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Cohomology of Graded Twisting of Hopf Algebras

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Abstract: Let A be a Hopf algebra and B a graded twisting of A by a finite abelian group Γ. Then, categories of comodules over A and B are equivalent (but they are not necessarily monoidally equivalent). We show the relation between the Hochschild cohomology of A and B explicitly. This partially answer a question raised by Bichon. As an application, we prove that A is a twisted Calabi–Yau Hopf algebra if and only if B is a twisted Calabi–Yau algebra, and give the relation between their Nakayama automorphisms.

Keywords: graded twisting; Hochschild cohomology; Calabi–Yau algebra

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1. Introduction

Hochschild cohomology was introduced by Hochschild in 1945 [1] for any associative algebra. Since then, many mathematicians have investigated the Hochschild cohomology $\mathrm{HH}^*(A)$ for various types of algebras A. In particular, the structure of the Hochschild cohomology ring of a Hopf algebra has been studied extensively. To calculate the cohomology ring of an algebra A, it is sometimes convenient to use an injective resolution for the coalgebra A^*. In [2], the authors constructed minimal injective resolutions for many well-known Hopf algebras, such as exterior algebras, truncated polynomial algebras, etc. The most intricate example is a subalgebra of the Steenrod algebra, its cohomology is given by 13 generators and 54 relations. May, in [3], constructed resolutions for computing the cohomology of the universal enveloping algebras of restricted Lie algebras. The structure of the Hochschild cohomology algebra of a group algebra was discussed in [4–6]. Later, Linckelmann generalized the result in [5] to the case of commutative Hopf algebras [7]. Recently, in [8], the author gave a general expression of the Gerstenhaber bracket on the Hochschild cohomology of a Hopf algebra A with bijective antipode.

Another interesting question about the Hochschild cohomology of Hopf algebras was raised by Bichon in [9]:

Question 1. *If A and B are Hopf algebras with equivalent tensor categories of comodules, how are their Hochschild cohomologies related?*

Let A and B be two such Hopf algebras; it is shown in [10] that their Hochschild cohomologies are indeed closely related. One can transport a free Yetter–Drinfeld resolution of the trivial module over A to the same kind of resolution of the trivial module over B. In some sense, the Gerstenhaber–Schack cohomology [11,12] is an invariance under the monoidal equivalence of tensor categories of comodules. In [9], Bichon proved that the Hochschild cohomology of a Hopf algebra is determined by its Gerstenhaber–Schack cohomology. Consequently, the Hochschild cohomology of A can be expressed by the Gerstenhaber–Schack cohomology of B. To be precise, there is a functor $F : {}_A\mathcal{M}_A \to {}^B_B\mathcal{M}^B_B$ from the category of A-bimodules to the categories of Hopf bimodules over B, such that for any A-bimodule M, $\mathrm{HH}^*(A, M) \cong \mathrm{HH}^*_{\mathrm{GS}}(B, F(M))$. However, so far, we do not know whether the Hochschld cohomology of a Hopf algebra can determine its Gerstenhaber–Schack cohomology. There is no explicit expression for the relation between the Hochschild cohomologies of A and B.

1.1. Motivation

The aim of this paper is to answer Question 1 when B is a graded twisting of A. In this case, the categories of comodules over A and B are also equivalent, but they are not necessarily monoidally equivalent. The graded twisting of Hopf algebras was introduced in [13], and is the formalization of a construction in [14] that solved the quantum group realization problem of the Kazhdan–Wenzl categories [15].

1.2. Main Results

As in Section 2, for a Hopf algebra A, the homological algebra over the enveloping algebra $A \otimes A^{op}$ can be described by that over A. Therefore, to describe the Hochschild cohomology of A, it is sufficient to discuss the Ext group over A. The following theorem describes the relation between the cohomology of a Hopf algebra and its graded twisting by a finite abelian group (Theorems 3 and 4).

Theorem 1. *Let Γ be a finite abelian group and (p, α) an invariant cocentral action of Γ on a Hopf algebra A with bijective antipode. Let $B = A^{t,\alpha}$ be the graded twisting of A. If A is homologically smooth, then*

(1) There is an isomorphism of left B-modules

$$\mathrm{Ext}^i_B(\Bbbk_B, B_B) = (\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A))^{\Gamma},$$

for $i \geqslant 0$.

(2) For a graded right A-module M, we have

$$\mathrm{Ext}^i_B(\Bbbk_B, M^{\alpha}) \cong M \otimes_A (\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A))^{\Gamma},$$

for $i \geqslant 0$.

In the above theorem, $(\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A))^{\Gamma}$ denotes the set of Γ-invariant elements of $\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A)$. The Γ-action on $\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A)$ will be defined in Section 3.1 and the B-action on it is induced by the $A \rtimes \Gamma$-action as defined in (4). For a graded right A-module M, M^{α} is the twisted module of M as defined in Section 3.3.

As an application, we prove that the Calabi–Yau (CY for short) property is preserved by graded twisting (Theorem 5, the definition of a twisted CY algebra will be recalled in Definition 3).

Theorem 2. *Let A be a Hopf algebra with a bijective antipode and Γ a finite abelian group. Let B be a graded twisting of A by Γ. The algebra A is a twisted CY algebra if and only if B is a twisted CY algebra. The Nakayama automorphisms of A and B satisfy the following equation:*

$$\mu_B(a \otimes h) = \mathrm{hdet}(h)\mu_A(a) \otimes h,$$

for any $a \otimes h \in B$, where hdet *denotes the homological determinant of the Γ-action.*

2. Notations and Preliminaries

We work over a fixed algebraically closed field $\Bbbk$ of characteristic 0. All algebras and vector spaces are over $\Bbbk$. The unadorned tensor $\otimes$ means $\otimes_{\Bbbk}$ and Hom means $\mathrm{Hom}_{\Bbbk}$.

Given an algebra A, we write A^e for the enveloping algebra $A \otimes A^{op}$, where A^{op} is the opposite algebra of A. The category of the right (resp. left) A-modules is denoted by $\mathcal{M}_A$ (resp. ${}_A\mathcal{M}$). An A-bimodule can be identified with a left (or right) A^e-module.

For an A-bimodule M and an algebra automorphism μ of A, we let M_μ denote the A-bimodule such that $M_\mu \cong M$ as vector spaces, and the bimodule structure is given by

$$a \cdot m \cdot b = am\mu(b),$$

for all $a, b \in A$ and $m \in M$. Similarly, we have ${}_\mu M$. It is well-known that $A_\mu \cong {}_{\mu^{-1}}A$ as A-bimodules, and $A_\mu \cong A$ as A-bimodules if and only if μ is an inner automorphism of A.

If A is a Hopf algebra, as usual, we use the symbols Δ, ε and S for its comultiplication, counit, and antipode, respectively. We use Sweedler's (sumless) notation for the comultiplication and coaction of A. The category of right A-comodules is denoted by $\mathcal{M}^A$. We write ${}_A\Bbbk$ (resp. $\Bbbk_A$) for the left (resp. right) trivial module defined by the counit ε of A.

2.1. Graded Hopf Algebras

To recall the definition of graded twisting of Hopf algebras, we need to first recall the definition and some properties of graded Hopf algebras.

Let A be a Hopf algebra and Γ be a group. From [16] (Lemma 1.3), there is a one-to-one correspondence between

(1) A cocentral Hopf algebra homomorphism $p : A \to \Bbbk\Gamma$, that is,

$$p(a_1) \otimes a_2 = p(a_2) \otimes a_1$$

for any $a \in A$;

(2) A direct sum decomposition $A = \bigoplus_{g\in\Gamma} A_g$ such that $A_g A_h \subset A_{gh}$ and $\Delta(A_g) \subset A_g \otimes A_g$ for all $g, h \in \Gamma$.

Assume we are given (1), the grading is given by

$$A_g = \{a \in A \mid p(a_1) \otimes a_2 = g \otimes a\} = \{a \in A \mid a_1 \otimes p(a_2) = a \otimes g\}.$$

If (2) is given, the map p is given by $p(a) = \varepsilon(a)g$ for $a \in A_g$. Note that we always have $1 \in A_e$ and $S(A_g) = A_{g^{-1}}$.

To state some properties of graded Hopf algebras, let us recall the definition of an exact sequence of Hopf algebras.

A sequence of Hopf algebra maps

$$\Bbbk \to B \xrightarrow{i} A \xrightarrow{p} L \to \Bbbk \tag{1}$$

is said to be exact if the following conditions hold:

(1) i is injective and p is surjective,
(2) $\ker p = Ai(B)^+ = i(B)^+A$, where $i(B)^+ = i(B) \cap \ker(\varepsilon)$,
(3) $i(B) = A^{\mathrm{co}L} = \{a \in A : (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1\}$
$\quad = {}^{\mathrm{co}L}A = \{a \in A : (p \otimes \mathrm{id})\Delta(a) = 1 \otimes a\}$.

An exact sequence as above and such that A is faithfully flat as a right B-module is called strict. If L is cosemisimple, then an exact sequence is automatically strict (cf. [9]).

The following Lemma is Proposition 2.2 in [13].

Lemma 1. *Let $p : A \to K\Gamma$ be a surjective cocentral Hopf algebra homomorphism. Then*

(1) the grading on A is strong, i.e., $A_g A_h = A_{gh}$ for all $g, h \in \Gamma$; we also have $A_g^+ A_h = A_g A_h^+ = A_{gh}^+$;
(2) A_g is a finitely generated projective left and right A_e-module for every $g \in \Gamma$;
(3) A is a faithfully flat left and right A_e-module, as well as a faithfully coflat left and right $\Bbbk\Gamma$-comodule;
(4) There is a Hopf algebra exact sequence $\Bbbk \to A_e \to A \to \Bbbk\Gamma \to \Bbbk$.

2.2. Graded Twisting of Hopf Algebras

Now we recall the graded twisting of Hopf algebras introduced in [16].

Let A be a Hopf algebra and Γ a group. An invariant cocentral action of Γ on A is a pair (p, α), where

(1) $p : A \to \mathbb{C}\Gamma$ is a surjective cocentral Hopf algebra map;
(2) $\alpha : \Gamma \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ is an action of Γ by Hopf algebra automorphisms on A, with $p\alpha = p$ for all $g \in \Gamma$.

In terms of grading, the condition in (2) is equivalent to $\alpha_g(A_h) = A_h$ for all $g, h \in \Gamma$.

Remark 1. *With the action α, the algebra A is obviously a left $\Bbbk\Gamma$-module with the action defined by*

$$h \cdot a = \alpha_h(a), \qquad \text{for } a \in A,\ h \in \Gamma.$$

The algebra A can also be viewed as a right $\Bbbk\Gamma$-module with right action:

$$a \cdot h = \alpha_{h^{-1}}(a), \qquad \textit{for } a \in A,\ h \in \Gamma.$$

Recall that the crossed product $A \rtimes \Gamma$ is the tensor product $A \otimes \Bbbk\Gamma$ with the product defined by

$$(a \otimes g)(b \otimes h) = a\alpha_g(b) \otimes gh,$$

for any $a, b \in A, g, h \in \Gamma$. It is a Hopf algebra with the coproduct

$$\Delta(a \otimes g) = a_1 \otimes g \otimes a_2 \otimes g,$$

the counit

$$\varepsilon(a \otimes g) = \varepsilon(a)$$

and antipode

$$S_{A\rtimes\Gamma}(a \otimes g) = S_A(\alpha_{g^{-1}}(a)) \otimes g^{-1},$$

for any $a \otimes g \in A \rtimes \Gamma$.

Definition 1. *Let A be a Hopf algebra and Γ a group. Let (p, α) be an invariant cocentral action of Γ on A, the graded twisting $A^{t,\alpha}$ of A is the Hopf subalgebra*

$$A^{t,\alpha} = \sum_{g\in\Gamma} A_g \otimes g \subseteq A \rtimes \Gamma,$$

of the crossed product Hopf algebra $A \rtimes \Gamma$.

Remark 2. *When the group Γ is abelian, this construction is symmetrical. That is, the algebra A is also a graded twisting of $A^{t,\alpha}$. It can be directly checked that the map $\tilde{p} = p \otimes \varepsilon : A^{t,\alpha} \to \Bbbk\Gamma$ is a surjective cocentral Hopf algebra homomorphism and the maps $\beta = \alpha^{-1} \otimes \mathrm{id}|_{A^{t,\alpha}}$ are Hopf algebra automorphisms. Then, A is isomorphic to $(A^{t,\alpha})^{t,\beta}$ as Hopf algebras, given by the map $a_g \mapsto a_g \otimes g \otimes g$ for $a_g \in A_g$.*

Lemma 2. *Let A be a Hopf algebra and Γ a finite abelian group. Assume that (p, α) is an invariant cocentral action of Γ on A. Then, we have the following:*

(1) The map $\bar{p} : A \rtimes \Gamma \to \Bbbk\Gamma$ defined by $\bar{p}(a \otimes g) = p(a)g^{-1}$ is a surjective cocentral map.
(2) There is a strict exact sequence of Hopf algebras

$$0 \to A^{t,\alpha} \to A \rtimes \Gamma \to \Bbbk\Gamma \to 0.$$

Proof. (1) Can be checked directly.

(2) From (1), the map $\bar{p}$ is a surjective cocentral map. It is obvious that $(A \rtimes \Gamma)_e = A^{t,\alpha}$, then there is an exact sequence of Hopf algebras

$$0 \to A^{t,\alpha} \to A \rtimes \Gamma \to \Bbbk\Gamma \to 0$$

by Lemma 1. It is strict, since $\Bbbk\Gamma$ is cosemisimple. $\square$

2.3. Hochschild Cohomology

We end this section by recalling the Hochschild cohomology of Hopf algebras.

Let A be an algebra and M an A-bimodule. The Hochschild cohomology of A with coefficients in M is defined as

$$\mathrm{HH}^*(A, M) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_{A^e}(A, M).$$

It is well-known that under the cup product, $\mathrm{HH}^*(A) = \mathrm{HH}^*(A, A)$ is a graded commutative algebra and $\mathrm{HH}^*(A, M)$ is a module over $\mathrm{HH}^*(A)$.

Let A be a Hopf algebra and N a right A-module. The cohomology of A with coefficients in N is defined as

$$\mathrm{H}^*(A, N) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(\Bbbk_A, N).$$

The space $\mathrm{H}^*(A, \Bbbk)$ is a graded algebra under the Yoneda product, and $\mathrm{H}^*(A, N)$ is a module over $\mathrm{H}^*(A, \Bbbk)$.

It is well-known that if $\Bbbk_A$ admits a finitely generated projective resolution, then there is an isomorphism

$$\mathrm{Ext}^n_A(\Bbbk_A, N) \cong N \otimes_A \mathrm{Ext}^n_A(\Bbbk_A, A_A),$$

for any right A-module N.

The Hochschild cohomology of a Hopf algebra can be calculated by its cohomology. Let A be a Hopf algebra, and M an A-bimodule. A right A-module structure on M can be defined by

$$m \leftharpoonup x = S(x_1)mx_2,$$

for any $x \in A$ and $m \in M$. We denote this right A-module by $R(M)$. Similarly, $L(M)$ is M having the left A-module structure defined by

$$x \rightharpoonup m = x_1 m S(x_2),$$

for any $x \in A$ and $m \in M$.

The following well-known lemma (see, e.g., [10,17]) shows that the homological algebra over A^e can be described by that over A.

Lemma 3. *Let A be a Hopf algebra and M an A-bimodule. Then,*

$$\mathrm{HH}^i(A, M) \cong \mathrm{Ext}^i_A({}_A\Bbbk, L(M)) \cong \mathrm{Ext}^i_A(\Bbbk_A, R(M))$$

for all $i \geqslant 0$.

3. Cohomology of Graded Twisting

In this section, we give our main results. Let A be a Hopf algebra and B a graded twisting of A by a finite abelian group Γ. Since B is a Hopf subalgebra of the crossed product $A \rtimes \Gamma$, the relation between the cohomology of A and B is achieved by discussing the cohomology of crossed products and the cohomology of Hopf subalgebras.

3.1. Cohomology of Crossed Products

In this subsection, we describe the cohomology of crossed products.

Let A be a Hopf algebra and Γ a finite group. Assume $\alpha : \Gamma \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ is an action of Γ by Hopf algebra automorphisms on A. For a more detailed account on the actions of Hopf algebras on algebras, we refer to the book by Montgomery [18] and the paper by Centrone [19]. Although the description of the Hochschild cohomology of $A \rtimes \Gamma$ can be derived from the results in [20], we give a complete and more direct proof for the results needed. Previous results about the cohomology of crossed products can also be found, for example, in [21–24] and the references therein.

Let M and N be two right $A \rtimes \Gamma$-modules. Then, $\mathrm{Hom}_A(M, N)$ is a right $\Bbbk\Gamma$-module with the adjoint action:

$$(f \leftharpoonup g)(m) = (f(m \cdot g^{-1})) \cdot g, \tag{2}$$

for $g \in \Gamma$, $f \in \mathrm{Hom}_A(M, N)$ and $m \in M$. For a right $\Bbbk\Gamma$-module X, let

$$X^{\Gamma} = \{x \in X | x \cdot g = x, \text{ for all } g \in \Gamma\}$$

be the set of Γ-invariant elements. It is clear that

$$\mathrm{Hom}_{A \rtimes \Gamma}(M, N) = \mathrm{Hom}_A(M, N)^{\Gamma}.$$

This isomorphism can be extended to the following isomorphisms (see, e.g., [22,25]),

$$\mathrm{Ext}^i_{A \rtimes \Gamma}(M, N) = \mathrm{Ext}^i_A(M, N)^{\Gamma}, \quad \text{for all } i \geqslant 0. \tag{3}$$

Let N be a left A-module. The vector space $\Bbbk\Gamma \otimes N$ is a left $A \rtimes \Gamma$-module defined by

$$(a \otimes h) \cdot (g \otimes n) = hg \otimes \alpha_{g^{-1}h^{-1}}(a)n, \tag{4}$$

for $n \in N$, $g, h \in \Gamma$ and $a \in A$.

It is easy to check that A is a right $A \rtimes \Gamma$-module with the action defined by

$$a \cdot (b \otimes h) = \alpha_{h^{-1}}(ab)$$

for all $a, b \in A$, $h \in \Gamma$. Then, we have the following lemma.

Lemma 4. *Let M be a right $A \rtimes \Gamma$-module. The left A-module structure of* $\mathrm{Hom}_A(M, A_A)$ *is compatible with the right Γ-action in the sense that*

$$(af) \leftharpoonup h = \alpha_{h^{-1}}(a)(f \leftharpoonup h), \tag{5}$$

for all $h \in \Gamma$, $f \in \mathrm{Hom}_A(M, A)$ and $a \in A$. Consequently, $\Bbbk\Gamma \otimes \mathrm{Hom}_A(M, A_A)$ is an $A \rtimes \Gamma$-$\Bbbk\Gamma$-bimodule, where the left $A \rtimes \Gamma$-module structure is given as in (4) and the right Γ-action is diagonal.

Proof. First, we show that Equation (5) holds. Indeed, for any $m \in M$, we have

$$\begin{aligned}
[\alpha_{h^{-1}}(a)(f \leftharpoonup h)](m) &= \alpha_{h^{-1}}(a)(f \leftharpoonup h)(m) \\
&= \alpha_{h^{-1}}(a)\left(f\left(m \cdot h^{-1}\right)\right) \cdot h \\
&= \alpha_{h^{-1}}(a)\alpha_{h^{-1}}\left(f\left(m \cdot h^{-1}\right)\right) \\
&= \alpha_{h^{-1}}\left(af\left(m \cdot h^{-1}\right)\right) \\
&= \alpha_{h^{-1}}\left((af)\left(m \cdot h^{-1}\right)\right) \\
&= (af)\left(m \cdot h^{-1}\right) \cdot h \\
&= [(af) \leftharpoonup h](m).
\end{aligned}$$

Then, we show that $\Bbbk\Gamma \otimes \mathrm{Hom}_A(M, A_A)$ is an $A \rtimes \Gamma$-$\Bbbk\Gamma$-bimodule. For all $g \otimes f \in k\Gamma \otimes \mathrm{Hom}_A(M, A)$, $h \in \Gamma$ and $a \otimes k \in A \rtimes \Gamma$, on one hand, we have

$$\begin{aligned} ((a \otimes k) \cdot (g \otimes f)) \cdot h &= \Big(kg \otimes \alpha_{g^{-1}k^{-1}}(a)f\Big) \cdot h \\ &= kgh \otimes \Big(\alpha_{g^{-1}k^{-1}}(a)f\Big) \leftharpoonup h \\ &\overset{(5)}{=} kgh \otimes \alpha_{h^{-1}}\Big(\alpha_{g^{-1}k^{-1}}(a)\Big)(f \leftharpoonup h) \\ &= kgh \otimes \alpha_{h^{-1}g^{-1}k^{-1}}(a)(f \leftharpoonup h). \end{aligned}$$

On the other hand,

$$\begin{aligned} (a \otimes k) \cdot ((g \otimes f) \cdot h) &= (a \otimes k) \cdot (gh \otimes f \leftharpoonup h) \\ &= kgh \otimes \alpha_{h^{-1}g^{-1}k^{-1}}(a)(f \leftharpoonup h). \end{aligned}$$

Therefore, $((a \otimes k) \cdot (g \otimes f)) \cdot h = a \otimes k \cdot ((g \otimes f) \cdot h)$. Therefore, $\Bbbk\Gamma \otimes \mathrm{Hom}_A(M, A)$ is an $A \rtimes \Gamma$-$\Bbbk\Gamma$-bimodule. □

Let M be a right $A \rtimes \Gamma$-module. There is a natural left $A \rtimes \Gamma$-module structure on $\mathrm{Hom}_{A\rtimes\Gamma}(M, A \rtimes \Gamma)$ induced by the left $A \rtimes \Gamma$-module structure of $A \rtimes \Gamma$. $\mathrm{Hom}_{A\rtimes\Gamma}(M, A \rtimes \Gamma)$ is also a right $\Bbbk\Gamma$-module (see (2)). Then, $\mathrm{Hom}_{A\rtimes\Gamma}(M, A \rtimes \Gamma)$ is an $A \rtimes \Gamma$-$\Bbbk\Gamma$-bimodule.

The following lemma may be well-known, and we conclude a proof here for the sake of completeness.

Lemma 5. *Let P be a finitely generated projective right $A \rtimes \Gamma$-module. Then,*

$$\Bbbk\Gamma \otimes \mathrm{Hom}_A(P, A) \cong \mathrm{Hom}_A(P, A \rtimes \Gamma)$$

as $A \rtimes \Gamma$-Γ-bimodules.

Proof. Let

$$\psi : \Bbbk\Gamma \otimes \mathrm{Hom}_A(P, A) \to \mathrm{Hom}_A(P, A \rtimes \Gamma)$$

be the morphism defined by

$$\psi(g \otimes f)(p) = \alpha_g(f(p)) \otimes g,$$

for all $g \in \Gamma$, $f \in \mathrm{Hom}_A(P, A)$ and $p \in P$. We check that ψ is an $A \rtimes \Gamma$-$\Bbbk\Gamma$-bimodule map. For any $g, h \in \Gamma$, $f \in \mathrm{Hom}_A(P, A)$ and $p \in P$, we have

$$\begin{aligned} \psi((g \otimes f) \cdot h)(p) &= \psi(gh \otimes f \leftharpoonup h)(p) \\ &= \alpha_{gh}((f \leftharpoonup h)(p)) \otimes g\,h \\ &= \alpha_{gh}\Big((f\Big(p \cdot h^{-1}\Big)) \cdot h\Big) \otimes g\,h \\ &= \alpha_{gh}\alpha_{h^{-1}}\Big(f\Big(p \cdot h^{-1}\Big)\Big) \otimes g\,h \\ &= \alpha_g\Big(f\Big(p \cdot h^{-1}\Big)\Big) \otimes g\,h \\ &= \psi(g \otimes f)\Big(p \cdot h^{-1}\Big)h \\ &= (\psi(g \otimes f) \leftharpoonup h)(p) \end{aligned}$$

and

$$\begin{aligned}\psi((a\otimes h)\cdot(g\otimes f))(p) &= \psi\Big(hg\otimes\alpha_{g^{-1}h^{-1}}(a)f\Big)(p)\\ &= \alpha_{hg}\Big(\alpha_{g^{-1}h^{-1}}(a)f\Big)(p)\otimes h\,g\\ &= \alpha_{hg}\Big(\alpha_{g^{-1}h^{-1}}(a)f(p)\Big)\otimes h\,g\\ &= a\alpha_{hg}(f(p))\otimes h\,g\\ &= (a\otimes h)(\alpha_g(f(p))\otimes g)\\ &= (a\otimes h)\psi(g\otimes f)(p)\\ &= ((a\otimes h)\psi(g\otimes f))(p).\end{aligned}$$

The vector space $\Bbbk\Gamma\otimes A$ is an algebra with the following multiplication:

$$(g\otimes a)(h\otimes b)=gh\otimes\alpha_{h^{-1}}(a)b \qquad \text{for} g,h\in\Gamma,a,b\in A.$$

There is algebra an isomorphism $A\rtimes\Gamma\to\Bbbk\Gamma\otimes A$ defined by $a\otimes g\mapsto g\otimes\alpha_{g^{-1}}(a)$. This algebra isomorphism induces an $A\rtimes\Gamma$-bimodule structure on $\Bbbk\Gamma\otimes A$, and $A\rtimes\Gamma$ is isomorphic to $\Bbbk\Gamma\otimes A$ as $A\rtimes\Gamma$-bimodules. Now, ψ is an isomorphism following the fact that P is a finitely generated projective right A-module, since Γ is a finite group. □

Definition 2. *An algebra A is called homologically smooth if A has a bounded resolution by finitely generated projective A^e-modules.*

A Hopf algebra A is homologically smooth is equivalent to that the trivial module $\Bbbk_A$ (or ${}_A\Bbbk$) that admits a bounded projective resolution with each term finitely generated (cf. [17] (Proposition A.2)).

For example, by [10] (Theorem 5.1), the coordinate algebras of quantum symmetry groups of non-degenerate bilinear forms introduced by M. Dubois-Violette and G. Launer [26] are homologically smooth.

The following proposition can be viewed as a slight generalization of [22] (Proposition 1.3).

Proposition 1. *Let A be a Hopf algebra and Γ a finite group. Assume $\alpha:\Gamma\to\mathrm{Aut}_{\mathrm{Hopf}}(A)$ is an action of Γ by Hopf algebra automorphisms on A and A is homologically smooth. Then,*

$$\mathrm{Ext}^i_{A\rtimes\Gamma}(\Bbbk_{A\rtimes\Gamma},A\rtimes\Gamma_{A\rtimes\Gamma})=(\Bbbk\Gamma\otimes\mathrm{Ext}^i_A(\Bbbk_A,A_A))^\Gamma$$

as left $A\rtimes\Gamma$-modules.

Proof. Since A is homologically smooth and Γ is a finite group, by [23] (Proposition 2.11), $A\rtimes\Gamma$ is also homologically smooth. Then, $\Bbbk_{A\rtimes\Gamma}$ admits a projective resolution

$$\cdots\to P_n\to P_{n-1}\to\cdots\to P_1\to P_0\to\Bbbk\to 0$$

such that each P_n is finitely generated as an $A\rtimes\Gamma$-module. The group Γ is a finite group, this resolution can also be regarded as a projective resolution of the trivial module $\Bbbk_A$. Applying the function $\mathrm{Hom}_A(-,A\rtimes\Gamma)$ to the above resolution, we obtain the following complex of $A\rtimes\Gamma$-$\Bbbk\Gamma$-bimodules

$$\begin{aligned}0\to\mathrm{Hom}_A(P_0,A\rtimes\Gamma)\to\mathrm{Hom}_A(P_1,A\rtimes\Gamma)\to\cdots\\ \to\mathrm{Hom}_A(P_n,A\rtimes\Gamma)\to\cdots\end{aligned}\tag{6}$$

By Lemma 5, this complex is isomorphic to the following complex of $A \rtimes \Gamma$-$\Bbbk\Gamma$-bimodules

$$0 \to \Bbbk\Gamma \otimes \mathrm{Hom}_A(P_0, A) \to \Bbbk\Gamma \otimes \mathrm{Hom}_A(P_1, A) \to \cdots \\ \to \Bbbk\Gamma \otimes \mathrm{Hom}_A(P_n, A) \to \cdots \tag{7}$$

After taking the cohomologies of the complexes (6) and (7), we obtain the isomorphisms of $A \rtimes \Gamma$-$\Bbbk\Gamma$-bimodules

$$\mathrm{Ext}^i_A(\Bbbk_A, A \rtimes \Gamma) \cong \Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A)$$

for all $i \geqslant 0$. From (3), we have left $A \rtimes \Gamma$-module isomorphisms

$$\begin{aligned} \mathrm{Ext}^i_{A\rtimes\Gamma}(\Bbbk_{A\rtimes\Gamma}, A \rtimes \Gamma_{A\rtimes\Gamma}) &\cong (\mathrm{Ext}^i_A(\Bbbk_A, A \rtimes \Gamma))^\Gamma \\ &\cong (\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A))^\Gamma. \end{aligned}$$

for all $i \geqslant 0$. □

3.2. Cohomology of Hopf Subalgebras

In this subsection, we show how the cohomologies of a Hopf algebra and its Hopf subalgebra are related. The discussion is based on Section 3.1 of [27].

Let $B \subset A$ be a Hopf subalgebra. Then, B^+A is a coideal in A, so that $L = A/B^+A$ is a coalgebra. L is also naturally a right A-module. Let $\mathcal{M}^L_A$ be the category defined as follows:

- The objects are both right A-modules and right L-comodules such that for any $v \in V$ and $a \in A$,
$$(v \cdot a)_{(0)} \otimes (v \cdot a)_{(1)} = v_{(0)} \cdot a_1 \otimes v_{(1)} \cdot a_2.$$
- The morphisms are A-linear and L-colinear maps.

If in addition, the Hopf subalgebra B satisfies $B^+A = AB^+$, then $L = A/B^+A$ is an A-A-bimodule. Let ${}_A\mathcal{M}^L_A$ be the category defined as:

- The objects are both A-A-bimodules and right L-comodules such that for any $v \in V$, $a, b \in A$,
$$(a \cdot vs. \cdot b)_{(0)} \otimes (a \cdot vs. \cdot b)_{(1)} = a_1 \cdot v_{(0)} \cdot b_1 \otimes a_2 \cdot v_{(1)} \cdot b_2.$$
- The morphisms are A-A-bilinear and L-colinear maps.

Since $p(b) = \varepsilon(b)p(1)$, for $b \in B$, if V is an object in $\mathcal{M}^L_A$ (resp. ${}_A\mathcal{M}^L_A$), then

$$V^{\mathrm{co}L} = \{v \in V | v_{(0)} \otimes v_{(1)} = v \otimes p(1)\}$$

is a sub-B-module (resp. sub-B-B-bimodule) of V.

The following proposition can be viewed as a refinement of Proposition 3.6 in [27].

Proposition 2. *Let $B \subset A$ be a Hopf subalgebra Assume that the antipode of A is bijective, that A is faithfully flat as a left or right B-module, that $B^+A = AB^+$ (so that $L = A/AB^+$ is a quotient Hopf algebra), and that L is finite-dimensional. Then, we have the following isomorphism for any $M \in \mathcal{M}_A$ and any $N \in \mathcal{M}^L_A$,*

$$\mathrm{Ext}^*_A(M, N) \cong \mathrm{Ext}^*_B(M_{|B}, N^{\mathrm{co}L}) \tag{8}$$

Moreover, if N is an object in ${}_A\mathcal{M}^L_A$, then the above isomorphism is an isomorphism of left B-modules. The left B-module structures on $\mathrm{Ext}^*_A(M, N)$ and $\mathrm{Ext}^*_B(M_{|B}, N^{\mathrm{co}L})$ are induced by the natural left B-action on N.

Proof. Since L is a finite dimensional Hopf algebra, it is well-known that there exists a left integral $\tau \in L$ and a right integral $h : L \to \Bbbk$ on L, such that $h(\tau) = 1$ and $hS(\tau) \neq 0$.

An element $t \in A$ is chosen such that $p(t) = \tau$. For $f \in \mathrm{Hom}_B(M, {}^{coL}N)$, it can be viewed as a B-linear map $M \to N$. Following from Example 3.3 and Lemma 3.4 in [27], there is a linear map

$$\begin{array}{rcl}\Psi : \mathrm{Hom}_B(M, N^{\mathrm{co}L}) & \to & \mathrm{Hom}_A(M, N) \\ f & \mapsto & \tilde{f},\ \tilde{f}(x) = f(x \cdot S(t_1)) \cdot t_2.\end{array}$$

By the proof of [27] (Proposition 3.6), the map Ψ is a linear isomorphism and induces a linear isomorphism $\mathrm{Ext}^*_A(M, N) \cong \mathrm{Ext}^*_B(M_{|B}, N^{\mathrm{co}}L)$. To complete the proof of this proposition, we only need to show that Ψ is an isomorphism of left B-modules when $N \in {}_A\mathcal{M}^L_A$.

In fact, for $f \in \mathrm{Hom}_B(M, N^{coL})$, $b \in B$ and $x \in M$, we have

$$\begin{aligned}\Psi(b \cdot f)(x) &= (b \cdot f)(x \cdot S(t_1)) \cdot t_2 \\ &= b \cdot f(x \cdot S(t_1)) \cdot t_2 \\ &= b \cdot (\tilde{f}(x)) \\ &= b \cdot (\Psi(f)(x)) \\ &= (b \cdot \Psi(f))(x).\end{aligned}$$

□

3.3. Cohomology of Graded Twisting

Now, we can prove the main results of this section.

Lemma 6. *Let A and B be Hopf algebras with bijective antipodes, and assume that B is a graded twisting of A by a finite abelian group Γ. Then, A is homologically smooth if and only if B is homologically smooth.*

Proof. Let (p, α) be the invariant cocentral action on A such that $B = A^{t,\alpha}$. If A is homologically smooth, then so is $A \rtimes \Gamma$ ([23] (Proposition 2.11)). By Lemma 2, there is a strict exact sequence of Hopf algebras

$$0 \to A^{t,\alpha} \to A \rtimes \Gamma \to \Bbbk\Gamma \to 0.$$

Hence, $A^{t,\alpha}$ is homologically smooth by Proposition 3.5 in [27].

Since Γ is an abelian group, A is also a graded twisting of $A^{t,\alpha}$ by Remark 2. Therefore, A is homologically smooth when $A^{t,\alpha}$ is too. □

Theorem 3. *Let Γ be a finite abelian group and (p, α) an invariant cocentral action of Γ on a Hopf algebra A with bijective antipode. Let $B = A^{t,\alpha}$ be the graded twisting of A. If A is homologically smooth, then there is an isomorphism of left B-modules*

$$\mathrm{Ext}^i_B(\Bbbk_B, B_B) = (\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A))^\Gamma,$$

for $i \geqslant 0$, where Γ acts on $\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A)$ diagonally and the B-action on $\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A)$ is induced by the $A \rtimes \Gamma$-action on it defined as in (4).

Proof. By Lemma 2, there is a strict exact sequence of Hopf algebras

$$0 \to A^{t,\alpha} \to A \rtimes \Gamma \to \Bbbk\Gamma \to 0.$$

It is easy to check that $A \rtimes \Gamma \in {}_{A\rtimes\Gamma}\mathcal{M}^L_{A\rtimes\Gamma}$. Now, we have the following isomorphisms of left B-modules:

$$\begin{array}{rcl}\mathrm{Ext}^i_B(\Bbbk_B, B_B) & \cong & \mathrm{Ext}^i_B(\Bbbk_B, (A \rtimes \Gamma)^{\mathrm{co}\Gamma}) \\ & \cong & \mathrm{Ext}^i_{A\rtimes\Gamma}(\Bbbk_{A\rtimes\Gamma}, A \rtimes \Gamma_{A\rtimes\Gamma}) \\ & \cong & (\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A))^\Gamma,\end{array}$$

for $i \geqslant 0$. The second and third isomorphisms follow from Propositions 1 and 2, respectively. $\square$

Remark 3. *In the above theorem, since the group Γ is abelian, as mentioned in Remark 2, A is isomorphic to a graded twisting of B. Then, conversely to Theorem 3, the cohomology of A can be expressed by that B. To be precise, $(\tilde{p}, \beta)$ is a cocentral invariant action of Γ on B, where $\tilde{p} = p \otimes \varepsilon$ and $\beta = \alpha^{-1} \otimes \mathrm{id}|_B$, and A is isomorphic to $B^{t,\beta}$ as Hopf algebras. Moreover, B is homologically smooth by Lemma 6. Hence, we have the following isomorphism of left A-modules:*

$$\mathrm{Ext}^i_A(\Bbbk_A, A_A) = (\Bbbk\Gamma \otimes \mathrm{Ext}^i_B(\Bbbk_B, B_B))^\Gamma,$$

for $i \geqslant 0$.

Example 1. *Let A be a Hopf algebra with a cocentral surjective Hopf algebra map $p : A \to \Bbbk\mathbb{Z}_2$. Let $A * A$ denote the free product Hopf algebra of A with itself. There is a cocentral Hopf algebra map $A * A \to \Bbbk\mathbb{Z}_2$ whose restriction to each copy is p. We still denote this map by p. Let $\alpha : \Bbbk\mathbb{Z}_2 \to \mathrm{Aut}_{\mathrm{Hopf}}(A * A)$ be the action such that α_g is the Hopf algebra automorphism of $A * A$ that exchanges the two copies of A, where g is the generator of $\mathbb{Z}_2$. We obtain an invariant cocentral action (p, α) of $\mathbb{Z}_2$ on $A * A$, and hence a graded twisting $(A * A)^{t,\alpha}$.*

Now, let $A = \mathcal{O}(\mathrm{SL}q(2))$, the quantum linear group. It is the algebra with generators a, b, c, d, subject to the relations

$$\begin{array}{lll} ab = qba & ac = qca & bc = cb \\ bd = qdb & cd = qdc & ad - qbc = da - q^{-1}bc = 1. \end{array}$$

This algebra is a special case of the Hopf algebra $\mathcal{B}(E)$ defined by Dubois-Violette and Launer [26] ($\mathcal{O}(\mathrm{SL}q(2)) = \mathcal{B}(E_q)$ for some matrix E_q). It can be deduced from Lemma 5.6 and Proposition 6.2 in [10] that

$$\mathrm{Ext}^i_A(\Bbbk_A, A_A) \cong \begin{cases} {}_\eta\Bbbk & i = d; \\ 0 & i \neq d, \end{cases}$$

where $\eta : A \to \Bbbk$ is the algebra map defined by $\eta(a) = q^{-2}a, \eta(d) = q^2 d$ and $\eta(b) = \eta(c) = 0$. By carefully checking the proof of [28] (Theorem 5.1), we obtain the following isomorphism of left $A * A$-modules for $i \leq 0$

$$\mathrm{Ext}^i_{A*A}(\Bbbk_{A*A}, (A * A)_{A*A}) \cong \mathrm{Ext}^i_A(\Bbbk_A, A * A) \oplus \mathrm{Ext}^i_A(\Bbbk_A, A * A),$$

where $A * A$ has the restricted A-module structure.

The trivial module $\Bbbk_A$ over A has a finitely generated projective resolution by [10], therefore

$$\begin{aligned} \mathrm{Ext}^i_A(\Bbbk_A, A * A) &\cong (A * A) \otimes_A \mathrm{Ext}^i_A(\Bbbk_A, A_A) \\ &\cong \begin{cases} (A * A) \otimes_A {}_\eta\Bbbk & i = d; \\ 0 & i \neq d. \end{cases} \end{aligned}$$

Notice that the action α induces an action on $(A * A) \otimes_A {}_\eta\Bbbk$. We still denote this action by α.

Write $B = (A * A)^{t,\alpha}$. By Theorem 3 and checking the $\Bbbk\mathbb{Z}_2$-module structure on Ext^i_{A*A} $(\Bbbk_{A*A}, (A * A)_{A*A})$, we obtain that $\mathrm{Ext}^i_B(\Bbbk_B, B_B) = 0$ for $i \neq d$, and as left B-modules,

$$\begin{aligned} \mathrm{Ext}^d_B(\Bbbk_B, B_B) &\cong (\Bbbk\mathbb{Z}_2 \otimes ((A * A) \otimes_A {}_\eta\Bbbk \bigoplus (A * A) \otimes_A {}_\eta\Bbbk))^{\mathbb{Z}_2} \\ &\cong (A * A) \otimes_A {}_\eta\Bbbk \bigoplus (A * A) \otimes_A {}_\eta\Bbbk, \end{aligned}$$

where the B-module structure on $(A * A) \otimes_A {}_\eta\Bbbk \bigoplus (A * A) \otimes_A {}_\eta\Bbbk$ is given by

$$(y \otimes g)(x_1, x_2) = (y\alpha_g(x_2), y\alpha_g(x_1)),$$

for $y \otimes g \in B$, $(x_1, x_2) \in (A * A) \otimes_A {}_\eta\Bbbk \bigoplus (A * A) \otimes_A {}_\eta\Bbbk$.

Keep the same notations from Theorem 3. Let $M = \bigoplus_{g\in\Gamma} M_g$ be a graded right A-module. Next, we will define a twisted module M^α, and compare the cohomology of A with coefficient M and the cohomology of $A^{t,\alpha}$ with coefficient M^α.

We define a twisted module of M as follows. The vector space $M \otimes \Bbbk\Gamma$ is a right $A \rtimes \Gamma$-module as the $A \rtimes \Gamma$-action is defined by

$$(m \otimes g)(a \otimes h) = m\alpha_g(a) \otimes gh,$$

for any $m \otimes g \in M \otimes \Bbbk\Gamma$ and $a \otimes h \in A \rtimes \Gamma$. We denote this module by $M \rtimes \Gamma$. The twisted module M^α is defined as the submodule $M^\alpha = \bigoplus_{g\in\Gamma} M_g \otimes g \subseteq M \rtimes \Gamma$. It is a right module over $A^{t,\alpha}$.

The right $A \rtimes \Gamma$-module $M \rtimes \Gamma$ is a $\Bbbk\Gamma$-comodule by the coaction

$$\begin{aligned} M \otimes \Bbbk\Gamma &\to M \otimes \Bbbk\Gamma \otimes \Bbbk\Gamma \\ m \otimes h &\mapsto m \otimes h \otimes gh^{-1} \end{aligned}$$

for $m \in M_g$. The space $(M \rtimes \Gamma)^{\mathrm{co}\Gamma}$ is just M^α. Now, we check that $M \rtimes \Gamma$ is an object in $\mathcal{M}^{\Bbbk\Gamma}_{A\rtimes\Gamma}$. For any $g, h, k, l \in \Gamma$ and $m \in M_g$, $a \in A_k$, we have

$$\begin{aligned} & ((m \otimes h)(a \otimes l))_{(0)} \otimes ((m \otimes h)(a \otimes l))_{(1)} \\ = \;& (m\alpha_h(a) \otimes hl)_{(0)} \otimes (m\alpha_h(a) \otimes hl)_{(1)} \\ = \;& m\alpha_h(a) \otimes hl \otimes gkh^{-1}l^{-1} \\ = \;& (m \otimes h)(a \otimes l) \otimes (gh^{-1})(kl^{-1}) \\ = \;& (m \otimes h)_{(0)}(a \otimes l)_1 \otimes (m \otimes h)_{(1)}\bar{p}((a \otimes l)_2). \end{aligned}$$

The last equation follows from the following equations

$$\begin{aligned} (a \otimes l)_1 \otimes \bar{p}((a \otimes l)_2) &= a_1 \otimes l \otimes p(a_2)l^{-1} \\ &= a \otimes l \otimes kl^{-1}, \end{aligned}$$

since $a \in A_k$.

Theorem 4. *Let Γ be a finite abelian group and (p, α) an invariant cocentral action of Γ on a Hopf algebra A with a bijective antipode. Assume that A is homologically smooth, then for a graded right A-module M, we have*

$$\mathrm{Ext}^i_B(\Bbbk_B, M^\alpha) \cong M \otimes_A (\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A))^\Gamma,$$

for $i \geqslant 0$.

Proof. The module $M \rtimes \Gamma$ is in $\mathcal{M}^{\Bbbk\Gamma}_{A\rtimes\Gamma}$ and $(M \rtimes \Gamma)^{\mathrm{co}\Gamma} = M^\alpha$. Then, by Proposition 2, for $i \geqslant 0$,

$$\begin{aligned} \mathrm{Ext}^i_B(\Bbbk_B, M^\alpha) &= \mathrm{Ext}^i_B(\Bbbk_B, (M \rtimes \Gamma)^{\mathrm{co}\Gamma}) \\ &\cong \mathrm{Ext}^i_{A\rtimes\Gamma}(\Bbbk_{A\rtimes\Gamma}, M \rtimes \Gamma). \end{aligned}$$

The algebra A is homologically smooth and Γ is a finite group. As mentioned in the proof of Proposition 1, $A \rtimes \Gamma$ is also homologically smooth. Then, $\Bbbk_{A\rtimes\Gamma}$ admits a projective resolution $\mathbf{P}_* \to \Bbbk \to 0$ with each term finitely generated. Therefore, for $i \geqslant 0$,

$$\begin{aligned} & \mathrm{Ext}^i_{A\rtimes\Gamma}(\Bbbk_{A\rtimes\Gamma}, M \rtimes \Gamma) \\ \cong \;& \mathrm{H}^i\mathrm{Hom}_{A\rtimes\Gamma}(\mathbf{P}_*, M \rtimes \Gamma) \\ \cong \;& (M \rtimes \Gamma) \otimes_{A\rtimes\Gamma} \mathrm{H}^i\mathrm{Hom}_{A\rtimes\Gamma}(\mathbf{P}_*, A \rtimes \Gamma) \\ \cong \;& M \otimes_A \mathrm{Ext}^i_{A\rtimes\Gamma}(\Bbbk_{A\rtimes\Gamma}, A \rtimes \Gamma) \\ \cong \;& M \otimes_A (\Bbbk\Gamma \otimes \mathrm{Ext}^*_A(\Bbbk_A, A_A))^\Gamma, \end{aligned}$$

where the last isomorphism follows from Proposition 1. This completes the proof. □

4. The Calabi–Yau Property

Let A be a Hopf algebra and B a graded twisting of A by a finite abelian group. As an application of Theorem 3, we show that A is a twisted Calabi–Yau algebra if and only if B is a twisted Calabi–Yau algebra.

Let us recall the definition of twisted Calabi–Yau algebras.

Definition 3. *A homologically smooth algebra A is called a twisted Calabi–Yau algebra of dimension d if there is an automorphism μ of A such that $\mathrm{Ext}^d_{A^e}(A, A^e) \cong A_\mu$ as A-bimodules and $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ for $i \neq d$.*

A Calabi–Yau algebra is a twisted Calabi–Yau algebra whose Nakayama automorphism is an inner automorphism.

In the following, Calabi–Yau is abbreviated as CY for short.

Twisted CY Hopf algebras are closely related to Artin–Schelter (AS for short) algebras. We first recall some facts about Hopf algebras. Let A be a Hopf algebra and $\eta : A \to \Bbbk$ an algebra map. There is an algebra automorphism $[\eta]^r$ of A defined by

$$[\eta]^r(a) = \eta(a_2)a_1.$$

Its inverse is just $[\eta S]^r$. This automorphism is usually called the right winding automorphism of A. Similarly, the left winding automorphism $[\eta]^l$ of A is defined by

$$[\eta]^l(a) = \eta(a_1)a_2.$$

It is also an algebra automorphism with its inverse $[\eta S]^l$. For an algebra map $\eta : A \to \Bbbk$, it is well-known that $\eta S^2 = \eta$ (see, e.g., [24]). Therefore, any winding automorphism commutes with S^2.

Let A be a Hopf algebra. For $i \geqslant 0$, $\mathrm{Ext}^i_A(\Bbbk_A, A_A)$ is a left A-module, and we define an A-bimodule structure on $\mathrm{Ext}^i_A(\Bbbk_A, A_A) \otimes A$ as follows

$$a \cdot (e \otimes x) \cdot b = eb_1 \otimes axS^2(b_2), \tag{9}$$

for any $e \in \mathrm{Ext}^i(\Bbbk_A, A_A)$ and $a, b, x \in A$. Similarly, since $\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA)$ is a right A-module, $\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \otimes A$ is an A-bimodule with the left and right A-action defined by

$$a \cdot (e \otimes x) \cdot b = a_2e \otimes S^2(a_1)xb, \tag{10}$$

for any $e \in \mathrm{Ext}^i_A({}_A\Bbbk, {}_AA)$ and $a, b, x \in A$.

Following from Proposition 2.1.3 in [17], we obtain the following lemma.

Lemma 7. *Let A be a Hopf algebra such that it is homologically smooth. There are isomorphisms of A-bimodules*

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \otimes A \cong \mathrm{Ext}^i_A(\Bbbk_A, A_A) \otimes A, \tag{11}$$

all $i \geqslant 0$, where the A-bimodules structures on $\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) \otimes A$ and on $\mathrm{Ext}^i_A(\Bbbk_A, A_A) \otimes A$ are induced by (9) and (10), respectively.

Now, we recall the definition of an AS–regular algebra.

A Hopf algebra A is said to be left AS–Gorenstein if

(1) $\mathrm{injdim}_A A = d < \infty$,
(2) $\mathrm{Ext}^i_A({}_A\Bbbk, A) = 0$ for $i \neq d$ and $\dim \mathrm{Ext}^d_A({}_A\Bbbk, A) = 1$.

A right AS–Gorenstein Hopf algebra can be defined similarly. If a Hopf algebra A is both left and right AS–Gorenstein, then A is called AS–Gorenstein. If, in addition, the global dimension of A is finite, then A is called AS–regular.

Remark 4. *Compared with [29] (Definition 1.2), we do not require the Hopf algebra H to be Noetherian. When A is AS–Gorenstein and homologically smooth, the right injective dimension always equals the left injective dimension, which are both given by the integer d such that* $\mathrm{Ext}^d_{A^e}(A, A^e) \neq 0$. *We refer to [17] (Remark 2.1.5) for an explanation.*

The following lemma follows from [17] (Proposition 2.1.6).

Lemma 8. *Let H be a Hopf algebra with a bijective antipode. Then, the following are equivalent:*

(1) A is a twisted CY algebra.

(2) A is a left AS–Gorenstein and the left trivial module ${}_A\Bbbk$ admits a bounded projective resolution with each term finitely generated.

(3) A is a right AS–Gorenstein and the right trivial module $\Bbbk_A$ admits a bounded projective resolution with each term finitely generated.

From the above lemma, if A is a twisted CY Hopf algebra of dimension d, then the vector space $\mathrm{Ext}^d_A(\Bbbk_A, A_A)$ is a one-dimensional left A-module. It is called the right homological integral of A and denoted by $\int^r_A$. Let $\mathbf{e}$ be a non-zero element in $\int^r_A$, the left A-action defines an algebra map $\eta : A \to \Bbbk$ by $a \cdot \mathbf{e} = \eta(a)\mathbf{e}$, for any $a \in A$. That is, $\int^r_A \cong {}_\eta\Bbbk$ as left A-modules. Similarly, the one-dimensional right A-module $\mathrm{Ext}^d_A({}_A\Bbbk, {}_AA)$ is called the left homological integral of A and denoted by $\int^l_A$. There is an algebra map $\xi : A \to \Bbbk$ such that $\int^l_A \cong \Bbbk_\xi$. Following from Lemma 7, we obtain the A-bimodule isomorphisms

$$\mathrm{Ext}^d_{A^e}(A, A^e) \cong A_{S^2[\xi]^l} \cong {}_{S^2[\eta]^r}A.$$

In conclusion, we obtain the following result (cf. [30] (Lemma 1.6)).

Lemma 9. *Let A be a twisted CY Hopf algebra. Let $\xi : A \to \Bbbk$ be an algebra map such that $\int^l_A \cong \Bbbk_\xi$ are right A-modules. Then, a Nakayama automorphism of A is given by $\mu = S^2[\xi]^l$. Alternatively, the algebra automorphism $S^{-2}[\eta S]^r$ is also a Nakayama automorphism A, where $\eta : A \to \Bbbk$ is the algebra map such that $\int^r_A \cong {}_\eta\Bbbk$ are left A-modules.*

Definition 4. *Let Γ be a group and A a twisted CY Hopf algebra such that there is an action $\alpha : \Gamma \to Aut_{Hopf}(A)$ of Γ by Hopf algebra automorphisms on A. Then, both $\Bbbk$ and A are right $A \rtimes \Gamma$-modules. Therefore, $\mathrm{Ext}^d_A(\Bbbk_A, A_A)$ is a one-dimensional $\Bbbk\Gamma$-module. Let $\mathbf{e}$ be a non-zero element in $\mathrm{Ext}^d_A(\Bbbk_A, A_A)$. Then, there exists an algebra homomorphism $\mathrm{hdet} : A \to \Bbbk$ satisfying*

$$\mathbf{e} \leftharpoonup g = \mathrm{hdet}(g)\mathbf{e},$$

for all $g \in \Gamma$. The map $\mathrm{hdet} : A \to \Bbbk$ is called the homological determinant of the Γ-action on A.

Remark 5. *The homological determinant of a Hopf action on a connected AS–Gorenstein algebra is already defined in [24,31,32]. In [20], the author defined the (weak) homological determinant of a Hopf action on a twisted CY algebra. Let A be a Hopf algebra as in the above definition. Note that both A and A^e are right $A^e \rtimes \Gamma$-modules, $\mathrm{Ext}^i_{A^e}(A, A^e)$ is a right $\Bbbk\Gamma$-module, $\mathrm{Ext}^i_A(\Bbbk_A, A_A) \otimes A$ is also a right $\Bbbk\Gamma$-module with diagonal action. Then, the isomorphism $\mathrm{Ext}^i_{A^e}(A, A^e) \cong \mathrm{Ext}^i_A(\Bbbk_A, A_A) \otimes A$ is actually an isomorphism of the right $\Bbbk\Gamma$-modules. Then, one can check that the above definition coincides with the (weak) homological determinant defined in [20].*

Theorem 5. *Let A be a Hopf algebra with a bijective antipode and Γ a finite abelian group. Let (p, α) be an invariant cocentral action of Γ on A. The algebra A is a twisted CY algebra if and only if its graded twisting $A^{t,\alpha}$ is also a twisted CY algebra. Let $\int^r_A \cong {}_\eta\Bbbk$ and $\int^r_{A^{t,\alpha}} \cong {}_{\bar\eta}\Bbbk$, where $\eta : A \to \Bbbk$ and $\bar\eta : A^{t,\alpha} \to \Bbbk$ are algebra maps. Then, $\bar\eta(a \otimes h) = \eta(a)\mathrm{hdet}^{-1}(h)$ for*

$a \otimes h \in A^{t,\alpha}$. Consequently, there are Nakayama automorphisms of A and $A^{t,\alpha}$, which satisfy the following equation

$$\mu_{A^{t,\alpha}}(a \otimes h) = \mathrm{hdet}(h)\mu_A(a) \otimes h,$$

for any $a \otimes h \in A^{t,\alpha}$.

Proof. First, we prove that if A is a twisted CY algebra, then so is $A^{t,\alpha}$. Let $B = A^{t,\alpha}$. It is homologically smooth by Lemma 6. If A is a twisted CY algebra of dimension d, then $\mathrm{Ext}^d_A(\Bbbk_A, A_A) \cong {}_\eta\Bbbk$ for some algebra map $\eta : A \to \Bbbk$ and $\mathrm{Ext}^i_A(\Bbbk_A, A_A) = 0$ for $i \neq d$. From Theorem 3, we have the following isomorphism

$$\begin{aligned} \mathrm{Ext}^i_B(\Bbbk_B, B_B) &\cong (\Bbbk\Gamma \otimes \mathrm{Ext}^i_A(\Bbbk_A, A_A))^\Gamma \\ &\cong \begin{cases} (\Bbbk\Gamma \otimes {}_\eta\Bbbk)^\Gamma & i = d; \\ 0 & i \neq d. \end{cases} \end{aligned}$$

Since Γ is a finite group, we have that $\dim \mathrm{Ext}^d_B(\Bbbk_B, B_B) = 1$ and $\dim \mathrm{Ext}^i_B(\Bbbk_B, B_B) = 0$ for $i \neq d$. The algebra B is a twisted CY by Lemma 8.

Let $t = \sum_{g\in\Gamma} \mathrm{hdet}(g)g \in \Gamma$. It satisfies that $th = \mathrm{hdet}^{-1}(h)t$ for any $h \in \Gamma$. Let $\mathbf{e}$ be a non-zero element in $\mathrm{Ext}^d_A(\Bbbk_A, A_A)$. Then, $\mathbf{e} \leftharpoonup h = \mathrm{hdet}(h)\mathbf{e}$, for all $h \in \Gamma$. Consequently, $t \otimes \mathbf{e}$ is a non-zero element in $\mathrm{Ext}^d_B(\Bbbk_B, B_B)$. The element t also satisfies that $ht = \mathrm{hdet}^{-1}(h)t$. We have that $\eta(\alpha_g(a)) = \eta(a)$ for any $g \in \Gamma$ and $a \in A$ by (5) in Lemma 4. Therefore, for any $a \otimes h \in B$,

$$\begin{aligned} (a \otimes h)(t \otimes \mathbf{e}) &= (a \otimes h)(\textstyle\sum_{g\in\Gamma} \mathrm{hdet}(g)g \otimes \mathbf{e}) \\ &= \textstyle\sum_{g\in\Gamma} \mathrm{hdet}(g)hg \otimes \alpha_{g^{-1}h^{-1}}(a)\mathbf{e} \\ &= \textstyle\sum_{g\in\Gamma} \mathrm{hdet}(g)\eta\left(\alpha_{g^{-1}h^{-1}}(a)\right)hg \otimes \mathbf{e} \\ &= \textstyle\sum_{g\in\Gamma} \mathrm{hdet}(g)\eta(a)hg \otimes \mathbf{e} \\ &= \eta(a)\mathrm{hdet}^{-1}(h)t \otimes \mathbf{e}. \end{aligned}$$

This shows that $\mathrm{Ext}^d_B(\Bbbk_B, B_B) \cong {}_{\bar\eta}\Bbbk$, where $\bar\eta$ is the algebra map defined by $\bar\eta(a \otimes h) = \eta(a)\mathrm{hdet}^{-1}(h)$ for $a \otimes h \in B$. From Lemma 9, a Nakayama automorphism of $A^{t,\alpha}$ is given by

$$\begin{aligned} \mu_{A^{t,\alpha}}(a \otimes h) &= [\bar\eta S_{A^{t,\alpha}}]^r S^{-2}_{A^{t,\alpha}}(a \otimes h) \\ &= \mathrm{hdet}(h)\eta(S_A(a_2))S_A^{-2}(a_1) \otimes h \\ &= \mathrm{hdet}(h)\mu_A(a) \otimes h, \end{aligned}$$

where μ_A is a Nakayama automorphism of A.

Conversely, when Γ is abelian, A is a graded twisting of $A^{t,\alpha}$ by the group Γ by Remark 2. Then, if $A^{t,\alpha}$ is a twisted CY algebra, then so is A. □

Example 2. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let $A, B \in \mathrm{GL}_m(\Bbbk)$. Let us recall the Hopf algebra $\mathcal{G}(A, B)$ defined in [33]. It is presented by generators $(u_{ij})_{1\leqslant i,j\leqslant m}, \mathbb{D}, \mathbb{D}^{-1}$ subject to relations*

$$u^t A u = A\mathbb{D}, \quad uBu^t = B\mathbb{D}, \quad \mathbb{D}\mathbb{D}^{-1} = 1 = \mathbb{D}^{-1}\mathbb{D}, \tag{12}$$

where u is the matrix $(u_{ij})_{1\leqslant i,j\leqslant m}$ and u^t denotes its transpose. There is a natural Hopf algebra structure on it (see [33] for details). Let $q \in \Bbbk^\times$ be a root of unity of order $n \leq 2$, and

$$A_q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} \quad A_{q^{-1}} = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix}.$$

The algebra $\mathcal{G}(A_{q^{-1}}, A_q)$ is just the coordinate algebra $\mathcal{O}(\mathrm{GL}_{q^{-1},q}(2))$. This algebra is a graded twisting of $\mathcal{O}(\mathrm{GL}(2))$. To be specific, let g be a generator of $\mathbb{Z}_n$. There is a cocentral Hopf algebra map

$$p : \mathcal{O}(\mathrm{GL}(2)) \to \mathbb{Z}_n, \quad \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \mapsto \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}.$$

Let α_g be the Hopf algebra automorphism of $\mathcal{O}(\mathrm{GL}(2))$ defined by

$$\alpha_g\left(\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}\right) = \begin{pmatrix} u_{11} & q^{-1}u_{12} \\ qu_{21} & u_{22} \end{pmatrix},$$

and $\alpha : \mathbb{Z}_n \to \mathcal{O}(\mathrm{GL}(2))$ the group action defined by $g \mapsto \alpha_g$. Then, (p, α) is an invariant cocentral action of $\mathbb{Z}_n$ on $\mathcal{O}(\mathrm{GL}(2))$. It can be checked that

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \mapsto \begin{pmatrix} u_{11} \otimes g & u_{12} \otimes g \\ u_{21} \otimes g & u_{22} \otimes g \end{pmatrix}$$

induces an isomorphism $\mathcal{O}\left(\mathrm{GL}_{q^{-1},q}(2)\right) \cong \mathcal{O}(\mathrm{GL}(2))^{t,\alpha}$.

The CY property of the algebras $\mathcal{G}(A, B)$ has been discussed in [30]. By [30] (Theorem 3.1), the algebra $\mathrm{GL}_{q^{-1},q}(2)$ is a twisted CY algebra with a Nayakama automorphism μ defined by

$$\mu\left(\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}\right) = \begin{pmatrix} u_{11} & q^{-2}u_{12} \\ q^2u_{21} & u_{22} \end{pmatrix}.$$

This algebra automorphism μ is an inner automorphism. Indeed, $\mu(u) = \mathbb{D}^{-1}u\mathbb{E}$ for any $u \in \mathrm{GL}_{q^{-1},q}(2)$, where $\mathbb{D} = u_{11}u_{22} - qu_{12}u_{21}$. Therefore, $\mathcal{O}(\mathrm{GL}_{q^{-1},q}(2))$ is a CY algebra. It can also be obtained by viewing $\mathcal{O}(\mathrm{GL}_{q^{-1},q}(2))$ as a graded twisting of $\mathcal{O}(\mathrm{GL}(2))$.

The $\mathcal{O}(\mathrm{GL}(2))$ is a CY algebra; hence, $\int^r_{\mathcal{O}(\mathrm{GL}(2))} \cong {}_\varepsilon\Bbbk$ ([30] (Lemma 1.6)). From Theorem 2.3 in [30], we can obtain a bounded finitely generated projective resolution of the right trivial module over $\mathcal{O}(\mathrm{GL}(2))$. It can be checked that the homological determinant of the action α is trivial, that is, $\mathrm{hdet} = \varepsilon$. From Theorem 5, $\int^r_{\mathcal{O}(\mathrm{GL}_{q^{-1},q}(2))}$ is the left trivial module over $\mathcal{O}(\mathrm{GL}_{q^{-1},q}(2))$. Moreover, $S^2_{\mathcal{O}(\mathrm{GL}_{q^{-1},q}(2))}$ is just the identity. Therefore, we can also obtain that $\mathcal{O}\left(\mathrm{GL}_{q^{-1},q}(2)\right)$ is a CY algebra by Theorem 5.

Remark 6. *As mentioned in [13] (Remark 2.4), the algebra structure on a graded twisting is a special case of the Zhang twist of a graded algebra constructed in [34].*

Let Γ be a group, $A = \bigoplus_g A_g$ a Γ-graded algebra, and $\tau = \left\{\tau_g \mid g \in \Gamma\right\}$ a set of a twisting system of A, namely a graded linear automorphisms of A, such that

$$\tau_g(a\tau_h(b)) = \tau_g(a)\tau_{gh}(b)$$

for all $g, h, l \in \Gamma$ and all $a \in A_h, b \in A_l$. Then, a new graded and associative multiplication on A is defined by

$$a \cdot_\tau b = a\tau_h(b)$$

for all $y \in A_h$, $z \in A_l$. The new graded algebra $(\bigoplus_g A_g, \cdot_\tau)$ is called the Zhang twist of A by τ, and is denoted by A^τ.

It has been proven in [34] that some homological properties are preserved under Zhang twisting for connected $\mathbb{Z}$-graded algebras.

Now, let (p, α) be an invariant cocentral action of a group Γ on a Hopf algebra A. It is easy to check that $\{\alpha_g | g \in \Gamma\}$ is a twisting system of A. The graded twisting $A^{t,\alpha}$ is just the twisted algebra A^α. In this paper, we have proven that the CY property is preserved under graded twisting by a finite abelian group. We conjecture that some other homological properties will be preserved under the Zhang twisting for Hopf algebras with invariant cocentral actions.

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