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Differential-Difference Elliptic Equations with Nonlocal Potentials in Half-Spaces

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Abstract: We investigate the half-space Dirichlet problem with summable boundary-value functions for an elliptic equation with an arbitrary amount of potentials undergoing translations in arbitrary directions. In the classical case of partial *differential* equations, the half-space Dirichlet problem for *elliptic* equations attracts great interest from researchers due to the following phenomenon: the solutions acquire qualitative properties specific for *nonstationary* (more exactly, parabolic) equations. In this paper, such a phenomenon is studied for nonlocal generalizations of elliptic differential equations, more exactly, for elliptic differential-difference equations with nonlocal potentials arising in various applications not covered by the classical theory. We find a Poisson-like kernel such that its convolution with the boundary-value function satisfies the investigated problem, prove that the constructed solution is infinitely smooth outside the boundary hyperplane, and prove its uniform power-like decay as the timelike independent variable tends to infinity.

Keywords: differential-difference equations; nonlocal potential elliptic equations; half-space Dirichlet problem; summable boundary-value functions

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1. Introduction

1.1. Elliptic Equations in Half-Spaces

It is well-known that for classical partial differential equations, the half-space problem with a single boundary-value condition is well posed both for the parabolic and elliptic cases (see, e.g., [1,2]). This is the Cauchy problem in the former case and the Dirichlet problem in the latter one. Though all independent variables are spatial in the elliptic case, the only independent variable varying on a semiaxis (unlike the other ones varying on whole real lines) acquires the so-called *timelike properties* (and the said variable itself is called the *timelike variable*): the resolving operator of the problem possesses the semigroup property with respect to that variable, and the behavior of the solutions for large values of that variable are similar to the large-time behavior of the solutions of the Cauchy problem for parabolic equations (see, e.g., [3]).

It turns out that in both cases (the parabolic one and the elliptic one), those qualitative properties of solutions substantially depend on the class of the boundary-value functions of the problem. If the boundary-value function belongs to $L_\infty(\mathbb{R}^n)$, then the well-known Repnikov–Ei'delman stabilization condition is valid (see [4]): depending on the limit properties of means of the boundary-value function, the solution either has a limit or does not have it. If the boundary-value function belongs to $L_1(\mathbb{R}^n)$, then the case qualitatively changes: the solution always has a limit, it is always equal to zero, and this decay is uniform.

1.2. Differential-Difference Equations

The phenomenon described in the previous section is quite far from being a specific property of two prototype equations (the Laplace one and the heat one). In particular, it occurs for differential-difference equations, i.e., equations where translation operators (apart

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from differential ones) act on the desired function. The unfailing worldwide interest in this generalization of classical differential equations started (within the contemporary mathematical approach) from the pioneering paper [5]) and is mainly caused by the following two reasons. The first one is purely theoretical: due to the *nonlocal nature* of differential-difference (and, more broadly, functional-differential) operators, not all research tools, methods, and approaches developed for differential equations can be helpful for functional-differential ones. For instance, no technique based on the maximum principle works in the differential-difference case. Thus, for functional-differential equations, one has to invent new research methods. Another reason is the existence of various applications of functional-differential equations in areas not covered by classical differential equations.

For the general theory, both aspects are comprehensively covered in [6–9] (also see references therein). It should be noted that non-differential operators contained in the studied equations might be quite diverse. For instance, they might be integrodifferential operators (see, e.g., [10–16] and references therein), operators of contractions and extensions of the independent variables (see, e.g., [17–21] and references therein), or others (see, e.g., [22,23] and references therein). In general, those operators are bounded (unlike differential ones), but due to their *nonlocal nature*, they cannot be treated as subordinate terms or small perturbations: their presence cause qualitatively new properties of the solutions.

The present paper is devoted to the timelike properties of elliptic differential-difference problems, as described in Section 1.1. The specified problem with essentially bounded boundary-value functions has already been studied for a relatively long time (see, e.g., [24] and references therein); the most-general result obtained up to now can be found in [25]. The investigation of this problem with summable boundary-value functions (i.e., the problem with finite-energy boundary data) started quite recently (see [26]). For differential-difference equations (regardless of their types), it is reasonable and natural to consider the following two cases separately: the case where differential and translation operators form *superpositions* and the case where they form *sums*. At the moment, the most general result for the former case is obtained in [27]. The present paper is devoted to the latter case. Its investigation started from [28], where a prototype equation is considered: the nonlocal term is single and the translation acts with respect to a coordinate direction. In [29], this result is generalized as follows: the translation operator acts in an *arbitrary* direction, but the nonlocal term is still *single*. Here, we present the most general result in this direction: the equation contains several nonlocal terms, and no restrictions for the directions of the translations are imposed.

Thus, in the half-space $\mathbb{R}^n \times (0, \infty)$, we consider the equation

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x, y) - \sum_{k=1}^{m} a_k u(x + h_k, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0, \tag{1}$$

where m and n are positive integers, $a_1, \dots, a_m$ are nonnegative constants, and $h_k := (h_{k1}, \dots, h_{kn})$, $k \in \overline{1, m}$ are vectors from $\mathbb{R}^n$ with real coordinates.

We introduce the nonnegative constants $a_0 := \sum\limits_{k=1}^{m} a_k$ and $h_0 := \max\limits_{k \in \overline{1,m}} |h_k|$. Note that both constants are strictly positive because we deal with classical differential equations (instead of differential-difference once); otherwise: if $a_0 = 0$, then Equation (1) is just the Laplace equation, while if $h_0 = 0$, then Equation (1) is the Laplace equation with a constant positive potential.

We impose the following restriction on the parameters a_0 and h_0:

$$h_0 \max\{a_0, \sqrt{a_0}\} < \frac{\pi}{2}. \tag{2}$$

Apart from Equation (1), we consider the boundary-value condition

$$u\Big|_{y=0} = u_0(x), x \in \mathbb{R}^n, \tag{3}$$

where $u_0 \in L_1(\mathbb{R}^n)$.

2. Integral Representations of Solutions

The following assertion is valid.

Theorem 1. *The function*

$$u(x,y) = \int_{\mathbb{R}^n} \mathcal{E}(x-\xi, y)u(\xi)d\xi, \tag{4}$$

where

$$\mathcal{E}(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-yG_1(\xi)} \cos[x\cdot\xi - yG_2(\xi)]d\xi, \tag{5}$$

$$G_{\left\{ {1 \atop 2} \right\}}(\xi) = \rho(\xi) \left\{ {\cos \atop \sin} \right\} \theta(\xi), \tag{6}$$

$$\rho(\xi) = \left(\left[|\xi|^2 + a(\xi) \right]^2 + b^2(\xi) \right)^{\frac{1}{4}}, \tag{7}$$

$$\theta(\xi) = \frac{1}{2} \arctan \frac{b(\xi)}{|\xi|^2 + a(\xi)}, \tag{8}$$

and

$$\left\{ {a \atop b} \right\}(\xi) = \sum_{k=1}^{m} a_k \left\{ {\cos \atop \sin} \right\} h_k \cdot \xi,$$

satisfies Equation (1) *in the half-space* $\mathbb{R}^n \times (0, \infty)$.

Proof. First, let us prove that all the introduced functions are well-defined.

We investigate the sign of the function $|\xi|^2 + a(\xi)$ in dependence on the relations between the vector $(a_1, \ldots, a_m)$ of the coefficients and the translation vectors $h_1, \ldots, h_m$ from $\mathbb{R}^n$.

If $|h_k \cdot \xi| < \frac{\pi}{2}$, then $\cos h_k \cdot \xi > 0$, and therefore, $\xi_k^2 + a_k \cos h_k \cdot \xi > 0$.

If $|h_k \cdot \xi| \geq \frac{\pi}{2}$, then $|h_k||\xi||\cos(\widehat{h_k, \xi})| \geq \frac{\pi}{2}$, which means that $|h_k||\xi| \geq \frac{\pi}{2}$, i.e., $|\xi| \geq \frac{\pi}{2|h_k|}$, and therefore,

$$|\xi|^2 + a(\xi) \geq \frac{\pi^2}{4|h_k|^2} + \sum_{l=1}^{m} a_l \cos h_l \cdot \xi.$$

The right-hand side of the last inequality is positive due to Condition (2). Hence, the function $|\xi|^2 + a(\xi)$ is positive everywhere.

Thus, the denominator in (8) is positive everywhere, which means that function (8) and, therefore, functions (6) are well-defined.

Now, to prove the well-definiteness of function (5), we estimate the function $G_1(\xi)$ from below. Since

$$2\theta(\xi) = \arctan \frac{b(\xi)}{|\xi|^2 + a(\xi)},$$

it follows that $-\frac{\pi}{2} < 2\theta(\xi) < \frac{\pi}{2}$ and $-\frac{\pi}{4} < \theta(\xi) < \frac{\pi}{4}$. Therefore, $\cos 2\theta(\xi) > 0$ and $\cos\theta(\xi) > \frac{\sqrt{2}}{2}$. Then

$$\cos 2\theta(\xi) = \frac{1}{\sqrt{1 + \tan^2 2\theta(\xi)}} = \left(1 + \frac{b^2(\xi)}{\left[|\xi|^2 + a(\xi) \right]^2} \right)^{-\frac{1}{2}} = \sqrt{\frac{\left[|\xi|^2 + a(\xi) \right]^2}{\left[|\xi|^2 + a(\xi) \right]^2 + b^2(\xi)}}.$$

Since the denominator of the last fraction is equal to $\rho^4(\xi)$ and the positivity of the function $|\xi|^2 + a(\xi)$ is guaranteed by Condition (2), it follows that $\cos 2\theta(\xi) = \dfrac{|\xi|^2 + a(\xi)}{\rho^2(\xi)}$. Further, since $\cos\theta(\xi) > 0$, it follows that $\cos\theta(\xi) = \sqrt{\dfrac{1+\cos 2\theta(\xi)}{2}}$. Therefore,

$$G_1(\xi) = \rho(\xi)\frac{1}{\sqrt{2}}\sqrt{1 + \frac{|\xi|^2 + a(\xi)}{\rho^2(\xi)}} = \sqrt{\frac{\rho^2(\xi) + |\xi|^2 + a(\xi)}{2}}. \tag{9}$$

Now, we take into account that

$$\rho^4(\xi) = \left[|\xi|^2 + a(\xi)\right]^2 + b^2(\xi) = |\xi|^4 + 2a(\xi)|\xi|^2 + a^2(\xi) + b^2(\xi) \geq |\xi|^4 - 2|a(\xi)||\xi|^2 + a^2(\xi) = \left[|\xi|^2 - |a(\xi)|\right]^2.$$

Since $|a(\xi)| \leq \sum\limits_{k=1}^{m} = a_0 > 0$, it follows that the inequality $\rho^2(\xi) \geq |\xi|^2 - |a(\xi)|$ is valid outside the ball $\{|\xi| < a_0\}$. Hence, the inequality $G_1(\xi) \geq \sqrt{|\xi|^2 - |a(\xi)|} \geq \sqrt{|\xi|^2 - a_0}$ is valid outside the same ball.

Thus, for each positive y, the absolute value of the integrand function in (5) is majorized by the function $e^{-y\sqrt{|\xi|^2 - a_0}}$ in $\mathbb{R}^n \backslash \{|\xi| < 2a_0\}$ and by the identical unit in $\{|\xi| < 2a_0\}$, which proves the well-definiteness of the function $\mathcal{E}(x,y)$ in the half-space $\mathbb{R}^n \times (0,\infty)$. The formal differentiating of function (5) inside the integral (with respect to each of its independent variables) causes the appearance of integrand factors that do not grow faster than the power functions of ξ. Hence, all derivatives of function (5) are well-defined in $\mathbb{R}^n \times (0,\infty)$.

Now, we have to prove that the function $\mathcal{E}(x,y)$ satisfies Equation (1). Taking into account that

$$(2\pi)^n \mathcal{E}_y(x,y) = -\int\limits_{\mathbb{R}^n} G_1(\xi)e^{-yG_1(\xi)}\cos[x\cdot\xi - yG_2(\xi)]d\xi + \int\limits_{\mathbb{R}^n} G_2(\xi)e^{-yG_1(\xi)}\sin[x\cdot\xi - yG_2(\xi)]d\xi$$

and, therefore,

$$\begin{aligned}(2\pi)^n \mathcal{E}_{yy}(x,y) &= \int\limits_{\mathbb{R}^n} G_1^2(\xi)e^{-yG_1(\xi)}\cos[x\cdot\xi - yG_2(\xi)]d\xi - \int\limits_{\mathbb{R}^n} G_1(\xi)G_2(\xi)e^{-yG_1(\xi)}\sin[x\cdot\xi - yG_2(\xi)]d\xi \\ &- \int\limits_{\mathbb{R}^n} G_1(\xi)G_2(\xi)e^{-yG_1(\xi)}\sin[x\cdot\xi - yG_2(\xi)]d\xi - \int\limits_{\mathbb{R}^n} G_2^2(\xi)e^{-yG_1(\xi)}\cos[x\cdot\xi - yG_2(\xi)]d\xi \\ &= \int\limits_{\mathbb{R}^n}\left[G_1^2(\xi) - G_2^2(\xi)\right]e^{-yG_1(\xi)}\cos[x\cdot\xi - yG_2(\xi)]d\xi - 2\int\limits_{\mathbb{R}^n} G_1(\xi)G_2(\xi)e^{-yG_1(\xi)}\sin[x\cdot\xi - yG_2(\xi)]d\xi,\end{aligned}$$

we compute

$$G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi)\cos 2\theta(\xi) = |\xi|^2 + a(\xi)$$

and

$$2G_1(\xi)G_2(\xi) = \rho^2(\xi)\cos 2\theta(\xi)\tan 2\theta(\xi) = \left[|\xi|^2 + a(\xi)\right]\frac{b(\xi)}{\left[|\xi|^2 + a(\xi)\right]} = b(\xi).$$

Then, we can substitute the function $\mathcal{E}(x,y)$ in Equation (1):

$$(2\pi)^n\left[\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)\right] = \int\limits_{\mathbb{R}^n}\left(-\sum_{j=1}^{n}\xi^2 + |\xi|^2\right)e^{-yG_1(\xi)}\cos[x\cdot\xi - yG_2(\xi)]d\xi$$

$$+\int_{\mathbb{R}^n} e^{-yG_1(\xi)}\Big(a(\xi)\cos[x\cdot\xi - yG_2(\xi)] - b(\xi)\sin[x\cdot\xi - yG_2(\xi)]\Big)d\xi$$

$$= \int_{\mathbb{R}^n} e^{-yG_1(\xi)}\sum_{k=1}^{m} a_k\Big(\cos h_k\cdot\xi\cos[x\cdot\xi - yG_2(\xi)] - \sin h_k\cdot\xi\sin[x\cdot\xi - yG_2(\xi)]\Big)d\xi$$

$$= \int_{\mathbb{R}^n} e^{-yG_1(\xi)}\sum_{k=1}^{m} a_k\cos\Big[\cos(x+h_k)\cdot\xi - yG_2(\xi)\Big]d\xi = \sum_{k=1}^{m} a_k\int_{\mathbb{R}^n} e^{-yG_1(\xi)}\cos\Big[\cos(x+h_k)\cdot\xi - yG_2(\xi)\Big]d\xi$$

$$= 2\pi\sum_{k=1}^{m} a_k\mathcal{E}(x+h_k, y).$$

Thus, function (5) satisfies Equation (1) in the half-space $\mathbb{R}^n \times (0,\infty)$.

Now, we have to prove that function (4) satisfies Equation (1) in the same half-space and that it can be differentiated inside the integral. To do that, we estimate the function $\mathcal{E}$ and its derivatives as follows.

First, we estimate the function $|\xi|^2 + a(\xi) = |\xi|^2 + a_1\cos h_1\cdot\xi + \cdots + a_m\cos h_m\cdot\xi$ from below.

If $h_k\cdot\xi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $k \in \overline{1,m}$, then $h_k\cdot\xi > 0$. Hence, $|\xi|^2 + a(\xi) \geq |\xi|^2$ provided that $|h_k||\xi| < \frac{\pi}{2}$ for each $k \in \overline{1,m}$.

Thus, in the ball $\left\{|\xi| < \frac{\pi}{2\max\limits_{k\in\overline{1,m}}|h_k|} = \frac{\pi}{2h_0}\right\}$, the function $|\xi|^2 + a(\xi)$ is bounded from below by the function $|\xi|^2$.

Outside this ball, the following estimate holds:

$$|\xi|^2 + a(\xi) \geq \frac{\pi^2}{4h_0^2} - \sum_{k=1}^{m} a_k = \frac{\pi^2}{4h_0^2} - a_0 > 0$$

by virtue of Condition (2).

Therefore, the function $|\xi|^2 + a(\xi)$ is nonnegative, which means that function (9) is estimated from below by the function $\frac{1}{\sqrt{2}}\sqrt{|\xi|^2 + a(\xi)}$. We use the estimate $G_1(\xi) \geq \frac{|\xi|}{\sqrt{2}}$ inside the ball $\left\{|\xi| < \frac{\pi}{2h_0}\right\}$ and the estimate $G_1(\xi) \geq \sqrt{|\xi|^2 - a_0}$ outside the ball $\{|\xi| < a_0\}$. By virtue of Condition (2), $a_0 < \frac{\pi}{2h_0}$, and therefore, function (5) satisfies the estimate

$$(2\pi)^n|\mathcal{E}(x,y)| \leq \int\limits_{\left\{|\xi| < \frac{\pi}{2h_0}\right\}} e^{-\frac{y}{\sqrt{2}}|\xi|}d\xi + \int\limits_{\left\{|\xi| > \frac{\pi}{2h_0}\right\}} e^{-y\sqrt{|\xi|^2 - a_0}}d\xi = \int_0^{\frac{\pi}{2h_0}} r^{n-1}e^{-\frac{y}{\sqrt{2}}r}dr + \int_{\frac{\pi}{2h_0}}^{\infty} r^{n-1}e^{-y\sqrt{r^2 - a_0}}dr.$$

The first term of the last sum is estimated from above by the following expression:

$$\int_0^{\infty} r^{n-1}e^{-\frac{y}{\sqrt{2}}r}dr = \frac{2^{\frac{n}{2}}}{y^n}\int_0^{\infty}\rho^{n-1}e^{-\rho}d\rho = \frac{2^{\frac{n}{2}}(n-1)!}{y^n}.$$

The second one is equal to

$$\frac{1}{2}\int\limits_{\frac{\pi^2}{4h_0^2}-a_0}^{\infty}(\tau+a_0)^{\frac{n}{2}-1}e^{-y\sqrt{\tau}}d\tau \leq \frac{1}{2}\int\limits_{\frac{\pi^2}{4h_0^2}-a_0}^{\infty}\left(\tau+\frac{\tau}{C}\right)^{\frac{n}{2}-1}e^{-y\sqrt{\tau}}d\tau = \frac{C+1}{2C}\int\limits_{\frac{\pi^2}{4h_0^2}-a_0}^{\infty}\tau^{\frac{n}{2}-1}e^{-y\sqrt{\tau}}d\tau,$$

where $C = \dfrac{\frac{\pi^2}{4h_0^2}-a_0}{a_0}$.

The last integral is estimated from above by the integral

$$\int\limits_0^{\infty}\tau^{\frac{n}{2}-1}e^{-y\sqrt{\tau}}d\tau = \frac{2}{y^2}\int\limits_0^{\infty}\left(\frac{\rho^2}{y^2}\right)^{\frac{n}{2}-1}e^{-\rho}\rho d\rho = \frac{2}{y^n}\int\limits_0^{\infty}\rho^{n-1}e^{-\rho}d\rho = \frac{2(n-1)!}{y^n}.$$

Thus, $|\mathcal{E}(x,y)|$ is estimated from above by the function $\frac{\text{const}}{y^n}$, which means that integral (4) absolutely converges for each positive y and that the function u defined by it satisfies the following estimate in $\mathbb{R}^n \times (0,\infty)$:

$$|u(x,y)| \leq \frac{\text{const}\|u_0\|_1}{y^n}. \tag{10}$$

Differentiating the function $\mathcal{E}(x,y)$ with respect to each its independent variable, we add one more regular integrand factor that does not increase faster than $|\xi|$. No finite amount of such factors affect the convergence of the integral, while the right-hand side of estimate (10) is affected as follows:

$$|D^l u(x,y)| \leq \frac{\text{const}\|u_0\|_1}{y^{n+l}}, \tag{11}$$

where l is an arbitrary positive integer and the left-hand side denotes an arbitrary partial derivative of order l of the function $u(x,y)$.

Therefore, the integral obtained after the formal differentiating of integral (4) with respect to each variable absolutely converges in $\mathbb{R}^n \times (0,\infty)$. Combining this fact with the fact that the function $\mathcal{E}(x,y)$ satisfies Equation (1) in $\mathbb{R}^n \times (0,\infty)$, we obtain that (4) is an infinitely smooth solution of Equation (1) in $\mathbb{R}^n \times (0,\infty)$. □

3. Operational Scheme

In this section, we show the way to find the Poisson-like kernel $\mathcal{E}(x,y)$. We apply the well-known Gel'fand–Shilov operational scheme (see, e.g., [30] (Sec. 10)), using the fact that translation operators are Fourier multipliers.

Thus, we (formally) apply the Fourier transformation with respect to the (n-dimensional) variable x to problem (1),(3). This operation takes the boundary-value problem for a *partial functional-differential* equation to an initial-value problem for an *ordinary differential* equation, i.e., to the problem:

$$\frac{d^2\widehat{u}}{dy^2} = \left(|\xi|^2 + \sum_{k=1}^{m} a_k\cos h_k\cdot\xi + i\sum_{k=1}^{m} a_k\sin h_k\cdot\xi\right)\widehat{u},\ y\in(0,+\infty), \tag{12}$$

$$\widehat{u}(0;\xi) = \widehat{u_0}(\xi). \tag{13}$$

The characteristic equation of Equation (12), which is a linear ordinary second-order differential equation with constant coefficients depending on the n-dimensional parameter ξ, is equal to $\pm\rho(\xi)[\cos\theta(\xi) + i\sin\theta(\xi)]$, where $\rho(\xi)$ and $\theta(\xi)$ are defined by relations (7) and (8), respectively. We solve problem (12) and (13), suitably select the value of the "free" arbitrary constant (it exists because the amount of boundary-value conditions is less than the order of the equation), and (formally) apply the inverse Fourier transformation to the obtained solution. This yields:

$$\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} e^{ix\cdot\xi-y\rho(\xi)[\cos\theta(\xi)+i\sin\theta(\xi)]}\int\limits_{\mathbb{R}^n} u_0(z)e^{iz\cdot\xi}dzd\xi=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} u_0(z)\int\limits_{\mathbb{R}^n} e^{i(x-z)\cdot\xi-y\rho(\xi)[\cos\theta(\xi)+i\sin\theta(\xi)]}d\xi dz$$

$$=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} u_0(z)\int\limits_{\mathbb{R}^n} e^{i[(x-z)\cdot\xi-y\rho(\xi)\sin\theta(\xi)]}e^{-y\rho(\xi)\cos\theta(\xi)}d\xi dz$$

$$=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} u_0(z)\int\limits_{\mathbb{R}^n} \cos[(x-z)\cdot\xi-y\rho(\xi)\sin\theta(\xi)]e^{-y\rho(\xi)\cos\theta(\xi)}d\xi dz$$

$$+\frac{i}{(2\pi)^n}\int\limits_{\mathbb{R}^n} u_0(z)\int\limits_{\mathbb{R}^n} \sin[(x-z)\cdot\xi-y\rho(\xi)\sin\theta(\xi)]e^{-y\rho(\xi)\cos\theta(\xi)}d\xi dz.$$

Taking into account the oddness of the function $b(\xi)$ with respect to each variable ξ_j, we obtain function (4).

Note that all actions undertaken in this section do not constitute a proof: we change the order of the integrating, apply the direct and inverse Fourier transformations, and nullify integrals of odd functions over symmetric regions, but we do not care about the convergence of the corresponding integrals. Thus, the function $u(x,y)$ obtained at this step is obtained *heuristically* (in the total correspondence of the specified Gel'fand–Shilov scheme). Once this function is obtained, we have to prove that it is well-defined, can be differentiated inside the integral, and satisfies the investigated equation. This strict proof is provided in Section 2 (see Theorem 1).

Remark 1. *By construction, the obtained solution of Equation* (1) *satisfies Condition* (3) *in the sense of generalized functions* (*according to the Gel'fand–Shilov definition, i.e.,* $u(\cdot,y)\to u_0(\cdot)$ *in the topology of generalized functions of the* n*-dimensional variable* x *as the real parameter* y *tends the zero from the right*)*. The proof is totally the same as in [31] (Remark 2).*

Combining this remark with estimate (11), we obtain the following main assertion of the paper.

Theorem 2. *If* $u_0\in L_1(\mathbb{R}^n)$ *and Condition* (2) *is satisfied, then function* (4) *satisfies problem* (1),(3) *in the sense of generalized functions. This solution is infinitely smooth in the open half-space* $\mathbb{R}^n\times(0,\infty)$ *and satisfies* (*together with all its derivatives*) *estimate* (11) *in the specified half-space, where* l *is an arbitrary positive integer and the constant depends only on* n, l, a_0, *and* h_0.

4. Conclusions

In this paper, we continue the investigation of half-space boundary-value problems for differential-difference elliptic equations with nonlocal potentials, extending the consideration to the most general case of the equation: the amount of the nonlocal potentials is arbitrary, no commensurability requirements are imposed on the coefficients at the potentials, and the directions of the translations of the potentials (and, therefore, the angles between them) are arbitrary. We construct a solution, express it by a Poisson-like integral representation, prove its infinite smoothness outside the boundary hyperplane, and show that the following general phenomenon (common for a quite broad class of half-space elliptic and parabolic problems) takes place in the considered case as well: if the boundary-value function is summable, then the constructed solution uniformly decays (with all its partial derivatives with respect to all independent variables) as the timelike independent variable tends to infinity. The rate of this decay is estimated by the power function of the timelike variable.

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