (1-\alpha)b_2 - \varepsilon(\alpha t_1 + (1-\alpha)t_2)v\\ &\subseteq \cup_{t>0} th(D) + intW_+.\end{aligned}$$

So, $\bigcup_{t>0} th(D) + intY_+$, is a convex set.

Similarly, $\bigcup_{t>0} tf(D) + intY_+$, and $\bigcup_{t>0} tg(D) + intZ_+$ are also convex. Therefore, the set

$$C = \left(\bigcup_{t>0} tf(D) + intY_+\right) \times \left(\bigcup_{t>0} tg(D) + intZ_+\right) \times \left(\bigcup_{t>0} th(D) + intW_+\right)$$

is convex.

From assumption (c), $intC \neq \varnothing$. We also have $(0_Y, 0_Z, 0_W) \notin B$ since (i) has no solution. Therefore, according to the separation theorem of convex sets of topological vector space, $\exists$ nonzero vector $(\xi, \eta, \varsigma) \in Y^* \times Z^* \times W^*$, such that

$$\xi\left(t_1 f(x) + y^0\right) + \eta\left(gt_2(x) + z^0\right) + \varsigma\left(t_3 h(x) + w^0\right) \geq 0,$$

for $\forall t_1, t_2, t_3 > 0, \forall x \in D, \forall y^0 \in intY_+, \forall z^0 \in intZ_+, \forall w^0 \in B$.

Since $intY_+, intZ_+$ are convex cones, and B is a linear space, we obtained

$$\xi\left(t_1 f(x) + \lambda_1 y^0\right) + \eta\left(t_2 g(x) + \lambda_2 z^0\right) + \varsigma\left(t_3 h(x) + \lambda_3 w^0\right) \geq 0$$

$$\forall x \in D, \forall y^0 \in intY_+, \forall z^0 \in intZ_+, \forall w^0 \in B, \forall \lambda_i > 0, (i = 1,2,3), \forall t_i > 0, (i = 1,2,3).$$

Let $\lambda_i \to 0$ $(i = 2,3), t_i \to 0$ $(i = 1,2,3)$; therefore,

$$\xi(y^0) \geq 0, \forall y^0 \in intY_+.$$

Therefore, $\xi(y) \geq 0, \forall y \in Y_+$. Hence, $\xi \in Y_+^*$. Similarly, $\eta \in intZ_+, \varsigma \in intW_+$.
Thus,

$$(\xi, \eta, \varsigma) \in Y_+^* \times Z_+^* \times W^*.$$

Therefore,

$$\xi(f(x)) + \eta(g(x)) + \varsigma(h(x)) \geq 0,\ x \in D,$$

which means that (ii) has solutions.

On the other hand, suppose that (ii) has a solution (ξ, η, ς) with $\xi \neq 0_{Y_+^*}$, i.e.,

$$\xi(f(x)) + \eta(g(x)) + \varsigma(h(x)) \geq 0,\ x \in D.$$

We are going to prove that (i) has no solution.

Otherwise, if (i) has a solution $\widetilde{x} \in D$, then $f(\widetilde{x}) \prec\prec 0,\ g(\widetilde{x}) \leq 0,\ h(\widetilde{x}) = 0$. Hence, one would have

$$\xi(f(\widetilde{x})) + \eta(g(\widetilde{x})) + \varsigma(h(\widetilde{x})) < 0,$$

which is a contradiction. The proof is completed. □

We considered the following optimization problem with set-valued maps:

$$(VP) \qquad Y_+\text{-min} \qquad f\,(x)$$

$$s.t.\ g_i(x) \cap (-Z_{i+}) \neq 0,\ i = 1, 2, \cdots, m,$$

$$0 \in h_j(x),\ j = 1, 2, \cdots, n,$$

$$x \in D,$$

where f: $X \to 2^Y$, $g_i : X \to 2^{Z_i}$, $h_j : X \to 2^{W_j}$ are set-valued maps, Z_{i+} is a closed convex cone in Z_i and D is a nonempty subset of X.

Definition 7 (weakly efficient solution). *A point $\overline{x} \in F$ is said to be a weakly efficient solution of (VP) if there exists no $x \in D$ satisfying $f(\overline{x}) \succ\succ f(x)$, where*

$$F := \{x \in D : g(x) \prec 0,\ h(x) = 0\}.$$

Let

$$P\ min[A, Y_+] = \{y \in A : (y - A) \cap intY_+ = \varnothing\},$$

$$P\ max[A, Y_+] = \{y \in A : (A - y) \cap intY_+ = \varnothing\}.$$

In the sequel, $B(W, Y)$ denotes the set of all continuous linear mappings T from W to Y; $B^+(Z, Y)$ denotes the set of all non-negative and continuous linear mappings S from Z to Y, where non-negative mapping S means that $S(z) \in Y_+, \forall z \in Z$, write

$$L(\overline{x}, \overline{S}, \overline{T}) = f(\overline{x}) + \overline{S}(g(\overline{x})) + \overline{T}(h(\overline{x})).$$

Definition 8 (vector saddle-point). *$(\overline{x}, \overline{S}, \overline{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is said to be a vector saddle-point of $L(\overline{x}, \overline{S}, \overline{T})$ if*

$$L(\overline{x}, \overline{S}, \overline{T}) \in P\ min[L(X, \overline{S}, \overline{T}), Y_+] \cap P\ max[L(\overline{x}, B^+(Z, Y), B(W, Y)), Y_+].$$

where

$$\begin{aligned} &P\ max[L(\overline{x}, B^+(Z, Y), B(W, Y)), Y_+] \\ &= \{\mu : \mu = P\ max[L(\overline{x}, S, T), Y_+], (S, T) \in B^+(Z, Y) \times B(W, Y)\}. \end{aligned}$$

Theorem 4. *$(\overline{x}, \overline{S}, \overline{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L(\overline{x}, \overline{S}, \overline{T})$ if and only if $\exists \overline{y} \in f(\overline{x}), \overline{z} \in g(\overline{x})$, such that*

(i) $\overline{y} \in P\ min[L(X, \overline{S}, \overline{T}), Y_+]$;
(ii) $g(\overline{x}) \subset -Z_+,\ h(\overline{x}) = \{0\}$;
(iii) $(f(\overline{x}) - \overline{y} - \overline{S}(\overline{z})) \cap intY_+ = \varnothing$.

Proof. The sufficiency. Suppose that the conditions (i)–(iii) are satisfied. Note that $-g(\overline{x}) \subseteq Z_+, h(\overline{x}) = \{0\}$ implies

$$-S(g(\overline{x})) \subseteq Y_+,\ T(h(\overline{x})) = \{0\},\ \forall (S, T) \in B^+(Z, Y) \times B(W, Y),$$

and the condition (i) states that

$$\{\overline{y} - [f(X) + \overline{S}(g(X)) + \overline{T}(h(X))]\} \cap intY_+ = \varnothing,$$

So, $Y_+ + intY_+ \subseteq Y_+$ and $-S(\overline{z}) \in Y_+$ imply

$$\{\overline{y} + \overline{S}(\overline{z}) + \overline{T}(\overline{w}) - [f(X) + \overline{S}(g(X)) + \overline{T}(h(X))]\} \cap intY_+ = \varnothing.$$

Hence,

$$\overline{y}+\overline{S}(\overline{z})+\overline{T}(\overline{w})\in Pmin[L(X,\overline{S},\overline{T}),Y_+].$$

On the other hand, since $(f(\overline{x})-[\overline{y}+S(\overline{z})])\cap intY_+=\varnothing$, from $intY_+ + Y_+\subseteq intY_+$, we conclude that

$$\left\{\bigcup_{(S,T)\in B^+(Z,Y)\times B(W,Y)}[f(\overline{x})+S(g(\overline{x}))+T(h(\overline{x}))\right]-[\overline{y}+\overline{S}(\overline{z})+\overline{T}(\overline{w})]\}\cap intY_+=\varnothing.$$

Hence,

$$\overline{y}+\overline{S}(\overline{z})+\overline{T}(\overline{w})\in Pmax[L(\overline{x},B^+(Z,Y),B(W,Y)),Y_+]$$

Consequently,

$$L(\overline{x},\overline{S},\overline{T})\cap Pmin[L(X,\overline{S},\overline{T}),Y_+]\cap Pmax[L(\overline{x},B^+(Z,Y),B(W,Y)),Y_+]\neq\varnothing.$$

Therefore, $(\overline{x},\overline{S},\overline{T})\in X\times B^+(Z,Y)\times B(W,Y)$ is a vector saddle-point of $L(\overline{x},\overline{S},\overline{T})$.

The necessity. Assume that $(\overline{x},\overline{S},\overline{T})\in X\times B^+(Z,Y)\times B(W,Y)$ is a vector saddle-point of $L(\overline{x},\overline{S},\overline{T})$. From Definition 8, one has

$$L(\overline{x},\overline{S},\overline{T})\cap Pmin[L(X,\overline{S},\overline{T}),Y_+]\cap Pmax[L(\overline{x},B^+(Z,Y)\times B(W,Y)),Y_+]\neq\varnothing.$$

So, $\exists\overline{y}\in f(\overline{x})$, $\overline{z}\in g(\overline{x})$, $\overline{w}\in h(\overline{x})$, i.e.,

$$\overline{y}+S(\overline{z})+T(\overline{w})\in L(\overline{x},\overline{S},\overline{T})=f(\overline{x})+S(g(\overline{x}))+T(h(\overline{x})),$$

such that

$$\{f(\overline{x})+S(g(\overline{x}))+T(h(\overline{x}))-[\overline{y}+\overline{S}(\overline{z})+\overline{T}(\overline{w})]\}\cap intY_+=\varnothing,$$
$$\forall(S,T)\in B^+(Z,Y)\times B(W,Y),$$

and

$$(\overline{y}+\overline{S}(\overline{z})+\overline{T}(\overline{w})-[f(X)+\overline{S}(g(X))+\overline{T}(h(X))])\cap intY_+=\varnothing.$$

Taking $T=\overline{T}$ we obtained

$$S(z)-\overline{S}(\overline{z})\notin intY_+,\forall z\in g(\overline{x}),\forall S\in B^+(Z,Y).$$

We aim to show that $-\overline{z}\in Z_+$.

Otherwise, since $0\in -Z_+$, if $-\overline{z}\notin Z_+$, we would have $-\overline{z}\neq 0$,

Because Z_+ is a closed convex set, by the separate theorem $\exists\eta\in Z^*\backslash\{0\}$,

$$\eta(tz_+)>\eta(-\overline{z}),\ \forall z\in Z_+,\ \forall t>0.$$

i.e.,

$$\eta(z_+)>\frac{1}{t}\eta(-\overline{z}),\ \forall z\in Z_+,\ \forall t>0.$$

Let $t\to\infty$; we obtained $\eta(z_+)\geq 0$, $\forall z\in Z_+$, which means that $\eta\in Z^*_+\backslash\{0\}$. Meanwhile, $0\in Z_+$ yields $\eta(\overline{z})>0$. Given $\widetilde{z}\in intZ_+$ and let

$$S(z)=\frac{\eta(z)}{\eta(\overline{z})}\widetilde{z}+\overline{S}(z).$$

Then, $\overline{S}\in B^+(Z,Y)$ and

$$(\overline{z})-\overline{S}(\overline{z})=\widetilde{z}\in intY_+.$$

This is a contradiction. Therefore,

$$-\overline{z}\in Z_+.$$

At this point, we aim to prove that $-g(\overline{x}) \subseteq Z_+$.

Otherwise, if $-g(\overline{x}) \not\subset Z_+$, then $\exists z_0 \in g(\overline{x})$, such that $0 \neq -z_0 \notin Z_+$. Similar to the above $\exists \eta_0 \in Z^* \backslash \{0\}$, such that $\eta_0 \in Z_+^* \backslash \{0\}$, $\eta_0(z_0) > 0$. Given $\widetilde{z} \in intZ_+$ and let

$$S_0(z) = \frac{\eta_0(z)}{\eta_0(z_0)}\widetilde{z}.$$

Then, $S_0 \in B^+(Z,Y)$ and $S_0(z_0) = \widetilde{z} \in intY_+$. We proved that $-\overline{z} \in Z_+$, so $-\overline{S}(\overline{z}) \in Y_+$. Therefore,

$$S_0(z_0) - \overline{S}(\overline{z}) \in intY_+ + Y_+ \subseteq intY_+.$$

Again, a contradiction.

Therefore, $-g(\overline{x}) \subseteq Z_+$. Similarly, one has $-h(\overline{x}) \subseteq W_+$. From (Lemma 2), we obtain

$$\left[T(h(\overline{x})) - \overline{T}(\overline{w})\right] \cap intY_+ = \varnothing.$$

Hence,

$$T(\overline{w}) - \overline{T}(\overline{w}) \notin intY_+,\ \forall T \in B(W,Y).$$

Similarly, we obtained

$$T(w) - \overline{T}(\overline{w}) \notin intY_+, \forall w \in h(\overline{x}),\ \forall T \in B(W,Y).$$

If $\overline{w} \neq 0$, since $-h(\overline{x}) \subseteq W_+$ and W_+ is a pointed cone, we have $\overline{w} \notin W_+$. Because Y_+ is a closed convex set, by the separation theorem $\exists \varsigma \in W^*$, such that

$$\varsigma(w) < \varsigma(\overline{w}), \forall w \in W_+.$$

So, $\varsigma(\overline{w}) \neq 0$ since $0 \in W_+$. Taking $y^0 \in intY_+$ and defining $T^0 \in B^+(W,Y)$ by

$$T^0(w) = \frac{\varsigma(w)}{\varsigma(\overline{w})}y^0 + \overline{T}(w).$$

Then,

$$T^0(\overline{w}) - \overline{T}(\overline{w}) = y^0 \in intY_+,$$

A contradiction. Therefore,$\overline{w} = 0$. Thus,

$$0 \in h(\overline{x}).$$

At this point, we aim to prove $h(\overline{x}) = \{0\}$.

Otherwise, if $w^0 \in h(\overline{x}) : w^0 \neq 0$, then $\exists \varsigma^0 \in W^*$, such that $\varsigma^0(w) < \varsigma^0\left(w^0\right), \forall w \in W_+$. So, $\varsigma^0\left(w^0\right) \neq 0$. Given $y_0 \in intY_+$ and defining $T_0 \in B(W,Y)$, by

$$T_0(w) = \frac{\varsigma^0(w)}{\varsigma^0(w^0)}y_0.$$

Then, $T_0(w^0) = y_0 \in intY_+$. This contradiction implies that we must have

$$h(\overline{x}) = \{0\}.$$

We conclude that

$$\overline{y} \in P\ min[L\left(X, \overline{S}, \overline{T}\right), Y_+],$$

and

$$\left(f(\overline{x}) - \overline{y} - \overline{S}(\overline{z})\right) \cap intY_+ = \varnothing.$$

We proved that, if $(\overline{x}, \overline{S}, \overline{T}) \in X \times B^+(Z,Y) \times B(W,Y)$ is a vector saddle-point of $L(\overline{x}, \overline{S}, \overline{T})$, then the conditions (i)–(iii) hold. □

Theorem 5. *If $(\overline{x}, \overline{S}, \overline{T}) \in X \times B^+(Z,Y) \times B(W,Y)$ is a vector saddle-point of $L(\overline{x}, \overline{S}, \overline{T})$, and if $0 \in \overline{S}(g(\overline{x}))$, then $\overline{x}$ is a weak efficient solution of (VP).*

Proof. Assume that $(\overline{x}, \overline{S}, \overline{T}) \in D \times B^+(Z,Y) \times B(W,Y)$ is a vector saddle-point of $L(\overline{x}, \overline{S}, \overline{T})$; from Theorem 2, we have

$$-S(g(\overline{x})) \subseteq Y_+,\ h(\overline{x}) = \{0\}.$$

So, $\overline{x} \in D$ (the feasible solution of (VP). $\exists \overline{y} \in f(\overline{x})$, such that $\overline{y} \in Pmin[L(X, \overline{S}, \overline{T}), Y_+]$, i.e.,

$$(\overline{y} - [f(X) + \overline{S}(g(X)) + \overline{T}(h(X)]) \cap intY_+ = \varnothing.$$

Thus,

$$(\overline{y} - [f(D) + \overline{S}(g(\overline{x})) + \overline{T}(h(\overline{x})]) \cap intY_+ = \varnothing.$$

Since $0 \in \overline{S}(g(\overline{x}))$, one has

$$(\overline{y} - f(D)) \cap intY_+ = \varnothing.$$

Therefore, $\overline{x}$ is a weakly efficient solution of (VP). □

5. Vector Lagrangian Theorems

Definition 9 (vector Lagrangian map)**.** *The vector Lagrangian map $L : X \times B^+(Z,Y) \times B(W,Y) \to 2^Y$ of (VP) is defined by the set-valued map*

$$L(x,S,T) = f(x) + S(g(x)) + T(h(x)).$$

Given $(S,T) \in B^+(Z,Y) \times B(W,Y)$, we considered the minimization problem induced by (VP):

$$\text{(VPST) } Y_+ - minL(x, S,\ T),$$
$$s.t., x \in D.$$

Definition 10 (slater constrained qualification (SC))**.** *Let $\overline{x} \in F$. We consider that (VP) satisfies the Slater Constrained Qualification at $\overline{x}$ if the following conditions hold:*

(1) $\exists x \in D$, *s.t.* $h_j(x) = 0,\ g_i(x) \prec\prec 0$;
(2) $0 \in inth_j(D)$ *for all j.*

According the following Theorem 6, (VPST) can also be considered as a dual problem of (VP).

Theorem 6. *Let $\overline{x} \in D$. Assume that $f(x) - f(\overline{x}),\ g(x),\ h(x)$ satisfies the generalized convexity condition (a), the generalized affineness condition (b) as well as the inner point condition (c), and (VP) satisfies the Slater Constrained Qualification (SC). Then, $\overline{x} \in D$ is a weakly efficient solution of (VP) if, and only if, $\exists (S,T) \in B^+(Z,Y) \times B(W,Y)$, such that $\overline{x} \in D$ is a weakly efficient solution of (VPST).*

Proof. Assume $\exists (S,T) \in B^+(Z,Y) \times B(W,Y)$, such that $\overline{x} \in D$ is a weakly efficient solution of (VPST). Then, there exist $\overline{y} \in f(\overline{x}),\ \overline{z} \in g(\overline{x}),\ \overline{w} \in h(\overline{x})$, such that

$$(\overline{y} + S(\overline{z}) + T(\overline{w}) - [f(D) + S(g(D)) + T(h(D))]) \cap intY_+ = \varnothing,$$

If $(\overline{y} - f(D)) \cap intY_+ \neq \emptyset$, then $\exists y \in f(D)$, such that $\overline{y} - y \in intY_+$, i.e.,

$$(\overline{y} + S(\overline{z}) + T(\overline{w}) - [y + S(\overline{z}) + T(\overline{w})]) \in intY_+.$$

This means that

$$(\overline{y} + S(\overline{z}) + T(\overline{w}) - [f(D) + S(g(D)) + T(h(D))]) \cap intY_+ \neq \emptyset,$$

which is a contradiction.

Therefore,

$$(\overline{y} - f(D)) \cap intY_+ = \emptyset.$$

Hence, $\overline{x} \in D$ is a weakly efficient solution of (VP).

Conversely, suppose that $\overline{x} \in D$ is a weakly efficient solution of (VP). So, $\exists \overline{y} \in f(\overline{x})$, such that there is not any $x \in D$ for which $f(x) - \overline{y} \in -intY_+$. That is to say, there is not any $x \in X$, such that

$$f(x) - \overline{y} \in -intY_+,\ g(x) \in -Z_+, 0_W \in h(x).$$

By Theorem 3, $\exists (\xi, \eta, \varsigma) \in Y_+^* \times Z_+^* \times W^* \backslash \{(0_{Y^*}, 0_{Z^*}, 0_{W^*})\}$, such that

$$\xi(f(x) - \overline{y}) + \eta(g(x)) + \varsigma(h(x)) \geq 0,\ \forall x \in D.$$

Since $\overline{y} \in f(\overline{x})$ and $0_W \in h(\overline{x})$, taking $x = \overline{x}$ in (1), we obtained

$$\eta(g(\overline{x})) \geq 0.$$

However, $\overline{x} \in D$ and $\eta \in Z_+^*$ imply that $\exists \overline{z} \in g(\overline{x}) \cap (-Z_+)$, for which

$$\eta(\overline{z}) \leq 0.$$

Hence, $\eta(\overline{z}) = 0$, which means

$$0 \in \eta(g(\overline{x}))$$

Since $x \in D$ implies $0_W \in h(x)$ and $g(x) \cap (-Z_+) \neq \emptyset$ implies $\exists z \in g(x) \cap (-Z_+)$, such that $\eta(z) \leq 0$, we have

$$\xi(f(x) - \overline{y}) \geq 0,\ \forall x \in D.$$

Because the Slater Constraint Qualification is satisfied, similar to the proof of Theorem 4, we have $\xi \neq 0_{Y^*}$. So, we may take $y_0 \in intY_+$, such that

$$\xi(y_0) = 1.$$

Define the operator $S : Z \to Y$ and $T : W \to Y$ by

$$S(z) = \eta(z)y_0, T(w) = \varsigma(w)y_0.$$

It is easy to see that

$$S \in B^+(Z, Y), S(Z_+) = \eta(Z_+)y_0 \subseteq Y_+,$$
$$T \in B(W, Y).$$

Therefore

$$S(g(\overline{x})) = \eta(g(\overline{x}))y_0 \in 0 \cdot Y_+ = 0_Y.$$

Since $\overline{x} \in D$, we have $0_W \in h(\overline{x})$. Hence,

$$0_Y \in T(h(\overline{x})).$$

Therefore,

$$\overline{y} \in f(\overline{x}) \subseteq f(\overline{x}) + S(g(\overline{x})) + T(h(\overline{x})).$$

And then

$$\begin{aligned} &\xi[f(x) + S(g(x)) + T(h(x))] \\ &= \xi(f(x)) + \eta((g(x))\xi(y_0) + \varsigma(h(x))\xi(y_0) \\ &= \xi(f(x)) + \eta(g(x)) + \varsigma(h(x)) \\ &\geq \xi(\overline{y}), \forall x \in D. \end{aligned}$$

i.e.,

$$\xi[f(x) - \overline{y}) + S(g(x)) + T(h(x))] \geq 0, \forall x \in D.$$

Taking $F(x) = f(x) + S(g(x)) + T(h(x))$, $G(x) = \{0_Z\}$ and $H(x) = \{0_W\}$ and applying Theorem 4 to the functions $F(x) - \overline{y},\ G(x),\ H(x)$, we have

$$(\overline{y} - [f(D) + S(g(D)) + T(h(D))] \cap intY_+ = \varnothing,$$

as well as

$$\overline{y} \in F(\overline{x}) = f(\overline{x}) + S(g(\overline{x})) + T(h(\overline{x})),$$

since $0_Y \in S(g(\overline{x})),\ 0_Y \in T(h(\overline{x}))$.

Consequently, $\overline{x} \in D$ is a weakly efficient solution of (VPST).

We complete the proof. □

Definition 11 (NNAMCQ)**.** *Let $\overline{x} \in F$. We say that (VP) satisfies the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) at $\overline{x}$, if there is no nonzero vector $(\eta, \varsigma) \in \Pi_{i=1}^{m} Z_i^* \times \Pi_{j=1}^{n} W_j^*$ satisfying the system*

$$\begin{aligned} &\min_{x \in D \cap U(\overline{x})} \left[\sum_{i=1}^{m} \eta_i g_i(x) + \sum_{j=1}^{n} \varsigma_j h_j(x) \right] = 0 \\ &\sum_{i=1}^{m} \eta_i g_i(\overline{x}) = 0, \end{aligned}$$

where $U(\overline{x})$ is some neighborhood of $\overline{x}$.

Similar to the proof of Theorem 6, we obtained Theorem 7.

Theorem 7. *Let $\overline{x} \in D$. Assume that $f(x) - f(\overline{x}), g(x), h(x)$ satisfies the generalized convexity condition (a), the generalized affineness condition (b), as well as the inner point condition (c). If $\overline{x}$ is a weakly efficient solution of (VP), then $\exists$ vector Lagrangian multiplier $(S, T) \in B^+(Z, Y) \times B(W, Y)$, such that $\overline{x} \in D$ is a weakly efficient solution of (VPST).* Inversely, if (NNAMCQ) holds at *$\overline{x} \in D$, and if $\exists$ vector Lagrangian multiplier $(S, T) \in B^+(Z, Y) \times B(W, Y)$, such that $\overline{x}$ is a weakly efficient solution of (VPST), then $\overline{x}$ is a weakly efficient solution of (VP).*

6. Conclusions

Jeyakumar [2] introduced the following definition of sub convexlike functions for single-valued functions.

Let Y be a topological vector space and $D \subseteq X$ be a nonempty set. A set-valued map $f\colon\ D \to 2^Y$ is said to be Y_+-sub convexlike on D if $\exists$ bounded set-valued map $u\colon\ D \to Y$, $\forall x_1, x_2 \in D, \forall \varepsilon > 0, \forall \alpha \in [0,\ 1], \exists x_3 \in D$, such that

$$\varepsilon u + \alpha f(x_1) + (1 - \alpha) f(x_2) \prec_{Y_+} f(x_3),$$

where the partial order is induced by a convex cone Y_+ of Y.

Jeyakumar [3] introduced the following subconvexlikeness.

A set-valued map $f\colon D \to 2^Y$ is said to be Y_+-subconvexlike on D if $\exists\theta \in intY_+$, such that $\forall x_1, x_2 \in D$, $\forall\varepsilon > 0$, $\forall\alpha \in [0,\ 1]$, $\exists x_3 \in D$ holds

$$\varepsilon\theta + \alpha f(x_1) + (1-\alpha)f(x_2) \prec_{Y_+} f(x_3).$$

In this paper, we proved that the above two generalized convexities are equivalent in locally convex topological spaces. Since Banach spaces are locally convex topological spaces (n-dimensional Euclidean spaces are Banach spaces), we proved that the two definitions of generalized convexities are the same. Then, we solved set-valued vector optimization problems and obtained vector saddle-point theorems and some vector Lagrangian theorems. Our optimization problems have inequality, equality as well as an abstract constraint. Our inequality constraints are generalized convex maps and the generalized convexities are defined by partial order relations.

A set-valued map $f\colon\ D \to 2^Y$ is said to be affine on D if $\forall x_1, x_2 \in D, \forall\beta \in R$; there holds

$$\beta f(x_1) + (1-\beta)f(x_2) = f(\beta x_1 + (1-\beta)x_2.$$

We defined the following sub affinelike maps in order to weaken the condition of the "equality constraints" for optimization problems.

A set-valued map $f\colon\ D \to 2^Y$ is said to be Y_+-sub affinelike on D if $\forall x_1, x_2 \in D$, $\forall\alpha \in (0,\ 1), \forall v \in intW_+$, $\exists x_3 \in D$; there holds

$$v + \alpha f(x_1) + (1-\alpha)f(x_2) = f(x_3).$$

Then, we considered the following optimization problem with set-valued maps:

$$(VP) \qquad Y_+\text{-min} \qquad f\,(x)$$

$$s.t.\ g_i(x) \cap (-Z_{i+}) \neq \varnothing,\ i = 1, 2, \cdots, m,$$

$$0 \in h_j(x),\ j = 1, 2, \cdots, n,$$

$$x \in D,$$

where $f\colon\ X \to 2^Y$ and $g_i : X \to 2^{Z_i}$ are sub convexlike and $h_j : X \to 2^{W_j}$ are sub affinelike.

For a single-valued situation, the above optimization problem (VP) may be written as follows.

$$Y_+\text{-min} \qquad f\,(x)$$

$$s.t.\ g_i(x) \prec 0,\ i = 1, 2, \cdots, m,$$

$$h_j(x) = 0,\ j = 1, 2, \cdots, n,$$

$$x \in D.$$

We obtained vector saddle-point theorems and vector Lagrangian theorems for the set-valued optimization problem (VP). Our Theorem 3 is a generalization of theorems of alternatives in [2,3] and a modification of theorems of alternatives in [4,7,8,11,13]. Our saddle-points theorems (Theorems 4 and 5) are generalizations of the saddle-point theorem in [4,14] and modifications of saddle-point theorems in [14,16]. Our Lagrangian theorems (Theorems 6 and 7) are generalizations of Lagrangian theorems in [14] and modifications of those in [10,15]. We can also extended the results in [12] according to our methods used in this paper.

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References

1. Fan, K. Minimax theorems. *Proc. Natl. Acad. Sci. USA* **1953**, *39*, 42–47. [CrossRef] [PubMed]
2. Jeyakumar, V. Convexlike Alternative Theorems and Mathematical Programming. *Optimization* **1985**, *16*, 643–652. [CrossRef]
3. Jeyakumar, V. A Generalization of a Minimax Theorem of Fan via a Theorem of the Alternative. *J. Optim. Theory Appl.* **1986**, *48*, 525–533. [CrossRef]
4. Gutiérrez, C.; Huerga, L.; Novo, V. Scalarization and Saddle Points of Approximate Proper Solutions in Nearly Subconvexlike Vector Optimization Problems. *J. Math. Anal. Appl.* **2012**, *389*, 1046–1058. [CrossRef]
5. Hern, E.; Novo, V. Weak and Proper Efficiency in Set-Valued Optimization on Real Linear Spaces. *J. Convex Anal.* **2007**, *14*, 275–296.
6. Xu, Y.D.; Li, S.J. Tightly Proper Efficiency in Vector Optimization with Nearly Cone-Subconvexlike Set-Valued Maps. *J. Inequal. Appl.* **2011**, *2011*, 839679. [CrossRef]
7. Yuan, C.H.; Rong, W.D. ε-Properly Efficiency of Multiobjective Semidefinite Programming with Set Valued Functions. *Math. Probl. Eng.* **2017**, *2017*, 5978130. [CrossRef]
8. Zhou, Z.-A.; Peng, J.-W. A Generalized Alternative Theorem of Partial and Generalized Cone Subconvexlike Set-Valued Maps and Its Applications in Linear Spaces. *J. Appl. Math.* **2012**, *2012*, 370654. [CrossRef]
9. Yosida, K. *Functional Analysis*; Springer: Berlin, Germany, 1978.
10. Chen, J.; Xu, Y.; Zhang, K. Approximate Weakly Efficient Solutions of Set-Valued Vector Equilibrium Problems. *J. Inequal. Appl.* **2018**, *2018*, 181. [CrossRef] [PubMed]
11. Huang, Y.W. A Farkas-Makowski Type Alternative Theorem and Its Applications to Set-Valued Equilibrium Problems. *J. Nonlinear Convex Anal.* **2002**, *3*, 100–118.
12. Zhang, C.-L. Set-valued equilibrium problems based on the concept of null set with applications. *Optimization* **2022**, *2022*, 2109969. [CrossRef]
13. Galán, M.R. A Theorem of the Alternative with an Arbitrary Number of Inequalities and Quadratic Programming. *J. Glob. Optim.* **2017**, *69*, 427–442. [CrossRef]
14. Li, Z.F.; Wang, S.Y. Lagrange Multipliers and Saddle Points in Multiobjective Programming. *J. Optim. Theory Appl.* **1994**, *83*, 63–81. [CrossRef]
15. Zhou, Y.Y.; Zhou, J.C.; Yang, X.Q. Existence of Augmented Lagrange Multipliers for Cone Constrained Optimization Problems. *J. Glob. Optim.* **2014**, *58*, 243–260. [CrossRef]
16. Zeng, R. Generalized Gordon Alternative Theorem with Weakened Convexity and Its Applications. *Optimization* **2002**, *51*, 709–717. [CrossRef]