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Analytic and Asymptotic Properties of the Generalized Student and Generalized Lomax Distributions

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Abstract: Analytic and asymptotic properties of the generalized Student and generalized Lomax distributions are discussed, with the main focus on the representation of these distributions as scale mixtures of the laws that appear as limit distributions in classical limit theorems of probability theory, such as the normal, folded normal, exponential, Weibull, and Fréchet distributions. These representations result in the possibility of proving some limit theorems for statistics constructed from samples with random sizes in which the generalized Student and generalized Lomax distributions are limit laws. An overview of known properties of the generalized Student distribution is given, and some simple bounds for its tail probabilities are presented. An analog of the 'multiplication theorem' is proved, and the identifiability of scale mixtures of generalized Student distributions is considered. The normal scale mixture representation for the generalized Student distribution is discussed, and the properties of the mixing distribution in this representation are studied. Some simple general inequalities are proved that relate the tails of the scale mixture with that of the mixing distribution. It is proved that for some values of the parameters, the generalized Student distribution is infinitely divisible and admits a representation as a scale mixture of Laplace distributions. Necessary and sufficient conditions are presented that provide the convergence of the distributions of sums of a random number of independent random variables with finite variances and other statistics constructed from samples with random sizes to the generalized Student distribution. As an example, the convergence of the distributions of sample quantiles in samples with random sizes is considered. The generalized Lomax distribution is defined as the distribution of the absolute value of the random variable with the generalized Student distribution. It is shown that the generalized Lomax distribution can be represented as a scale mixture of folded normal distributions. The convergence of the distributions of maximum and minimum random sums to the generalized Lomax distribution is considered. It is demonstrated that the generalized Lomax distribution can be represented as a scale mixture of Weibull distributions or that of Fréchet distributions. As a consequence, it is demonstrated that the generalized Lomax distribution can be limiting for extreme statistics in samples with random size. The convergence of the distributions of mixed geometric random sums to the generalized Lomax distribution is considered, and the corresponding extension of the famous Rényi theorem is proved. The law of large numbers for mixed Poisson random sums is presented, in which the limit random variable has a generalized Lomax distribution.

Keywords: generalized Student distribution; generalized Lomax distribution; exponential power distribution; scale mixture; limit theorem; random sum; statistic constructed from sample with random size

MSC: 60F05; 60G50; 60G55; 62E20; 62G30

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1. Introduction

1.1. History of the Problem and Motivation

The *t*-distribution, which is more often called the Student distribution, was proposed in 1908 in the fundamental paper [1] by William Sealy Gosset published in *Biometrika* under the pseudonym 'Student'. Originally, this distribution played only a technical role in the so-called theory of errors. In the paper [2], R. Fisher gave a detailed description of the application of the Student distribution in problems related to the statistical analysis of normal samples. However, when, in the middle of the 20th century, it was noticed that the distributions of various financial data (e.g., increments of stock prices) do not meet the normal model and have noticeably heavier tails with power-type decreases, some specialists turned to the Student distribution as a heavy-tailed alternative to the normal distribution. Now, the Student distribution is one of the most popular models for economic and financial data [3]. In the paper [4], an attempt was made to explain the adequacy of the Student model from the viewpoint of limit theorems of probability theory, and it was demonstrated that, in descriptive statistics, this distribution can be used as an asymptotic approximation since it appears as the limit law for statistics constructed from samples when the sample size obeys the negative binomial distribution.

In recent years, many generalizations of the Student distribution have been proposed, including those that are purely analytic [5] and purely artificial [6]. A comprehensive review of generalizations of the Student distribution was presented in [3]. Unfortunately, many generalizations are in some sense formal, not-so-well theoretically justified, and are based on the reasons of convenience of fitting to particular data. In the present paper, primary attention is paid to the generalization of the Student distribution that is based on the representation of a so-distributed random variable as a quotient of two independent random variables. The numerator in this quotient is the random variable with the exponential power distribution, whereas the denominator is the power of a gamma-distributed random variable with identical shape and scale parameters. This generalization is due to Mcdonald and Newey [7] (see also [8,9]), who noticed that the generalized Student distribution as defined can be obtained as the scale mixture of a power exponential distribution where the mixing law is the inverse generalized gamma distribution. The scale mixture representation opens the way to construct rather simple asymptotic settings in which the appropriately defined generalized Student distribution appears as a limit law. Consequently, the generalized Student distribution obtains a theoretic foundation as an asymptotic approximation. Apparently, it is this property that makes the generalized Student distribution an attractive model for financial data [10–13]. This approach is also very promising for the construction of multivariate and asymmetric generalizations, e.g., see [14].

Since heavy-tailed distributions are widely encountered in many practical problems, they are under serious theoretic study. For example, there are developments in the context of the Tsallis entropy that result in power-law distributions and fractional differential operators. In both cases, we also have a connection with stable distributions and Lévy processes (see, e.g., [15]). Although stable Lévy processes with power-type tails have very serious theoretic grounds, they are not so easily statistically treated because, with four rather trivial exceptions, stable densities cannot be represented in terms of elementary functions. Simple representations for the generalized Student densities make them promising alternatives to stable laws. Moreover, the analytic properties (e.g., the infinite divisibility) of the generalized Student distributions and limit theorems for sums of independent random variables with the generalized Student distributions as the limit laws presented below, together with the functional limit theorems for compound Cox processes proved in [16], guarantee the possibility to construct a Lévy process (more exactly, a subordinated Wiener process) whose finite-dimensional distributions are of the generalized Student type.

Another benefit of the approach based off the scale mixture representation is that it makes it possible to easily trace the relationship of the generalized Student distribution with the generalized Lomax distribution, which is a popular power-type heavy-tailed model that was used in many applied problems after it was introduced in [17], where it was used to analyze business failure data. The Lomax distribution appeared to be a convenient heavy-tailed alternative to exponential, gamma, and Weibull distributions [18]. Possible applications of the Lomax distribution and its generalizations involve many fields, from modelling business records [19] to reliability and lifetime testing [20]. An extensive bibliography can be found in [21]. Various generalizations of the Lomax distribution were used in [22–26] and many other studies; see the extensive bibliography in [21].

In accordance with the approach that is used in the present paper, the generalized Lomax distribution is just the distribution of the absolute value of a random variable with the generalized Student distribution. This definition makes it possible to study the important analytic properties of the so-generalized Lomax distribution, such as its infinite divisibility, identifiability, and mixture representability. In turn, these properties open the way to proving limit theorems in rather simple asymptotic settings in which the generalized Lomax distribution appears to be the limit law. These limit theorems may serve as a theoretical foundation for the adequacy of the generalized Lomax distribution as an asymptotic approximation in descriptive statistics and an explanation of the excellent fit of this distribution to real data in many cases.

In the present paper, we study analytic and asymptotic properties of the generalized Student and generalized Lomax distributions, paying main attention to the representation of these distributions as scale mixtures of the laws that appear as limit distributions in classical limit theorems of probability theory, such as the normal, folded normal, exponential, Weibull, and Fréchet distributions. These representations result in the possibility of proving some limit theorems for statistics constructed from samples with random sizes in which the generalized Student and generalized Lomax distributions are limit laws. Unlike the conventional analytical approach used in most papers on generalized Student or generalized Lomax distributions, in the present paper, we use a kind of 'arithmetic' way of reasoning within the space of random variables. According to this approach, instead of the operation of scale mixing distributions, we consider the operation of multiplication of random variables, provided the multipliers are independent. Nevertheless, speaking of random variables, we actually deal with their distributions. This approach makes the reasoning substantially simpler, the proofs shorte, and reveals some general features of the distributions under consideration.

The paper is organized as follows. Section 1.2 contains auxiliary definitions and introduces some basic properties of the distributions involved in the subsequent reasoning. In Section 2.1, an overview of known the properties of the generalized Student distribution is given, and some simple bounds for its tail probabilities are presented; furthermore, an analog of the 'multiplication theorem' is proved, and the identifiability of scale mixtures of generalized Student distributions is considered. In Section 2.2, the normal scale mixture representation for the generalized Student distribution is discussed, and the properties of the mixing distribution in this representation are studied. In particular, in order to study the tail probabilities of the mixing distributions, some simple general inequalities are proved here that relate the tails of the scale mixture with those of the mixing distribution. It is proved here that for some values of the parameters, the generalized Student distribution is infinitely divisible and admits a representation as a scale mixture of Laplace distributions. In Section 2.3, necessary and sufficient conditions are presented that provide the convergence of the distributions of sums of a random number of independent random variables with finite variances to the generalized Student distribution. Section 2.4 presents necessary and sufficient conditions that provide the convergence of the distributions of 'asymptotically normal' statistics constructed from samples with random sizes to the generalized Student distribution. As an example, the convergence of the distributions of sample quantiles in samples with random sizes is considered. Section 3.1 contains the definition and basic properties of the generalized Lomax distribution. In Section 3.2, it is shown that the generalized Lomax distribution can be represented as a scale mixture of the folded normal distribution (the distribution of the maximum of the standard Wiener process on the unit interval). In Section 3.3, the convergence of the distributions of maximum and minimum random sums to the generalized Lomax distribution is considered. In Section 3.4, it is demonstrated that the generalized Lomax distribution can be represented as a scale mixture of Weibull distributions or as a mixture of Fréchet distributions. These representations make it possible to demonstrate in Section 3.5 that the generalized Lomax distribution can be limiting for extreme statistics in samples with a random size. Finally, in Section 3.6, the convergence of the distributions of mixed geometric random sums to the generalized Lomax distribution is considered, and the corresponding extension of the famous Rényi theorem is proved.

1.2. Auxiliary Definitions and Notation

All the random variables are assumed to be defined on one and the same probability space $(\Omega, \mathcal{A}, \mathsf{P})$.

The product of *independent*random elements will be denoted by the symbol $\circ$. The symbols $\stackrel{d}{=}$ and $\Longrightarrow$ will stand for the coincidence of distributions and convergence in distribution, respectively. The symbol $\square$ marks the end of the proof. The indicator function of a set A will be denoted $\mathbb{I}_A(z)$: if $z \in A$, then $\mathbb{I}_A(z) = 1$; otherwise, $\mathbb{I}_A(z) = 0$.

A random variable with the standard exponential distribution will be denoted W_1, as follows:

$$\mathsf{P}(W_1 < x) = \left[1 - e^{-x}\right]\mathbb{I}_{[0,\infty)}(x).$$

For $x > 0$ and $r > 0$, the (lower) incomplete gamma function will be denoted as $\Gamma(r; x)$:

$$\Gamma(r; x) = \int_0^x z^{r-1} e^{-z} dz.$$

Let $\Gamma(r) \stackrel{\text{def}}{=} \Gamma(r; \infty)$ be the 'usual' Euler's gamma function.

A random variable having a gamma distribution with a shape parameter $r > 0$ and a scale parameter $\lambda > 0$ will be denoted as $G_{r,\lambda}$, where

$$\mathsf{P}(G_{r,\lambda} < x) = \int_0^x g(z; r, \lambda) dz, \ \text{ with } \ g(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \mathbb{I}_{[0,\infty)}(x),$$

Obviously, in this notation, $G_{1,1}$ is a random variable with the standard exponential distribution $G_{1,1} = W_1$.

A generalized gamma distribution is an absolutely continuous distribution defined by the density

$$gg_{r,\alpha,\mu}(x) = \frac{|\alpha|\mu^r}{\Gamma(r)} x^{\alpha r-1} e^{-\mu x^{\alpha}} \mathbb{I}_{[0,\infty)}(x)$$

with $\alpha \in \mathbb{R}$, $\mu > 0$, and $r > 0$. A random variable with the density $gg_{r,\alpha,\mu}(x)$ will be denoted as $\overline{G}_{r,\alpha,\mu}$. It is easy to see that

$$\overline{G}_{r,\alpha,\mu} \stackrel{d}{=} G_{r,\mu}^{1/\alpha} \stackrel{d}{=} \mu^{-1/\alpha} G_{r,1}^{1/\alpha} \stackrel{d}{=} \mu^{-1/\alpha} \overline{G}_{r,\alpha,1}. \tag{1}$$

Let $\gamma > 0$. The distribution of the random variable W_γ:

$$\mathsf{P}\big(W_\gamma < x\big) = \left[1 - e^{-x^{\gamma}}\right]\mathbb{I}_{[0,\infty)}(x),$$

is called the Weibull distribution with a shape parameter γ. It is easy to see that

$$W_1^{1/\gamma} \stackrel{d}{=} W_\gamma \stackrel{d}{=} \overline{G}_{1,\gamma,1}. \tag{2}$$

The random variable W_α^{-1} is said to have an inverse Weibull or Fréchet distribution, as follows:

$$\mathsf{P}(W_\alpha^{-1} < x) = \mathsf{P}(W_\alpha \geq \tfrac{1}{x}) = \exp\{x^{-\alpha}\}, \quad x \geq 0.$$

The standard normal distribution function and its density will be denoted by $\Phi(x)$ and $\phi(x)$, where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \Phi(x) = \int_{-\infty}^{x} \phi(z)dz,$$

respectively. A random variable with the standard normal distribution will be denoted by X.

A random variable with the strictly stable characteristic function

$$\mathfrak{g}_{\alpha,\theta}(t) = \exp\Big\{ -|t|^{\alpha} \exp\Big\{ -\frac{i\pi\theta\alpha}{2} \mathrm{sign} t \Big\}\Big\}, \quad t \in \mathbb{R}, \tag{3}$$

where $0 < \alpha \leq 2$, $|\theta| \leq \theta_{\alpha} = \min\{1, \frac{2}{\alpha} - 1\}$, will be denoted by $S_{\alpha,\theta}$. The probability density of the random variable $S_{\alpha,\theta}$ will be denoted by $S_{\alpha,\theta}$. For the properties of stable distributions with characteristic functions (3), see, e.g., [15,27,28].

It is easy to see that $S_{2,0} \stackrel{d}{=} \sqrt{2}X$.

If $\theta = 1$ and $0 < \alpha \leq 1$, the corresponding strictly stable random variable takes only nonnegative values. If $\alpha = 1$ and $\theta = \pm 1$, then the corresponding stable distributions are degenerate in ± 1, respectively. All the other strictly stable distributions are absolutely continuous. There are no explicit representations for stable distributions in terms of elementary functions with four exceptions: the normal distribution ($\alpha = 2$, $\theta = 0$), the Cauchy distribution ($\alpha = 1$, $\theta = 0$), the Lévy distribution ($\alpha = \frac{1}{2}$, $\theta = 1$) and the distribution symmetric to the Lévy law ($\alpha = \frac{1}{2}$, $\theta = -1$). Expressions for stable densities in terms of generalized Meijer G-functions (Fox functions) can be found in [29,30].

According to the 'multiplication theorem' (see, e.g., Theorem 3.3.1 in [27]), for any admissible pair of parameters (α, θ) and any $\alpha' \in (0, 1]$, the product representation

$$S_{\alpha\alpha',\theta} \stackrel{d}{=} S_{\alpha',1}^{1/\alpha} \circ S_{\alpha,\theta}$$

holds. In particular, for any $\alpha \in (0, 2]$,

$$S_{\alpha,0} \stackrel{d}{=} \sqrt{2S_{\alpha/2,1}} \circ X,$$

that is, any symmetric strictly stable distribution is a scale mixture of the normal distributions.

Let $\alpha > 0$. The symmetric exponential power distribution is an absolutely continuous distribution defined by its Lebesgue probability density

$$p_{\alpha}(x) = \frac{\alpha}{2\Gamma(\frac{1}{\alpha})} \cdot e^{-|x|^{\alpha}}, \quad -\infty < x < \infty. \tag{4}$$

To simplify the notation and calculation, here and in what follows, we will use a single parameter α in Representation (4) since this parameter is, in some sense, characteristic of and determines the shape of the distribution (4). With $\alpha = 1$, Relationship (4) defines the classical Laplace distribution as

$$p_1(x) = \tfrac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}$$

with zero mean and a variance of 2. With $\alpha = 2$, Relationship (4) defines the normal (Gaussian) distribution with a zero mean and a variance of $\frac{1}{2}$. Any random variable with a probability density $p_{\alpha}(x)$ will be denoted by Q_{α}.

The class of distributions (4) was introduced and studied in 1923 in the paper [31] by M. T. Subbotin. For more details concerning the properties of exponential power distributions, see [32,33] and the references therein.

It is easy to make sure that

$$|Q_{\alpha}|^{\alpha} \stackrel{d}{=} G_{1/\alpha,1}. \tag{5}$$

In our further reasoning, we will exploit the following properties of exponential power distributions. For convenience, we present them as lemmas.

Lemma 1 (e.g., see [32]). *For $\delta > -1$, we have*

$$\mathsf{E}|Q_\alpha|^\delta = \frac{\alpha}{\Gamma(\frac{1}{\alpha})}\int_0^\infty x^\delta e^{-x^\alpha}dx = \frac{\Gamma(\frac{\delta+1}{\alpha})}{\Gamma(\frac{1}{\alpha})}.$$

Lemma 2 ([32]). *Let $\alpha \in (0,2]$, $\alpha' \in (0,1]$. Then,*

$$Q_{\alpha\alpha'} \stackrel{d}{=} Q_\alpha \circ U_{\alpha,\alpha'}^{-1/\alpha}, \tag{6}$$

where $U_{\alpha,\alpha'}$ is a random variable such that if $\alpha' = 1$, then $U_{\alpha,\alpha'} = 1$ for any $\alpha \in (0,2]$, and if $0 < \alpha' < 1$, then $U_{\alpha,\alpha'}$ is absolutely continuous with a probability density

$$u_{\alpha,\alpha'}(x) = \frac{\alpha'\Gamma(\frac{1}{\alpha})}{\Gamma(\frac{1}{\alpha\alpha'})} \cdot \frac{s_{\alpha',1}(x)}{x^{1/\alpha}} \cdot \mathbb{I}_{(0,\infty)}(x).$$

Corollary 1 ([34]). *Any symmetric exponential power distribution with $\alpha \in (0,2]$ is a scale mixture of normal laws:*

$$Q_\alpha \stackrel{d}{=} \sqrt{\tfrac{1}{2}U_{2,\alpha/2}^{-1}} \circ X.$$

Corollary 2 (e.g., see [32]). *Any symmetric exponential power distribution with $\alpha \in (0,1]$ is a scale mixture of Laplace laws:*

$$Q_\alpha \stackrel{d}{=} U_{1,\alpha}^{-1} \circ Q_1.$$

Lemma 3 ([32]). *For any $\alpha \in (0,1]$, the distribution of the random variable $U_{2,\alpha/2}^{-1}$ is a mixed exponential:*

$$U_{2,\alpha/2}^{-1} \stackrel{d}{=} 4U_{1,\alpha}^{-2} \circ W_1.$$

Recall that a distribution function $F(x)$ whose characteristic function is denoted $f(t)$ is infinitely divisible if, for each $n \in \mathbb{N}$, there exists a characteristic function $f_n(t)$ such that $f(t) = f_n^n(t)$ and $t \in \mathbb{R}$. In terms of random variables (if the probability space $(\Omega, \mathcal{A}, \mathsf{P})$ is rich enough), this means that for each $n \in \mathbb{N}$, there exist independent identically distributed random variables $Y_{n,1}, Y_{n,2}, \dots, Y_{n,n}$ such that a random variable Y whose distribution function is $F(x)$ admits the representation $Y = Y_{n,1} + Y_{n,2} + \dots + Y_{n,n}$. The property of infinite divisibility is very important in some problems. For example, infinite divisible distributions exist, and only they can be limiting for sums of independent asymptotically negligible (in particular, identically distributed) random variables (see [35]. Moreover, this is crucial in the construction of Lévy processes (see, e.g., [15,16]).

Corollary 3 ([32]). *For any $\alpha \in (0,1]$, the distribution of the random variable $U_{2,\alpha/2}^{-1}$ is infinitely divisible.*

In the present paper, we consider the generalizations of the Student and Lomax distributions.

The Student distributionwas introduced in [1] and is defined as the distribution of the random variable

$$T_r \stackrel{d}{=} X \circ G_{r,r}^{-1/2},$$

where $r > 0$ is the shape parameter usually called 'the degrees of freedom'. The probability density of the Student distribution, up to scale and location transformation, has the form

$$f_r(x) = \frac{\Gamma(r+\frac{1}{2})}{\sqrt{\pi r}\Gamma(r)}\Big(1+\frac{x^2}{r}\Big)^{-(r+1/2)}, \quad x \in \mathbb{R}.$$

The Lomax distribution, also called the Pareto Type II distribution, was introduced in [17]. The probability density of the Lomax distribution, up to scale and location transformation, has the form

$$f_r^*(x) = \frac{r}{(1+x)^{r+1}}, \quad x \geq 0,$$

where $r > 0$ is the shape parameter.

2. The Generalized Student Distribution

2.1. The Definition and Elementary Properties of the Generalized Student Distribution

Let $\alpha \in (0,2]$ and $r \in \mathbb{R}$ be such that $\alpha r \geq 1$. Assume that the random variables Q_α and $G_{r,r}$ are independent. Consider the random variable $T_{r,\alpha}$, defined as the product

$$T_{r,\alpha} \stackrel{\text{def}}{=} Q_\alpha \circ G_{r,r}^{-1/\alpha}. \tag{7}$$

The distribution of the random variable $T_{r,\alpha}$ will be called a generalized Student distribution with parameters α and r. (It should be noted that in [14], instead of $-\frac{1}{\alpha}$, the exponent of $G_{r,r}$ is $-\frac{1}{2}$, which does not restrict generality but leads to more complicated notation).

Find the probability density function $f_{r,\alpha}(x)$ of $T_{r,\alpha}$. Since Q_α and $G_{r,r}$ are independent, by the Fubini theorem, we have

$$f_{r,\alpha}(x) = \frac{\alpha r^r}{2\Gamma(r)\Gamma(\frac{1}{\alpha})}\int_0^\infty u^{1/\alpha}e^{-u|x|^\alpha}u^{r-1}e^{-ru}du =$$

$$= \frac{\alpha r^r}{2\Gamma(r)\Gamma(\frac{1}{\alpha})(r+|x|^\alpha)^{r+1/\alpha}}\int_0^\infty u^{r+1/\alpha-1}e^{-u}du =$$

$$= \frac{\alpha\Gamma(r+\frac{1}{\alpha})}{2r^{1/\alpha}\Gamma(r)\Gamma(\frac{1}{\alpha})}\Big(1+\frac{|x|^\alpha}{r}\Big)^{-(r+1/\alpha)} = \frac{\alpha}{2r^{1/\alpha}\mathrm{B}(r,\frac{1}{\alpha})}\Big(1+\frac{|x|^\alpha}{r}\Big)^{-(r+1/\alpha)}, \quad x \in \mathbb{R}. \tag{8}$$

Here and in what follows, $\mathrm{B}(a,b)$ is the beta-function:

$$\mathrm{B}(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a > 0,\ b > 0.$$

It is easily seen that with $\alpha = 2$, the generalized Student distribution turns into the classical Student distribution up to the re-parametrization. If, in addition, $r = 1$, the generalized Student distribution is a Cauchy distribution.

When $\alpha = 1$, the generalized Student distribution turns into a two-sided Lomax distribution.

We see that the family of generalized Student distributions is wide enough and contains popular power-type-tailed laws.

Moreover, this family is flexible enough since it contains distributions with various shapes of their vertices. Consider this variety in more detail. First, from (8), it follows that the densities of all the generalized Student distributions are finite:

$$\max_x f_{r,\alpha}(x) = f_{r,\alpha}(0) = \frac{\alpha\mathrm{B}(r,\frac{1}{\alpha})}{2r^{1/\alpha}}.$$

Second, consider the behavior of the derivative of the density $f_{r,\alpha}(x)$ in the neighborhood of zero. Since $f_{r,\alpha}(x)$ is symmetric, it suffices to consider $x > 0$. For such x, we have

$$\frac{d}{dx}f_{r,\alpha}(x) = -\frac{\alpha^2 \mathrm{B}(r, \frac{1}{\alpha})x^{\alpha-1}}{2r^{1/\alpha}}\Big(1+\frac{x^\alpha}{r}\Big)^{-(r+1)}.$$

Therefore, if $\alpha > 1$, then

$$\lim_{x\to 0+}\frac{d}{dx}f_{r,\alpha}(x) = 0;$$

that is, the vertex of $f_{r,\alpha}(x)$ is smooth and rather flat.

If $\alpha = 1$, then

$$\lim_{x\to 0+}\frac{d}{dx}f_{r,\alpha}(x) = -\frac{\alpha^2}{2r^{1/\alpha}\mathrm{B}(r, \frac{1}{\alpha})};$$

that is, the vertex of $f_{r,\alpha}(x)$ looks like a non-zero angle.

If $\alpha < 1$, then

$$\lim_{x\to 0+}\frac{d}{dx}f_{r,\alpha}(x) = -\infty;$$

that is, in this case, the vertex of $f_{r,\alpha}(x)$ is 'infinitely' sharp.

The two last cases noticeably differ from the traditional Student density shape.

As is demonstrated by the two following statements, when r increases, the tails of a generalized Student distribution become less heavy, so that finally, a generalized Student distribution turns into an exponential power distribution.

Proposition 1. *The following asymptotic relationship holds:*

$$\lim_{r\to\infty}\sup_x\Big|f_{r,\alpha}(x) - \frac{\alpha e^{-|x|^\alpha}}{2\Gamma(\frac{1}{\alpha})}\Big| = 0. \tag{9}$$

Proof. Note that the relationships

$$\lim_{r\to\infty}\Big(1+\frac{|x|^\alpha}{r}\Big)^r = e^{|x|^\alpha} \quad \text{and} \quad \lim_{r\to\infty}\frac{\Gamma(r+\frac{1}{\alpha})}{r^{1/\alpha}\Gamma(r)} = 1$$

imply the point-wise convergence of the densities. Since the limit exponential power density function is monotone on each semi-axis, as well as bounded and continuous, by the Dini theorem, the convergence is uniform in $x \in \mathbb{R}$. □

This property of the generalized Student distributions can be mathematically formulated in terms of distribution functions as well. For $\alpha \in (0,2]$ and $r > 0$, denote $F_{r,\alpha}(x) = \mathsf{P}(T_{r,\alpha} < x)$, $x \in \mathbb{R}$,

$$H_\alpha(x) \stackrel{\text{def}}{=} \mathsf{P}(Q_\alpha < x) = \begin{cases} \dfrac{1}{2} + \dfrac{\Gamma(\frac{1}{\alpha};\, x^\alpha)}{2\Gamma(\frac{1}{\alpha})}, & x \geq 0, \\[2ex] \dfrac{1}{2} - \dfrac{\Gamma(\frac{1}{\alpha};\, |x|^\alpha)}{2\Gamma(\frac{1}{\alpha})}, & x < 0. \end{cases}$$

Corollary 4. *For any $\alpha \in (0,2]$, as $r \to \infty$, the distribution functions of the random variables $T_{r,\alpha}$ converge to the exponential power distribution function $H_\alpha(x)$ uniformly in $x \in \mathbb{R}$:*

$$\lim_{r\to\infty}\sup_x|F_{r,\alpha}(x) - H_\alpha(x)| = 0.$$

Proof. This statement follows from Proposition 4 by the Lebesgue-dominated convergence theorem and the Dini theorem mentioned above.

Another way of proving this result is as follows. Let $[a]$ and $\{a\}$, correspondingly, denote the integer part and the fractional part of a real number a. Represent r as $r = [r] + \{r\}$. Then, the random variable $G_{r,r}$ can be represented as

$$G_{r,r} \stackrel{d}{=} \tfrac{1}{r}G_{r,1} \stackrel{d}{=} \frac{1}{r}\sum_{j=1}^{[r]} G_{1,1} + \frac{G_{\{r\},1}}{r}.$$

As $r \to \infty$, the first summand on the right-hand side of this relation almost surely converges to 1 by the strong law of large numbers, whereas the second summand almost surely converges to zero. This means that $G_{r,r} \longrightarrow 1$ almost surely converges to 1. Now, by the Slutsky theorem [36] (see also [37], Sect. 20.6), it follows from the definition of $T_{r,\alpha}$ that $T_{r,\alpha} \Longrightarrow Q_\alpha$. Since the limit function $H_\alpha(x)$ is monotone, bounded, and continuous, by the Dini theorem, the convergence of distribution functions is uniform in $x \in \mathbb{R}$. □

Now consider the moments of the generalized Student distribution.

Proposition 2. *For any $\delta \in (-1, \alpha r)$*

$$\mathsf{E}|T_{r,\alpha}|^\delta = \mathsf{E}G_{r,r}^{-\delta/\alpha} \cdot \mathsf{E}|Q_\alpha|^\delta = \frac{r^{\delta/\alpha}\Gamma(r - \frac{\delta}{\alpha})\Gamma(\frac{\delta+1}{\alpha})}{\Gamma(r)\Gamma(\frac{1}{\alpha})}.$$

Proof. This relationship follows from (7) and Lemma 1. □

The distribution function of $T_{r,\alpha}$, in general, cannot be expressed in terms of elementary functions. The integral of $f_{r,\alpha}(x)$ can be written (e.g., see [38], item 3.194) in terms of the hypergeometric function ${}_2F_1(\cdot,\cdot,\cdot,\cdot)$ (e.g., see [38], item 9.111):

$$F_{r,\alpha}(x) = \begin{cases} \dfrac{1}{2} + \dfrac{\alpha x}{2r^{1/\alpha}\mathrm{B}(r, \frac{1}{\alpha})}\,{}_2F_1\big(r + \frac{1}{\alpha}, \frac{1}{\alpha}; 1 + \frac{1}{\alpha}; -\frac{x^\alpha}{r}\big), & x \geq 0, \\[2ex] \dfrac{1}{2} - \dfrac{\alpha |x|}{2r^{1/\alpha}\mathrm{B}(r, \frac{1}{\alpha})}\,{}_2F_1\big(r + \frac{1}{\alpha}, \frac{1}{\alpha}; 1 + \frac{1}{\alpha}; -\frac{|x|^\alpha}{r}\big), & x < 0. \end{cases}$$

Nevertheless, we can obtain very simple two-sided bounds for the tail probabilities of $T_{r,\alpha}$.

Proposition 3. *For any $x > 0$, we have*

$$\frac{r^{r-1}}{\mathrm{B}(r, \frac{1}{\alpha})x^{\alpha r}} \cdot \frac{x^{\alpha r+1}}{(r + x^\alpha)^{r+1/\alpha}} \leq \mathsf{P}(|T_{r,\alpha}| \geq x) \leq \frac{r^{r-1}}{\mathrm{B}(r, \frac{1}{\alpha})x^{\alpha r}}.$$

Proof. For any $x > 0$, we obviously have

$$\mathsf{P}(|T_{r,\alpha}| \geq x) = 2\int_x^\infty f_{r,\alpha}(y)dy = \frac{\alpha}{r^{1/\alpha}\mathrm{B}(r, \frac{1}{\alpha})}\int_x^\infty \Big(1 + \frac{|y|^\alpha}{r}\Big)^{-(r+1/\alpha)}dy. \tag{10}$$

For the integral on the right-hand side of (10), we easily obtain the following lower bound:

$$\int_x^\infty \Big(1 + \frac{|y|^\alpha}{r}\Big)^{-(r+1/\alpha)}dy = r^{r+1/\alpha}\int_x^\infty \Big(\frac{r}{y^\alpha} + 1\Big)^{-(r+1/\alpha)}\frac{dy}{y^{\alpha r+1}} \geq$$

$$\geq \frac{r^{r+1/\alpha}x^{\alpha r+1}}{(r + x^\alpha)^{r+1/\alpha}}\int_x^\infty \frac{dy}{y^{\alpha r+1}} = \frac{r^{r+1/\alpha-1}}{\alpha x^{\alpha r}} \cdot \frac{x^{\alpha r+1}}{(r + x^\alpha)^{r+1/\alpha}}. \tag{11}$$

The upper bound for this integral is obvious:

$$\int_x^\infty \Big(1+\frac{|y|^\alpha}{r}\Big)^{-(r+1/\alpha)}dy = r^{r+1/\alpha}\int_x^\infty \Big(\frac{r}{y^\alpha}+1\Big)^{-(r+1/\alpha)}\frac{dy}{y^{\alpha r+1}} \leq$$

$$\leq r^{r+1/\alpha}\int_x^\infty \frac{dy}{y^{\alpha r+1}} = \frac{r^{r+1/\alpha-1}}{\alpha x^{\alpha r}}. \tag{12}$$

Now, the desired statement easily follows from (11), (12), and (10). □

Since

$$\lim_{x\to\infty}\frac{x^{\alpha r+1}}{(r+x^\alpha)^{r+1/\alpha}} = 1, \tag{13}$$

we immediately obtain the following statement.

Corollary 5. *The tailprobabilities of $T_{r,\alpha}$ satisfy the following asymptotic relation:*

$$\lim_{x\to\infty} x^{\alpha r}\mathsf{P}(|T_{r,\alpha}|\geq x) = \frac{r^{r-1}}{\mathrm{B}(r,\,\frac{1}{\alpha})}.$$

Lemma 2 was proved in [32] with the application of the 'multiplication theorem' for stable distributions (Theorem 3.3.1 in [27]). Therefore, this lemma can be regarded as a 'multiplication theorem' for exponential power distributions. This lemma can be used to establish a kind of an analog of 'multiplication theorem' for generalized Student distributions.

Proposition 4. *For any $0<\alpha\leq\beta\leq 2$ and any $r>\frac{1}{\beta}$, we have*

$$G_{r,r}^{-1/\beta}\circ T_{r,\alpha} \stackrel{d}{=} G_{r,r}^{-1/\alpha}\circ T_{r,\beta}\circ U_{\beta,\alpha/\beta}^{-1/\beta}.$$

Proof. The assertion of Lemma 2 can be rewritten as

$$Q_\alpha \stackrel{d}{=} Q_\beta\circ U_{\beta,\alpha/\beta}^{-1/\beta}.$$

Now, the desired statement follows from the definition of $T_{r,\alpha}$. □

One more representation of a random variable with the generalized Student distribution is possible.

Proposition 5. *The following relationship holds:*

$$T_{r,\alpha} \stackrel{d}{=} r^{1/\alpha}Q_\alpha\circ|Q_{1/r}|^{-1/\alpha r}.$$

Proof. According to (5), we have

$$G_{r,r} \stackrel{d}{=} \tfrac{1}{r}G_{r,1} \stackrel{d}{=} \tfrac{1}{r}|Q_{1/r}|^{1/r},$$

whence follows the desired result. □

Now consider the property of the identifiability of scale mixtures of generalized Student distributions. Recall the definition of the identifiability of scale mixtures. Let T be a random variable with the distribution function $F_T(x)$ and let V_1 and V_2 be two nonnegative random variables. The family of scale mixtures of F_T is said to be identifiable if the equality $T\circ V_1 \stackrel{d}{=} T\circ V_2$ implies $V_1 \stackrel{d}{=} V_2$.

Proposition 6. *For any fixed $\alpha \in (0,2]$ and $r > \frac{1}{\alpha}$, the family of scale mixtures of generalized Student distributions is identifiable; that is, if V_1 and V_2 are two nonnegative random variables, then the equality $T_{r,\alpha} \circ V_1 \stackrel{d}{=} T_{r,\alpha} \circ V_2$ implies $V_1 \stackrel{d}{=} V_2$.*

Proof. In [32], it was proved that the family of scale mixtures of exponential power distributions is identifiable. Hence, if V_1 and V_2 are two nonnegative random variables, then the equality $T_{r,\alpha} \circ V_1 \stackrel{d}{=} T_{r,\alpha} \circ V_2$ implies $V_1 \circ G_{r,r}^{-1/\alpha} \stackrel{d}{=} V_2 \circ G_{r,r}^{-1/\alpha}$ or, which is the same, $G_{r,1} \circ V_1^{-\alpha} \stackrel{d}{=} G_{r,1} \circ V_2^{-\alpha}$. As was proved in [39], the family of scale mixtures of gamma distributions is identifiable. Hence, the last relationship implies $V_1^{-\alpha} \stackrel{d}{=} V_2^{-\alpha}$ or $V_1 \stackrel{d}{=} V_2$, which is the same. □

2.2. *Mixture Representation for the Generalized Student Distribution and Related Topics*

2.2.1. Normal Mixture Representation

Proposition 7. *For any $\alpha \in (0,2]$ and any $r > \frac{1}{\alpha}$ the generalized Student distribution is a scale mixture of normal distributions:*

$$T_{r,\alpha} \stackrel{d}{=} \sqrt{D_{r,\alpha}} \circ X, \tag{14}$$

where

$$D_{r,\alpha} \stackrel{\text{def}}{=} \tfrac{1}{2}\big(U_{2,\alpha/2} \circ G_{r,r}^{2/\alpha}\big)^{-1} \stackrel{d}{=} \tfrac{1}{2}\big(U_{2,\alpha/2} \circ \overline{G}_{r,\alpha/2,r}\big)^{-1},$$

so that

$$\mathsf{P}(T_{r,\alpha} < x) = \int_0^\infty \Phi\Big(\frac{x}{\sqrt{y}}\Big) d\mathsf{P}(D_{r,\alpha} < y). \tag{15}$$

This statement directly follows from (7) and Corollary 1.

In accordance with Lemma 2, for $\alpha \in (0,2)$, the probability density $u^*_{2,\alpha/2}(x)$ of the random variable $U_{2,\alpha/2}^{-1}$ has the form

$$u^*_{2,\alpha/2}(x) = \frac{\alpha\sqrt{\pi}}{2\Gamma(\frac{1}{\alpha})} \cdot \frac{s_{\alpha/2,1}(\frac{1}{x})}{x^{3/2}}, \quad x > 0.$$

If $\alpha = 2$, then the distribution of $U_{2,\alpha/2}^{-1}$ is degenerate at Point 1.

The generalized gamma probability density $gg_{r,\alpha/2,r}(x)$ of the random variable $G_{r,r}^{2/\alpha}$ has the form

$$gg_{r,\alpha/2,r}(x) = \frac{r^r}{\Gamma(r)} u^{\alpha(r+1)/2-2} e^{-ru^{\alpha/2}}, \quad x > 0.$$

Therefore, the mixing random variable $D_{r,\alpha}$ in (15) has the probability density

$$q_{r,\alpha}(x) = \frac{r^r \alpha\sqrt{2\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(r)x^{3/2}} \int_0^\infty s_{\alpha/2,1}\big(\tfrac{2}{ux}\big) u^{\alpha(r+1)/2-5/2} e^{-ru^{\alpha/2}} du, \quad x > 0.$$

This expression is cumbersome and can hardly be used either for the purpose of clarifying the analytic and asymptotic properties of the mixing distribution or its statistical analysis. However, as will be shown in the next subsection, it is possible to obtain rather accurate (asymptotic) two-sided bounds for the tail probability of the distribution of $D_{r,\alpha}$.

2.2.2. The Properties of the Mixing Distribution And Inequalities for the Tail Probabilities

Proposition 8. *There exist finite positive constants $\underline{C} = \underline{C}(r,\alpha)$ and $\overline{C} = \overline{C}(r,\alpha)$ such that for any $\delta \in (0,1)$*

$$\liminf_{x\to\infty} x^{\alpha r/2+\delta}\mathsf{P}(D_{r,\alpha} \geq x) \geq \underline{C} \tag{16}$$

and

$$\limsup_{x\to\infty} x^{\alpha r/2}\mathsf{P}(D_{r,\alpha} \geq x) \leq \overline{C}. \tag{17}$$

For example, as $\underline{C}$ and $\overline{C}$, one can take

$$\underline{C} = \frac{r^{r-1}}{\mathrm{B}(r, \frac{1}{\alpha})}, \quad \overline{C} = \frac{r^{r-1}}{2\mathrm{B}(r, \frac{1}{\alpha})[1 - \Phi(1)]}.$$

Roughly speaking, Proposition 8 states that the distribution of the mixing random variable $D_{r,\alpha}$ in Proposition 4 has the power-type tails decreasing such that $O(x^{-\alpha r/2})$ as $x \to \infty$.

In order to prove this proposition, we need to formulate and prove some general inequalities relating the tails of a scale mixture with that of the mixing distribution. These inequalities will be formulated as lemmas.

Lemma 4. *Let Y be a random variable with a symmetric distribution. Let U be a positive random variable. Then, for any $x > 0$ and $u > 0$,*

$$\mathsf{P}(|Y \circ U| > x) \geq \mathsf{P}\Big(|Y| > \frac{x}{u}\Big)\mathsf{P}(U > u).$$

Proof. Denote the distribution function of Y as $F(x)$. Then, for any $x > 0$ and $u > 0$, due to the monotonicity of F, we have

$$\mathsf{P}(|Y \circ U| > x) = 2\int_0^\infty \Big[1 - F\Big(\frac{x}{y}\Big)\Big] d\mathsf{P}(U < y) \geq 2\int_u^\infty \Big[1 - F\Big(\frac{x}{y}\Big)\Big] d\mathsf{P}(U < y) \geq$$

$$\geq 2\Big[1 - F\Big(\frac{x}{u}\Big)\Big] \int_u^\infty d\mathsf{P}(U < y) = \mathsf{P}\Big(|Y| > \frac{x}{u}\Big)\mathsf{P}(U \geq u).$$

□

Now, if we set $Y = X$ (that is, $F = \Phi$), $U = \sqrt{D_{r,\alpha}}$, and $u = x^\epsilon$ with arbitrary $\epsilon \in [0, 2]$, then for any $x > 0$, Proposition 2, Lemma 4, and Proposition 3 yield the bound

$$\frac{r^{r-1}}{\mathrm{B}(r, \frac{1}{\alpha})x^{\alpha r}} \geq \mathsf{P}(|T_{r,\alpha}| > x) \geq \mathsf{P}(D_{r,\alpha} \geq x^\epsilon)\mathsf{P}(|X| \geq x^{1-\epsilon/2}). \tag{18}$$

Additionally, if $\epsilon = 2$, then (18), in turn, implies

$$x^{\alpha r}\mathsf{P}(D_{r,\alpha} \geq x^2) \leq \frac{r^{r-1}}{2\mathrm{B}(r, \frac{1}{\alpha})[1 - \Phi(1)]}, \tag{19}$$

thus proving (17).

Lemma 4 generalizes a result of [40].

Lemma 5. *Let Y be a random variable independent of a positive random variable U. Then, for any $x > 0$ and $\delta \in (0, 1)$,*

$$\mathsf{P}(|Y \circ U| \geq x) \leq \mathsf{P}(|Y| \geq x^{1-\delta}) + \mathsf{P}(U \geq x^\delta)\mathsf{P}(|Y| < x^{1-\delta}) =$$

$$= \mathsf{P}(|Y| \geq x^{1-\delta})\mathsf{P}(U < x^\delta) + \mathsf{P}(U \geq x^\delta) \leq \mathsf{P}(|Y| \geq x^{1-\delta}) + \mathsf{P}(U \geq x^\delta).$$

Proof. It is not difficult to verify that for any $\delta \in (0, 1)$,

$$\big\{\omega : \ln|Y(\omega)| + \ln U(\omega) \geq \ln x\big\} \subseteq \big\{\omega : \ln|Y(\omega)| \geq (1 - \delta)\ln x\big\} \cup \big\{\omega : \ln U(\omega) \geq \delta \ln x\big\}.$$

Therefore,

$$\mathsf{P}(|Y \circ U| \geq x) = \mathsf{P}(\ln|Y \circ U| \geq \ln x) = \mathsf{P}(\ln|Y| + \ln U \geq \ln x) \leq$$

$$\leq \mathsf{P}\big(\{\ln|Y| \geq (1 - \delta)\ln x\} \cup \{\ln U \geq \delta \ln x\}\big) =$$

$$= \mathsf{P}(\ln|Y| \geq (1-\delta)\ln x) + \mathsf{P}(\ln U \geq \delta \ln x) - \mathsf{P}(\ln|Y| \geq (1-\delta)\ln x) \cdot \mathsf{P}(\ln U \geq \delta \ln x) \leq$$

$$= \mathsf{P}(|Y| \geq x^{1-\delta}) + \mathsf{P}(U \geq x^{\delta}) - \mathsf{P}(|Y| \geq x^{1-\delta}) \cdot \mathsf{P}(U \geq x^{\delta})\} =$$

$$= \mathsf{P}(|Y| \geq x^{1-\delta}) + \mathsf{P}(U \geq x^{\delta})\mathsf{P}(|Y| < x^{1-\delta}) = \mathsf{P}(|Y| \geq x^{1-\delta})\mathsf{P}(U < x^{\delta}) + \mathsf{P}(U \geq x^{\delta}) \leq$$

$$\leq \mathsf{P}(|Y| \geq x^{1-\delta}) + \mathsf{P}(U \geq x^{\delta}).$$

The lemma is proved. □

It should be noted that in Lemma 5, no conditions were imposed on the distribution of the random variable Y.

Now, if we set $Y = X$ (that is, $F = \Phi$) and $U = \sqrt{D_{r,\alpha}}$, then for any $x > 0$ and $\epsilon \in (0,2)$, Proposition 4 and Lemma 5 yield the bound

$$\mathsf{P}(|T_{r,\alpha}| > x) \leq \mathsf{P}(|X| \geq x^{1-\epsilon/2}) + \mathsf{P}(D_{r,\alpha} \geq x^{\epsilon}),$$

which is valid for any $\epsilon \in (0,2)$. Hence, in turn, it follows that

$$\frac{\mathsf{P}(D_{r,\alpha} \geq x^{\epsilon})}{\mathsf{P}(|T_{r,\alpha}| > x)} \geq 1 - \frac{\mathsf{P}(|X| \geq x^{1-\epsilon/2})}{\mathsf{P}(|T_{r,\alpha}| > x)}. \tag{20}$$

It is well-known that for any $y > 0$,

$$\mathsf{P}(|X| \geq y) \leq \frac{\sqrt{2}}{\sqrt{\pi}y} \exp\Big\{-\frac{y^2}{2}\Big\}. \tag{21}$$

From the left inequality of Proposition 3 and (21), it follows that for any $\epsilon \in (0,2)$,

$$\lim_{x\to\infty} \frac{\mathsf{P}(|X| \geq x^{1-\epsilon/2})}{\mathsf{P}(|T_{r,\alpha}| > x)} \leq \frac{\sqrt{2}\mathrm{B}(r, \frac{1}{\alpha})}{\sqrt{\pi}r^{r-1}} \cdot \lim_{x\to\infty}(r + x^{\alpha})^{r+1/\alpha}x^{\epsilon/2-2}\exp\Big\{-\frac{x^{2-\epsilon}}{2}\Big\} = 0.$$

Hence, with the account of (13), from (20) and the left inequality of Proposition 3, it follows that for any $\epsilon \in (0,2)$

$$\liminf_{x\to\infty} x^{\alpha r}\mathsf{P}(D_{r,\alpha} \geq x^{\epsilon}) \geq \frac{r^{r-1}}{\mathrm{B}(r, \frac{1}{\alpha})},$$

thus proving (35). Thus, Proposition 8 is completely proved. □

Proposition 9. *If $\alpha \in (0,1]$ and $r > \frac{1}{\alpha}$, then the random variable $D_{r,\alpha}$ has the mixed exponential distribution*

$$D_{r,\alpha} \stackrel{d}{=} 2\big(G_{r,r}^{1/\alpha} \circ U_{1,\alpha}\big)^{-2} \circ W_1.$$

Proof. From Corollary 1, Lemma 3, and the definition of the generalized Student distribution, we obtain the representation

$$T_{r,\alpha} \stackrel{d}{=} Q_{\alpha} \circ G_{r,r}^{-1/\alpha} \stackrel{d}{=} \sqrt{2\big(G_{r,r}^{1/\alpha} \circ U_{1,\alpha}\big)^{-2} \circ W_1} \circ X.$$

Now the desired result follows from the identifiability of scale mixtures of normal distributions (see, e.g., [39]). □

Corollary 6. *For $\alpha \in (0,1] \cup \{2\}$ and any $r > \frac{1}{\alpha}$, the generalized Student distribution is infinitely divisible.*

Proof. According to Proposition 6, for $\alpha \in (0,1]$ in Representation (14), the scaling (mixing) distribution is mixed exponential and, hence, in accordance with the result of [41], infinitely divisible. In turn, if the mixing distribution in a normal scale mixture is infinitely divisible, then, in accordance with [42], Ch. XVII, Sect. 3, the normal scale mixture is infinitely divisible itself.

In the case that $\alpha = 2$, the infinite divisibility of the generalized Student distribution (in this case, the conventional Student distribution) for any $r > 0$ was proved in [43]. $\square$

Proposition 10. *If $\alpha \in (0, 1]$ and $r > \frac{1}{\alpha}$, then the generalized Student distribution is a scale mixture of the Laplace laws,*

$$T_{r,\alpha} \stackrel{d}{=} Y_{r,\alpha} \circ Q_1,$$

where

$$Y_{r,\alpha} \stackrel{d}{=} \left(G_{r,r}^{1/\alpha} \circ U_{1,\alpha}\right)^{-1}.$$

Proof. This statement follows from Corollary 2 and the definition of the random variable $T_{r,\alpha}$. $\square$

2.3. Convergence of the Distributions of Random Sums to the Generalized Student Law

In applied probability, it is a convention, probably based on some topics of [35], that to make sure that a probability distribution can serve as a well-justified model of a real phenomenon, one should construct a limit setting where this distribution is a limit distribution or asymptotic approximation (say, a scheme of maximum or summation of random variables). The existence of such a limit setting with specific conditions providing the convergence to the assumed distribution can provide a better understanding of real mechanisms that generate observed statistical regularities.

The representation for the generalized Student distribution as a scale mixture of normals obtained in Proposition 4 opens the way for the construction in this section of an 'if and only if' version of the random-sum central limit theorem with the generalized Student distribution as the limit law.

Consider independent not necessarily identically distributed random variables $X_1, X_2, \ldots$ with $\mathsf{E}X_i = 0$ and $0 < \sigma_i^2 = \mathsf{E}X_i^2 < \infty$, $i \geq 1$. For $n \in \mathbb{N}$, denote

$$S_n = X_1 + \ldots + X_n, \quad B_n^2 = \sigma_1^2 + \ldots + \sigma_n^2.$$

Assume that the random variables $X_1, X_2, \ldots$ satisfy the Lindeberg condition such that for any $\tau > 0$,

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{i=1}^{n} \int_{|x| \geq \tau B_n} x^2 d\mathsf{P}(X_i < x) = 0. \tag{22}$$

It is well-known that under these assumptions,

$$\mathsf{P}\big(S_n < B_n x\big) \Longrightarrow \Phi(x)$$

(this is the classical central limit theorem due to Lindeberg).

Let $N_1, N_2, \ldots$ be a sequence of integer-valued nonnegative random variables defined on the same probability space so that for each $n \in \mathbb{N}$, the random variable N_n is independent of the sequence $X_1, X_2, \ldots$. Denote $S_{N_n} = X_1 + \ldots + X_{N_n}$. For definiteness, in what follows, we assume that $\sum_{j=1}^{0} = 0$. In what follows, the convergence will be meant as $n \to \infty$.

Recall that a random sequence $N_1, N_2, \ldots$ is said to be infinitely increasing in probability if $\mathsf{P}(N_n \leq m) \longrightarrow 0$ for any $m \in (0, \infty)$.

Let $\{d_n\}_{n\geq 1}$ be an infinitely increasing sequence of positive numbers.

The following version of the central limit theorem for random sums is the base for the proof of the main result of this section.

Lemma 6 ([44])**.** *Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let the Lindeberg condition* (22) *hold. Moreover, let $N_n \to \infty$ in* *probability. The distribution functions of appropriately normalized random sums S_{N_n} converge to some distribution function $F(x)$,*

$$\mathsf{P}\Big(\frac{S_{N_n}}{d_n} < x\Big) \Longrightarrow F(x),$$

if and only if there exists a distribution function $H(x)$ satisfying the conditions

$$H(0) = 0, \quad F(x) = \int_0^\infty \Phi\Big(\frac{x}{\sqrt{y}}\Big) dH(y), \quad x \in \mathbb{R},$$

and $\mathsf{P}(B_{N_n}^2 < x d_n^2) \Longrightarrow H(x)$.

Proof. This statement is a particular case of a result proved in [44]; also see Theorem 3.3.2 in [45]. □

The main result of this section is the following statement presenting *necessary and sufficient* conditions for the convergence of the distributions of random sums of independent random variables with finite variances to the generalized Student distribution.

Proposition 11. *Let $\alpha \in (0,2]$, $r > \frac{1}{\alpha}$. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let the Lindeberg condition (22) hold. Moreover, let $N_n \to \infty$ in probability. Then, the distributions of the normalized random sums S_{N_n} converge to the generalized Student law with parameters r and α; that is,*

$$\frac{S_{N_n}}{d_n} \Longrightarrow T_{r,\alpha}$$

with some $d_n > 0$, $d_n \to \infty$, if and only if

$$\frac{B_{N_n}^2}{d_n^2} \Longrightarrow D_{r,\alpha} \stackrel{d}{=} \tfrac{1}{2}\big(U_{2,\alpha/2} \circ \overline{G}_{r,\alpha/2,r}\big)^{-1}. \tag{23}$$

Proof. This statement is a direct consequence of Lemma 4 with $H(x) = \mathsf{P}(D_{r,\alpha} < x)$ and Proposition 4. □

Note that if the random variables $X_1, X_2, \ldots$ are identically distributed, then $\sigma_i = \sigma$, $i \in \mathbb{N}$, and the Lindeberg condition holds automatically. In this case, it is reasonable to take $d_n = \sigma\sqrt{n}$. Hence, from Proposition 11, in this case, it follows that for the convergence

$$\frac{S_{N_n}}{\sigma\sqrt{n}} \Longrightarrow T_{r,\alpha}$$

to take place, it is necessary and sufficient that

$$\frac{N_n}{n} \Longrightarrow D_{r,\alpha}. \tag{24}$$

It should be especially noted that despite the requirement that the summands in the sum have finite variances, the resulting generalized Student distribution in Proposition 11 may have *arbitrarily heavy* tails. The parameters of the limit-generalized Student distribution are entirely defined by the asymptotic behavior of the random index N_n (see Relationship (24)).

One more remark concerns the curious form of the random variable $D_{r,\alpha}$ due to which the realization of Conditions (23) and (24) in practical situations may seem doubtful. However, in many practical problems, the flow of informative events producing observations can be successfully modelled by a doubly stochastic Poisson process (also called a Cox process). Such a process is defined as a Poisson process with stochastic intensity.

Namely, a doubly stochastic Poisson process is a stochastic point process of the form $N(t) \stackrel{\text{def}}{=} \Pi(L(t))$, where $\Pi(t)$, where $t \geq 0$, is a homogeneous Poisson process with unit intensity, and the stochastic process $L(t)$, where $t \geq 0$, is independent of $\Pi(t)$ and possesses the following properties: $L(0) = 0$, $\mathsf{P}(L(t) < \infty) = 1$ for any $t > 0$, and the sample paths of $L(t)$ do not decrease and are right-continuous. In this context, the Cox process $N(t)$ is said to be lead by the process $L(t)$. For more details concerning Cox and more general subordinated processes, see, e.g., [46–48].

In real problems, the process $L(t)$ characterizing the cumulative intensity of the flow of informative events depends on many factors whose influence is hardly predictable, and it is quite likely that the statistical regularities in its behavior can be approximated by the distribution of the random variable $D_{r,\alpha}$. Now, if $N_n \stackrel{\text{def}}{=} N(n)$, then for Condition (24) to hold, it is necessary and sufficient that $n^{-1}L(n) \Longrightarrow D_{r,\alpha}$ [49]. This means that actually, Conditions (23) and (24) are not as artificial as it may seem at the first sight.

2.4. Convergence of the Distributions of Statistics Constructed from Samples with Random Sizes to the Generalized Student Distribution

In practice, rather often, the data are collected or registered during a certain period of time so that the sequence (flow) of informative events, each of which brings the next observation, is a random point process. Hence, the number of available observations may be unknown until the termination of the process of their registration. Therefore, the number of accumulated observations (sample size) should also be treated as a (random) observation. This means that the problems and results of the classical mathematical statistics, in which the size of the available sample is usually assumed to be deterministic, deals with conditional distributions given the concrete value of the sample size. In the asymptotic settings, this value plays the role of an infinitely increasing known parameter. However, the asymptotic behavior of the (unconditional) distributions of statistics constructed from samples with random sizes noticeably differs from that of the distributions of statistics in the classical case, which are actually conditional distributions given the particular value of the sample size. For a more detailed motivation for the consideration of statistics constructed from samples with random sizes, see, e.g., [32].

The randomness of the sample size usually leads to the limit distributions for the corresponding statistics being heavy-tailed, even in situations where the conditional distributions of the same statistics given a non-random sample size are asymptotically normal; see, e.g., [4,45,50].

Consider a traditional setting of mathematical statistics. As in the preceding section, consider the random variables $N_1, N_2, \ldots, X_1, X_2, \ldots$ defined on one and the same probability space so that for each $n \geq 1$, the random variable N_n takes only natural values and is independent of the 'observations' $X_1, X_2, \ldots$. Let $t_n = t_n(X_1, \ldots, X_n)$ be a statistic, that is, a measurable function of $X_1, \ldots, X_n$. For every $n \geq 1$ and $\omega \in \Omega$, define the random variable $t_{N_n} = t_{N_n(\omega)}(\omega)$ as

$$t_{N_n} = t_{N_n(\omega)}\Big(X_1(\omega), \ldots, X_{N_n(\omega)}(\omega)\Big).$$

A statistic t_n is said to be *asymptotically normal* if there exist $\delta > 0$ and $\theta \in \mathbb{R}$ such that

$$\mathsf{P}\big(\delta\sqrt{n}(t_n - \theta) < x\big) \Longrightarrow \Phi(x). \tag{25}$$

Lemma 7 ([51])**.** *Assume that $N_n \longrightarrow \infty$ in probability and the statistic t_n is asymptotically normal in the sense of* (25)*. A distribution function $F(x)$ such that*

$$\mathsf{P}\big(\delta\sqrt{n}(t_{N_n} - \theta) < x\big) \Longrightarrow F(x),$$

exists if and only if there exists a distribution function $H(x)$ satisfying the conditions

$$H(0) = 0, \quad F(x) = \int_0^\infty \Phi(x\sqrt{y})dH(y), \quad x \in \mathbb{R}, \quad \mathsf{P}(N_n < nx) \Longrightarrow H(x).$$

The following theorem presents necessary and sufficient conditions for the convergence of the distributions of statistics, which are suggested to be asymptotically normal in the traditional sense but are constructed from samples with random sizes, to the generalized Student distribution.

Proposition 12. *Let $\alpha \in (0,2]$, $r > \frac{1}{\alpha}$. Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. Moreover, let $N_n \to \infty$ in probability and let the statistic t_n be asymptotically normal in the sense of (25). Then, the distribution of the statistic t_{N_n} constructed from samples with random sizes N_n converges to the generalized Student law $F_{r,\alpha}(x)$; that is,*

$$\mathsf{P}\big(\delta\sqrt{n}(t_{N_n} - \theta) < x\big) \Longrightarrow F_{r,\alpha}(x),$$

if and only if

$$\frac{N_n}{n} \Longrightarrow D_{r,\alpha}^{-1} \stackrel{d}{=} 2U_{2,\alpha/2} \circ \overline{G}_{r,\alpha/2,r}. \tag{26}$$

Proof. This statement is a direct consequence of (14) and Lemma 7 with $H(x) = \mathsf{P}(D_{r,\alpha}^{-1} < x)$. □

As an example of an application of Proposition 12, consider the following statement establishing necessary and sufficient conditions for the sample quantiles to have the generalized Student asymptotic distribution.

In addition to the notation introduced above, for each $n \in \mathbb{N}$, let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be order statistics constructed from the sample $X_1, X_2, \ldots, X_n$ so that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$. Assume that the common distribution of X_j is absolutely continuous and denote the corresponding probability density as $p(x)$. Let $q \in (0,1)$. The quantile of order q of the random variable X_1 will be denoted ξ_q. For a fixed $n \in \mathbb{N}$, define the sample quantile as $X_{([nq]+1)}$, where $[a]$ stands for the integer part of a real number a. The following Lemma is a particular case of a result from [52].

Lemma 8. *Assume that the density $p(x)$ is differentiable in the neighborhood of ξ_q and $p(\xi_q) \neq 0$. Then, as $n \to \infty$,*

$$\frac{p(\xi_q)}{\sqrt{q(1-q)}} \cdot \sqrt{n}\big(X_{([nq]+1)} - \xi_q\big) \Longrightarrow X.$$

This statement means that the sample quantile $X_{([nq]+1)}$ is asymptotically normal in the sense of (25) with $\delta = p(\xi_q)/\sqrt{q(1-q)}$ and $\theta = \xi_q$.

In [4], an example was presented of the convergence of the distributions of some statistics constructed from samples with random sizes to the classical Student distribution. In that paper, it was assumed that the sample size had a negative binomial distribution. Here, we will present a generalization of this result. As is known, the negative binomial distribution considered in [4] is a mixed Poisson distribution with a mixing gamma distribution. A random variable N with a negative binomial distribution can be represented as $N = \Pi(G_{r,\lambda})$, where $r > 0$, $\lambda > 0$, and $\Pi(t)$ is the Poisson process with the unit intensity independent of the gamma-distributed random variable $G_{r,\lambda}$. Here, we will use the same construction and assume that for each $n \in \mathbb{N}$, the random sample size N_n has the mixed Poisson distribution of the form

$$N_n = \Pi(nD_{r,\alpha}^{-1}). \tag{27}$$

With $\alpha = 2$ the random variable $D_{r,\alpha}^{-1}$ obviously turns into $G_{r,r}$ so that, as in this case, we deal with the negative binomially distributed sample size considered in [4].

Corollary 7. *Let $\alpha \in (0,2]$ where $r > \frac{1}{\alpha}$. Let the random variable N_n be defined as (27) and be independent of the sequence $X_1, X_2, \ldots$ for each $n \in \mathbb{N}$. Then, the distribution of the sample quantiles constructed from samples with random sizes N_n converges to the generalized Student law $F_{r,\alpha}(x)$; that is,*

$$\mathsf{P}\Big(\frac{p(\xi_q)}{\sqrt{q(1-q)}} \cdot \sqrt{n}\big(X_{([qN_n]+1)} - \xi_q\big) < x\Big) \Longrightarrow F_{r,\alpha}(x).$$

Proof. It is easy to verify that the random variables N_n defined as (27) satisfy Condition (26) so that the desired result follows from Proposition 12. □

It should be noted that in Proposition 12 and Corollary 6, a non-random normalization and centering was used for the statistics constructed from samples with random sizes. This was performed because a reasonable approximation to the distribution of the basic statistics can be constructed only if both centering and normalizing values are non-random. Otherwise (that is, if normalization is random depending on the random sample size), the approximate asymptotic distribution function becomes random itself. For example, random normalization makes the problem of the evaluation of significance levels from the asymptotic distribution of the test statistic senseless.

3. Generalized Lomax Distribution

3.1. Definition and Basic Properties of the Generalized Lomax Distribution

The distribution of the random variable

$$|T_{r,\alpha}| \stackrel{d}{=} |Q_\alpha| \circ G_{r,r}^{-1/\alpha}$$

will be called a generalized Lomax distribution. When $\alpha = 1$, this distribution is known as Lomax distribution. In general, with an arbitrary $\alpha \in (0,2]$, the distribution of $|T_{r,\alpha}|$ can just as well be called folded generalized Student or one-sided generalized Student distribution. However, in what follows, we will keep to the term generalized Lomax distribution.

From (8), it is easy to see that the probability density $f^*_{r,\alpha}(x)$ of the generalized Lomax distribution has the form

$$f^*_{r,\alpha}(x) = \frac{\alpha}{r^{1/\alpha}\mathrm{B}(r, \frac{1}{\alpha})}\Big(1 + \frac{x^\alpha}{r}\Big)^{-(r+1/\alpha)}, \quad x \geq 0.$$

Recall that here, $\alpha \in (0,2]$ and $r > 0$ so that $\alpha r > 1$.

The expression for the moments of the generalized Lomax distribution is given by Proposition 2.

Proposition 13. *For $\alpha \in (0,1]$ and $r > \frac{1}{\alpha}$, the generalized Lomax distribution is mixed exponential.*

Proof. Since $|Q_1| \stackrel{d}{=} W_1$, from Proposition 10, it directly follows that

$$|T_{r,\alpha}| \stackrel{d}{=} \big(U_{1,\alpha} \circ G_{r,r}^{1/\alpha}\big)^{-1} \circ W_1. \tag{28}$$

□

Corollary 8. *For $\alpha \in (0,1]$ and $r > \frac{1}{\alpha}$, the generalized Lomax distribution is infinitely divisible.*

Proof. The statement follows from Proposition 13 and the result of [41], according to which it is sufficient that F is mixed exponential in order for a distribution function $F(x)$ such that $F(0) = 0$ to be infinitely divisible. □

Proposition 14. *For $\alpha \in (0,2]$ and $r > \frac{1}{\alpha}$, the scale mixtures of generalized Lomax distributions are identifiable; that is, if V_1 and V_2 are two nonnegative random variables, then the equality $|T_{r,\alpha}| \circ V_1 \stackrel{d}{=} |T_{r,\alpha}| \circ V_2$ implies $V_1 \stackrel{d}{=} V_2$.*

Proof. The proof is similar to that of Proposition 6. □

The generalized Lomax distribution can be just as well defined in terms of only (generalized) gamma distributions or only exponential power distributions, as is demonstrated in the following statement implied by Relationship (5).

Proposition 15. *For $\alpha \in (0,2]$ and $r > \frac{1}{\alpha}$, the following relationships hold:*

$$|T_{r,\alpha}| \stackrel{d}{=} \left(r|Q_\alpha| \circ |Q_{1/r}|^{-1/r}\right)^{1/\alpha} \stackrel{d}{=} \left(rG_{1/\alpha,1} \circ |Q_{1/r}|^{-1/r}\right)^{1/\alpha} \stackrel{d}{=}$$

$$\stackrel{d}{=} \left(G_{1/\alpha,r} \circ |Q_{1/r}|^{-1/r}\right)^{1/\alpha} \stackrel{d}{=} \left(G_{1/\alpha,r} \circ G_{r,1}^{-1}\right)^{1/\alpha}. \tag{29}$$

3.2. Generalized Lomax Distribution as a Scale Mixture of Folded Normal Distributions

From Proposition 7, we obviously obtain the following statement.

Corollary 9. *For any $\alpha \in (0,2]$ and any $r > \frac{1}{\alpha}$, the generalized Lomax distribution is a scale mixture of folded normal distributions:*

$$|T_{r,\alpha}| \stackrel{d}{=} \sqrt{D_{r,\alpha}} \circ |X|, \tag{30}$$

where

$$D_{r,\alpha} \stackrel{\text{def}}{=} \tfrac{1}{2}\left(U_{2,\alpha/2} \circ G_{r,r}^{2/\alpha}\right)^{-1} \stackrel{d}{=} \tfrac{1}{2}\left(U_{2,\alpha/2} \circ \overline{G}_{r,\alpha/2,r}\right)^{-1},$$

so that

$$\mathsf{P}(|T_{r,\alpha}| < x) = 2\int_0^\infty \Phi\Big(\frac{x}{\sqrt{y}}\Big) d\mathsf{P}(D_{r,\alpha} < y) - 1. \tag{31}$$

Moreover, if $\alpha \in (0,1]$, then $D_{r,\alpha} \stackrel{d}{=} 2W_1 \circ \left(U_{1,\alpha} \circ G_{r,r}^{1/\alpha}\right)^{-2}$.

3.3. Convergence of the Distributions of Maximum and Minimum Random Sums to the Generalized Lomax Distribution

In this section, it will be demonstrated that the generalized Lomax distribution can be the limit law for maximum sums of a random number of independent random variables (maximum random sums), minimum random sums, and absolute values of random sums.

In addition to the notation $S_n = X_1 + \ldots + X_n$ introduced in Section 2.3, for $n \in \mathbb{N}$, denote $\overline{S}_n = \max_{1 \le i \le n} S_i$, where $\underline{S}_n = \min_{1 \le i \le n} S_i$. The random variables $X_1, X_2, \ldots$ will be assumed to satisfy the Lindeberg condition (22). It is well-known that under these assumptions, not only does $\mathsf{P}\big(S_n < B_n x\big) \Longrightarrow \Phi(x)$ (see Section 2.3), but also $\mathsf{P}\big(\overline{S}_n < B_n x\big) \Longrightarrow 2\Phi(x) - 1$, $x \ge 0$, and $\mathsf{P}\big(\underline{S}_n < B_n x\big) \Longrightarrow 2\Phi(x)$, $x \le 0$.

Let, as usual, $N_1, N_2, \ldots$ be a sequence of nonnegative random variables such that for each $n \in \mathbb{N}$ the random variables $N_n, X_1, X_2, \ldots$ are independent. For $n \in \mathbb{N}$, let $S_{N_n} = X_1 + \ldots + X_{N_n}$, $\overline{S}_{N_n} = \max_{1 \le i \le N_n} S_i$, and $\underline{S}_{N_n} = \min_{1 \le i \le N_n} S_i$ (for definiteness, assume that $S_0 = \overline{S}_0 = \underline{S}_0 = 0$). Let $\{d_n\}_{n \ge 1}$ be an arbitrary infinitely increasing sequence of positive numbers. Here, the convergence is meant as $n \to \infty$.

Lemma 9 ([44])**.** *Assume that the random variables $X_1, X_2, \ldots$ and $N_1, N_2, \ldots$ satisfy the conditions specified above. In particular, let the Lindeberg Condition (22) hold and let $N_n \to \infty$ in probability. Then, the distributions of normalized random sums weakly converge to some distribution; that is, there exists a random variable Y such that $d_n^{-1} S_{N_n} \Longrightarrow Y$ if and only if there exists a nonnegative random variable U such that $Y \stackrel{d}{=} \sqrt{U} \circ X$ and if any of the following conditions holds:*

(i) $d_n^{-1}|S_{N_n}| \Longrightarrow |Y|$;
(ii) *There exists a random variable* $\overline{Y}$ *such that* $d_n^{-1}\overline{S}_{N_n} \Longrightarrow \overline{Y}$;
(iii) *There exists a random variable* $\underline{Y}$ *such that* $d_n^{-1}\underline{S}_{N_n} \Longrightarrow \underline{Y}$;
(iv) *There exists a nonnegative random variable* U *such that* $d_n^{-2}B_{N_n}^2 \Longrightarrow U$.

Moreover,

$$\mathsf{P}\big(\underline{Y} < x\big) = 2\mathsf{E}\Phi\big(xU^{-1/2}\big),\ x \le 0;\ \ \mathsf{P}\big(\overline{Y} < x\big) = \mathsf{P}\big(|Y| < x\big) = 2\mathsf{E}\Phi\big(xU^{-1/2}\big) - 1,\ x \ge 0.$$

Lemma 9 and Corollary 8 imply the following statement.

Proposition 16. *Let* $\alpha \in (0, 2]$. *Assume that the random variables* $X_1, X_2, \ldots$ *and* $N_1, N_2, \ldots$ *satisfy the conditions specified above. In particular, let the Lindeberg Condition* (22) *hold. Moreover, let* $N_n \to \infty$ *in probability. Then the following five statements are equivalent:*

$$d_n^{-1}S_{N_n} \Longrightarrow T_{r,\alpha};\ \ d_n^{-1}\overline{S}_{N_n} \Longrightarrow |T_{r,\alpha}|;\ \ d_n^{-1}\underline{S}_{N_n} \Longrightarrow -|T_{r,\alpha}|;$$

$$d_n^{-1}|S_{N_n}| \Longrightarrow |T_{r,\alpha}|;\ \ d_n^{-2}B_{N_n}^2 \Longrightarrow D_{r,\alpha}.$$

3.4. Generalized Lomax Distribution as a Mixed Weibull Distribution (with $1 \le \alpha \le 2$*) and as a Mixed Fréchet Distribution (with* $0 < \alpha \le 1$*)*

In addition to the auxiliary information presented in the Introduction, in this section, we will need some more definitions and auxiliary results.

In the paper [53], it was shown that any gamma distribution with a shape parameter no greater than one is mixed exponential. Namely, the density $g(x; r, \mu)$ of a gamma distribution with $0 < r < 1$ can be represented as

$$g(x; r, \mu) = \int_0^\infty ze^{-zx}p(z; r, \mu)dz,$$

where

$$p(z; r, \mu) = \frac{\mu^r}{\Gamma(1-r)\Gamma(r)} \cdot \frac{\mathbb{I}_{[\mu,\infty)}(z)}{(z-\mu)^r z}. \tag{32}$$

Moreover, a gamma distribution with a shape parameter $r > 1$ cannot be represented as a mixed exponential distribution.

In [54], it was proved that if $r \in (0, 1)$, $\mu > 0$, and $G_{r,1}$ and $G_{1-r,1}$ are independent gamma-distributed random variables, then the density $p(z; r, \mu)$ defined by (32) corresponds to the random variable

$$Z_{r,\mu} = \frac{\mu(G_{r,1} + G_{1-r,1})}{G_{r,1}} \stackrel{d}{=} \mu Z_{r,1} \stackrel{d}{=} \mu\big(1 + \tfrac{1-r}{r}R_{1-r,r}\big), \tag{33}$$

where $R_{1-r,r}$ is a random variable with the Snedecor–Fisher distribution defined by the probability density

$$f(x; 1-r, r) = \frac{(1-r)^{1-r}r^r}{\Gamma(1-r)\Gamma(r)} \cdot \frac{\mathbb{I}_{(0,\infty)}(x)}{x^r[r + (1-r)x]}. \tag{34}$$

In other words, if $r \in (0, 1)$, then

$$G_{r,\mu} \stackrel{d}{=} Z_{r,\mu}^{-1} \circ W_1. \tag{35}$$

In [32], it was proved that if $\alpha \ge 1$, then the one-sided EP distribution is a scale mixture of Weibull distributions:

$$|Q_\alpha| \stackrel{d}{=} Z_{1/\alpha,1}^{-1/\alpha} \circ W_\alpha. \tag{36}$$

Recall that the random variable W_α^{-1} is said to have an *inverse Weibull* or *Fréchet* distribution:

$$\mathsf{P}(W_\alpha^{-1} < x) = \mathsf{P}(W_\alpha \geq \tfrac{1}{x}) = \exp\{x^{-\alpha}\}, \quad x \geq 0.$$

From (5) and Gleser's result (35), we obtain the following statement.

Proposition 17. *(i). If $1 < \alpha \leq 2$ and $\frac{1}{\alpha} < r < 1$, then the generalized Lomax distribution is a scale mixture of Fréchet distributions:*

$$|T_{r,\alpha}| \stackrel{d}{=} |Q_\alpha| \circ Z_{r,r}^{1/\alpha} \circ W_\alpha^{-1} \stackrel{d}{=} \big(G_{1/\alpha,r} \circ Z_{r,1}\big)^{1/\alpha} \circ W_\alpha^{-1}. \tag{37}$$

(ii). If $1 < \alpha \leq 2$ and $r > \frac{1}{\alpha}$, then the generalized Lomax distribution is a scale mixture of Weibull distributions:

$$|T_{r,\alpha}| \stackrel{d}{=} \big(Z_{1/\alpha,1} \circ G_{r,r}\big)^{-1/\alpha} \circ W_\alpha. \tag{38}$$

Proof. Relationship (37) follows from Proposition 15 and (35). Relationship (38) follows from Proposition 15, (36), and (2) with $\gamma = \alpha$. □

3.5. Some Limit Theorems for Extreme Order Statistics in Samples with Random Sizes

Proposition 17 states that the generalized Lomax distributions with $\alpha \geq 1$ can be represented as scale mixtures of the Weibull distribution or as scale mixtures of the Fréchet distribution. In other words, Relationship (38) can be expressed in the following form: for any $x \geq 0$,

$$\mathsf{P}(|T_{r,\alpha}| < x) = \int_0^\infty (1 - e^{-zx^\alpha})d\,\mathsf{P}(Z_{1/\alpha,1} \circ G_{r,r} < z), \tag{39}$$

whereas Relationship (37) can be rewritten as

$$\mathsf{P}(|T_{r,\alpha}| < x) = \int_0^\infty e^{-zx^{-\alpha}}d\,\mathsf{P}(Z_{r,1} \circ G_{1/\alpha,r} < z). \tag{40}$$

At the same time, in the case that $0 < \alpha \leq 1$, Relationship (28) can be written in the form

$$\mathsf{P}(|T_{r,\alpha}| < x) = \int_0^\infty (1 - e^{-zx})d\,\mathsf{P}(U_{1,\alpha} \circ G_{r,r}^{1/\alpha} < z). \tag{41}$$

As is well known, all the parent distributions in these mixtures can be limiting for extreme-order statistics.

From (39) and (40), it follows that the generalized Lomax distribution with $\alpha \geq 1$ can appear as a limit distribution in limit theorems for extreme-order statistics in samples with random sizes. To illustrate this, we will consider the limit setting dealing with the max-compound and min-compound doubly stochastic Poisson processes.

Recall that the definition of a doubly stochastic Poisson process (Cox process) was given in Section 2.3.

Now, let $N(t)$, where $t \geq 0$, be the a doubly stochastic Poisson process (Cox process) lead by the process $L(t)$. Let $T_1, T_2, \ldots$ be the jump points of the process $N(t)$. Consider a marked Cox point process $\{(T_i, X_i)\}_{i\geq 1}$, where $X_1, X_2, \ldots$ are independent identically distributed random variables independent of the process $N(t)$. Most studies dealing with the point process $\{(T_i, X_i)\}_{i\geq 1}$ deal with a traditional *compound Cox process* $S(t)$ defined for each $t \geq 0$ as the *sum* of all marks X_i of the points T_i of the marked Cox point process that do not exceed the time t. In $S(t)$, the compounding operation is *summation*. In many applied problems, of no less importance are the other functions of the marked Cox point process $\{(T_i, X_i)\}_{i\geq 1}$: the so-called max-compound Cox process or min-compound Cox process that differ from $S(t)$ in that the compounding operation of summation is replaced by the operation of taking the maximum or minimum of the marking random variables, respectively. The analytic and asymptotic properties of max-compound and min-compound Cox processes were considered in [55–57].

Let $N(t)$ be a Cox process. The process $M(t)$ defined as

$$M(t) = \begin{cases} -\infty, & \text{if } N(t) = 0, \\ \max\limits_{1 \le k \le N(t)} X_k, & \text{if } N(t) \ge 1, \end{cases}$$

where $t \ge 0$, is called a *max-compound Cox process*.

The process $m(t)$ defined as

$$m(t) = \begin{cases} +\infty, & \text{if } N(t) = 0, \\ \min\limits_{1 \le k \le N(t)} X_k, & \text{if } N(t) \ge 1, \end{cases}$$

where $t \ge 0$, is called a *min-compound Cox process*.

The common distribution function of the random variables X_j will be denoted $F(x)$. In what follows, we will use the conventional notation

$$\text{lext}(F) = \inf\{x : \ F(x) > 0\}, \quad \text{rext}(F) = \sup\{x : \ F(x) < 1\}.$$

Lemma 10. *Assume that there exists a positive infinitely increasing function $d(t)$ and a positive random variable L such that*

$$\frac{L(t)}{d(t)} \Longrightarrow L \tag{42}$$

as $t \to \infty$. Let us also assume that $\text{lext}(F) > -\infty$ and the distribution function $P_F(x) \equiv F\big(\text{lext}(F) - x^{-1}\big)$ satisfies the condition that there exists a number $\gamma > 0$ such that for any $x > 0$

$$\lim_{y \to \infty} \frac{P_F(yx)}{P_F(y)} = x^{-\gamma}. \tag{43}$$

Then, there exist functions $a(t)$ and $b(t)$ such that

$$\mathsf{P}\Big(\frac{m(t) - a(t)}{b(t)} < x\Big) \Longrightarrow H(x)$$

as $t \to \infty$, where

$$H(x) = \begin{cases} \displaystyle\int_0^\infty (1 - e^{-zx^\gamma})d\mathsf{P}(L < z), & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Moreover, the functions $a(t)$ and $b(t)$ can be defined as

$$a(t) = \text{lext}(F), \quad b(t) = \sup\Big\{x : \ F(x) \le \frac{1}{d(t)}\Big\} - \text{lext}(F). \tag{44}$$

Proof. This lemma can be proved in the same way as Theorem 2 in [55] dealing with max-compound Cox processes using the fact that

$$\min\{X_1, \ldots, X_{N(t)}\} = -\max\{-X_1, \ldots, -X_{N(t)}\}.$$

□

Proposition 18. *Let $0 < \alpha \le 1$, $r > \frac{1}{\alpha}$. Assume that there exists a positive infinitely increasing function $d(t)$ such that condition (42) holds with*

$$L \stackrel{d}{=} U_{1,\alpha} \circ G_{r,r}^{1/\alpha}.$$

Let us also assume that $\mathrm{lext}(F) > -\infty$ *and the distribution function* $P_F(x) \equiv F\big(\mathrm{lext}(F) - x^{-1}\big)$ *satisfies Condition (43) with* $\gamma = 1$*. Then, there exist functions* $a(t)$ *and* $b(t)$ *such that*

$$\frac{m(t) - a(t)}{b(t)} \Longrightarrow |T_{r,\alpha}| \tag{45}$$

as $t \to \infty$*. Moreover, the functions* $a(t)$ *and* $b(t)$ *can be defined by (44).*

Proof. This statement directly follows from Lemma 10 with the account of (41). □

Proposition 19. *Let* $1 \le \alpha \le 2$*, where* $r > \frac{1}{\alpha}$*. Assume that there exists a positive infinitely increasing function* $d(t)$ *such that Condition (42) holds with*

$$L \stackrel{d}{=} Z_{1/\alpha,1} \circ G_{r,r}.$$

Let us also assume that $\mathrm{lext}(F) > -\infty$ *and the distribution function* $P_F(x) \equiv F\big(\mathrm{lext}(F) - x^{-1}\big)$ *satisfies Condition (43) with* $\gamma = \alpha$*. Then, there exist functions* $a(t)$ *and* $b(t)$ *such that*

$$\frac{m(t) - a(t)}{b(t)} \Longrightarrow |T_{r,\alpha}| \tag{46}$$

as $t \to \infty$*. Moreover, the functions* $a(t)$ *and* $b(t)$ *can be defined by (44).*

Proof. This statement directly follows from Lemma 10 with the account of (39). □

Lemma 11. *Assume that there exist a positive infinitely increasing function* $d(t)$ *and a nonnegative random variable* L *such that Condition (42) holds. Let us also assume that* $\mathrm{rext}(F) = \infty$ *and there exists a positive number* γ *such that*

$$\lim_{y \to \infty} \frac{1 - F(yx)}{1 - F(y)} = x^{-\gamma} \tag{47}$$

for any $x > 0$*. Then, there exist a positive function* $b(t)$ *and a distribution function* $H_1(x)$ *such that*

$$\mathsf{P}\Big(\frac{M(t)}{b(t)} < x\Big) \Longrightarrow H_1(x)$$

as $t \to \infty$*. Moreover,*

$$H_1(x) = \begin{cases} 0, & x < 0, \\ \displaystyle\int_0^\infty e^{-zx^{-\gamma}} d\mathsf{P}(L < z), & x \ge 0, \end{cases}$$

and the function $b(t)$ *can be defined as*

$$b(t) = \inf\Big\{x : \ 1 - F(x) \le \frac{1}{d(t)}\Big\}. \tag{48}$$

Proposition 20. *Let* $1 \le \alpha \le 2$*,* $r > \frac{1}{\alpha}$*. Assume that there exists a positive infinitely increasing function* $d(t)$ *such that Condition (42) holds with*

$$L \stackrel{d}{=} Z_{r,1} \circ G_{1/\alpha,r}.$$

Let us also assume that $\mathrm{rext}(F) = \infty$ *and Condition (47) holds with* $\gamma = \alpha$*. Then, there exists a positive function* $b(t)$ *such that*

$$\frac{M(t)}{b(t)} \Longrightarrow |T_{r,\alpha}| \tag{49}$$

as $t \to \infty$*. Moreover, the function* $b(t)$ *can be defined by (48).*

Proof. This statement directly follows from Lemma 10 with the account of (40). □

It is very simple to give examples of processes satisfying the conditions described in Propositions 18 and 19. Let $L(t) \equiv Ut$ and $d(t) \equiv t$, where $t \geq 0$, where U is a positive random variable. Then, choosing an appropriately distributed U, we can provide the validity of the corresponding condition for the convergence of $L(t)/d(t)$. Moreover, the parameter t may not have the meaning of physical time. For example, it may be some location parameter of $L(t)$, so that the statements of this section concern the case of the large mean intensity of the Cox process.

3.6. Convergence of the Distributions of Mixed Geometric Random Sums to the Generalized Lomax Distribution And Extensions of the Rényi Theorem

In the preceding section, we made sure that the generalized Lomax distribution can be limiting for extreme-order statistics in samples of random sizes. Here, it will be demonstrated that this distribution can also be used as an asymptotic approximation for the distributions of sums of independent random variables.

According to Proposition 13, if $\alpha \in (0,1]$ and $r > \frac{1}{\alpha}$, then the generalized Lomax distribution is mixed exponential. According to Corollary 7, it is infinitely divisible and hence, by the Lévy–Khintchin theorem, can be limiting for sums of independent random variables in the double array limit scheme under the condition of the uniform negligibility of summands.

However, the classical summation scheme is far from the only summation model within which the generalized Lomax distribution can be an asymptotic distribution. To be sure of this, consider two limit settings dealing with mixed geometric and mixed Poisson random sums. In both of these settings, we will deal with versions of the law of large numbers for random sums where, unlike the classical situation, the limit may be random [45].

First, consider mixed geometric random sums.

Let $p \in (0,1)$ and let V_p be a random variable having a geometric distribution with the parameter p: $\mathsf{P}(V_p = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$. This means that

$$\mathsf{P}(V_p > m) = \sum_{k=m+1}^{\infty} p(1-p)^{k-1} = (1-p)^m$$

for any $m \in \mathbb{N}$. Let $(\pi_n)_{n\geq 1}$ be a sequence of positive random variables taking values in the interval $(0,1)$, and, moreover, for each $n \geq 1$ and all $p \in (0,1)$, the random variables π_n and V_p are independent.

For each $n \in \mathbb{N}$, let $N_n = V_{\pi_n}$. Hence,

$$\mathsf{P}(N_n > m) = \int_0^1 (1-z)^m \, d\mathsf{P}(\pi_n < z) \tag{50}$$

for any $m \in \mathbb{N}$. The distribution of the random variable N_n will be called π_n*-mixed geometric* (for more details, see [58]).

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables such that the expectation $\mathsf{E}X_1$ exists. Assume that $\mathsf{E}X_1 \equiv a \neq 0$. According to the Kolmogorov strong law of large numbers, this condition implies that

$$\frac{1}{na} \sum_{j=1}^{n} X_j \longrightarrow 1 \tag{51}$$

almost surely as $n \to \infty$.

For $n \in \mathbb{N}$, let $S_n = X_1 + \cdots + X_n$. Let N_n be a random variable with a π_n-mixed geometric distribution (50). Assume that for each $n \in \mathbb{N}$, the random variable N_n is independent of the sequence $X_1, X_2, \ldots$. Our nearest aim is to study the asymptotic behavior of the random sum S_{N_n} as $n \to \infty$.

Lemma 12 ([58]). *Assume that the random variables $X_1, X_2, \ldots$ satisfy Condition (51). Let for each $n \in \mathbb{N}$ the random variable N_n have a π_n-mixed geometric distribution (50) and be independent of the sequence $X_1, X_2, \ldots$. Assume that there exists a positive random variable N such that*

$$n\pi_n \Longrightarrow N$$

as $n \to \infty$. Then

$$\frac{S_{N_n}}{n} \Longrightarrow aN^{-1} \circ W_1 \quad (n \to \infty).$$

Proposition 21. *Let $\alpha \in (0,1]$ and $r > \frac{1}{\alpha}$. Assume that the random variables $X_1, X_2, \ldots$ satisfy Condition (51). Let for each $n \in \mathbb{N}$ the random variable N_n have a π_n-mixed geometric distribution (50) and be independent of the sequence $X_1, X_2, \ldots$. Assume that*

$$n\pi_n \Longrightarrow U_{1,\alpha} \circ G_{r,r}^{1/\alpha} \tag{52}$$

as $n \to \infty$. Then,

$$\lim_{n\to\infty} \sup_{x\geq 0} \big|\mathsf{P}(S_{N_n} < na \cdot x) - \mathsf{P}(|T_{r,\alpha}| < x)\big| = 0.$$

Proof. By Lemma 12 with $N \stackrel{d}{=} U_{1,\alpha} \circ G_{r,r}^{1/\alpha}$ and (28), Condition (52) implies

$$\frac{S_{N_n}}{na} \Longrightarrow \big(U_{1,\alpha} \circ G_{r,r}^{1/\alpha}\big)^{-1} \circ W_1 \stackrel{d}{=} |T_{r,\alpha}|. \tag{53}$$

Now it remains for us to refer to the Dini theorem, according to which, since the distribution function of the limit random variable is continuous, convergence in distribution (53) implies the uniform convergence of the distribution functions. □

Proposition 21 is an example of extension of the famous Rényi theorem on the asymptotic behavior of the distributions of geometric sums (or rarefied renewal processes) [59] to the case of mixed geometric sums. In turn, the Rényi theorem can be regarded as an example of the law of large numbers for geometric random sums.

Now, we turn to mixed Poisson random sums. For each $n \in \mathbb{N}$, define the random variable N_n as $N_n = \Pi(L_n)$, where $\Pi(t)$, with $t \geq 0$, is the Poisson process with unit intensity and L_n is a positive random variable independent of the process $\Pi(t)$. The distribution of N_n is a mixed Poisson distribution, as follows:

$$\mathsf{P}(N_n = k) = \frac{1}{k!}\int_0^\infty e^{-u}u^k d\,\mathsf{P}(L_n < u), \quad k = 0, 1, 2, \ldots \tag{54}$$

Proposition 22. *Let $\alpha \in (0,2]$ and let $r > \frac{1}{\alpha}$. Assume that the random variables $X_1, X_2, \ldots$ satisfy Condition (51). Let for each $n \in \mathbb{N}$ the random variable N_n have a mixed Poisson distribution (54). Then,*

$$\lim_{n\to\infty} \sup_{x} \big|\mathsf{P}(S_{N_n} < na \cdot x) - \mathsf{P}(|T_{r,\alpha}| < x)\big| = 0$$

if and only if

$$\frac{L_n}{n} \Longrightarrow D_{r,\alpha} \stackrel{d}{=} \tfrac{1}{2}\big(U_{2,\alpha/2} \circ \overline{G}_{r,\alpha/2,r}\big)^{-1}.$$

Proof. This statement is the direct consequence of Theorem 1 in [60] and the Dini theorem mentioned above. □

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