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Fuzzy Domination Graphs in Decision Support Tasks

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Abstract: In decision support tasks, one often has to deal with uncertainty due to fuzzy judgments of the decision maker or the expert. This paper proposes methods that allow you to rank the alternatives based on fuzzy evaluations. This is achieved by using fuzzy weighted summation, fuzzy implication, a computation graph showing the criteria, and a fuzzy dominance graph showing the alternatives. If the criteria have equal importance, then fuzzy graphs corresponding to the dominance of each of the criteria are used. An algorithm that is used for both the transition from fuzzy dominance graphs and the ranking of alternatives is proposed. This algorithm is based on the idea of constructing Kemeny medians or other concordant rankings at a given confidence level in the existence of corresponding arcs. Computational experiments have shown the performance of these approaches. It is reasonable to apply them in problems that require complex expert evaluations with a large number of alternatives and criteria.

Keywords: fuzzy number; multicriteria decision analysis; fuzzy graph; weighted sum model; ranking

MSC: 90B50; 05C72

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1. Introduction

Despite the fact that the theory of fuzzy sets has existed for quite a long time, its practical application faces a number of difficulties. One of the difficulties is the definition of membership functions for fuzzy model parameters. The solution of such problems strongly depends on the specific subject area. One article [1] considered the situation with air transit when fuzzy arises due to the incompleteness of our knowledge about the world around us. Another problem is related to the fact that fuzzy judgments are inherent in human nature. This includes fuzzy judgments in decision support tasks. Often, a decision maker (DM) cannot be fully confident in the priorities of the criteria, on the basis of which a choice is made. Uncertainty arises in the analysis of points of multicriteria space and appears in many problems [2,3]. If fuzzy estimates of their usefulness are calculated, how do these points compare to these estimates? To answer this question, a technique for constructing a fuzzy ranking of alternatives, based on fuzzy preference domains, fuzzy judgement matching methods, and fuzzy graphs, was developed. This allowed us to take into account the required level of confidence in the dominance of alternatives and thus move to a more or less strict ordering of decisions.

Decision making is a complex process which seeks to determine the desired outcome from a variety of perspectives. This process may use implicit or explicit assumptions, which are influenced by several factors, such as physiological, biological, cultural, social, and factors, among others. Nowadays, complex decision-making problems can be solved using statistical data, economic theories, mathematical models, and special algorithms that help to calculate and evaluate decisions. The interest in these methods is now extensive. In Ref. [4], it is said that, every year, thousands of papers are published on the multicriteria

analysis of alternatives, and their number is growing every year. Modern web technologies allow new approaches to be applied for visualization purposes in decision support systems [5]. Decision support tasks arise in different areas, including aviation [6], medicine [7], enterprise resource management [8], and many other industries [9].

2. Literature Review

A comparison of fuzzy values is a classical decision-making problem. This rating and ranking method is very often used in a situation that does not allow a more structured decision approach. Such problems are usually characterized by a lack of objective and reliable information. If the described rating and ranking method is used in such a situation, quite frequently, there will be doubt and uncertainty about the exact values that are to be assigned to various ratings and weights.

This paper contends that the sort of uncertainty that comes into play here is better represented by the notion of fuzziness than that of chance. The idea of fuzziness is explained by Zadeh [10].

It is a simple exercise in the theory of fuzzy sets to reformulate Kahne's basic procedure [11]. Kahne's ideas are somewhat elaborated by introducing measures that make it explicit how much better the preferred alternative is compared to the other alternatives, and how the initial uncertainties manifest themselves in a final evaluation.

Many fuzzy ordering methods are proposed in the literature [12–15]. Wang and Kerre [16] propose three classes of fuzzy ordering methods. In the first class, each method converts fuzzy numbers by estimation [17]. They are then compared on their respective values, as demonstrated by [18–20]. The second class sets the reference sets and ranks the fuzzy numbers that compare the reference sets [21,22]. In the last class, a fuzzy relation is constructed for pairwise comparisons between the fuzzy values that are involved. In this class of ordering methods, the relation can be modeled using probabilities [23] or a combination of indices [24–26]. V. Nogin develops methods of fuzzy choice based on the Pareto principle and methods of fuzzy qualitative significance of criteria [27]. However, Pareto dominance can only be used for a small number of criteria.

In Ref. [28], a fuzzy interval graph is introduced. On the basis of this fuzzy structure, fuzzy orgraphs of possibilities and preferences are defined, and their properties are investigated. The transitivity and completeness properties of preference orgraphs induce a non-empty kernel (a fuzzy non-dominated subset) and propose a procedure for ranking fuzzy numbers. However, this procedure can lead to a large number of alternatives that are indistinguishable in preference, which makes it difficult to analyze them further. It may be relevant if there is a graph that describes the subject area and if optimization problems are solved on it [29].

The use of fuzzy weighted sum is demonstrated in another article [30]. However, the obtained fuzzy ranks are not always easy to use in practice. Furthermore, it will be shown that the defuzzification of fuzzy pre-mining relations can lead to an incoherent graph and to non-transitive judgments.

In spite of these articles, as well as a number of others, the problem under consideration has not yet received a comprehensive solution. Defuzzification applied by many authors leads to the loss of essential information. Therefore, the development of decision-making methods in the case of fuzzy weights, fuzzy preference areas, and fuzzy expert evaluations, as well as an analysis of the possibility of adapting the use of "classical" approaches to make decisions on fuzzy evaluations, is of great interest.

3. Materials and Methods

For the basic notions of the theory of fuzzy sets, we refer to Zadeh's papers. Either of the articles [10] or [30] contains all the material needed in this paper. A fuzzy set will be indicated as $(X, \mu(x))$, where X is the space on which the fuzzy set is defined, and $\mu(x)$ and $x \in X$ are the membership functions of the set.

The rules that will be applied in this paper for defining product fuzzy sets and composing conditional fuzzy sets are based on the minimum rule. Although a case can be made for other rules, in this paper, attention will be restricted to the minimum rule, which is considered in most of the literature on fuzzy sets.

As before, let $A_1, A_2, A_3, \ldots A_m$. Am denotes the alternatives that are compared and $c_1, c_2, c_3, \ldots c_n$ denote the different criteria that the alternatives are to be judged upon. It is assumed that the fuzzy rating of criterion c_j of alternative A_i is represented by a membership function $\mu_{ij}(x)$, where x takes its values on the real line $\mathbb{R}$. Similarly, the relative importance of criterion c_j will be a fuzzy variable as well, represented by the membership function $\omega_j(w)$, where w also takes its values on $\mathbb{R}$. All membership functions will take values in the interval [0, 1]. Furthermore, we shall normally assume that all membership functions have finite support, and that the membership functions $\omega_j(w), j = \overline{1,n}$ have their support in the positive real line.

Let the evaluated alternative A_i be characterized by a vector of fuzzy criteria values:

$$(X_1, X_2, X_3, \ldots X_n). \tag{1}$$

They correspond to the following membership functions:

$$\mu_1(x), \mu_2(x), \mu_3(x), \ldots \mu_n(x). \tag{2}$$

Next, it is important to consider the fixed alternative A_i. Therefore, we will not specify index i in the formulas.

Figure 1 shows what the membership functions for five fuzzy variables might look like for one of the alternatives. The data and code are located in an open repository at: https://github.com/sudakov/fuzzy-graphs/ (accessed on 1 June 2023).

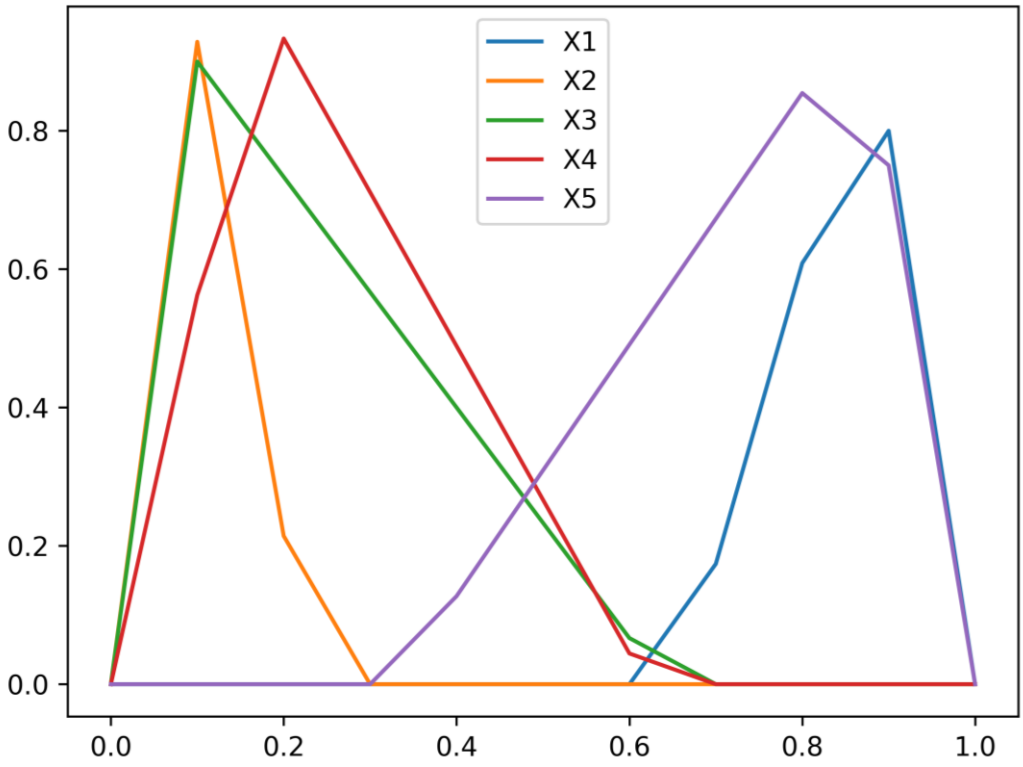


Figure 1. Examples of membership functions for fuzzy variables.

Criterion weights take fuzzy values:

$$(W_1, W_2, W_3, \ldots W_n). \tag{3}$$

We denote the membership functions of fuzzy weights as:

$$\omega_1(w), \omega_2(w), \omega_3(w), \ldots \omega_n(w). \tag{4}$$

Figure 2 shows what the functions for the five fuzzy weights used in the test case might look like.

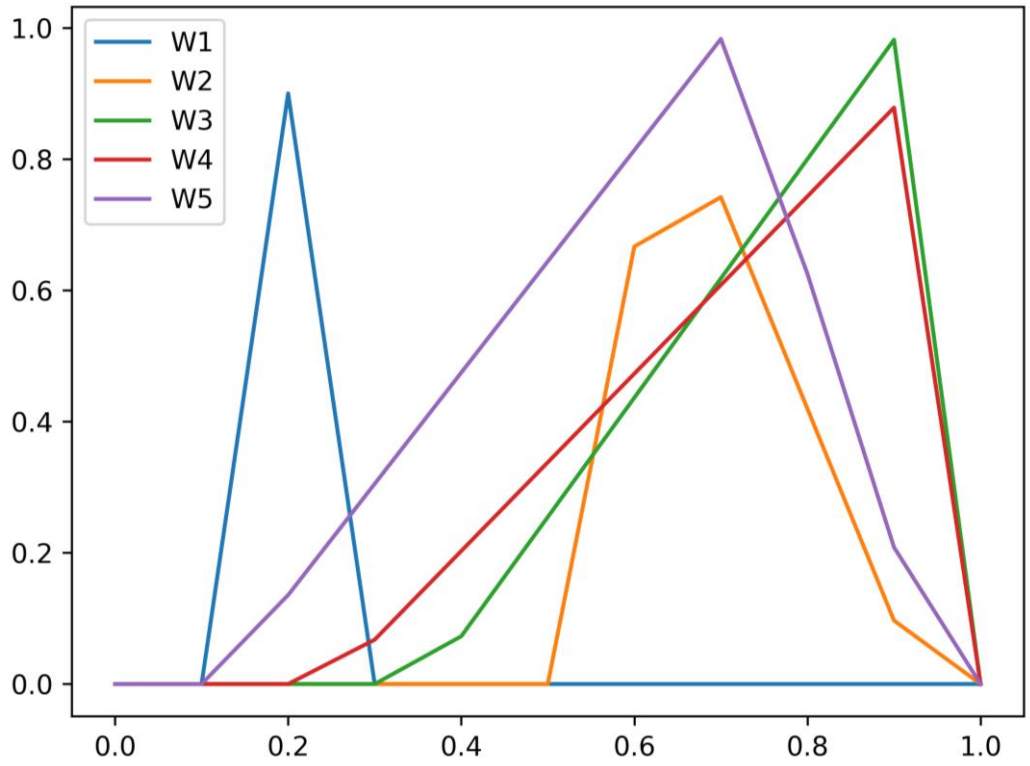


Figure 2. Examples of membership functions for fuzzy weights.

The classical weighted sum will look like this:

$$Y = \sum_{i=1}^{n} W_i X_i. \tag{5}$$

In order to determine the fuzzy evaluation of alternative A_i, based on the fuzzy ratings and weights, let us consider the function g, mapping $\mathbb{R}^{2n}$ into R, defined by:

$$g(z) = \sum_{i=1}^{n} w_i x_i. \tag{6}$$

where $z = (x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_n)$. On the product space $\mathbb{R}^{2n}$, we define a membership function $\zeta(z)$ given by

$$\zeta(z) = \left[\bigwedge_{i=1}^{n} \mu_i(x_i)\right] \bigwedge \left[\bigwedge_{i=1}^{n} \omega_i(w_i)\right]. \tag{7}$$

The symbol $\bigwedge$, which is used both in the prefix and infix notation, denotes the operation of taking the minimum. Through the mapping g: $\mathbb{R}^{2n} \to \mathbb{R}$, the fuzzy set $(\mathbb{R}^{2n}, \zeta(z))$ induces a fuzzy set $(\mathbb{R}, \vartheta(y))$, with membership function

$$\vartheta(y) = \sup_{z:g(z)=y} \zeta(z), y \in \mathbb{R}. \tag{8}$$

In a fuzzy sense, this membership function characterizes the final rating of alternative A. This approach in fuzzy logic is called the principle of generalization.

To calculate the membership function of a fuzzy number Y, we substitute Formulas (7) and (6) into Formula (8):

$$\vartheta(y) = \sup_{\substack{x_1, x_2, \ldots, x_n \\ w_1, w_2, \ldots, w_n \\ y = \sum_{i=1}^{n} w_i x_i}} \min\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n), \omega_1(w_1), \omega_2(w_2), \ldots, \omega_n(w_n)\}. \quad (9)$$

The use of a weighted sum is good during the initial stage of setting up the decision maker's preference model, but problems, such as the following, can arise later on:

- it is not clear how the membership functions of the weights should be set;
- the result of a fuzzy weighted sum is difficult to explain to the DM and it is difficult to carry out the correction of weights in case inadequate estimates of alternatives are obtained [31];
- the degree of "fuzzy" evaluations of some criteria may depend on what values other criteria have taken.

This paper proposes a new method for determining the fuzzy judgments of the DM, combining the approaches used so far in fuzzy automatic control, based on expert judgments [32], and the idea of partitioning the criteria space into regions, as proposed by the authors in the combined method of decision support [33].

The method of partitioning the definition areas of all criteria into fuzzy intervals is proposed. At the same time, it is used to intersect such intervals, taking into account the restrictions on the maximum and minimum degree of belonging of any criterion value. Then, for some combinations of fuzzy criterion values, the DM expresses his or her judgments in a fuzzy preference scale. We can consider the definition of fuzzy domains as a fuzzy implication of the form:

IF a point of the criteria space $(X_1, X_2, X_3, \ldots X_n)$. belongs to the fuzzy region D_k, THEN the preference of alternative Y is equal to the fuzzy preference level T_k given for the region:

$$(X_1, X_2, X_3, \ldots X_n) \in D_k \rightarrow Y := T_k, k = \overline{1, K}. \quad (10)$$

The fuzzy preference region T_k is given by the membership function:

$$T_k : \tau_k(y). \quad (11)$$

The resulting model of the DM value system is checked for the coverage of all points of the criterion space with a preference level not lower than the specified one. Then, the resulting model can be used to evaluate an arbitrary number of alternatives in automatic mode. Both explicit and implicit criterion values can be fed to the input of the model, and then the method allows the membership functions of the alternatives to be constructed to all preference areas.

Figure 3 shows an example of membership functions for three possible preference levels:

- T_1—bad;
- T_2—acceptable;
- T_3—good.

In this case, the lexical values (bad, acceptable, and good) characterize the preference of alternatives falling into some fuzzy area. However, to rank the alternatives, it is necessary to compare them with numbers. The type of membership function of lexical value describes the degree of certainty in those numerical values of preference.

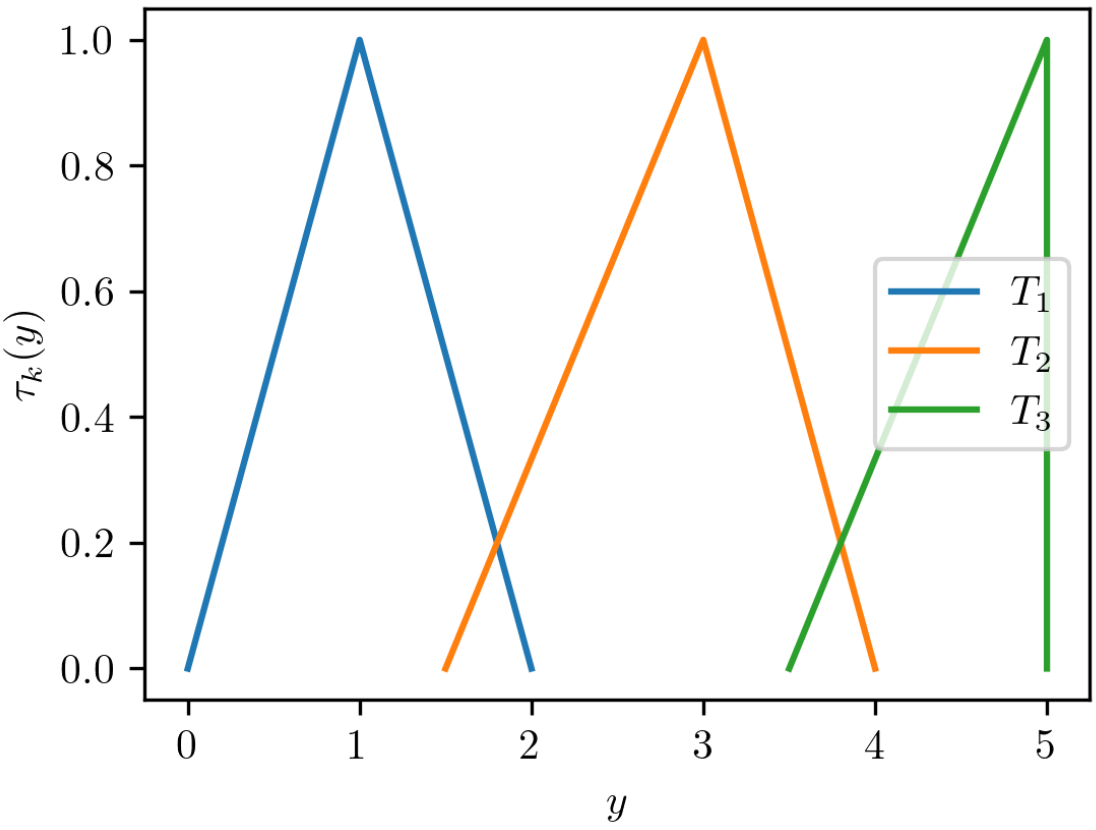


Figure 3. Examples of membership functions for fuzzy preference levels.

The membership calculation for a particular alternative is performed as follows. First, the membership level of the alternative is determined for all fuzzy regions D_k:

$$\rho_k = \min_{i=\overline{1,n}} \sup_{x_i} \min\{\mu_i(x_i), d_{ik}(x_i)\}, \tag{12}$$

where $d_{ik}(x_i)$ is a function that determines whether the value x_i belongs to the region D_k.

Then, all areas D_k are combined and the final fuzzy rank Y is determined by the membership function:

$$\vartheta(y) = \max_k \min\{\tau_k(y), \rho_k\}. \tag{13}$$

An example of setting fuzzy domains of preferences on real data is given in another article [34]. The abilities of the ranking procedure based on the preferences specified in fuzzy domains are demonstrated using the important problem of selecting an electronic flight bag for flight crews. In this study, the alternatives were different to iPad models, and the criteria included technical and economic characteristics. The next task was to determine their fuzzy ranks. However, in another article [34], the ranks were simply defuzzied by the center of gravity method. Thus, information about the uncertainty of the obtained ranking was lost.

For the case of more than 9 criteria ($n > 9$), it was quite difficult to describe the area D_k. Furthermore, practical problems with a large number of criteria were found to occur quite often. In this case, the criteria were combined into groups, usually using the semantic feature. Let us denote the subset of criterion numbers I_h. A criterion is a part of at least one group:

$$\forall i \exists h : i \in I_h. \tag{14}$$

Combining all subsets of criteria numbers must include all criteria:

$$\left| \bigcup_{h} I_h \right| = n. \tag{15}$$

To calculate the criteria, we again used a weighted sum:

$$U_h = \sum_{i \in I_h} W_i X_i \tag{16}$$

or expert rules:

$$\rho_k = \min_{i \in I_h} \sup_{x_i} \min\{\mu_i(x_i), d_{ik}(x_i)\}. \tag{17}$$

The final score can also be calculated using a weighted sum of:

$$Y = \sum_{h} W_h U_h. \tag{18}$$

It was acceptable to combine these rules using different approaches for different criteria. This choice depended on how many criteria were combined and whether they were preference-dependent.

The results can be presented on an acyclic graph, but not necessarily a tree. An example of such a graph is shown in Figure 4. If it is necessary to take into account dependences between the criteria, leading to a graph that is not acyclic, and then one can apply the approach stated in [35]. As applied to the task of selecting an electronic flight bag, criterion U_1 can characterize technical characteristics and criterion U_2 can characterize economic ones.

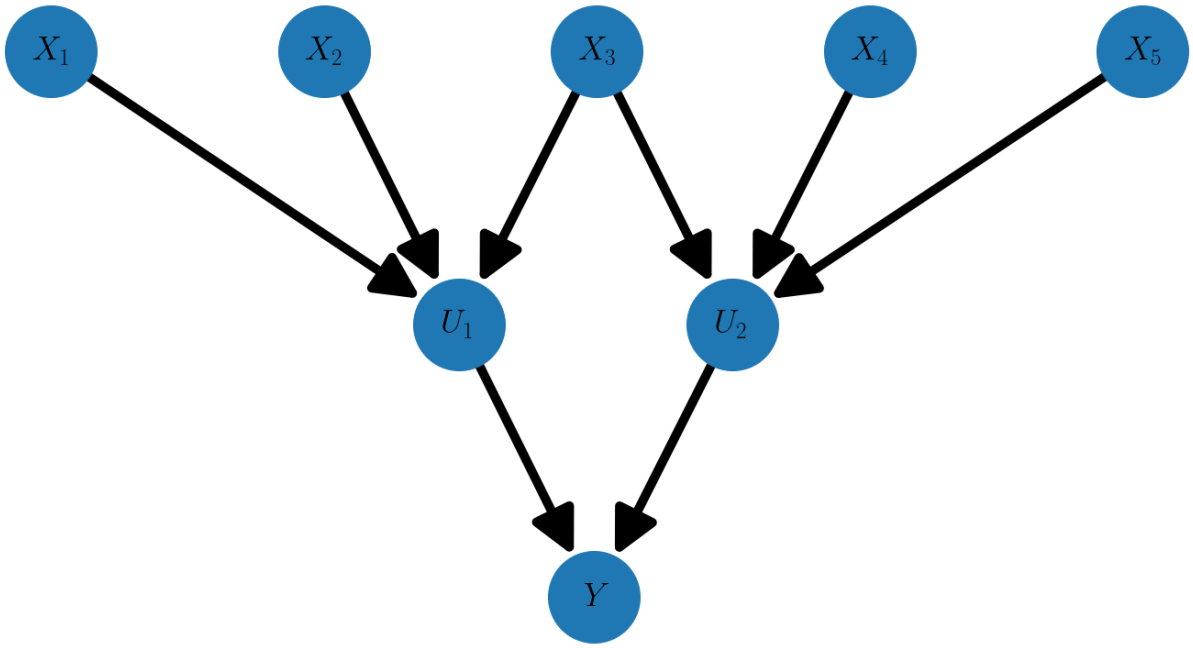


Figure 4. An example of an acyclic graph of criteria aggregation.

This graph is not fuzzy. It has a well-defined structure. The set I_h defines specific ancestor vertex numbers. A fuzzy graph occurs when a transition is made from the evaluation of a single alternative to a set of alternatives. Earlier, we considered the computation of the priority of a single alternative defined by the vector $(X_1, X_2, X_3, \ldots X_n)$. If many alternatives were found, calculations were performed according to similar formulas, as well as an additional index with a number of alternatives appears, which does not affect the calculations.

An example of the membership functions of criterion U_1 for the two alternatives is shown in Figure 5. The weights are shown in Figure 2. The values of the criteria X_1, X_2, X_3 for the fifth alternative are shown in Figure 1, and the other alternatives are given in the supporting materials.

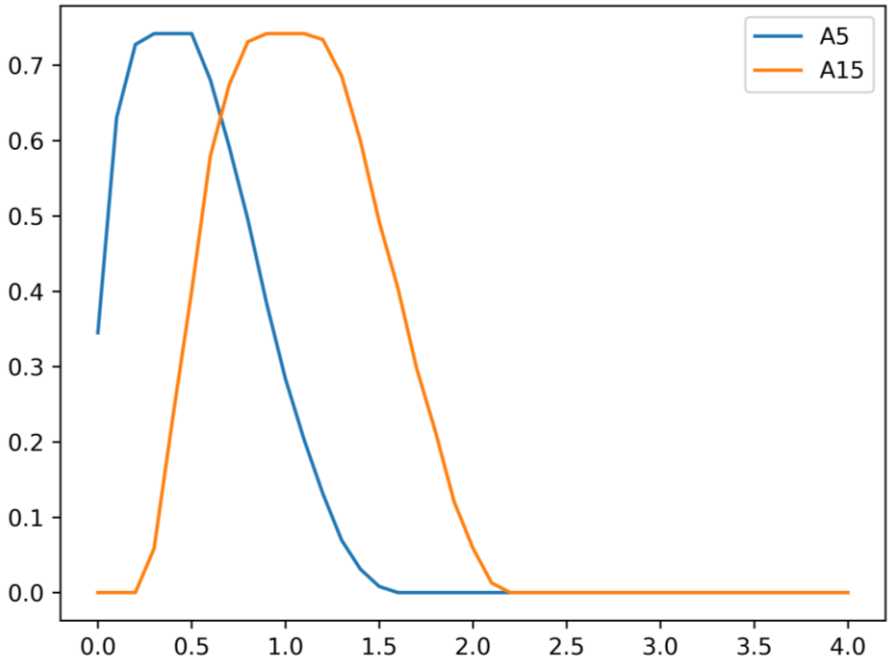


Figure 5. An example of the membership functions for the two alternatives.

Let a set of alternatives be given:

$$\{A_1, A_2, A_3, \ldots A_m\}. \tag{19}$$

The dominance ratio of the alternatives should be determined so that:

$$A_i \succcurlyeq A_j \leftrightarrow Y_i \geq Y_j. \tag{20}$$

In an oriented graph, the vertices are alternatives, and the arcs are dominance relations. The problem arises because the numbers Y_i and Y_j are fuzzy. So, there can only be some degree of certainty in the existence of an arc. Even if an arc exists with a degree of certainty 1, this does not exclude the existence of the opposite arc with a degree of certainty different from 0. A graph with such fuzzy dominance relations can be called fuzzy. The degree of certainty that the alternative A_i is not worse than A_j is calculated with the following formula:

$$r_{ij} = \sup_{\substack{y_i, y_j \\ y_i \geq y_j}} \min\{\vartheta_i(y_i), \vartheta_j(y_j)\}. \tag{21}$$

This formula is a special case of Formula (8), where the function $g(z)$ is the predicate $y_i \geq y_j$. The function $g(z)$, in this case, maps the $\mathbb{R}^2$ set to the $\mathbb{Z}_2$ set.

The values of r_{ij} can either be calculated by the above formulas or obtained from experts. The expert reports how confident he or she is that alternative A_i is no worse than A_j.

To illustrate this, Formula (17) in Figure 5 shows that alternative 5 is as good as alternative 15 with a confidence level of $r_{5,15} = 0.63$. However, the opposite relation of dominance is also possible: alternative 15 is not worse than alternative 5 with a confidence level of $r_{15,5} = 0.74$.

The matrix consisting of elements r_{ij} is a fuzzy adjacency matrix:

$$M = \|r_{ij}\|. \tag{22}$$

Thus, the fuzzy graph $G = (V, E)$ in this article is the set of vertices $V = \{A_i, i = \overline{1,n}\}$ corresponding to alternatives and the set of fuzzy arcs $E = \{A_i \succcurlyeq A_j\}$. The arcs correspond to the dominance of alternative A_i over A_j. The confidence of existence of arc ij is equal to r_{ij}.

One way of obtaining unambiguous dominance is to set a threshold θ. If $r_{ij} \geq \theta$, we considered that the arc exists; otherwise, we deduced that it does not exist. There are several reasonable considerations for choosing the threshold θ. The dominance graph must not violate transitivity, that is, the graph must be acyclic.

Figure 6 shows the graph obtained for $\theta = 0.7$ in a computational experiment with 30 alternatives. The alternatives were evaluated using a weighted sum. The graph has a cycle. It is shown with red arrows. Moreover, the graph is not connected. This is bad since alternatives from different connectivity components cannot be distinguished by preference. For $\theta = 0.8$, the graph becomes acyclic, but the number of disconnected graph components increases.

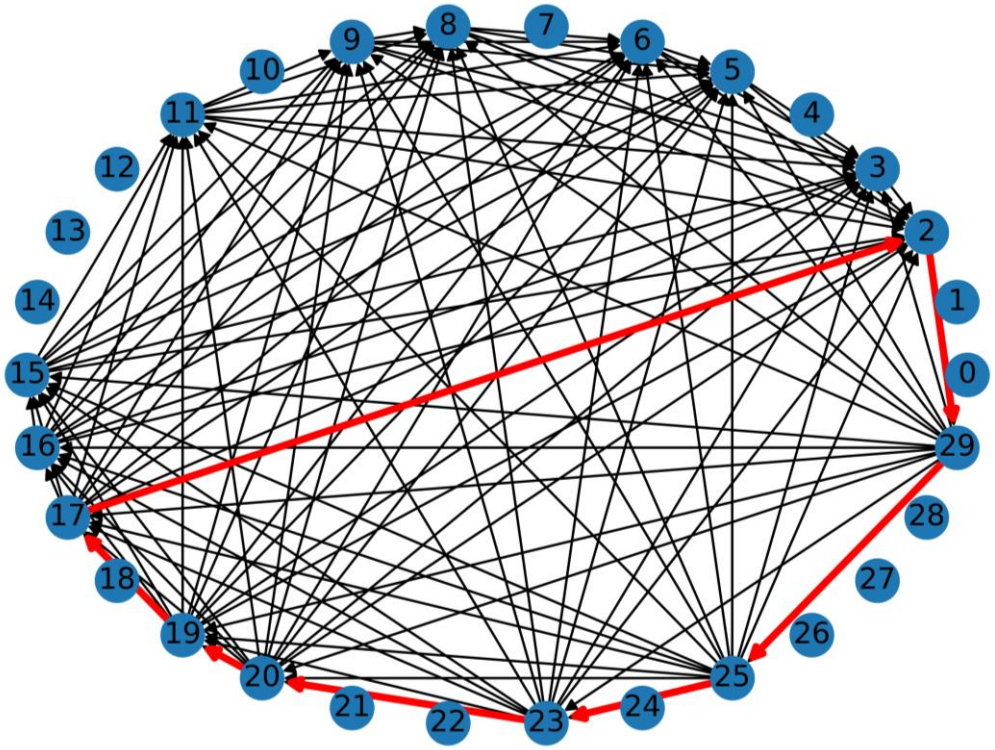


Figure 6. An example dominance graph of alternatives.

One can choose the minimum value of θ for which this condition is satisfied and several larger values of θ. As a result, each level will have its own dominance graph. To move to a single graph, we can apply the methods used to reconcile expert judgments. In this case, different graphs act as experts, including Kemeney's median, Condorcet's method, and many others. As a result, we can obtain a ranking that reflects both the preferences of the decision maker and the fact that there is some degree of uncertainty in the judgments.

In practical problems, there may be more than one fuzzy dominance graph. For example, if the criteria are of equal importance, then the fuzzy graphs corresponding to the dominance of each of the criteria can be obtained. Aggregating such graphs into a single graph will enable us to arrive at a single fuzzy evaluation of alternatives without using weights $W_1, W_2, W_3, \ldots W_n$ or fuzzy estimates of preference areas $T_1, T_2, T_3, \ldots T_K$. This method may be acceptable and simpler in some problems, since obtaining W_i, D_k, and D_k is a rather labor-intensive procedure.

The general algorithm for aggregating fuzzy graphs is as follows:

1. The values of the required criteria are obtained for all alternatives. These criteria can be calculated using the weighted sum formulas above (9) or preference areas (13)

based on the values of the primary criteria (leaves in the criterion aggregation tree). Otherwise, the criteria can be directly evaluated by the DM or the expert.

2. Fuzzy adjacency matrices $M_l = \left\| r_{ij}^l \right\|$ are obtained for each criterion $l = \overline{1,n}$ using Formula (21).
3. The threshold levels $\theta = 0$ to 1 are cycled through with a given step.
4. For each criterion $l = \overline{1,n}$, binary adjacency matrices are obtained:

$$M_l(\theta) = \left\| r_{ij}^l(\theta) = \begin{cases} 1, r_{ij}^l \geq \theta \\ 0, r_{ij}^l < \theta \end{cases} \right\|. \tag{23}$$

5. Using Kemeny's median, the average ranking $M_*(\theta) = \left\| r_{ij}^*(\theta) \right\|$ is found. This step will be explained further below.
6. A combination of index values i, j is determined, as well as the maximum threshold t_{ij} at which $r_{ij}^*(t_{ij}) = 1$. It is this value of t_{ij} that will be written in the adjacency matrix of the aggregated graph.
7. Figure 7 shows the algorithm in the UML notation.

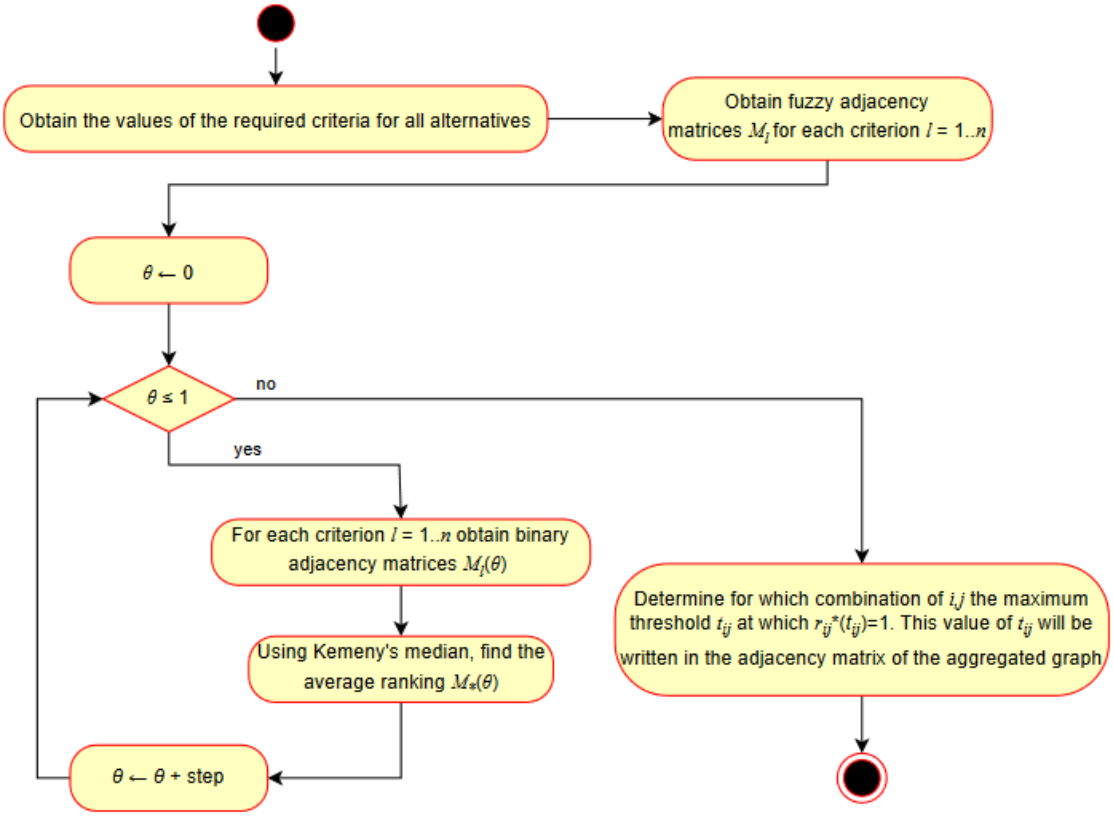


Figure 7. The general algorithm for aggregating fuzzy graphs (the symbol * means the optimal value of the rank).

Kemeny's median is distinguished by the fact that it is the unique voting system that is neutral, consistent, and Condorcet. The voting scheme here is a rule for choosing winning alternatives based on voter preference rankings. The criteria serve as the voter in this case. Let us consider the procedure of finding the Kemeny median:

It is necessary to find such values of $r^*_{ij}(\theta)$ so that they provide the minimum deviation from the given dominance relations $r^l_{ij}(\theta)$. Since the problem is solved for a fixed θ, (θ) is not specified any further.

$$\min_{r^*_{ij}\epsilon\{0,1\}} \sum_{i,j,l} \left| r^l_{ij} - r^*_{ij} \right| \tag{24}$$

The ranking must be unambiguous and not lead to cycles in the domination graph. For this purpose, additional constraints are introduced:

$$r^*_{ij} + r^*_{ji} = 1 \forall i \neq j, \tag{25}$$

$$r^*_{ij} + r^*_{jk} + r^*_{ki} \geq 1 \forall i \neq j \neq k. \tag{26}$$

Formula (25) guarantees connectivity and the absence of cycles for any pair of two vertices. Formula (26) ensures the absence of cycles of three or more vertices. These statements are demonstrated in this paper [36]. Unfortunately, it is NP-hard to compute Kemeny rankings. Moreover, the target function contains a nonlinear modulus operation. In another paper [36], a method of transforming the target function is proposed, which allows the calculations to be sped up. For this purpose, for each possible arc (i, j), we introduce its weight w_{ij} by defining disagreement that $A_i \succcurlyeq A_j$. Let us define h_{ij} as the number of criteria for which $A_i \succcurlyeq A_j$. Then, w_{ij} is calculated with the following formula:

$$w_{ij} = \begin{cases} h_{ji} - h_{ij}, h_{ji} > h_{ij} \\ 0, h_{ji} \leq h_{ij} \end{cases} \tag{27}$$

Then, the target function will take the form:

$$\min_{r^*_{ij}\epsilon\{0,1\}} \sum_{i,j,l} w_{ij} r^*_{ij} \tag{28}$$

This problem can be solved using standard mixed-integer linear programming (MILP) solvers. In this paper, the SCIP Optimization Suite [37] was used as the optimization package. It is fast and free for non-commercial use and comes with an open-source code.

4. Results

To investigate the abilities of the proposed ranking algorithm, a number of computational experiments were conducted. In the first experiment, 10 alternatives were considered. In the second experiment, 50 alternatives were considered. The first experiment allowed the defuzzied dominance relations to be visualized and analyzed in the form of graphs. The second experiment allowed the possibilities of the method to be investigated on substantial dimensions. The dimensions over 50 were not analyzed because they require a lot of time and powerful computing resources. This is explained by the fact that the search for the Kemeny median is an NP-hard problem.

For each computational experiment, three fuzzy ranking matrices were used. These ranks were assumed to be derived from the simplest model for the DM by comparing the alternatives. Initially, three random ranks were obtained. Then, an oriented dominance graph was constructed for them. Normally distributed random noise $\mathcal{N}(0, 0.1)$ was imposed on the adjacency matrix of the graph. The random noise simulated the uncertainty of the DM in their estimates. The resulting matrix corresponded to a fuzzy graph showing the dominance of alternatives.

The use of synthetic data was explained by the fact that it was necessary to estimate the accuracy of the obtained solution. The real DM estimates will not allow us to make these estimations, as it is difficult to say whether there are errors in the proposals of the system. However, in modeling, we ourselves know what kind of error is embedded in the judgments, so we can check how the proposed method copes with it.

The resulting fuzzy dominance matrices of alternatives for the first computational experiment are shown in Table 1.

Table 1. Fuzzy ranking matrices r_{ij}^{l}.

l	$i\backslash j$	0	1	2	3	4	5	6	7	8	9
1	0	1.00	0.96	0.00	0.92	0.93	1.00	0.89	1.00	1.00	1.00
1	1	0.00	1.00	0.00	0.17	0.01	0.94	0.02	0.21	0.01	0.06
1	2	1.00	0.96	1.00	0.97	0.98	1.00	1.00	1.00	1.00	1.00
1	3	0.00	1.00	0.05	1.00	1.00	0.99	0.11	0.15	0.22	0.86
1	4	0.00	0.95	0.02	0.09	1.00	0.80	0.00	0.08	0.02	1.00
1	5	0.00	0.00	0.02	0.04	0.02	1.00	0.00	0.04	0.01	0.11
1	6	0.12	1.00	0.00	0.94	1.00	1.00	1.00	0.00	0.00	1.00
1	7	0.00	1.00	0.04	1.00	0.89	1.00	1.00	1.00	0.97	1.00
1	8	0.00	1.00	0.08	0.91	1.00	0.87	1.00	0.00	1.00	1.00
1	9	0.00	0.92	0.13	0.20	0.00	1.00	0.16	0.03	0.00	1.00
2	0	1.00	0.00	0.88	0.06	0.08	0.00	0.05	0.00	1.00	0.00
2	1	0.98	1.00	1.00	0.08	1.00	1.00	1.00	0.06	1.00	1.00
2	2	0.00	0.00	1.00	0.22	0.04	0.00	0.00	0.00	0.00	0.00
2	3	0.95	0.90	1.00	1.00	1.00	0.98	1.00	1.00	1.00	0.81
2	4	0.94	0.09	1.00	0.00	1.00	0.00	0.13	0.00	1.00	0.00
2	5	0.87	0.03	1.00	0.13	0.99	1.00	0.06	0.02	1.00	0.99
2	6	1.00	0.04	1.00	0.11	1.00	0.84	1.00	0.04	1.00	1.00
2	7	0.96	1.00	0.94	0.20	0.87	0.82	0.83	1.00	0.89	1.00
2	8	0.00	0.00	1.00	0.00	0.00	0.09	0.06	0.00	1.00	0.01
2	9	1.00	0.05	1.00	0.01	0.88	0.24	0.00	0.00	0.93	1.00
3	0	1.00	1.00	0.07	1.00	0.00	0.88	0.88	0.00	0.00	0.08
3	1	0.00	1.00	0.00	0.00	0.00	0.07	0.06	0.00	0.18	0.00
3	2	0.97	0.98	1.00	1.00	0.04	0.90	1.00	0.06	1.00	0.00
3	3	0.01	0.85	0.08	1.00	0.01	0.12	0.03	0.03	0.03	0.00
3	4	1.00	1.00	1.00	1.00	1.00	0.91	0.86	1.00	0.97	1.00
3	5	0.00	0.96	0.01	1.00	0.01	1.00	1.00	0.02	0.00	0.01
3	6	0.00	0.99	0.05	0.93	0.00	0.00	1.00	0.08	0.00	0.06
3	7	1.00	0.85	0.92	0.91	0.00	1.00	1.00	1.00	1.00	0.00
3	8	1.00	1.00	0.14	0.93	0.30	1.00	1.00	0.00	1.00	0.00
3	9	0.99	0.92	1.00	0.92	0.06	1.00	0.93	1.00	0.98	1.00

The algorithm described in the previous section was applied to obtain the final graph. There was a value analysis of $r_{ij}^{*}(\theta)$ carried out for θ = 0.1, 0.2, 0.3,...,1.0. Up to a level of θ = 0.8 values, $r_{ij}^{*}(\theta)$ did not change. The changes in the dominance graph are shown in Figures 8–10. The obtained graphs are acyclic.

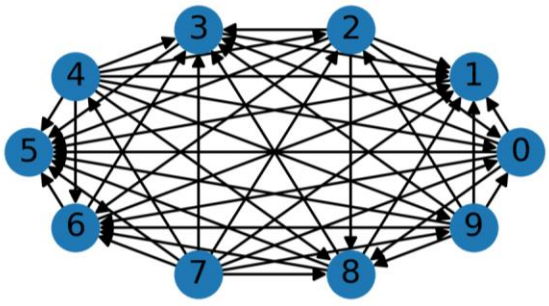


Figure 8. Dominance graph for θ = 0.8.

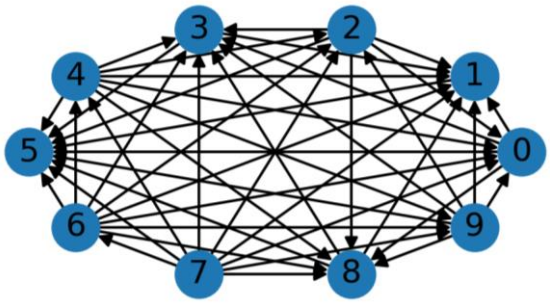


Figure 9. Dominance graph for $\theta = 0.9$.

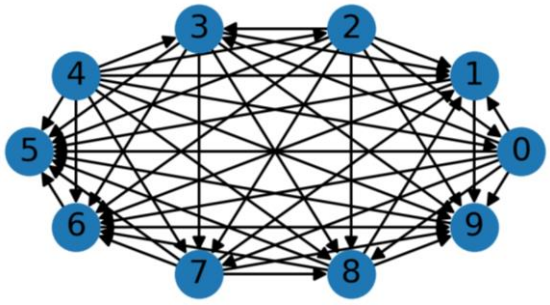


Figure 10. Dominance graph for $\theta = 1.0$.

Table 2 shows how the arcs in the graph changed. Up to a level of $\theta = 0.8$, the graph did not change and it can be taken as the final ranking. At a threshold level of 0.8, it matched the original graph corresponding to the primary Kemeny ranking before random noise was applied.

Table 2. Changed arcs of the resulting ranking.

i	**0**	**2**	**4**	**6**	**6**	**6**	**6**	**6**	**8**	**9**
j	**6**	**6**	**6**	**0**	**2**	**4**	**8**	**9**	**6**	**6**
θ	$r^*_{ij}(\theta)$									
0.1	1	1	1	0	0	0	0	0	1	1
...					...					
0.8	1	1	1	0	0	0	0	0	1	1
0.9	0	0	0	1	1	1	1	1	0	0
1.0	1	1	1	0	0	0	0	1	1	0

After finding the median Kemeny at each of the levels, a generalized fuzzy dominance graph was obtained. Each arc ij was stored in the fuzzy resulting graph with maximum θ corresponding to $r^*_{ij}(\theta) = 1$.

The fuzzy dominance matrix is shown in Table 3.

In the second experiment with 50 alternatives, we also obtained nine graphs for the dominance matrices $M_*(\theta) = \left\| r^*_{ij}(\theta) \right\|$, $\theta = 0.1, 0.2, 0.3, \ldots, 1.0$. For such dimensionality at each step of change θ, there were changes in the graph. The accuracy of the obtained solution was estimated as the number of elements differing from the true Kemeni median before the imposition of noise (see Figure 11).

As shown in Figure 11, the levels $\theta = 0.4$, 0.5, and 0.6 provided the maximum accuracy. However, in real problems, we do not know what the true exact solution is, as it was in the simulation. However, we can estimate at what levels of θ the solution will be most stable. For this, we need to compare the number of changed elements $r^*_{ij}(\theta)$ in the matrix $M_*(\theta)$ at each change in θ. In this case, it makes sense to focus on the most stable solution, understanding that the confidence in it is equal to θ.

An example of such a graph for an experiment with 50 alternatives is shown in Figure 12.

Table 3. The final fuzzy dominance matrix of alternatives.

$i \backslash j$	0	1	2	3	4	5	6	7	8	9
0	1	1	0	1	0	1	1	1	1	1
1	0	1	0	0	0	1	1	0	0	1
2	1	1	1	1	0	1	1	1	1	1
3	0	1	0	1	0	1	1	1	1	1
4	1	1	1	1	1	1	1	1	1	1
5	0	0	0	0	0	1	0	0	0	0
6	0.9	0.9	0.9	0.9	0.9	1	1	0	0.9	1
7	0.9	1	0.9	0.9	0.9	1	1	1	1	1
8	0	1	0	0.9	0	1	1	0	1	1
9	0.9	0.9	0.9	0.9	0	1	0.8	0	0.9	1

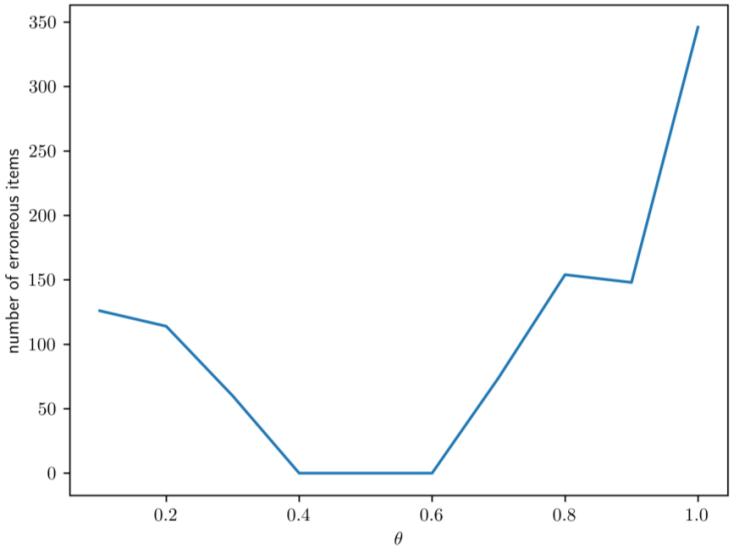


Figure 11. Dependence graph showing the number of errors on the level of θ.

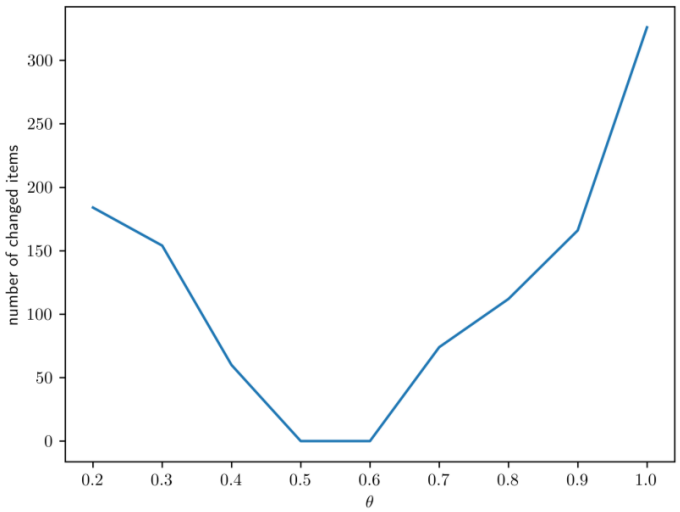


Figure 12. Dependence of the number of changed elements in the matrix $M_*(\theta)$ on level θ.

The solution time with 50 alternatives took about 3 min on an Apple M1 MAX 10-core CPU, with a 32-core GPU and 64 GB RAM. Problems of dimensions 60–70 were solved in

10–15 min. The exact time to solve a 100-dimensional problem exceeded 40 min. It was possible to speed up the calculation procedure by switching from exact Kemeny median search methods to heuristic and metaheuristic optimization methods [38]. These methods do not guarantee an exact solution. They were not used in this paper since, in order to verify the method, it was necessary to ensure that there was no error. In practice, the error may not be significant, so the application of heuristic and metaheuristic optimization was quite possible.

The Decision Support System is designed to help the DM in situations where he or she is having difficulty making a decision. For this reason, the DM has difficulty saying whether there are errors in the system's sentences. In other words, the difficulties are not caused by the proposed method, but instead by the nature (essence) of the problem to be solved. With the further practical use of the method, its evaluation in a real environment is quite possible. For this purpose, it is necessary for several DMs to use it in different situations, and, after some time, on a set of precedents, DMs can qualitatively estimate its work and estimate consequences of decision making on this basis. Further research on the approbation of the proposed method is planned in the future.

5. Conclusions

The proposed method of decision support allows DMs to rank the alternatives based on the fuzzy preferences of the DM, as well as sets with a fuzzy weighted sum and fuzzy rules, taking into account the fuzzy judgments of the DM. The use of a fuzzy graph ranking of alternatives allows the DM to coordinate the experts or use individual criteria as experts. For this purpose, the use of the Kemeney median at different levels of confidence is proposed, but other methods of ranking matching are also possible, e.g., Bordeaux, Condorcet, and Schulze, among others. Computational experiments have shown that the fuzzy graph model is used to find the ranking close to the initial judgments of the decision maker, provided that there is little distortion in the judgments. The method makes sense if the decision maker has difficulty in evaluating the criteria and alternatives, or the evaluation is carried out by experts who want to reflect a certain degree of uncertainty in their evaluations. The use of a graph of dependencies between criteria allows problems of high dimensionality to be solved with a large number of criteria and alternatives.

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