mathematics

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Sakaguchi Type Starlike Functions Related with Miller-Ross-Type Poisson Distribution in Janowski Domain

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Abstract: In this research, using the Poisson-type Miller-Ross distribution, we introduce new subclasses Sakaguchi type of star functions with respect to symmetric and conjugate points and discusses their characteristic properties and coefficient estimates. Furthermore, we proved that the class is closed by an integral transformation. In addition, we pointed out some new subclasses and listed their geometric properties according to specializing in parameters that are new and no longer studied in conjunction with a Miller-Ross Poisson distribution.

Keywords: Miller-Ross-type Poisson distribution; convolution; symmetric points; conjugate points; starlike functions; univalent functions

MSC: 30C45; 30C50; 33E12

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Citation: El-Deeb, S.M.; Alharbi, A.; Murugusundaramoorthy, G. Sakaguchi Type Starlike Functions Related with Miller-Ross-Type Poisson Distribution in Janowski Domain. *Mathematics* **2023**, *11*, 2918. https://doi.org/10.3390/math11132918

Academic Editors: Ioannis K. Argyros and Clemente Cesarano

Received: 1 May 2023
Revised: 15 June 2023
Accepted: 26 June 2023
Published: 29 June 2023

1. Introduction and Definitions

In recent years, the distribution of random variables has attracted excessive interest. Probability density functions perform an essential role in statistics and the concept of probability, particularly for distributions. There are numerous forms of distribution from situations of real existence, together with the binomial distribution, Poisson distribution and hypergeometric distribution. In the theory of geometric functions, simple distribution, along with Pascal, Poisson, logarithmic, binomial, beta negative binomial has been partially studied from a theoretical point of view (see [1–4]) and two parameters of the Mittag–Leffler-type probability distribution (see [5–8]).

Miller and Ross [9] proposed the special characteristic as the basis of the solution of fractional order initial value problem, that is known as the Miller-Ross functions (**MRF**) described as

$$\mathbf{E}_{\nu,\wp}(\xi) = \xi^{\nu} e^{\wp\xi} \Upsilon^{*}(\nu, \wp\xi), \quad (\nu, \wp, \xi \in \mathbb{C}, \text{ with } \Re(\nu) > 0, \ \Re(\wp) > 0)$$

where Υ^{*} is incomplete gamma function ([9], p. 314). Using the properties of the incomplete gamma function, the Miller-Ross function (**MRF**) can easily be written as

$$\mathbf{E}_{\nu,\wp}(\xi) := \xi^{\nu} \sum_{k=0}^{\infty} \frac{(\wp\xi)^{k}}{\Gamma(k+\nu+1)}, \quad \nu, \wp, \xi \in \mathbb{C}, \text{ with } \Re(\nu) > 0, \ \Re(\wp) > 0 \tag{1}$$

which can be stated as

$$\mathbf{E}_{\nu,\wp}(\xi) \equiv \xi^{\nu} \mathbf{E}_{1,1+\nu}(\wp\xi)$$

where in right hand side member $\mathbf{E}_{1,1+\nu}(\wp\xi)$ is the Mittag-Leffler function (**MLF**) of two parameter [10]. Some of special values of the **MRF** can be given as follows:

$$\begin{aligned} \mathbf{E}_{\nu,\wp}(0) &= 0, \quad \Re(\nu) > 0 \\ \mathbf{E}_{0,\wp}(\xi) &= e^{\wp\xi}, \\ \mathbf{E}_{0,1}(\xi) &= e^{\xi} \\ \mathbf{E}_{1,1}(\xi) &= e^{\xi} - 1. \end{aligned}$$

Miller-Ross and Mittag-Leffler function and eigen-functions, which play an imperative role in fractional calculus. These functions are the main tool in solving non-integer differential equations. Recently, Srivastava et al. [5] presented a study on Poisson distributions based on two parameters Mittag-Leffler type function Poisson distribution and the resulting moments, the moment generating function. Motivated by results on connections between various subclasses of analytic univalent functions using special functions and distribution series. This became the beginning of studies on several classes of analytical functions using the Miller-Ross Poisson distribution [11–14]. Lately Eker and Ece [11], normalized $\mathbf{E}_{\nu,\wp}$ and for $\wp > 0$ with $\nu > 2\wp - 1$ they presented **MRF** is univalent and starlike in $\mathbb{D}_{\frac{1}{2}} = \{\xi \in \mathbb{C} : |\xi| < \frac{1}{2}\}$. Further established if $\nu > (2 + \sqrt{2})\wp - 1$ then normalized **MRF** is univalent and convex in $\mathbb{D}_{\frac{1}{2}}$ (see [9]). The probability mass function of the Miller Ross-type Poisson distribution is given by

$$\mathcal{P}_{\nu,\wp}(\ell, n) := \frac{(\ell\wp)^n \, \ell^\nu}{\mathbf{E}_{\nu,\wp}(\ell)\Gamma(\nu + n + 1)}, \quad n = 0, 1, 2, 3, \cdots \tag{2}$$

where $\nu > -1, \wp > 0$ and $\mathbf{E}_{\nu,\wp}$ is **MRF** given in Equation (1). Miller-Ross-type Poisson distribution is given by

$$\mathcal{M}^{\ell}_{\nu,\wp}(\xi) = \xi + \sum_{k=2}^{\infty} \frac{(\ell\wp)^{k-1} \, \ell^\nu}{\mathbf{E}_{\nu,\wp}(\ell)\Gamma(\nu + k)} \xi^k, \qquad \xi \in \mathbb{D} \tag{3}$$

where

$$\mathbb{D} := \{\xi \in \mathbb{C} : |\xi| < 1\},$$

the unit disc.

Subclasses of Holomorphic (Analytic) Function $\mathcal{H}$:

Let $\mathcal{H}$ represents the class of all holomorphic (analytic) functions in $\mathbb{D}$ is given by

$$f(\xi) = \xi + \sum_{k=2}^{\infty} a_k \xi^k, \ \xi \in \mathbb{D}. \tag{4}$$

If $g \in \mathcal{H}$ is assuemed as

$$g(\xi) = \xi + \sum_{k=2}^{\infty} b_k \xi^k, \ \xi \in \mathbb{D}, \tag{5}$$

then, the convolution (or Hadamard) product of f and g is given by

$$f(\xi) * g(\xi) = (f * g)(\xi) := \xi + \sum_{k=2}^{\infty} a_k b_k \xi^k, \qquad \xi \in \mathbb{D}. \tag{6}$$

Let Ω be the family of functions *Schwarz function* given by

$$\Omega = \{\varpi \in \mathcal{H} : w(0) = 0 \quad and \quad |\varpi(z)| < 1, \ \xi \in \mathbb{D}\}.$$

If $F_1, F_2 \in \mathcal{H}$, we say that F_1 *is subordinate to* F_2, written as $F_1 \prec F_2$ or $F_1(\xi) \prec F_2(\xi)$ if there exists $\varpi \in \Omega$, such that $F_1(\xi) = F_2(\varpi(\xi))$, $\xi \in \mathbb{D}$. Moreover, if F_2 is univalent in $\mathbb{D}$, then equivalently(see [15,16]), we have

$$F_1(\xi) \prec F_2(\xi) \Leftrightarrow F_1(0) = F_2(0) \text{ and } F_1(\mathbb{D}) \subset F_2(\mathbb{D}). \tag{7}$$

Definition 1. *Let $w \in \Omega$ and for arbitrary fixed numbers $\mathcal{M}$ and $\mathcal{N}(-1 \leq \mathcal{N} < \mathcal{M} \leq 1)$, denote the family by $\mathcal{P}[\mathcal{M}, \mathcal{N}]$ consisting functions of the form*

$$p(\xi) = 1 + b_1\xi + b_2\xi^2 + \cdots \tag{8}$$

is analytic in $\mathbb{D}$ and then

$$p(\xi) \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi} \text{ or } p(\xi) = \frac{1+\mathcal{M}w(\xi)}{1+\mathcal{N}w(\xi)}, \qquad \xi \in \mathbb{D} \tag{9}$$

holds.

Note that $\dfrac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}$ conformably maps $\mathbb{D}$ onto a disc symmetric with respect to the real axis, centered at $\frac{1-\mathcal{M}\mathcal{N}}{1-\mathcal{N}^2}$ $(\mathcal{N} \neq \pm 1)$ with radius $\frac{\mathcal{M}-\mathcal{N}}{1-\mathcal{N}^2}$ $(\mathcal{N} \neq \pm 1)$.

Janowski [17] defined a subclass of starlike functions as:

$$\mathcal{S}^*(\mathcal{M},\mathcal{N}) = \left\{ f \in \mathcal{H} : \frac{\xi f'(\xi)}{f(\xi)} \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi} \ (-1 \leq \mathcal{N} < \mathcal{M} \leq 1; \xi \in \mathbb{D}) \right\} \tag{10}$$

and convex functions as

$$\mathcal{K}(\mathcal{M},\mathcal{N}) = \left\{ f \in \mathcal{H} : \frac{(\xi f'(\xi))'}{f'(\xi)} \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi} \ (-1 \leq \mathcal{N} < \mathcal{M} \leq 1; \xi \in \mathbb{D}) \right\}. \tag{11}$$

For example, taking $p(\xi) \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}$ where $\mathcal{M} \in \mathbb{C}; -1 \leq \mathcal{N} \leq 0$ and $\mathcal{M} \neq \mathcal{N}$, we get the classes $\mathcal{S}^*(\mathcal{M},\mathcal{N})$ and $\mathcal{K}(\mathcal{M},\mathcal{N})$ respectively. These classes with the restriction $-1 \leq \mathcal{N} \leq \mathcal{M} \leq 1$ are popularly named as Janowski starlike and Janowski convex functions, respectively. By fixing $\mathcal{M} = 1 - 2\varepsilon$ and $\mathcal{N} = -1$, where $0 \leq \varepsilon \leq 1$, we obtain the classes $\mathcal{S}^*(\mathcal{M},\mathcal{N}) = \mathcal{S}^*(\varepsilon)$ and $\mathcal{K}(\mathcal{M},\mathcal{N}) = \mathcal{K}(\varepsilon)$ of the starlike functions of order ε and convex functions of order ε, respectively. In particular, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ are the class of starlike functions and of convex functions in the open unit disk $\mathbb{D}$, respectively. Nasr and Aouf [18] defined a class of starlike functions of complex order as below:

$$\mathcal{S}(\hbar) = \left\{ f \in \mathcal{H} : \Re\left\{ 1 + \frac{1}{\hbar}\left(\frac{\xi f'(\xi)}{f(\xi)} - 1 \right) \right\} > 0; \quad \hbar \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ \xi \in \mathbb{D} \right\}.$$

Sakaguchi [19] gave a new direction of study by introducing a class functions starlike with respect to symmetric points, as

$$\mathcal{S}_s^* = \left\{ f \in \mathcal{H} : \Re\left(\frac{2\xi f'(\xi)}{f(\xi) - f(-\xi)} \right) > 0,\ \xi \in \mathbb{D} \right\}.$$

and starlike functions with regard to conjugate the points given by

$$\mathcal{S}_c^* = \left\{ f \in \mathcal{H} : \Re\left(\frac{2\xi f'(\xi)}{f(\xi) + \overline{f(\bar{\xi})}} \right) > 0, \xi \in \mathbb{D} \right\}.$$

Apparently a class of univalent functions, star-shaped with respect to symmetric points include classes of convex functions and odd functions starlike due to origin (see [19]).

Lately, many authors [20–22] study some new subclasses of Sakaguchi-type functions defined by using the concept of Janowski functions. Goel and Mehrok [20] introduced a subclass of $\mathcal{S}_s^{\star}$ as

$$\mathcal{S}_s^{\star}(\mathcal{M},\mathcal{N}), = \left\{ f \in \mathcal{H} : \frac{2zf'(\xi)}{f(\xi)-f(-\xi)} \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}; \quad -1 \le \mathcal{N} < \mathcal{M} \le 1, \xi \in \mathbb{D} \right\}.$$

In addition, for $-1 \le \mathcal{N} < \mathcal{M} \le 1$, $\hbar \in \mathbb{C}^*, \xi \in \mathbb{D}$. new subclasses of $\mathcal{S}_s^{\star}$ are defined as below

$$\mathcal{S}_s^*(\hbar,\mathcal{M},\mathcal{N}) = \left\{ f \in \mathcal{H} : 1 + \frac{1}{\hbar}\left(\frac{2\xi f'(\xi)}{f(\xi)-f(-\xi)} - 1 \right) \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}; \quad \xi \in \mathbb{D} \right\}, \tag{12}$$

and,

$$\mathcal{C}_s(\hbar,\mathcal{M},\mathcal{N}) = \left\{ f \in \mathcal{H} : 1 + \frac{1}{\hbar}\left(\frac{2(\xi f'(\xi))'}{(f(\xi)-f(-\xi))'} - 1 \right) \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}; \quad \xi \in \mathbb{D} \right\}. \tag{13}$$

By fixing $\hbar = 1$ the above classes yields the definition given in Aouf et al. [21]. The above classes $\mathcal{S}_s^*(\hbar,\mathcal{M},\mathcal{N})$ and $\mathcal{C}_s^*(\hbar,\mathcal{M},\mathcal{N})$ have been generalized by Arif et al. [22] based on Sălăgean Operator [23] and its properties have been discussed extensively. Many interesting subfamilies of $\mathcal{S}$ associated with circular domain have been studied in the literature from different perspectives closely related to $\mathcal{S}_s^{\star}$ (see [24–29] and references here). Inspired by aforementioned works, by using the convolution product as specified in Equation (6), we consider the linear operator

$$\mathcal{Q}_{\nu,\wp}^{\ell} : \mathcal{H} \to \mathcal{H}$$

as below:

$$\begin{aligned} \mathcal{Q}_{\nu,\wp}^{\ell} f(\xi) &= f(\xi) * \mathcal{M}_{\nu,\wp}^{\ell}(\xi) \\ &= \xi + \sum_{k=2}^{\infty} \Theta_k a_k \xi^k, \end{aligned} \tag{14}$$

where

$$\Theta_k = \Theta_k(\ell,\wp,\nu) = \frac{(\ell\wp)^{k-1}\,\ell^{\nu}}{\mathbf{E}_{\nu,\wp}(\ell)\Gamma(\nu+k)}. \tag{15}$$

Inspired by the study on $\mathcal{S}_s^*$ and $\mathcal{S}_C^*$ by Sakaguchi [19] and recent studies in [20–22], in this article using the Miller-Ross poisson distribution [5,11–14], we define two new classes $\mathcal{SS}_{\nu,\wp}^{\ell}(\hbar,\mathcal{M},\mathcal{N})$ and $\mathcal{SC}_{\nu,\wp}^{\ell}(\hbar,\mathcal{M},\mathcal{N})$ as given in Definition 2, over the Janowski domain. We investigated its characteristic properties and also determined the bounds for $|a_{2n}|$ and $|a_{2n+1}|$ for f in these newly defined classes. We further discussed the closure property under the integral transformation given by $F(\xi) = \frac{2}{\xi}\int_0^{\xi} f(t)dt$ for functions in these classes.

Now we define a new subclasses of Sakaguchi type starlike functions with respect to symmetric and conjugate symmetric points associated with the Miller-Ross poisson distribution operator $\mathcal{Q}_{\nu,\wp}^{\ell}$.

Definition 2. *For $-1 \le \mathcal{M} \le \mathcal{N} \le 1$, $\hbar \in \mathbb{C}^*$, let $f \in \mathcal{H}$ is said to be in the class*

(1) $\mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$ *if and only if*

$$1+\frac{1}{\hbar}\left[\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)-\mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)}-1\right]\prec\frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}, \tag{16}$$

and
(2) $\mathcal{SC}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$ *if and only if*

$$1+\frac{1}{\hbar}\left[\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)+\overline{\mathcal{Q}^{\ell}_{\nu,\wp}f(\overline{\xi})}}-1\right]\prec\frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}. \tag{17}$$

We note that $\overline{f(\overline{\xi})}=f(\xi)$, the $\mathcal{SC}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})=\mathcal{ST}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$. By fixing $\hbar=1$,we have following new classes:

Example 1. *For* $-1\le\mathcal{M}\le\mathcal{N}\le 1,$ *let* $f\in\mathcal{H}$ *is said to be in the class*

(1) $\mathcal{SS}^{\ell}_{\nu,\wp}(\mathcal{M},\mathcal{N})$ *if and only if*

$$\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)-\mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)}\prec\frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi},$$

and
(2) $\mathcal{SC}^{\ell}_{\nu,\wp}(\mathcal{M},\mathcal{N})$ *if and only if*

$$\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)+\overline{\mathcal{Q}^{\ell}_{\nu,\wp}f(\overline{\xi})}}\prec\frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}.$$

We note that $\overline{f(\overline{\xi})}=f(\xi)$, the $\mathcal{SC}^{\ell}_{\nu,\wp}(\mathcal{M},\mathcal{N})=\mathcal{ST}^{\ell}_{\nu,\wp}(\mathcal{M},\mathcal{N})$. Note that the functions

$$f(\xi)=\int_0^{\xi}\frac{(1-Mt)}{(1-Nt)(1+t^2)}dt \text{ and } f(\xi)=\int_0^{\xi}\frac{(1+Mt)}{(1+Nt)(1-t^2)}dt$$

which gives distortion bounds and extreme points of the function class studied for different perspective (details see [30]).

To prove our results, we will need the following lemmas.

Lemma 1 ([20], Lemma 2)**.** *If* $p(\xi)=1+p_1\xi+p_2\xi^2+\cdots\in\mathcal{P}[\mathcal{M},\mathcal{N}]$, *then*

$$|p_n|\le\mathcal{M}-\mathcal{N}.$$

Lemma 2 ([20], Lemma 2)**.** *If* R *be analytic and* S *starlike functions in* $\mathbb{D}$ *with* $R(0)=S(0)=0$. *then*

$$\frac{|R'(\xi)/S'(\xi)-1|}{|\mathcal{M}-\mathcal{N}(R'(\xi)/S'(\xi))|}<1,\ \ -1\le\mathcal{M}\le\mathcal{N}\le 1.$$

implies

$$\frac{|R(\xi)/S(\xi)-1|}{|\mathcal{M}-\mathcal{N}(R(\xi)/S(\xi))|}<1,\ \ (\xi\in\mathbb{D}).$$

2. Properties of the Subclass $\mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$

Unless otherwise specified, we let $-1 \le \mathcal{N} \le \mathcal{M} \le 1, \hbar \in \mathbb{C}^*$, and the powers are understood as principle values. Throughout this work, we use the notation

$$\prod_{i=1}^{k-1} A(i) = 1.$$

Theorem 1. *Let $f \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$, then the following condition*

$$1 + \frac{1}{\hbar}\left[\frac{\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi)} - 1\right] \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}, \tag{18}$$

is satisfied for ψ, the odd function given by

$$\psi(z) := \frac{f(\xi) - f(-\xi)}{2}. \tag{19}$$

Proof. If $f \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$, then there exists $h \in \mathcal{P}[\mathcal{M},\mathcal{N}]$, such that

$$h(\xi) = 1 + \frac{1}{\hbar}\left[\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi) - \mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)} - 1\right]. \tag{20}$$

It follows that

$$\begin{aligned} \hbar(h(\xi)-1) &= \tfrac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi) - \mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)} - 1, \\ \hbar(h(-\xi)-1) &= \tfrac{-2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi) - \mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)} - 1, \end{aligned} \tag{21}$$

which implies that

$$\frac{h(\xi)+h(-\xi)}{2} = 1 + \frac{1}{\hbar}\left[\frac{\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi)} - 1\right]. \tag{22}$$

On the other hand,

$$h(\xi) \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi},$$

and $\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ is univalent, then by Equation (7), we have

$$\frac{h(\xi)+h(-\xi)}{2} \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi},$$

it yield Equation (18). Thus the proof is complete. □

Theorem 2. *Let $f \in \mathcal{H}$ and is in the class $\mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$, if and only if there exists $h \in \mathcal{P}[\mathcal{M},\mathcal{N}]$ such that*

$$\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)' = (\hbar(h(\xi)-1)+1)\exp\left(\frac{\hbar}{2}\int_0^{\xi}\frac{h(t)+h(-t)-2}{t}dt\right). \tag{23}$$

Proof. From Theorem 1, we have

$$h(\xi) = 1 + \frac{1}{\hbar}\left[\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi) - \mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)} - 1\right] = 1 + \frac{1}{\hbar}\left[\frac{\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi)} - 1\right] \prec \frac{1+\mathcal{M}\xi}{1+\mathcal{N}\xi}.$$

But Equation (22), it implies

$$\frac{\left(\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi)} = \frac{1}{\xi} + \frac{\hbar}{2}\left(\frac{h(\xi)+h(-\xi)-2}{\xi}\right).$$

The above equation yield,

$$\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi) = \xi \exp\left(\frac{\hbar}{2}\int_0^{\xi}\frac{h(t)+h(-t)-2}{t}dt\right). \tag{24}$$

Since $f \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$, then from Equation (20) we obtain

$$\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)' = (\hbar(h(\xi)-1)+1)\mathcal{Q}^{\ell}_{\nu,\wp}\psi(\xi).$$

Using Equation (24) and above equation, we get Equation (23). This completes the proof. □

The foremost idea of the study on coefficient problems in several classes of $\mathcal{H}$ is to investigate the coefficients of functions in the hypothesized class using the coefficients of consistent functions with a positive real part. Thus, the coefficient functional can be predicted in advance using inequalities known for the $\mathcal{P}$ class. In the following theorem, we present estimates for f in the classes given in the Definition 2.

Theorem 3. *If $f \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$, then for all $n \geq 1$,*

$$|a_{2n}| \leq \frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^n n!|\Theta_{2n}|}\prod_{k=1}^{n-1}(|\hbar|(\mathcal{M}-\mathcal{N})+2k), \tag{25}$$

and

$$|a_{2n+1}| \leq \frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^n n!|\Theta_{2n+1}|}\prod_{k=1}^{n-1}(|\hbar|(\mathcal{M}-\mathcal{N})+2k) \tag{26}$$

where $\Theta_k; \forall\, k \geq 2$ as in Equation (15).

Proof. Since $f \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$, Definition 2 yields

$$1 + \frac{1}{\hbar}\left[\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi) - \mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)} - 1\right] = \frac{1+\mathcal{M}w(\xi)}{1+\mathcal{N}w(\xi)}. \tag{27}$$

Assuming that

$$h(\xi) = 1 + \sum_{k=1}^{\infty} c_k\xi^k = \frac{1+\mathcal{M}w(\xi)}{1+\mathcal{N}w(\xi)}. \tag{28}$$

In view of Equations (27) and (28), we get

$$2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)' = \left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi) - \mathcal{Q}^{\ell}_{\nu,\wp}f(-\xi)\right)\left(1 + \hbar\sum_{k=1}^{\infty}c_k\xi^k\right).$$

It follows from Equation (14) that

$$\begin{aligned} & \xi + 2\Theta_2 a_2 \xi^2 + 3\Theta_3 a_3 \xi^3 + 4\Theta_4 a_4 \xi^4 + \cdots + 2n\Theta_{2n} a_{2n} \xi^{2n} \\ + & (2n+1)\Theta_{2n+1} a_{2n+1} \xi^{2n+1} + \cdots \\ = & \left(z + \Theta_3 a_3 \xi^3 + \Theta_5 a_5 \xi^5 + \cdots + \Theta_{2n-1} a_{2n-1} \xi^{2n-1} + \Theta_{2n+1} a_{2n+1} \xi^{2n+1} + \cdots \right) \\ \times & \left(1 + \hbar c_1 \xi + \hbar c_2 \xi^2 + \cdots \right). \end{aligned}$$

Equating the coefficients of like powers of ξ, we obtain

$$2\Theta_2 a_2 = \hbar c_1 \tag{29}$$

$$2\Theta_3 a_3 = \hbar c_2, \tag{30}$$

$$4\Theta_4 a_4 = \hbar c_3 + \hbar c_1 \Theta_3 a_3, \tag{31}$$

$$4\Theta_5 a_5 = \hbar c_4 + \hbar c_2 \Theta_3 a_3, \tag{32}$$

$$2n\Theta_{2n} a_{2n} = \hbar c_{2n-1} + \hbar c_{2n-3} \Theta_3 a_3 + \hbar c_{2n-5} \Theta_5 a_5 + \ldots\ldots + \hbar c_1 \Theta_{2n-1} a_{2n-1}, \tag{33}$$

$$2n\Theta_{2n+1} a_{2n+1} = \hbar c_{2n} + \hbar c_{2n-2} \Theta_3 a_3 + \hbar c_{2n-4} \Theta_5 a_5 + \ldots\ldots + \hbar c_2 \Theta_{2n-1} a_{2n-1}. \tag{34}$$

We prove Equations (25) and (26) using mathematical induction.

Using Lemma 1, Equations (29)–(32) respectively, we get

$$\begin{aligned} |a_2| &\le \frac{|\hbar|}{2|\Theta_2|}(\mathcal{M} - \mathcal{N}), \\ |a_3| &\le \frac{|\hbar|}{2|\Theta_3|}(\mathcal{M} - \mathcal{N}), \\ |a_4| &\le \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{8|\Theta_4|}(2 + |\hbar|(\mathcal{M} - \mathcal{N})), \\ |a_5| &\le \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{8|\Theta_5|}(2 + |\hbar|(\mathcal{M} - \mathcal{N})). \end{aligned}$$

It trails that Equations (25) and (26) hold for $n = 1,\ 2$. Equation (33) in concurrence with Lemma 1 yields

$$|a_{2n}| \le \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{2n|\Theta_{2n}|} \left(1 + \sum_{r=1}^{n-1} |\Theta_{2r+1}||a_{2r+1}| \right).$$

Subsequently, we assume that Equations (25) and (26) hold for $3, 4, \ldots, n-1$. Accordingly the above inequality yields

$$|a_{2n}| \le \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{2n|\Theta_{2n}|} \left(1 + \sum_{r=1}^{n-1} \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{2^r r!} \prod_{k=1}^{r-1} (|\hbar|(\mathcal{M} - \mathcal{N}) + 2k) \right). \tag{35}$$

To complete the proof it is appropriate to show that

$$\begin{aligned} & \tfrac{|\hbar|(\mathcal{M} - \mathcal{N})}{2m|\Theta_{2m}|} \left(1 + \sum_{r=1}^{m-1} \tfrac{|\hbar|(\mathcal{M} - \mathcal{N})}{2^r r!} \prod_{k=1}^{r-1} (|\hbar|(\mathcal{M} - \mathcal{N}) + 2k) \right) \\ = & \tfrac{|\hbar|(\mathcal{M} - \mathcal{N})}{2^m m! |\Theta_{2m}|} \prod_{k=1}^{m-1} (|\hbar|(\mathcal{M} - \mathcal{N}) + 2k), \ m = 3, 4, \cdots n. \end{aligned} \tag{36}$$

It easy to perceive that Equation (36) is valid for $m = 3$.

Now, presume that Equation (36) is true for $4, \cdots, m-1$, then right hand side of Equation (35) is

$$\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2m|\Theta_{2m}|}\left(1+\sum_{r=1}^{m-1}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^r r!}\prod_{k=1}^{r-1}(|\hbar|(\mathcal{M}-\mathcal{N})+2k)\right)$$

$$\begin{aligned}
&= \frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2m|\Theta_{2m}|}\left(1+\sum_{r=1}^{m-2}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^r r!}\prod_{k=1}^{r-1}(|\hbar|(\mathcal{M}-\mathcal{N})+2k)\right. \\
&\quad \left.+\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^{m-1}(m-1)!}\prod_{k=1}^{m-2}(|\hbar|(\mathcal{M}-\mathcal{N})+2k)\right), \\
&= \frac{(m-1)|\Theta_{2m-2}|}{m|\Theta_{2m}|}\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^{m-1}(m-1)!|\Theta_{2m-2}|}\prod_{k=1}^{m-2}(|\hbar|(\mathcal{M}-\mathcal{N})+2k)\right. \\
&\quad \left.+\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2m|\Theta_{2m}|}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^{m-1}(m-1)!}\prod_{k=1}^{m-2}(|\hbar|(\mathcal{M}-\mathcal{N})+2k)\right), \\
&= \frac{(m-1)|\hbar|(\mathcal{M}-\mathcal{N})}{2^{m-1}m!|\Theta_{2m}|}\prod_{k=1}^{m-2}(|\hbar|(\mathcal{M}-\mathcal{N})+2k) \\
&\quad +\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2|\Theta_{2m}|}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^{m-1}m!}\prod_{k=1}^{m-2}(|\hbar|(\mathcal{M}-\mathcal{N})+2k), \\
&= \frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^{m-1}m!|\Theta_{2m}|}\prod_{k=1}^{m-2}(|\hbar|(\mathcal{M}-\mathcal{N})+2k)\left((m-1)+\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2}\right), \\
&= \frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^{m-1}m!|\Theta_{2m}|}\prod_{k=1}^{m-2}(|\hbar|(\mathcal{M}-\mathcal{N})+2k)\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})+2(m-1)}{2}\right), \\
&= \frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2^{m}m!|\Theta_{2m}|}\prod_{k=1}^{m-1}(|\hbar|(\mathcal{M}-\mathcal{N})+2k.
\end{aligned}$$

That is, Equation (36) is holds for $m=n$. From Equations (35) and (36) we get Equation (25). Correspondingly we can prove Equation (26). This concludes the proof of Theorem 3. □

Theorem 4. *If $f \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$, then $F \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar,\mathcal{M},\mathcal{N})$, where*

$$F(\xi)=\frac{2}{\xi}\int_0^{\xi} f(t)dt. \tag{37}$$

Proof. From Equation (37) it easy to see that

$$\begin{aligned}
&1+\frac{1}{\hbar}\left[\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}F(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp}F(\xi)-\mathcal{Q}^{\ell}_{\nu,\wp}F(-\xi)}-1\right] \\
&= \frac{2\xi\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)+(\hbar-3)\int_0^{\xi}\mathcal{Q}^{\ell}_{\nu,\wp}f(t)dt+(\hbar-1)\int_0^{\xi}\mathcal{Q}^{\ell}_{\nu,\wp}f(-t)dt}{\hbar\left(\int_0^{\xi}\mathcal{Q}^{\ell}_{\nu,\wp}f(t)dt+\int_0^{\xi}\mathcal{Q}^{\ell}_{\nu,\wp}f(-t)dt\right)}=\frac{R}{S}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\xi S'(\xi)}{S(\xi)} &= \frac{\xi \mathcal{Q}^{\ell}_{\nu,\wp} f(\xi) - \xi \mathcal{Q}^{\ell}_{\nu,\wp} f(-\xi)}{\int_0^{\xi} \mathcal{Q}^{\ell}_{\nu,\wp} f(t)dt + \int_0^{\xi} \mathcal{Q}^{\ell}_{\nu,\wp} f(-t)dt} \\ &= \frac{1}{2}\left(\frac{2\xi G'(\xi)}{G(\xi) - G(-\xi)} + \frac{2(-\xi)G'(-\xi)}{G(-\xi) - G(\xi)}\right), \end{aligned} \tag{38}$$

where $G(\xi) = \int_0^{\xi} \mathcal{Q}^{\ell}_{\nu,\wp} f(t)dt$. Since $f \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar, \mathcal{M}, \mathcal{N})$, it follows that

$$1 + \frac{1}{\hbar}\left[\frac{2\xi G''(\xi)}{G'(\xi) - G'(-\xi)} - 1\right] \prec \frac{1 + \mathcal{M}\xi}{1 + \mathcal{N}\xi},$$

and $G(\xi) \in \mathcal{C}^*_s(\hbar, \mathcal{M}, \mathcal{N}) \subset \mathcal{S}^*_s(\hbar, \mathcal{M}, \mathcal{N}) \subset \mathcal{S}^*_s$. From Equation (38), it follows that $S(\xi)$ is starlike function. In addition to

$$\frac{R'(\xi)}{S'(\xi)} = 1 + \frac{1}{\hbar}\left[\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp} f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp} f(\xi) - \mathcal{Q}^{\ell}_{\nu,\wp} f(-\xi)} - 1\right].$$

Thus

$$\frac{R'(\xi)}{S'(\xi)} = \frac{1 + \mathcal{M}w(\xi)}{1 + \mathcal{N}w(\xi)},$$

it follows that

$$\left|\left(\frac{R'(\xi)}{S'(\xi)} - 1\right)\right| < \left|\mathcal{M} - \mathcal{N}\left(\frac{R'(\xi)}{S'(\xi)}\right)\right|.$$

From Lemma 2, we have

$$\left|\left(\frac{R(\xi)}{S(\xi)} - 1\right)\right| < \left|\mathcal{M} - \mathcal{N}\left(\frac{R(\xi)}{S(\xi)}\right)\right|,$$

and this implies that $F \in \mathcal{SS}^{\ell}_{\nu,\wp}(\hbar, \mathcal{M}, \mathcal{N})$. □

3. The Subclass of $\mathcal{C}^{\ell}_{\nu,\wp}(\hbar, \mathcal{M}, \mathcal{N})$

Theorem 5. *Let $f \in \mathcal{SC}^{\ell}_{\nu,\wp}(\hbar, \mathcal{M}, \mathcal{N})$, then for all $n \geq 1$,*

$$|a_{2n}| \leq \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{(2n-1)!|\Theta_{2n}|} \prod_{k=1}^{2n-2} (|\hbar|(\mathcal{M} - \mathcal{N}) + k), \tag{39}$$

and

$$|a_{2n+1}| \leq \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{2n!|\Theta_{2n+1}|} \prod_{k=1}^{2n-1} (|\hbar|(\mathcal{M} - \mathcal{N}) + k). \tag{40}$$

where Θ_k; $\forall\, k \geq 2$ are given by Equation (15).

Proof. Since $f \in \mathcal{SC}^{\ell}_{\nu,\wp}(\hbar, \mathcal{M}, \mathcal{N})$, Definition 2 of 2, yields

$$1 + \frac{1}{\hbar}\left[\frac{2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp} f(\xi)\right)'}{\mathcal{Q}^{\ell}_{\nu,\wp} f(\xi) + \mathcal{Q}^{\ell}_{\nu,\wp} \overline{f(\overline{\xi})}} - 1\right] = \frac{1 + \mathcal{M}w(\xi)}{1 + \mathcal{N}w(\xi)}. \tag{41}$$

Assuming that

$$h(\xi) = 1 + \sum_{k=1}^{\infty} c_k \xi^k = \frac{1 + \mathcal{M}w(\xi)}{1 + \mathcal{N}w(\xi)}. \tag{42}$$

From Equations (41) and (42), we obtain

$$2\xi\left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi)\right)' = \left(\mathcal{Q}^{\ell}_{\nu,\wp}f(\xi) + \mathcal{Q}^{\ell}_{\nu,\wp}\overline{f(\overline{\xi})}\right)\left(1 + \hbar\sum_{k=1}^{\infty} c_k \xi^k\right).$$

It follows from Equation (14) that

$$\begin{aligned} & \xi + 2a\Theta_2\xi^2 + 3\Theta_3 a_3\xi^3 + 4\Theta_4 a_4\xi^4 + \cdots . + 2n\Theta_{2n}a_{2n}\xi^{2n} \\ + \; & (2n+1)\Theta_{2n+1}a_{2n+1}\xi^{2n+1} + \cdots \\ = \; & (\xi + \Theta_2 a_2\xi^2 + \Theta_3 a_3\xi^3 + \Theta_4 a_4\xi^4 + \cdots + \Theta_{2n}a_{2n}\xi^{2n} + \Theta_{2n+1}a_{2n+1}\xi^{2n+1} + \cdots) \\ \times \; & \left(1 + \hbar c_1\xi + \hbar c_2\xi^2 + \cdots\right). \end{aligned}$$

Equating like powers of ξ, we obtain

$$\Theta_2 a_2 = \hbar c_1, \tag{43}$$

$$2\Theta_3 a_3 = \hbar c_2 + \hbar c_1\Theta_2 a_2, \tag{44}$$

$$3\Theta_4 a_4 = \hbar c_3 + \hbar c_2\Theta_2 a_2 + \hbar c_1\Theta_3 a_3, \tag{45}$$

$$4\Theta_5 a_5 = \hbar c_4 + \hbar c_3\Theta_2 a_2 + \hbar c_2\Theta_3 a_3 + \hbar c_1\Theta_4 a_4, \tag{46}$$

$$(2n-1)\Theta_{2n}a_{2n} = \hbar c_{2n-1} + \hbar c_{2n-2}\Theta_2 a_2 + \ldots\ldots + \hbar c_2\Theta_{2n-2}a_{2n-2} + \hbar c_1\Theta_{2n-1}a_{2n-1}, \tag{47}$$

$$2n\Theta_{2n+1}a_{2n+1} = \hbar c_{2n} + \hbar c_{2n-1}\Theta_2 a_2 + \ldots\ldots + \hbar c_2\Theta_{2n-1}a_{2n-1} + \hbar c_1\Theta_{2n}a_{2n}. \tag{48}$$

Applying Lemma 1, to Equations (43)–(46) respectively, we get

$$\begin{aligned} |a_2| &\leq \frac{|\hbar|}{|\Theta_2|}(\mathcal{M} - \mathcal{N}), \\ |a_3| &\leq \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{2|\Theta_3|}(1 + |\hbar|(\mathcal{M} - \mathcal{N})), \\ |a_4| &\leq \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{2.3|\Theta_4|}(1 + |\hbar|(\mathcal{M} - \mathcal{N}))(2 + |\hbar|(\mathcal{M} - \mathcal{N})) \end{aligned}$$

and

$$|a_5| \leq \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{2.3.4|\Theta_5|}(1 + |\hbar|(\mathcal{M} - \mathcal{N}))(2 + |\hbar|(\mathcal{M} - \mathcal{N}))(3 + |\hbar|(\mathcal{M} - \mathcal{N})).$$

It tails that Equations (39) and (40) hold for $n = 1,\ 2$. Equation (47) in conjunction with Lemma 1 yields

$$|a_{2n}| \leq \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{(2n-1)|\Theta_{2n}|}\left(1 + \sum_{r=1}^{n-1}|\Theta_{2r}||a_{2r}| + \sum_{r=1}^{n-1}|\Theta_{2r+1}||a_{2r+1}|\right). \tag{49}$$

Subsequently, we accept that Equations (39) and (40) hold for $3, 4, \ldots, n-1$. Thus inequality Equation (49) yields

$$\begin{aligned} c|a_{2n}| \leq \; & \frac{|\hbar|(\mathcal{M} - \mathcal{N})}{(2n-1)|\Theta_{2n}|}\left(1 + \sum_{r=1}^{n-1}\frac{|\hbar|(\mathcal{M} - \mathcal{N})}{(2r-1)!}\prod_{i=1}^{2r-2}(i + |\hbar|(\mathcal{M} - \mathcal{N}))\right. \\ & \left. + \sum_{r=1}^{n-1}\frac{|\hbar|(\mathcal{M} - \mathcal{N})}{(2r)!}\prod_{i=1}^{2r-1}(i + |\hbar|(\mathcal{M} - \mathcal{N}))\right). \end{aligned} \tag{50}$$

In order to complete the proof it is enough to prove that

$$\begin{aligned}
&\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)|\Theta_{2m}|}\left(1+\sum_{r=1}^{m-1}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2r-1)!}\prod_{i=1}^{2r-2}(i+|\hbar|(\mathcal{M}-\mathcal{N}))\right.\\
&\left.+\sum_{r=1}^{m-1}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2r)!}\prod_{i=1}^{2r-1}(i+|\hbar|(\mathcal{M}-\mathcal{N}))\right)\\
&=\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)!|\Theta_{2m}|}\prod_{i=1}^{2m-2}(i+|\hbar|(\mathcal{M}-\mathcal{N})).
\end{aligned} \tag{51}$$

For $m=3$ we readily show Equation (51) is valid. Now, assume that Equation (51) is true for $4,\cdots,m-1$. Then Equation (50) is computed as below

$$\begin{aligned}
&\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)|\Theta_{2m}|}(1+\sum_{r=1}^{m-1}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2r-1)!}\prod_{i=1}^{2r-2}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\\
&+\sum_{r=1}^{m-1}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{2r!}\prod_{i=1}^{2r-1}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\\
=&\frac{(2m-3)|\Theta_{2m-2}|}{(2m-1)|\Theta_{2m}|}\left[\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-3)|\Theta_{2m-2}|}(1+\sum_{r=1}^{m-2}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2r-1)!}\prod_{i=1}^{2r-2}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\right.\\
&\left.+\sum_{r=1}^{m-2}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2r)!}\prod_{i=1}^{2r-1}(|\hbar|(\mathcal{M}-\mathcal{N})+i))\right]\\
&+\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)|\Theta_{2m}|}\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-3)!}\prod_{i=1}^{2m-4}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\right)\\
&+\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)|\Theta_{2m}|}\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-2)!}\prod_{i=1}^{2m-3}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\right),\\
=&\frac{(2m-3)|\Theta_{2m-2}|}{(2m-1)|\Theta_{2m}|}\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-3)!|\Theta_{2m-2}|}\prod_{i=1}^{2m-4}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\right)\\
&+\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)|\Theta_{2m}|}\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-3)!}\prod_{i=1}^{2m-4}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\right)\\
&+\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)|\Theta_{2m}|}\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-2)!}\prod_{i=1}^{2m-3}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\right),\\
=&\frac{1}{(2m-1)|\Theta_{2m}|}\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-3)!}\prod_{i=1}^{2m-4}(|\hbar|(\mathcal{M}-\mathcal{N})+i))(|\hbar|(\mathcal{M}-\mathcal{N})+(2m-3)\\
&+\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{|\Theta_{2m}|}\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)!}\prod_{i=1}^{2m-3}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\right),\\
=&\left(\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)!|\Theta_{2m}|}\prod_{i=1}^{2m-3}(|\hbar|(\mathcal{M}-\mathcal{N})+i)\right)(|\hbar|(\mathcal{M}-\mathcal{N})+2m-2),\\
=&\frac{|\hbar|(\mathcal{M}-\mathcal{N})}{(2m-1)!|\Theta_{2m}|}\prod_{i=1}^{2m-2}(|\hbar|(\mathcal{M}-\mathcal{N})+i).
\end{aligned}$$

That is, Equation (51) is holds for $m=n$. From Equations (50) and (51) we get Equation (39). On lines similar to above, we can prove Equation (40). Thus the proof is complete. □

4. Conclusions

The interaction of geometry and analysis is a key ingredient in the study of complex function theory. This speedy development is strongly related to the relationship between the geometric behavior and the analytical structure. In the current study, we got acquainted with a new one star functions with respect to symmetric points and symmetric of conjugate points that are related to the of the Miller-Ross type Poisson distribution function in the Janowski domain. We studied certain characteristic properties, coefficient bindings and closure properties in the integral transformation. Furthermore, by setting $\hbar = 1$, we can derive results for the function class given in the Example 1. Further one can extend the study by defining some new subclasses of Sakaguchi-type functions involving Miller-Ross type Poisson distribution function, by using the concept of Janowski functions in conic regions and investigate various interesting properties such as sufficiency criteria, coefficient estimates and distortion result in addition logarithmic coefficient [27,31], Fekete-Szegö inequalities [24,29,32] and coefficients of inverse functions can be obtained. As an application,of subordination concept one can provide an explicit construction for the complex potential (the complex velocity) and the stream function of two-dimensional fluid flow problems over a circular cylinder using both vortex and source/sink. Further determine the fluid flow produced by a single source and construct a univalent function so that the image of a source is also a source for a given complex potential (see [33]). This method can be applied to other important classes of functions such as meromorphic, bi -univalent, and harmonic functions and many interesting aspects of this work have been studied, such as corresponding appropriate formulas, coefficient bounds, distortion theorems, closure theorems, and the endpoint theorem (see [34]).

Author Contributions: Conceptualization, S.M.E.-D., A.A. and G.M.; methodology, S.M.E.-D., A.A. and G.M.; validation, S.M.E.-D., G.M. and A.A.; formal analysis, A.A., S.M.E.-D. and G.M.; investigation, S.M.E.-D., G.M. and A.A.; resources, S.M.E.-D., A.A. and G.M.; writing—original draft preparation, S.M.E.-D., A.A. and G.M.; writing—review and editing, S.M.E.-D., A.A. and G.M.; supervision, A.A., S.M.E.-D. and G.M.; project administration, S.M.E.-D., A.A. and G.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The researchers would like to thank the Deanship of Scientific Research, Qassim University, for funding the publication of this project.

Conflicts of Interest: The authors declare no conflict of interest.

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