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Pairs of Associated Yamabe Almost Solitons with Vertical Potential on Almost Contact Complex Riemannian Manifolds

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Abstract: Almost contact complex Riemannian manifolds, also known as almost contact B-metric manifolds, are, in principle, equipped with a pair of mutually associated pseudo-Riemannian metrics. Each of these metrics is specialized as a Yamabe almost soliton with a potential collinear to the Reeb vector field. The resulting manifolds are then investigated in two important cases with geometric significance. The first is when the manifold is of Sasaki-like type, i.e., its complex cone is a holomorphic complex Riemannian manifold (also called a Kähler–Norden manifold). The second case is when the soliton potential is torse-forming, i.e., it satisfies a certain recurrence condition for its covariant derivative with respect to the Levi–Civita connection of the corresponding metric. The studied solitons are characterized. In the three-dimensional case, an explicit example is constructed, and the properties obtained in the theoretical part are confirmed.

Keywords: Yamabe soliton; almost contact B-metric manifold; almost contact complex Riemannian manifold; Sasaki-like manifold; torse-forming vector field

MSC: 53C25; 53D15; 53C50; 53C44; 53D35; 70G45

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1. Introduction

The concept of Yamabe flow has been known since 1988, first introducted by R. S. Hamilton [1,2] to construct metrics with constant scalar curvature.

A time-dependent family of (pseudo-)Riemannian metrics ($g(t)$) considered on a smooth manifold ($\mathcal{M}$) is said to evolve by *Yamabe flow* if $g(t)$ satisfies the following evolution equation:

$$\frac{\partial}{\partial t}g(t) = -\tau(t)g(t), \qquad g(0) = g_0,$$

where $\tau(t)$ denotes the scalar curvature corresponding to $g(t)$.

A self-similar solution of the Yamabe flow on $(\mathcal{M}, g)$ is called a *Yamabe soliton* and is determined by the following equation:

$$\frac{1}{2}\mathcal{L}_\vartheta g = (\tau - \lambda)g, \tag{1}$$

where $\mathcal{L}_\vartheta g$ denotes the Lie derivative of g along the vector field (ϑ) called the soliton potential, and λ is the soliton constant (e.g., [3]). We denote this soliton as $(g; \vartheta, \lambda)$. In the case that λ is a differential function on $\mathcal{M}$, the solution is called a *Yamabe almost soliton*.

Many authors have studied Yamabe (almost) solitons on different types of manifolds in recent years (see e.g., [4–10]). The study of this kind of flow and the corresponding (almost) solitons has attracted the interest od mathematical physics because the Yamabe flow corresponds to the fast diffusion of the porous medium equation [11].

The author of [9] began the study of Yamabe solitons on almost contact complex Riemannian manifolds (abbreviated as accR manifolds) called almost contact B-metric manifolds. These manifolds are classified in [12] by G. Ganchev, V. Mihova, and K. Gribachev.

The pair of B metrics, which are related to each other by the almost contact structure, determine the geometry of the investigated manifolds. In [9,10], I studied Yamabe solitons obtained by contact conformal transformations for some interesting classes of manifolds. In the former paper, the studied manifold was cosymplectic or Sasaki-like, and in the latter, the soliton potential was torse-forming. Contact conformal transformations of an almost contact B-metric structure transform the two B metrics, the Reeb vector field, and its dual contact 1 form using this pair of metrics and a triplet of differentiable functions on the manifold (see e.g., [13]). These transformations generalize the $\mathcal{D}$-homothetic deformations of the considered manifolds introduced in [14].

In the present work, instead of these naturally occurring transformed Yamabe solitons involving the two B metrics, we use a condition for two Yamabe almost solitons for each of the metrics. Again, one of the simplest types of non-cosymplectic manifolds among those investigated, which is of interest to us, is precisely the Sasaki-like manifold introduced in [15]. This means that a warped product of a Sasaki-like accR manifold with a positive real axis gives rise to a complex cone, which is a Kähler manifold with a pair of Norden metrics. Note that the intersection of the classes of Sasaki-like manifolds and cosymplectic manifolds is an empty set. Different types of solitons on Sasaki-like manifolds were studied in [9,16,17].

Another interesting type of the studied manifold with Yamabe solitons is (as in [9,10]) the object of consideration in the present article. This is the case when the soliton potential is a torse-forming vertical vector field. Vertical means it has the same direction as the Reeb vector field. Torse-forming vector fields are defined by a certain recurrence condition for their covariant derivative regarding the Levi–Civita connection of the basic metric. These vector fields were first defined and studied by K. Yano [18], then investigated by various authors for manifolds with different tensor structures (e.g., [19–21]) and for the manifolds studied e.g., in [10,16,17].

The present paper is organized as follows. After the present introduction to the topic, in Section 2, we recall some known facts about the investigated manifolds. In Section 3, we set ourselves the task of equipping the considered manifolds with a pair of associated Yamabe almost solitons. In Section 4, we prove that there does not exist a Sasaki-like manifold equipped with a pair of Yamabe almost solitons with the vertical potential generated by each of the two fundamental metrics. A successful solution to the problem posed in Section 3 is proposed in Section 5 in the case in which the vertical potentials of the pair of Yamabe almost solitons are torse-forming. Section 6 provides an explicit example of the smallest dimension of the type of manifold constructed in the previous section.

2. accR Manifolds

A differentiable manifold ($\mathcal{M}$) of dimensions $(2n+1)$ and equipped with an almost contact structure (φ, ξ, η) and a B metric (g) is called an *almost contact B-metric manifold* or an *almost contact complex Riemannian* (abbr. *accR*) *manifold* and is denoted by $(\mathcal{M}, \varphi, \xi, \eta, g)$. More concretely, φ is an endomorphism of the tangent bundle $T\mathcal{M}$, ξ is a Reeb vector field, η is its dual contact 1 form, and g is a pseudo-Riemannian metric of signature $(n+1, n)$ satisfying the following conditions:

$$\begin{gathered} \varphi\xi = 0, \qquad \varphi^2 = -\iota + \eta \otimes \xi, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1, \\ g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \end{gathered} \tag{2}$$

where ι stands for the identity transformation on $\Gamma(T\mathcal{M})$ [12].

In the latter equality and beyond, x, y, and z represent arbitrary elements of $\Gamma(T\mathcal{M})$ or vectors in the tangent space ($T_p\mathcal{M}$) of $\mathcal{M}$ at an arbitrary point (p) in $\mathcal{M}$.

The following equations are immediate consequences of (2).

$$g(\varphi x, y) = g(x, \varphi y), \qquad g(x, \xi) = \eta(x), \qquad g(\xi, \xi) = 1, \qquad \eta(\nabla_x \xi) = 0,$$

where ∇ denotes the Levi–Civita connection of g.

The associated metric ($\tilde{g}$) of g on $\mathcal{M}$ is also a B metric and is defined by

$$\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y). \tag{3}$$

In [12], accR manifolds are classified with respect to the (0,3)-tensor F defined by

$$F(x, y, z) = g\big((\nabla_x \varphi) y, z\big). \tag{4}$$

It has the following basic properties:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y) F(x, \xi, z) + \eta(z) F(x, y, \xi), \tag{5}$$

$$F(x, \varphi y, \xi) = (\nabla_x \eta) y = g(\nabla_x \xi, y). \tag{6}$$

The Ganchev–Mihova–Gribachev classification of the studied manifolds cited in Section 1 consists of eleven basic classes ($\mathcal{F}_i$, $i \in \{1, 2, \dots, 11\}$) determined by conditions for F.

3. Pair of Associated Yamabe Almost Solitons

Let us consider an accR manifold ($(\mathcal{M}, \varphi, \xi, \eta, g)$) with a pair of associated Yamabe almost solitons generated by the pair of B metrics (g and $\tilde{g}$), i.e., $(g; \vartheta, \lambda)$ and $(\tilde{g}; \tilde{\vartheta}, \tilde{\lambda})$, which are mutually associated by the (φ, ξ, η) structure. Then, along with (1), the following identity also holds:

$$\frac{1}{2} \mathcal{L}_{\tilde{\vartheta}} \tilde{g} = (\tilde{\tau} - \tilde{\lambda}) \tilde{g}, \tag{7}$$

where $\tilde{\vartheta}$ and $\tilde{\lambda}$ are the soliton potential and the soliton function, respectively, and $\tilde{\tau}$ is the scalar curvature of the manifold with respect to $\tilde{g}$. We suppose that the potentials ϑ and $\tilde{\vartheta}$ are vertical, i.e., there exist differentiable functions (k and $\tilde{k}$ on $\mathcal{M}$), such that we have

$$\vartheta = k\xi, \qquad \tilde{\vartheta} = \tilde{k}\xi, \tag{8}$$

where $k(p) \neq 0$ and $\tilde{k}(p) \neq 0$ at every point p of M. We denote these potentials as (ϑ, k) and $(\tilde{\vartheta}, \tilde{k})$, respectively.

In this case, for the Lie derivatives of g and $\tilde{g}$ along ϑ and $\tilde{\vartheta}$, respectively, we obtain the following expressions:

$$\begin{aligned} (\mathcal{L}_{\vartheta} g)(x, y) &= g(\nabla_x \vartheta, y) + g(x, \nabla_y \vartheta) \\ &= \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x) + k\big\{g(\nabla_x \xi, y) + g(x, \nabla_y \xi)\big\}, \end{aligned} \tag{9}$$

$$\begin{aligned} \big(\mathcal{L}_{\tilde{\vartheta}} \tilde{g}\big)(x, y) &= \tilde{g}(\tilde{\nabla}_x \tilde{\vartheta}, y) + \tilde{g}(x, \tilde{\nabla}_y \tilde{\vartheta}) \\ &= \mathrm{d}\tilde{k}(x)\eta(y) + \mathrm{d}\tilde{k}(y)\eta(x) + \tilde{k}\big\{\tilde{g}(\tilde{\nabla}_x \xi, y) + \tilde{g}(x, \tilde{\nabla}_y \xi)\big\}. \end{aligned} \tag{10}$$

4. The Case When the Underlying accR Manifold Is Sasaki-like

The authors of [15], introduced a *Sasaki-like* manifold among accR manifolds. This type of manifold is defined by the condition that its complex cone is a Kähler–Norden manifold, i.e., the derived almost complex manifold ($\mathcal{M} \times \mathbb{R}^-$) equipped with a Norden metric ($r^2 g + \eta \otimes \eta - \mathrm{d}r^2$) for $r \in \mathbb{R}^-$ to have a parallel complex structure. A Sasaki-like accR manifold is determined by the following condition

$$(\nabla_x \varphi) y = \eta(y) \varphi^2 x + g(\varphi x, \varphi y) \xi.$$

Therefore, the fundamental tensor (F) of such a manifold has the following form:

$$F(x,y,z) = g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y). \tag{11}$$

Obviously, Sasaki-like accR manifolds form a subclass of the $\mathcal{F}_4$ class of the Ganchev–Mihova–Gribachev classification. Moreover, the following identities are valid:

$$\begin{aligned} &\nabla_x \xi = -\varphi x, && (\nabla_x \eta)(y) = -g(x, \varphi y), \\ &R(x,y)\xi = \eta(y)x - \eta(x)y, && \rho(x,\xi) = 2n\,\eta(x), \\ &R(\xi,y)z = g(y,z)\xi - \eta(z)y, && \rho(\xi,\xi) = 2n, \end{aligned} \tag{12}$$

where R and ρ represent the curvature tensor and the Ricci tensor of ∇, respectively, usually defined as $R = [\nabla, \nabla] - \nabla_{[\,,\,]}$, and ρ is the result of the contraction of R by its first index [15].

If the considered accR manifold ($(\mathcal{M}, \varphi, \xi, \eta, g)$) is Sasaki-like, due to the first equality of (12), (9) takes the following form:

$$(\mathcal{L}_\vartheta g)(x,y) = \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x) - 2kg(x, \varphi y). \tag{13}$$

We then input the result of (13) into (1) and obtain the following:

$$\frac{1}{2}\{\mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x)\} - kg(x, \varphi y) = (\tau - \lambda)g(x,y). \tag{14}$$

Replacing x and y with ξ in (14) yields

$$\mathrm{d}k(\xi) = \tau - \lambda. \tag{15}$$

The trace of (14) in an arbitrary basis ($\{e_i\}$ $(i = 1, 2, \ldots, 2n+1)$) implies

$$\mathrm{d}k(\xi) = (2n+1)(\tau - \lambda). \tag{16}$$

Combining (15) and (16) leads to $k = 0$, which contradicts the conditions; therefore, we find the following to be true:

Theorem 1. *There does not exist a Sasaki-like manifold ($(\mathcal{M}, \varphi, \xi, \eta, g)$) equipped with a g-generated Yamabe almost soliton with a vertical potential.*

Now, let us consider a similar situation but with respect to the associated B metric ($\tilde{g}$) and the corresponding Levi–Civita connection ($\tilde{\nabla}$).

First, similarly to (4), we define the fundamental tensor ($\tilde{F}$) for $\tilde{g}$ as follows:

$$\tilde{F}(x,y,z) = \tilde{g}\big((\tilde{\nabla}_x \varphi)y, z\big).$$

Since $\tilde{g}$ is also a B metric like g, it is obvious that properties (5) and (6) also hold for $\tilde{F}$, i.e.,

$$\begin{aligned} \tilde{F}(x,y,z) = \tilde{F}(x,z,y) = \tilde{F}(x,\varphi y, \varphi z) + \eta(y)\tilde{F}(x,\xi,z) + \eta(z)\tilde{F}(x,y,\xi), \\ \tilde{F}(x,\varphi y,\xi) = (\tilde{\nabla}_x \eta)y = \tilde{g}(\tilde{\nabla}_x \xi, y). \end{aligned} \tag{17}$$

Then, the well-known Koszul formula is used for $\tilde{g}$, i.e.,

$$\begin{aligned} 2\tilde{g}\Big(\tilde{\nabla}_x y, z\Big) = {} & x\big(\tilde{g}(y,z)\big) + y\big(\tilde{g}(x,z)\big) - z\big(\tilde{g}(x,y)\big) \\ & + \tilde{g}\big([x,y],z\big) + \tilde{g}\big([z,y],x\big) + \tilde{g}\big([z,x],y\big), \end{aligned}$$

After lengthy but standard calculations, we obtain the following relationship between $\tilde{F}$ and F [22]:

$$\begin{aligned} 2\tilde{F}(x,y,z) = {} & F(\varphi y, z, x) - F(y, \varphi z, x) + F(\varphi z, y, x) - F(z, \varphi y, x) \\ & + \{F(x,y,\xi) + F(\varphi y, \varphi x, \xi) + F(x, \varphi y, \xi)\}\eta(z) \\ & + \{F(x,z,\xi) + F(\varphi z, \varphi x, \xi) + F(x, \varphi z, \xi)\}\eta(y) \\ & + \{F(y,z,\xi) + F(\varphi z, \varphi y, \xi) + F(z,y,\xi) + F(\varphi y, \varphi z, \xi)\}\eta(x). \end{aligned} \tag{18}$$

Lemma 1. *For a Sasaki-like manifold $((\mathcal{M}, \varphi, \xi, \eta, g))$ with associated B metric $\tilde{g}$, the following holds:*

$$\tilde{\nabla}_x \xi = -\varphi x. \tag{19}$$

Proof. Due to (17) and (18), we obtain the following:

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_x \xi, y) = {} & F(\varphi^2 y, \xi, x) - F(\xi, \varphi^2 y, x) + \{F(\varphi y, \xi, \xi) + F(\xi, \varphi y, \xi)\}\eta(x) \\ & + F(x, \varphi y, \xi) + F(\varphi^2 y, \varphi x, \xi) + F(x, \varphi^2 y, \xi). \end{aligned} \tag{20}$$

In deriving the last equality, we use the properties in (2). We then apply the expression of φ^2 from (2) and some properties of F in this case. The first is $F(\xi, \xi, x) = 0$, which is a consequence of (11), and the second is the general identity $F(x, \xi, \xi) = 0$, which comes from (5). Thus, the relation in (20) simplifies to the following form:

$$2\tilde{g}(\tilde{\nabla}_x \xi, y) = F(\xi, x, y) + F(x, \varphi y, \xi) - F(y, \varphi x, \xi) - F(x, y, \xi) - F(y, x, \xi). \tag{21}$$

Thereafter, we compute the various components in the above formula by exploiting the fact that the given manifold is Sasaki-like, i.e., (11) is valid, and we obtain:

$$F(\xi, x, y) = 0, \qquad F(x, y, \xi) = g(\varphi x, \varphi y), \qquad F(x, \varphi y, \xi) = -g(x, \varphi y).$$

As a result, given the symmetry of $g(x, \varphi y)$ with respect to x and y, as well as (3), the equality in (21) simplifies to the following form:

$$\tilde{g}(\tilde{\nabla}_x \xi, y) = -\tilde{g}(\varphi x, y),$$

which is an equivalent expression of (19). □

We now apply (19) to (10) and use (3) to obtain:

$$(\mathcal{L}_{\tilde{\vartheta}} \tilde{g})(x,y) = \mathrm{d}\tilde{k}(x)\eta(y) + \mathrm{d}\tilde{k}(y)\eta(x) - 2\tilde{k}\, g(\varphi x, \varphi y). \tag{22}$$

We then substitute the expression from (22) into (7) and obtain the following:

$$\frac{1}{2}\{\mathrm{d}\tilde{k}(x)\eta(y) + \mathrm{d}\tilde{k}(y)\eta(x)\} - \tilde{k}\, g(\varphi x, \varphi y) = (\tilde{\tau} - \tilde{\lambda})\tilde{g}(x,y). \tag{23}$$

Contracting (23), we infer

$$\mathrm{d}\tilde{k}(\xi) + 2n\tilde{k} = \tilde{\tau} - \tilde{\lambda}. \tag{24}$$

On the other hand, we replace x and y in (23) with ξ and obtain

$$\mathrm{d}\tilde{k}(\xi) = \tilde{\tau} - \tilde{\lambda}. \tag{25}$$

Then, (24) and (25) imply $\tilde{k} = 0$, which is not admissible for the potential; therefore, the following holds:

Theorem 2. *There does not exist a Sasaki-like manifold* $((\mathcal{M}, \varphi, \xi, \eta, g))$ *equipped with a* $\tilde{g}$*-generated Yamabe almost soliton with a vertical potential.*

5. The Case of a Torse-Forming Vertical Potential

Let us recall a vector field (ϑ) on a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$ called a *torse-forming vector field* if the following identity is true:

$$\nabla_x \vartheta = f\, x + \gamma(x)\vartheta, \tag{26}$$

where f is a differentiable function, and γ is a 1 form [18,23]. The 1 form γ is called the *generating form*, and the function (f) is called the *conformal scalar* of ϑ [20].

Remark 1. *Some special types of torse-forming vector fields have been considered in various studies. A vector field (ϑ) determined by* (26) *is called:*

- torqued *if* $\gamma(\vartheta) = 0$ [21];
- concircular *if* $\gamma = 0$ [24];
- concurrent *if* $f - 1 = \gamma = 0$ [25];
- recurrent *if* $f = 0$ [26];
- parallel *if* $f = \gamma = 0$ (e.g., [27]).

Furthermore, if the potential (ϑ) is vertical, i.e., $\vartheta = k\,\xi$, then (26) yields the following:

$$\mathrm{d}k(x)\xi + k\nabla_x\xi = f\, x + k\,\gamma(x)\xi. \tag{27}$$

Since $\eta(\nabla_x\xi)$ vanishes identically, (27) implies the following:

$$\mathrm{d}k(x) = f\,\eta(x) + k\,\gamma(x),$$

which, due to the nowhere-vanishing k, yields the following expression for the generating form of ϑ:

$$\gamma(x) = \frac{1}{k}\{\mathrm{d}k(x) - f\,\eta(x)\}. \tag{28}$$

Then, the torse-forming vertical potential is determined by f and k, as denoted by $\vartheta(f, k)$.

Plugging (28) into (26), we obtain

$$\nabla_x\vartheta = -f\,\varphi^2 x + \mathrm{d}k(x)\xi, \tag{29}$$

which, together with $\nabla_x\vartheta = \nabla_x(k\,\xi) = \mathrm{d}k(x)\xi + k\nabla_x\xi$, yields the following form in the considered case:

$$\nabla_x\xi = -\frac{f}{k}\varphi^2 x. \tag{30}$$

By virtue of (30), for the curvature tensor of g, we obtain

$$R(x,y)\xi = -\Big\{\mathrm{d}h(x) + h^2\eta(x)\Big\}\varphi^2 y + \Big\{\mathrm{d}h(y) + h^2\eta(y)\Big\}\varphi^2 x, \tag{31}$$

where the following shorter notation is used for the function that is the coefficient in (30).

$$h = \frac{f}{k}. \tag{32}$$

As an immediate consequence of (31), we obtain the following expressions:

$$
\begin{aligned}
R(\xi, y)z &= g(\varphi y, \varphi z)\,\mathrm{grad}\,h - \mathrm{d}h(z)\varphi^2 y + h^2\{\eta(z)y - g(y,z)\xi\}, \\
\rho(y, \xi) &= -(2n-1)\mathrm{d}h(y) - \Big\{\mathrm{d}h(\xi) + 2nh^2\Big\}\eta(y), \\
\rho(\xi, \xi) &= -2n\Big\{\mathrm{d}h(\xi) + h^2\Big\}.
\end{aligned}
$$

Given (6) and (32), Equation (30) can be rewritten in the following form:

$$F(x, \varphi y, \xi) = -h\, g(\varphi x, \varphi y). \tag{33}$$

Bearing in mind that $F(x, \xi, \xi) = 0$, following from (5), the expression of (33) is equivalent to the following equality:

$$F(x, y, \xi) = -h\, g(x, \varphi y). \tag{34}$$

Then, (9) and (1) imply

$$\frac{1}{2}\{\mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x)\} - f g(\varphi x, \varphi y) = (\tau - \lambda)g(x, y). \tag{35}$$

Contracting (35) yields

$$\mathrm{d}k(\xi) + 2n\, f = (2n+1)(\tau - \lambda), \tag{36}$$

and substituting $x = y = \xi$ into (35) yields

$$\mathrm{d}k(\xi) = \tau - \lambda. \tag{37}$$

Then, combining (36) and (37) leads to an expression for the conformal scalar of ϑ as follows:

$$f = \tau - \lambda. \tag{38}$$

This means that the following statement is valid:

Theorem 3. *Let an accR manifold $((\mathcal{M}, \varphi, \xi, \eta, g))$ be equipped with a Yamabe almost soliton $(g; \vartheta(f,k), \lambda)$, where ϑ is a vertical torse-forming potential. Then, the scalar curvature (τ) of this manifold is the sum of the conformal scalar (f) of ϑ and the soliton function (λ), i.e., $\tau = f + \lambda$.*

Equations (37) and (38) yield

$$f = \mathrm{d}k(\xi). \tag{39}$$

Substituting (39) into (32), we obtain the following expression of the function h:

$$h = \mathrm{d}(\ln k)(\xi).$$

Corollary 1. *The potential $(\vartheta(f,k))$ of any Yamabe almost soliton $(g; \vartheta, \lambda)$ on $(\mathcal{M}, \varphi, \xi, \eta, g)$ is a torqued vector field.*

Proof. According to (39) and (28), $\gamma(\xi)$ vanishes. Hence, $\gamma(\vartheta) = 0$ is true, i.e., the potential (ϑ) is torqued, given Remark 1. □

The authors of [16] showed that class $\mathcal{F}_5$ is the only basic class in the considered classification of accR manifolds in which ξ or its collinear vector field can be torse-forming. Furthermore, the general class of accR manifolds with a torse-forming ξ is $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus$

$\mathcal{F}_5 \oplus \mathcal{F}_{10}$. Note that $\mathcal{F}_5$ manifolds are counterparts of β-Kenmotsu manifolds in the case of almost contact metric manifolds. The definition of class $\mathcal{F}_5$ is expressed as follows [12]:

$$F(x,y,z) = -\frac{\theta^*(\xi)}{2n}\{g(x,\varphi y)\eta(z) + g(x,\varphi z)\eta(y)\}, \tag{40}$$

where $\theta^*(\cdot) = g^{ij}F(e_i, \varphi e_j, \cdot)$ with respect to the basis $\{e_1, \ldots, e_{2n}, \xi\}$ of $T_p\mathcal{M}$. Moreover, on an $\mathcal{F}_5$ manifold, the Lee form (θ^*) satisfies the property $\theta^* = \theta^*(\xi)\eta$.

Then, in addition to the component in (34), we have

$$F(\xi,y,z) = 0, \qquad \omega = 0. \tag{41}$$

Let the potential $(\tilde{\vartheta})$ of the Yamabe almost soliton $(\tilde{g};\tilde{\vartheta},\tilde{\lambda})$ also be torse-forming and vertical, i.e.,

$$\tilde{\nabla}_x\tilde{\vartheta} = \tilde{f}\,x + \tilde{\gamma}(x)\tilde{\vartheta}, \qquad \tilde{\vartheta} = \tilde{k}\,\xi.$$

Similarly, we obtain analogous equalities of (29) and (30) for $\tilde{g}$ and its Levi–Civita connection $(\tilde{\nabla})$ in the following form:

$$\tilde{\nabla}_x\tilde{\vartheta} = -\tilde{f}\,\varphi^2 x + \mathrm{d}\tilde{k}(x)\xi, \tag{42}$$

$$\tilde{\nabla}_x\xi = -\tilde{h}\,\varphi^2 x, \tag{43}$$

where

$$\tilde{h} = \frac{\tilde{f}}{\tilde{k}}.$$

Moreover, we also have $\tilde{f} = \mathrm{d}\tilde{k}(\xi)$ and $\tilde{h} = \mathrm{d}(\ln\tilde{k})(\xi)$.

Thus, the following analogous assertions are valid.

Theorem 4. *Let an accR manifold $((\mathcal{M},\varphi,\xi,\eta,g))$ be equipped with a Yamabe almost soliton $(\tilde{g};\tilde{\vartheta}(\tilde{f},\tilde{k}),\tilde{\lambda})$, where $\tilde{\vartheta}$ is a vertical torse-forming potential. Then, the scalar curvature $(\tilde{\tau})$ of this manifold is the sum of the conformal scalar $(\tilde{f})$ of $\tilde{\vartheta}$ and the soliton function $(\tilde{\lambda})$, i.e., $\tilde{\tau} = \tilde{f} + \tilde{\lambda}$.*

Corollary 2. *The potential $(\tilde{\vartheta}(\tilde{f},\tilde{k}))$ of any Yamabe almost soliton $(\tilde{g};\tilde{\vartheta},\tilde{\lambda})$ on $(\mathcal{M},\varphi,\xi,\eta,g)$ is a torqued vector field.*

The following equality is given in [12] and expresses the relation between ∇ and $\tilde{\nabla}$ for the pair of B metrics of an arbitrary accR manifold:

$$\begin{aligned} 2g(\tilde{\nabla}_x y,z) = {} & 2g(\nabla_x y,z) - F(x,y,\varphi z) - F(y,x,\varphi z) + F(\varphi z,x,y) \\ & + \{F(y,z,\xi) + F(\varphi z,\varphi y,\xi) - \omega(\varphi y)\eta(z)\}\eta(x) \\ & + \{F(x,z,\xi) + F(\varphi z,\varphi x,\xi) - \omega(\varphi x)\eta(z)\}\eta(y) \\ & - \{F(\xi,x,y) - F(y,x,\xi) - F(x,\varphi y,\xi) \\ & \qquad -F(x,y,\xi) - F(y,\varphi x,\xi)\}\eta(z). \end{aligned} \tag{44}$$

By setting $y = \xi$, the last equality implies the following:

$$\begin{aligned} 2g(\tilde{\nabla}_x\xi,z) = {} & 2g(\nabla_x\xi,z) - F(x,\varphi z,\xi) - F(\xi,x,\varphi z) + F(\varphi z,x,\xi) \\ & + \omega(z)\eta(x) + F(x,z,\xi) + F(\varphi z,\varphi x,\xi). \end{aligned} \tag{45}$$

Taking into account (30), (33), (34), and (43), the relation (44) takes the following form:

$$2(\tilde{h} - h)g(\varphi x,\varphi z) = F(\xi,x,\varphi z) - \eta(x)\omega(z),$$

which, for an $\mathcal{F}_5$ manifold, due to (41), implies $\tilde{h} = h$, i.e.,

$$\frac{f}{k} = \frac{\tilde{f}}{\tilde{k}}. \tag{46}$$

To express some curvature properties of accR manifolds, an associated quantity (τ^*) of the scalar curvature (τ) of g is used in [28]. It is defined by the following trace of the Ricci tensor: ρ: $\tau^* = g^{ij}\rho_{is}\varphi_j^s$ with respect to the basis $\{e_1, \ldots, e_{2n}, \xi\}$. The relation between $\tilde{\tau}$ and τ^* for a manifold belonging to $\mathcal{F}_5^0 \subset \mathcal{F}_5$ is given in ([28], Corollary 2) as follows:

$$\tilde{\tau} = -\tau^* - \frac{2n+1}{2n}(\theta^*(\xi))^2 - 2\xi(\theta^*(\xi)). \tag{47}$$

The $\mathcal{F}_5^0$ subclass of $\mathcal{F}_5$ is introduced in [13] by the condition that the Lee form (θ^*) of the manifold be closed, i.e., $\mathrm{d}\,\theta^* = 0$. The last equality is equivalent to the following condition:

$$\mathrm{d}(\theta^*(\xi)) = \xi(\theta^*(\xi))\eta. \tag{48}$$

Using (33), we compute that

$$\theta^*(\xi) = 2nh, \qquad \xi(\theta^*(\xi)) = 2n\,\mathrm{d}h(\xi).$$

Therefore, (47) takes the following form:

$$\tilde{\tau} = -\tau^* - 2n(2n+1)h^2 - 4n\,\mathrm{d}h(\xi). \tag{49}$$

6. Example: A Cone over a Two-Dimensional Complex Space Form with Norden Metric

In this section, we consider the accR manifold construction given in [29].

First, let $(\mathcal{N}, J, g')$ be a two-dimensional almost complex manifold with Norden metric, i.e., J is an almost complex structure, and g' is a pseudo-Riemannian metric with a neutral signature such that $g'(Jx', Jy') = -g'(x', y')$ for arbitrary $x', y' \in \Gamma(T\mathcal{N})$. It is then known that $(\mathcal{N}, J, g')$ is a complex space form with constant sectional curvature, denoted, e.g., by k'.

Second, let $\mathcal{C}(\mathcal{N})$ be the cone over $(\mathcal{N}, J, g')$, i.e., $\mathcal{C}(\mathcal{N})$ is the warped product ($\mathbb{R}^+ \times_t \mathcal{N}$) with a generated metric (g) as follows:

$$g\left(\left(x', a\tfrac{\mathrm{d}}{\mathrm{d}t}\right), \left(y', b\tfrac{\mathrm{d}}{\mathrm{d}t}\right)\right) = t^2 g'(x', y') + ab,$$

where t is the coordinate on the set of positive reals ($\mathbb{R}^+$), and a and b are differentiable functions on $\mathcal{C}(\mathcal{N})$. Moreover, $\mathcal{C}(\mathcal{N})$ is equipped with an almost contact structure (φ, ξ, η) by

$$\varphi|_{\ker\eta} = J, \quad \xi = \tfrac{\mathrm{d}}{\mathrm{d}t}, \quad \eta = \mathrm{d}t, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{50}$$

Then, $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ is a three-dimensional accR manifold belonging to the $\mathcal{F}_1 \oplus \mathcal{F}_5$ class. In particular, this manifold can be of $\mathcal{F}_5$ if and only if J is parallel with respect to the Levi–Civita connection of g', but the constructed manifold cannot belong to $\mathcal{F}_1$ or $\mathcal{F}_0$ [29].

Let the considered manifold $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ belong to $\mathcal{F}_5$. Using the result ($\theta^*(\xi) = \frac{2}{t}$) from [29], we verify that the condition in (48) holds; therefore, $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ belongs to $\mathcal{F}_5^0$.

Let $\{e_1, e_2, e_3\}$ be a basis in any tangent space at an arbitrary point of $\mathcal{C}(\mathcal{N})$ such that

$$\begin{gathered} \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad e_3 = \xi, \\ g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_i, e_j) = 0, \ i \neq j. \end{gathered} \tag{51}$$

In [29], it is shown that the nonzero components of R of the constructed three-dimensional manifold with respect to the basis $\{e_1, e_2, e_3\}$ are determined by the equality

($R_{1212} = \frac{1}{t^2}(k' - 1)$) and the well-known properties of R. Obviously, $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ is flat if and only if $k' = 1$ for $(\mathcal{N}, J, g')$. The nonzero components of the Ricci tensor of $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ in the general case are then calculated as $\rho_{11} = -\rho_{22} = \frac{1}{t^2}(k' - 1)$. Furthermore, the scalar curvature (τ) and the associated quantity (τ^*) of $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$ are given by

$$\tau = \frac{2}{t^2}(k' - 1), \qquad \tau^* = 0. \tag{52}$$

Then, taking into account the vanishing of τ^*, the expression

$$\theta^*(\xi) = \frac{2}{t}, \tag{53}$$

and $n = 1$, we calculate $\tilde{\tau}$ according to (47) as

$$\tilde{\tau} = -\frac{2}{t^2}. \tag{54}$$

Using the results ($\nabla_{e_1} e_3 = \frac{1}{t} e_1$, $\nabla_{e_2} e_3 = \frac{1}{t} e_2$, and $\nabla_{e_3} e_3 = 0$) from [29] and $e_3 = \xi$ from (51), we derive the following formula for any x on $\mathcal{C}(\mathcal{N})$.

$$\nabla_x \xi = -\frac{1}{t}\varphi^2 x. \tag{55}$$

Comparing the last equality with (30), we conclude that

$$\frac{f}{k} = \frac{1}{t}, \tag{56}$$

i.e., $h = \frac{1}{t}$ holds due to (32), and (49) is also valid.

According to (39) and (56) and the expression of ξ in (50), we obtain the differential equation $t\mathrm{d}k = k\mathrm{d}t$, the solution of which for the function $k(t)$ is

$$k = ct, \tag{57}$$

where c is an arbitrary constant. Hence, (56) and (57) imply

$$f = c. \tag{58}$$

Taking into account (9), (55), and (57), we obtain

$$\mathcal{L}_\vartheta g = 2cg. \tag{59}$$

Let us define the following differentiable function on $\mathcal{C}(\mathcal{N})$

$$\lambda = \frac{2}{t^2}(k' - 1) - c. \tag{60}$$

Then, bearing in mind (52), (59), and (60), we check that the condition in (1) is satisfied and that $(g; \vartheta, \lambda)$ is a Yamabe almost soliton with vertical potential (ϑ).

Due to (8) and (57), the soliton potential (ϑ) is determined by $\vartheta = ct\xi$. Then, because $\mathrm{d}t = \eta$ according to (50) and (55), we obtain $\nabla_x \vartheta = cx$. This means that ϑ is torse-forming with conformal scalar $f = c$ and zero-generating form γ. According to Remark 1, the torse-forming vector field (ϑ) is concircular in the general case of our example, and, in particular, when $c = 1$, it is concurrent. Obviously, every concircular vector field is torqued, which supports Corollary 1.

Taking into account (52), (58), and (60), we check the truthfulness of Theorem 3.

In [30], a relation between the Levi–Civita connections (∇ and $\tilde{\nabla}$) of g and $\tilde{g}$, respectively, is given for $\mathcal{F}_5$ as follows:

$$\tilde{\nabla}_x y = \nabla_x y - \frac{\theta^*(\xi)}{2n}\{g(x,\varphi y) + g(\varphi x, \varphi y)\}\xi.$$

This relation for $(\mathcal{C}(\mathcal{N}), \varphi, \xi, \eta, g)$, and $y = \xi$ implies $\tilde{\nabla}_x \xi = \nabla_x \xi$, which, due to (55), yields

$$\tilde{\nabla}_x \xi = -\frac{1}{t}\varphi^2 x. \tag{61}$$

The expression in (61) also follows from (40), (45), and (53).

Then, using (43) and (61), we obtain

$$\frac{\tilde{f}}{\tilde{k}} = \frac{1}{t}, \tag{62}$$

which supports (46) and (56).

In a manner similar to obtaining (57) and (58), starting with (62), we find

$$\tilde{k} = \tilde{c}t, \qquad \tilde{c} = \text{const}, \tag{63}$$

$$\tilde{f} = \tilde{c}. \tag{64}$$

By virtue of (10), (61), and (63), we have

$$\mathcal{L}_{\tilde{\vartheta}}\tilde{g} = 2\tilde{c}\tilde{g}. \tag{65}$$

We define the following differentiable function on $\mathcal{C}(\mathcal{N})$:

$$\tilde{\lambda} = -\frac{2}{t^2} - \tilde{c}, \tag{66}$$

which, together with (54) and (65) shows, that the condition in (7) holds. Therefore, $(\tilde{g}; \tilde{\vartheta}, \tilde{\lambda})$ is a Yamabe almost soliton with vertical potential $(\tilde{\vartheta})$.

Using (42), (63), (64), and $\mathrm{d}t = \eta$ from (50), we obtain $\nabla_x \tilde{\vartheta} = \tilde{c}x$, which shows that $\tilde{\vartheta}$ is torse-forming with conformal scalar $\tilde{f} = \tilde{c}$ and zero-generating form $\tilde{\gamma}$. Therefore, $\tilde{\vartheta}$ is concircular for arbitrary $\tilde{c}$ and concurrent for $\tilde{c} = 1$. Obviously, every concircular vector field is torqued, which supports Corollary 2. Furthermore, the results of (54), (64), and (66) support Theorem 4.

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